Universidad Michoacana de San Nicolás de Hidalgo

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Off-shell four photon amplitudes in the worldline formalism

Thesis

to obtain the title of

Master of Sciences in the Area of Physics

presents

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Contents

Resumen

En esta tesis obtenemos el resultado exacto para la amplitud cuatro-fotón a un-lazo fuera de la capa de masa en D dimensiones y con uno, dos y cuatro fotones en el limite de bajas energías en electrodinámica cuántica escalar y espinorial. Para este proposito, usamos las técnicas y metodos del formalismo línea de mundo. Además, verificamos nuestros resultados al calcular una corrección del propagador del fotón en un campo externo y de la función-β a dos-lazos.

Palabras clave: Electrodinámica cuántica, un-lazo, escalar, espinorial, hipergeométrica.

Abstract

In this thesis, we obtain the exact result for the one-loop off-shell four-photon amplitude in D dimension and with one, two and four photons in the low energy limit in both, scalar and spinor quantum electrodynamics. For this purpose, we use the techniques and methods of the worldline formalism. Furthermore, we test our results by computing a correction to the photon propagator in an external field and the two-loop β-function.

Key words: Quantum electrodynamics, one-loop, scalar, spinor, hypergeometric.

Introduction

The study of light has been of big interest in physics, because the study of this, in the long term, could help us to understand in a more precise way the behavior, the structure and basic properties of matter. Classically, the description of light propagation is given by Maxwell equations that are a set of linear equations both in the sources and in the fields. For this reason, Maxwell equations do not predict photon-photon interactions since it is a nonlinear effect and a purely quantum-mechanical process.

The photon-photon interaction is possible due to the production of virtual electron-positron pairs, for example Delbrück scattering (1933) or the elastic scattering of a photon by the Coulomb field of nuclei is one of the first nonlinear effects that has been observed experimentally [1, 2, 3, 4]. This process is shown in Figure 1, diagrammatically, where the crossed lines in the diagram mean that there is an external interaction in the corresponding leg; in the case of Delbrück scattering this interaction is with a Coulomb field. This diagram also can be associated to the birefringence, but in this case the interaction is with a magnetic field, since birefringence refers to the change in the polarization of photons when they travel through a region of intense magnetic field. This effect was first pointed out by Adler in 1971 [5]. Unfortunately, as to date, there is no experimental observation of this effect even with the big effort that has been done to observe it, for example, the PVLAS (Polarisation of Vacuum with LASer) experiment in Ferrara [6, 7, 8, 9, 10] that dates back to the end of the '70s and now is in its fourth generation. However, photon splitting or the process where a photon splits into two photons after the interaction with an external field [11], is another good example of an observed nonlinear effect [12].

In this work, we focus in the photon-photon scattering, particularly

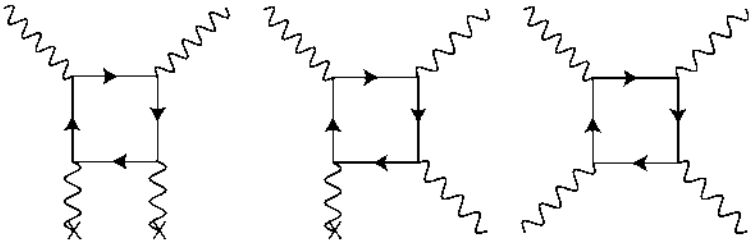

Figure 1: *From left to right, Delbrück scattering or birefringence, photon splitting and light by light scattering.*

in the interaction between four photons mediated by an electron-positron loop, also known as light by light scattering. We analyze in more detail this effect in Chapter 1 starting with the Euler-Heisenberg Lagrangian and ending with a short review of the most recent experiments related to this process. In Chapter 2, we present a brief explanation of the formalism we implement to carry out our calculations, the "worldline formalism". In Chapter 3, we present the integral form of the one-loop four-photon amplitude for none, one, two and four legs in the low energy limit for scalar/spinor quantum electrodynamics (QED). In Chapter 4, we present the analytical result for the amplitude with one photon in the low energy limit and we connect it with some theoretical results. In Chapter 5, we present the explicit result for the amplitude with two photons in the low energy limit and we verify this result by comparing with the photon propagator in an external constant field and with the corresponding two-loop corrections to the β-functions of scalar and spinor QED. In Chapter 6, we present the result for the amplitude with all photons in the low energy limit. In Chapter 7 we present some conclusions. In the appendices we present some conventions, formulas and a lot of details of our calculations that were too lengthy to present in the main text.

Chapter 1

History of the four photon amplitude

In 1936, Euler and Heisenberg inspired by Dirac's theory of the positron computed the nonperturbative, renormalized, one-loop effective Lagrangian for quantum electrodynamics in a classical electromagnetic background of constant field strength for spinor QED [13]

$$L_{spin} = -\frac{1}{hc}\int_0^\infty \frac{d\eta}{\eta^3} e^{-\eta e\varepsilon_c}\left\{\frac{e^2 ab\eta^2}{\tanh(eb\eta)\tan(ea\eta)} - 1 - \frac{e^2\eta^2}{3}(b^2-a^2)\right\}. \tag{1.1}$$

where ε_c is the critical field strength

$$\varepsilon_c = \frac{m^2c^3}{e\hbar} \tag{1.2}$$

and

$$a = \sqrt{\sqrt{\mathcal{F}^2+\mathcal{G}^2}-\mathcal{F}}, \qquad b = \sqrt{\sqrt{\mathcal{F}^2+\mathcal{G}^2}+\mathcal{F}}, \tag{1.3}$$

with $\mathcal{F}$ and $\mathcal{G}$ two Lorentz invariants

$$\mathcal{F} = -\frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right), \qquad \mathcal{G} = -\left(\vec{E}\cdot\vec{B}\right). \tag{1.4}$$

This effective Lagrangian, equation (1.1), generates all the perturbative diagrams in Figure 1.1, only when all external photon lines are in the low

energy limit, since in this limit, the background field strength tensor $F_{\mu\nu}$ can be taken to be constant. This allowed Euler and Heisenberg to write (1.1) in a relatively simple closed form. Nowadays we know this expression

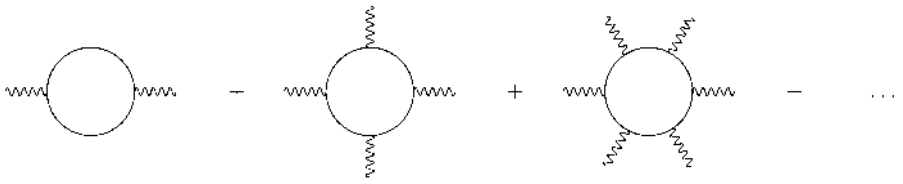

Figure 1.1: *Perturbative expansion of the Euler-Heisenberg effective Lagrangian in terms of Feynman diagrams.*

as the Euler-Heisenberg Lagrangian.

In the same year, Weisskopf computed the analogous Lagrangian for scalar QED [14]

$$L_{scal} = \frac{1}{2hc}\int_0^\infty \frac{d\eta}{\eta^3} e^{-\eta e\varepsilon_c}\left\{\frac{e^2 ab\eta^2}{\sinh(eb\eta)\sin(ea\eta)} - 1 + \frac{e^2\eta^2}{6}(b^2-a^2)\right\}. \tag{1.5}$$

A detailed review of Euler-Heisenberg Lagrangian and its applications can be found in [15]. Furthermore, one consequence that they found from Dirac's theory and from this Lagrangian is the possibility of light by light scattering. In Figure 1.1 the second diagram, left to right, corresponds to the first contribution to photon-photon scattering. This diagram is, schematically, the process where four photons interact mediated by a virtual electron positron loop. The cross section for this process with the four external photons in the low energy limit was first computed by Euler [16] in the same year, 1936. After them, Akhiezer (1937) computed the same cross section but considering all photons in the high energy limit [17]. A good review of both cases can be found in [18, 19].

Later, in 1951, Karplus and Neuman obtained the analytical result for this cross section with an arbitrary energy in each photon [20]. After them, in 1971 Costantini, De Tollis, and Pistoni calculated the four-photon scattering amplitude but now with the assumption that two of the four photons were off-shell and the other two on-shell [21].

Nevertheless, the propagation of light in an external magnetic field has been studied in [22, 23] and in the presence of an external electromagnetic field in [24, 25]. Also in [26] it is obtained an integral representation for the dimensionally regularized one-loop vacuum polarization tensor for scalar and spinor QED. In [18], there is a detailed review of such propagator in a constant electromagnetic field and some physical applications like photon-splitting, birefringence, speed of light in presence of an electromagnetic or gravitational background, light by light scattering, pair production and some others.

In the context of photon amplitudes, it was possible to compute a closed expression for the one-loop N-photon amplitude in scalar and spinor QED [27] (Bern-Kosower master formula, 1992), where they used the methods and techniques of the worldline formalism to obtain this result. A few years later, in 1997, from this master formula, Schubert obtained an integral representation for the scalar/spinor QED, four-photon amplitude totally off-shell [28]. Moreover, the only case with more than 4 photon and with arbitrary energy is the on-shell six-photon amplitude in scalar QED obtained in [29] for the massless case. However, in [30] it was possible to compute the explicit result for the on-shell N-photon amplitude with all external legs in the low energy limit.

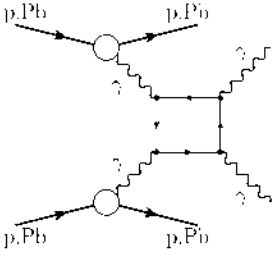

Figure 1.2: *Schematic diagram of elastic photon-photon collisions. Adapted from [35].*

In the experimental side, many laser experiments have been performed at low energies and unfortunately this effect of light by light scattering remains still unobservable in laser experiments; some examples of this are [31, 32, 33]. Even though, in 2017, ATLAS collaboration, finally, observed

light by light scattering in a high energy experiment [34], in the Large Hadron Collider (LHC), where they followed the approach proposed in [35]. In Figure 1.2, it is shown, schematically, the process in which photon-photon scattering ($\gamma\gamma \to \gamma\gamma$) occurs. The experiment is carried out in the following way; beams of lead (Pb) nuclei are accelerated to velocities close to the speed of light c, in the LHC and then a Pb nuclei beam collide with other Pb nuclei beam that is running in the opposite direction. In this interaction, a lot of processes can happen, in particular the one that is shown in Figure 1.2, where each of the two Pb nuclei emit a photon of high energy with a coherent initial state and then these two photons interact mediated by an electron-positron loop. Finally, the data that signals of the possible diphoton events is analyzed and compared with a Monte Carlo (MC) simulation. In Figure 1.3 it is shown the experimental data compared with the Monte Carlo simulation. Here the red line is for photon-photon scattering ($\gamma\gamma \to \gamma\gamma$), blue bars are for dielecton production ($\gamma\gamma \to e^+e^-$) and the gray bars refer to some other backgrounds that could mimic the diphoton event signatures (central exclusive production, CEP), for example the production of quark-antiquark pairs.

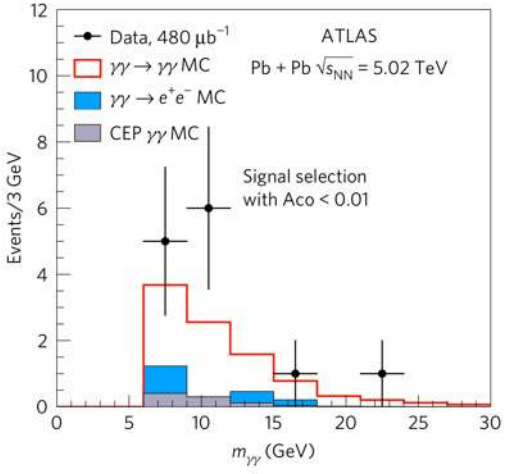

Figure 1.3: *Kinematic distributions for $\gamma\gamma \to \gamma\gamma$ event candidates. Adapted from [34].*

Laser experiments are in the region of low energies and in this region, the cross section is extremely small ($\sigma_{\gamma\gamma} \approx 0.7 \times 10^{-65} cm^2$, [32]). This is the main reason why it is very difficult to observe this process in a laser experiment. Therefore, new proposals of laser experiments have been offered in the last few years, for example [36, 37], where one of the proposed

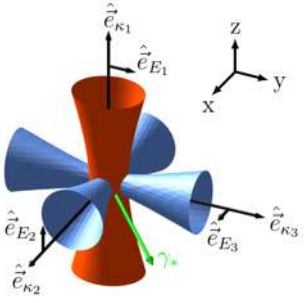

Figure 1.4: *Schematic light by light interaction in a laser experiment. Adapted from [37].*

examples consist in a triple collision of high-intensity laser beams (Figure 1.4), with the propagation directions $\hat{\vec{e}}_{\kappa_i}$, polarization vectors $\hat{\vec{e}}_{E_i}$ and two of them with the same wavelength, giving as a result a signal of dispersed photons (green arrow in Figure 1.4, $\gamma*$). The difficulty here is that laser of such intensity are still in development.

Moreover, as to date, there is no result for the one-loop four-photon amplitude totally off-shell. For this reason, the purpose of the present work is to give a first step in this direction by computing the analytical result for the one-loop four-photon amplitude totally off-shell and with one of its four external photon lines in the low energy limit. Then, in order to compute this amplitude, we start from the representation given in [28] and after some mathematical manipulations, we are able to give the exact result for the desired amplitude in terms of derivatives of known integrals [41]. Further, we are able to compute a more general and explicit expression for the four-photon amplitude with two legs in the low energy limit in terms of the hypergeometric function ${}_2F_1$. **These are new results.** However, we use the worldline formalism to compute such amplitudes that offers many advantages in the calculation and also we test our results by comparing

them with particular cases already known like the photon propagator in an external field and the two-loop β-function.

Chapter 2

Worldline formalism

It was a long time ago (around 30's) when people like Dirac, Klein, Gordon and others were constructing theories in order to understand the propagation of relativistic particles. They used to write differential equations to describe these theories; an example of this is the Schrödinger equation. However, people used to construct those theories based on differential equations and the "canonical quantization", where the first quantization of a classical theory is usually for single particles and the second quantization is usually considered when the number of particles is not constant.

Then, in the 40's Feynman realized that working with differential equations was too hard in the relativistic case and also he realized that there is another way to obtain the same results but with an easier method, path integral method. Feynman developed this integral representation of the relativistic quantum theory focusing on integrals over the fields of the theory, but also he realized that it was possible to work with the particle path integral, and he did not consider that this particle path integral was easier than the path integral over fields. Thus, he and many others after him preferred to work with fields.

Many years later, in the 90's Bern, Kosower and Strassler [38, 39] recognized the advantages of this particle path integral or like it is known now "worldline formalism" or "string-inspired formalism". It is called string-inspired formalism because its construction, the one that Bern, Kosower and Strassler developed, has its origins in string theory, but we do not refer to it anymore because this formalism is not really part of string theory. In

this chapter, we present a short review of the worldline formalism and we focus on the case of the one-loop effective action in QED and the associated photon amplitudes.

2.1 Scalar QED

The one-loop effective action for scalar QED in the worldline repreesentation is given by

$$\Gamma[A] = \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_{x(T)=x(0)} \mathcal{D}x \; e^{-\int_0^T d\tau \left(\frac{\dot{x}^2}{4} + ie\dot{x}\cdot A(x)\right)}, \tag{2.1}$$

where A is an arbitrary background field, m is the electron mass, e is the magnitude of electric charge carried by a single electron and $x = x(\tau)$. The symbol $\mathcal{D}x$ in (2.1); tell us that this integral is over all possible trajectories that satisfy the boundary condition $x(T) = x(0)$, in fact, an infinite number of trajectories. Our sign convention for the Minkowski metric is $\eta^{\mu\nu} = (-+++)$ but in the following, we work in the Euclidean space, $\eta^{\mu\nu} = (++++)$.

However, we can work with (2.1) directly as the effective action or also we can fix the field A to work with an amplitude either in position space or in momentum space. If we chose to work with the effective action, what we usually do is the following: We perform a Taylor expansion on the field A. Then, we perform another expansion, but in the exponential order by order respect to the field A and its derivatives. Finally, we see that it is possible to write these expansions in terms of the field strength tensor $F^{\mu\nu}$ and its derivatives. In this work, we are more interested in the amplitude, more precisely in those of photons. For this reason, we choose the background field to be a sum of plane waves

$$A_\mu(x) = \sum_{i=1}^{N} \varepsilon_{i\mu} \; e^{ik_i\cdot x}, \tag{2.2}$$

with definite polarization ε_i and momentum k_i for each photon. Now, we expand the exponential in (2.1) and we only take the terms multilinear in ε_i of this expansion, because those are the only terms that contribute to the one-loop N-photon amplitude. Then we exponentiate the remaining

integrals using the following trick

$$\int_0^T d\tau_i\ \varepsilon_i \cdot \dot{x} e^{ik_i \cdot x} = \int_0^T d\tau_i\ e^{ik_i \cdot x + \varepsilon_i \cdot \dot{x}} \Big|_{lin\ \varepsilon_i}, \tag{2.3}$$

in such way that it can be possible to write the amplitude as a Gaussian integral in the variable "x". Consequently, we perform this integral and it leaves the amplitude in momentum space (for more details, see Appendix C)

$$\Gamma[k_1, \varepsilon_1; ...; k_N, \varepsilon_N] = (-ie)^N (2\pi)^D \delta\left(\sum_{i=1}^N k_i\right) \int_0^\infty \frac{dT}{T} (4\pi T)^{-\frac{D}{2}} e^{-m^2 T}$$
$$\times \prod_{k=1}^N \int_0^T d\tau_k \exp\left[\sum_{i,j=1}^N \left(\frac{1}{2} G_{Bij} k_i \cdot k_j - i\dot{G}_{Bij} \varepsilon_i \cdot k_j + \frac{1}{2} \ddot{G}_{Bij} \varepsilon_i \cdot \varepsilon_j\right)\right]\Bigg|_{lin\ \varepsilon_i}, \tag{2.4}$$

where $G_{Bij} = G_B(\tau_i - \tau_j)$ is the bosonic Green's function of the operator $\left(\frac{-1}{4}\frac{d^2}{d\tau^2}\right)$ with periodic boundary conditions and $\dot{G}_{Bij}$, $\ddot{G}_{Bij}$ its first and second derivatives respect to τ_i

$$\langle \tau_i | \left(\frac{-1}{4}\frac{d^2}{d\tau^2}\right)^{-1} | \tau_j \rangle = -2G_B(\tau_i - \tau_j), \tag{2.5}$$

$$G_B(\tau_i - \tau_j) = |\tau_i - \tau_j| - \frac{(\tau_i - \tau_j)^2}{T} - \frac{T}{6}, \tag{2.6}$$

$$\begin{aligned} \dot{G}_B(\tau_i - \tau_j) &= \text{sign}(\tau_i - \tau_j) - 2\frac{(\tau_i - \tau_j)}{T}, \\ \ddot{G}_B(\tau_i - \tau_j) &= 2\delta(\tau_i - \tau_j) - \frac{2}{T}. \end{aligned} \tag{2.7}$$

The equation (2.4) is also known as the *Bern-Kosower master formula*. Since no assumptions have been made on the momenta k_i or polarizations ε_i, this master formula gives us an off-shell amplitude. Hence, we can manipulate our results to obtain corrections of higher order (multi-loop). For reasons of symmetry, we must take all photons to be incoming or

outgoing. In this work, we take all as ingoing. Thus the energy-momentum conservation law is written as

$$\sum_{i=1}^{N} k_i = 0. \tag{2.8}$$

Furthermore, if we look carefully at the equation (2.4), we can see that it is possible to write the exponential in the integral as

$$P_N(\dot{G}_{Bij}, \ddot{G}_{Bij}) \exp\left[\frac{1}{2}\sum_{i,j=1}^{N} G_{Bij} k_i \cdot k_j\right], \tag{2.9}$$

since (2.4) is linear in ε_i. Here P_N is a polynomial of $\dot{G}_{Bij}$ and $\ddot{G}_{Bij}$ and it can be turned into another polynomial Q_N depending only on the $\dot{G}_{Bij}$'s through an integration by parts (IBP) (see [38, 39, 40])

$$Q_N(\dot{G}_{Bij}) \exp\left[\frac{1}{2}\sum_{i,j=1}^{N} G_{Bij} k_i \cdot k_j\right]. \tag{2.10}$$

As a result of this IBP, one finds that certain combinations of the k_i's and ε_i's will appear in Q_N as "Lorentz cycles" Z_n, [39]

$$\begin{aligned} Z_2(ij) &\equiv \varepsilon_i \cdot k_j \varepsilon_j \cdot k_i - \varepsilon_i \cdot \varepsilon_j k_i \cdot k_j, \\ Z_n(i_1 i_2 ... i_n) &\equiv \mathrm{tr} \prod_{j=1}^{n} \left(k_{i_j} \otimes \varepsilon_{i_j} - \varepsilon_{i_j} \otimes k_{i_j}\right), \quad (n \geq 3), \end{aligned} \tag{2.11}$$

that also can be written in terms of the corresponding field strength tensor f_i, defined by

$$f_i^{\mu\nu} \equiv k_i^\mu \varepsilon_i^\nu - \varepsilon_i^\mu k_i^\nu, \tag{2.12}$$

as

$$\begin{aligned} Z_2(ij) &= \frac{1}{2}\mathrm{tr}\left(f_i f_j\right), \\ Z_n(i_1 i_2 ... i_n) &= \mathrm{tr}\left(f_{i_1} f_{i_2} \cdots f_{i_n}\right), \quad (n \geq 3). \end{aligned} \tag{2.13}$$

Notice that $f_i^{\mu\nu}$ is not the same as $F^{\mu\nu}$, in fact

$$F^{\mu\nu} = i \sum_{i=1}^{N} f_i^{\mu\nu} \, e^{ik_i \cdot x}, \tag{2.14}$$

since we have defined $f_i^{\mu\nu}$ as the field strength tensor for each photon. It is easy to check (2.14) with the definition of $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and (2.2).

2.2 Spinor QED

For spinor QED, the one-loop effective action can be written in a similar way as in (2.1) but at this point and for this thesis, we do not need to work directly with this spinor effective action because we can use the result above in (2.4) to reach the spinor amplitude via a "replacement rule" (see [38, 39, 40]). In order to perform this replacement rule, we only need to replace every τ-cycle appearing in Q_N in the following way

$$\begin{aligned} &Q_N \text{ scalar} \qquad \longrightarrow \qquad Q_N \text{ spinor} \\ &\dot{G}_{Bi_1i_2}\dot{G}_{Bi_2i_3}\cdots\dot{G}_{Bi_ki_1} \longrightarrow \dot{G}_{Bi_1i_2}\dot{G}_{Bi_2i_3}\cdots\dot{G}_{Bi_ki_1} - \dot{G}_{Fi_1i_2}\dot{G}_{Fi_2i_3}\cdots\dot{G}_{Fi_ki_1}, \end{aligned}$$

where

$$G_{Fij} = G_F(\tau_i, \tau_j) \equiv \langle \tau_i | \left(\frac{1}{2} \frac{d}{d\tau} \right)^{-1} | \tau_j \rangle = \text{sign}(\tau_i - \tau_j), \tag{2.15}$$

is the fermionic Green function of the operator $\left(\frac{1}{2}\frac{d}{d\tau}\right)$ with anti-periodic boundary conditions. Further, we need to multiply the scalar amplitude by an overall factor of "-2". This rule of replacement is related to the worldline supersymmetry.

2.3 Advantages of worldline formalism

There are some advantages of the worldline formalism compared with the standard Feynman calculations.

- It is a first quantized approach to quantum field theory.

- We can compute amplitudes in QED, QCD and quantum gravity.
- We do not need to calculate any trace of gamma matrices in a one-loop calculation. This is possible because of the "replacement rule".
- We can write the effective action or the amplitude in terms of the field strength tensor making the gauge invariance manifest.
- We do not need to compute any momentum integral over the loop.
- We do not need to fix any order to perform the integrals (for QED).
- In the specific case of the four-photon amplitudes, the Feynman diagram calculation suffers from spurious UV divergences that in the worldline formalism are eliminated by the standard integration-by-parts procedure.
- We do not need to look at different diagrams of the same kind to construct the amplitude. As we have already seen, there is a master formula that gives us the amplitude; we only need to fix the number of external legs and perform the integrals.

Chapter 3

Integral representation of the four photon amplitude

The four photon amplitude can be obtained from the Bern-Kosower master formula Eq. (2.4), by taking $N = 4$, this looks like

$$\Gamma_{\text{scal}}[k_1, \varepsilon_1; ...; k_4, \varepsilon_4] = \frac{e^4}{(4\pi)^{\frac{D}{2}}} \int_0^\infty \frac{dT}{T} T^{4-\frac{D}{2}} e^{-m^2 T} I(Q_4), \tag{3.1}$$

with

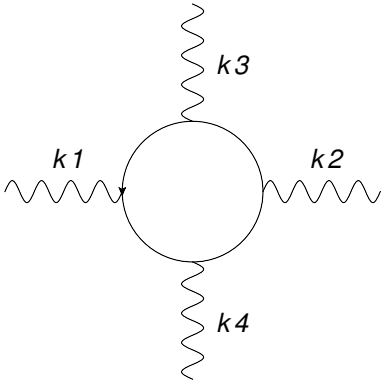

Figure 3.1: *Feynman diagram for the four-photon amplitude.*

$$I(Q_4) = \int_0^1 \int_0^1 \int_0^1 \int_0^1 du_1 du_2 du_3 du_4 Q_4 e^{\frac{T}{2}\sum_{i,j=1}^4 G_{Bij} k_i \cdot k_j}, \tag{3.2}$$

for scalar QED, and

$$\Gamma_{\text{spin}}[k_1, \varepsilon_1; ...; k_4, \varepsilon_4] = \frac{-2e^4}{(4\pi)^{\frac{D}{2}}} \int_0^\infty \frac{dT}{T} T^{4-\frac{D}{2}} e^{-m^2 T} \tilde{I}(\tilde{Q}_4), \tag{3.3}$$

with

$$\tilde{I}(\tilde{Q}_4) = \int_0^1 \int_0^1 \int_0^1 \int_0^1 du_1 du_2 du_3 du_4 \tilde{Q}_4 e^{\frac{T}{2}\sum_{i,j=1}^4 G_{Bij} k_i \cdot k_j}, \tag{3.4}$$

for spinor QED, where the factor "-2" comes from the "replacement rule" described in section 2.2, and the "tilde" helps us to differentiate the terms that correspond to the spinor case. In this expressions, Q_4 comes from the IBP procedure described in Chapter 2, Eq. (2.10). In fact, Q_4 has already been worked out in [40], so we take it from there and we use the "replacement rule" to obtain $\tilde{Q}_4$ from it.

In the following sections, we write the explicit result for Q_4 and $\tilde{Q}_4$. Then, we take the low energy limit in the fourth photon and we integrate over u_4. Successively, we do the same for the other three photons, one by one until we have all photons in the low energy limit and all the integrals over u_i have been performed. We define the low energy limit in a photon as the linearization of the amplitude respect to the momentum k_i. This means, for example, in the case of the fourth photon, that we expand the exponential in k_4. Then, we put this together with Q_4 and from this big expression we only take the linear part in k_4, since Q_4 contains terms that are of order zero, linear and some others of higher order that we do not consider anymore. Hence, sometimes we take the linear term of the exponential when the term in Q_4 is of order zero in k_4. Similarly, when the term in Q_4 is linear, we take the zeroth order term from the exponential and we never take the zeroth order term in both, Q_4 and the exponential, since these terms cancel because of the gauge invariance. We can see this in the integral of (2.3) if we take only the terms up to linear in k_i. After the expansion of the exponential, we will have

$$\int_0^T d\tau_i \dot{x} \cdot \varepsilon_i e^{ik_i \cdot x} = \int_0^T d\tau_i \dot{x} \cdot \varepsilon_i (1 + ik_i \cdot x), \tag{3.5}$$

where we can see that the constant term in the previous integral is a total derivative and it is canceled because of the boundary conditions, $x(T) = x(0)$. In the remaining integral, if we perform an integration by parts, it can be rewritten as

$$\int_0^T d\tau_i \dot{x} \cdot \varepsilon_i k_i \cdot x = -\frac{1}{2} \int_0^T d\tau_i \dot{x} \cdot f_i \cdot x. \tag{3.6}$$

It means that, when we take the low energy limit in one or more photons, it is possible to write the full amplitude in terms of the field strength tensor f_i, as we will see. This comes from the fact that the master formula (2.4) has been constructed from integrals like those in (2.3). In order to write all the amplitude in terms of f_i's, sometimes, it is necessary to introduce some arbitrary vectors r_j's that must satisfy the condition $r_j \cdot k_i \neq 0$. It is important to mention that each time we take the low energy limit in a photon, the integrals that we need to perform to obtain the amplitude become easier. In fact, if we take the low energy limit in the photon with momentum k_i, we can perform the integral over u_i easily. It is important to mention that, in our case, we can not fix $k_i = 0$, because, if we do it, we lose the amplitude completely. (For more details, see Appendices D and E.)

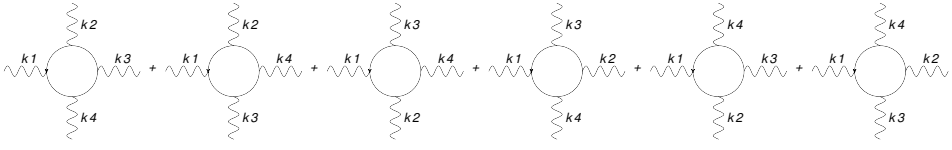

Figure 3.2: *Sum of Feynman diagrams for the four-photon amplitude.*

Now, let's compare with standard Feynman calculations. If we want to compute the four-photon amplitude with the standard procedure, first, we need to construct, using Feynman rules, the integral representation for the amplitude of each diagram in Figure 3.2. Then, we need to perform all these integrals and finally we need to combine all these contributions in a nontrivial way. While, in the worldline formalism we only need to solve the integrals in (3.1) for scalar or (3.3) for spinor amplitude that is much easier than the Feynman procedure, because we have all the information of the diagrams in Figure 3.2 in a single expression.

Moreover, in this chapter we proceed as follow: First, we present the full expressions for Q_4 and $\tilde{Q}_4$. Second, we present the integral form of the four-photon amplitude with one of its four legs in the low energy limit. In this case, Q_4 ($\tilde{Q}_4$) transforms into $I_{(4)}$ ($\tilde{I}_{(4)}$) when we perform the integral over u_4. Here the subscript "(4)" says that the integral is performed with the assumption of low energy in the fourth photon. Furthermore, we present this result in terms of G_{Bij} only, because in this form we can compute the remaining integrals easily. Third, we present the integral form of the four-photon amplitude with two of its legs in the low energy limit. Similarly, $I_{(4)}$ ($\tilde{I}_{(4)}$) transforms into $I_{(34)}$ ($\tilde{I}_{(34)}$) and all is in terms of G_{Bij} also. Finally, we present the integral form of the four-photon amplitude with its four legs in the low energy limit, and now $I_{(34)}$ ($\tilde{I}_{(34)}$) transforms into $I_{(1234)}$ ($\tilde{I}_{(1234)}$). Here and in the following, Q_4 without superscript means the sum of all Q_4's with superscript, and similarly for $\tilde{Q}_4$ and the different I's, $\tilde{I}$'s that we will use.

3.1 Four arbitrary momenta

In this section we present every term that must be integrated in (3.1) and (3.3) to obtain the most general four-photon scalar and spinor amplitude respectively (Figure 3.1). As these integrals are hard to attack, we leave this as a future work.

3.1.1 Scalar Q_4

In order to be precise, we take Q_4 from section 4.8 of [40] and we write the full expression as follows (the sum of all the terms in this section is Q_4)

$$Q_4^4(1234) = \dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B34}\dot{G}_{B41}Z_4(1234), \tag{3.7}$$

$$Q_4^4(1243) = \dot{G}_{B12}\dot{G}_{B24}\dot{G}_{B43}\dot{G}_{B31}Z_4(1243), \tag{3.8}$$

$$Q_4^4(1324) = \dot{G}_{B13}\dot{G}_{B32}\dot{G}_{B24}\dot{G}_{B41}Z_4(1324), \tag{3.9}$$

$$\begin{aligned} Q_4^3(123) = {} & \dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31}Z_3(123)\left(\dot{G}_{B41}\varepsilon_4\cdot k_1 + \dot{G}_{B42}\varepsilon_4\cdot k_2\right) \\ & + \dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31}Z_3(123)\dot{G}_{B43}\varepsilon_4\cdot k_3, \end{aligned} \tag{3.10}$$

$$\begin{aligned} Q_4^3(234) =& \dot{G}_{B23}\dot{G}_{B34}\dot{G}_{B42}Z_3(234)\left(\dot{G}_{B12}\varepsilon_1\cdot k_2+\dot{G}_{B13}\varepsilon_1\cdot k_3\right)\\ &+\dot{G}_{B23}\dot{G}_{B34}\dot{G}_{B42}Z_3(234)\dot{G}_{B14}\varepsilon_1\cdot k_4, \end{aligned} \tag{3.11}$$

$$\begin{aligned} Q_4^3(341) =& \dot{G}_{B34}\dot{G}_{B41}\dot{G}_{B13}Z_3(341)\left(\dot{G}_{B21}\varepsilon_2\cdot k_1+\dot{G}_{B23}\varepsilon_2\cdot k_3\right)\\ &+\dot{G}_{B34}\dot{G}_{B41}\dot{G}_{B13}Z_3(341)\dot{G}_{B24}\varepsilon_2\cdot k_4, \end{aligned} \tag{3.12}$$

$$\begin{aligned} Q_4^3(412) =& \dot{G}_{B41}\dot{G}_{B12}\dot{G}_{B24}Z_3(412)\left(\dot{G}_{B31}\varepsilon_3\cdot k_1+\dot{G}_{B32}\varepsilon_3\cdot k_2\right)\\ &+\dot{G}_{B41}\dot{G}_{B12}\dot{G}_{B24}Z_3(412)\dot{G}_{B34}\varepsilon_3\cdot k_4, \end{aligned} \tag{3.13}$$

$$\begin{aligned} Q_4^2(12) =& \dot{G}_{B12}\dot{G}_{B21}Z_2(12)\\ &\times\Big[\dot{G}_{B31}\varepsilon_3\cdot k_1\dot{G}_{B41}\varepsilon_4\cdot k_1+\dot{G}_{B31}\varepsilon_3\cdot k_1\dot{G}_{B42}\varepsilon_4\cdot k_2\\ &+\dot{G}_{B31}\varepsilon_3\cdot k_1\dot{G}_{B43}\varepsilon_4\cdot k_3+\dot{G}_{B32}\varepsilon_3\cdot k_2\dot{G}_{B41}\varepsilon_4\cdot k_1\\ &+\dot{G}_{B32}\varepsilon_3\cdot k_2\dot{G}_{B42}\varepsilon_4\cdot k_2+\dot{G}_{B32}\varepsilon_3\cdot k_2\dot{G}_{B43}\varepsilon_4\cdot k_3\\ &+\dot{G}_{B34}\varepsilon_3\cdot k_4\dot{G}_{B41}\varepsilon_4\cdot k_1+\dot{G}_{B34}\varepsilon_3\cdot k_4\dot{G}_{B42}\varepsilon_4\cdot k_2\\ &+\frac{1}{2}\dot{G}_{B34}\varepsilon_3\cdot\varepsilon_4\left(\dot{G}_{B31}k_3\cdot k_1-\dot{G}_{B41}k_4\cdot k_1\right)\\ &+\frac{1}{2}\dot{G}_{B34}\varepsilon_3\cdot\varepsilon_4\left(\dot{G}_{B32}k_3\cdot k_2-\dot{G}_{B42}k_4\cdot k_2\right)\Big], \end{aligned} \tag{3.14}$$

$$\begin{aligned} Q_4^2(13) =& \dot{G}_{B13}\dot{G}_{B31}Z_2(13)\\ &\times\Big[\dot{G}_{B21}\varepsilon_2\cdot k_1\dot{G}_{B41}\varepsilon_4\cdot k_1+\dot{G}_{B21}\varepsilon_2\cdot k_1\dot{G}_{B42}\varepsilon_4\cdot k_2\\ &+\dot{G}_{B21}\varepsilon_2\cdot k_1\dot{G}_{B43}\varepsilon_4\cdot k_3+\dot{G}_{B23}\varepsilon_2\cdot k_3\dot{G}_{B41}\varepsilon_4\cdot k_1\\ &+\dot{G}_{B23}\varepsilon_2\cdot k_3\dot{G}_{B42}\varepsilon_4\cdot k_2+\dot{G}_{B23}\varepsilon_2\cdot k_3\dot{G}_{B43}\varepsilon_4\cdot k_3\\ &+\dot{G}_{B24}\varepsilon_2\cdot k_4\dot{G}_{B41}\varepsilon_4\cdot k_1+\dot{G}_{B24}\varepsilon_2\cdot k_4\dot{G}_{B43}\varepsilon_4\cdot k_3\\ &+\frac{1}{2}\dot{G}_{B24}\varepsilon_2\cdot\varepsilon_4\left(\dot{G}_{B21}k_2\cdot k_1-\dot{G}_{B41}k_4\cdot k_1\right)\\ &+\frac{1}{2}\dot{G}_{B24}\varepsilon_2\cdot\varepsilon_4\left(\dot{G}_{B23}k_2\cdot k_3-\dot{G}_{B43}k_4\cdot k_3\right)\Big], \end{aligned} \tag{3.15}$$

$$
\begin{aligned}
Q_4^2(14) =& \dot{G}_{B14}\dot{G}_{B41}Z_2(14) \\
&\times \Big[\dot{G}_{B21}\varepsilon_2\cdot k_1\dot{G}_{B31}\varepsilon_3\cdot k_1 + \dot{G}_{B21}\varepsilon_2\cdot k_1\dot{G}_{B32}\varepsilon_3\cdot k_2 \\
&\quad + \dot{G}_{B21}\varepsilon_2\cdot k_1\dot{G}_{B34}\varepsilon_3\cdot k_4 + \dot{G}_{B23}\varepsilon_2\cdot k_3\dot{G}_{B31}\varepsilon_3\cdot k_1 \\
&\quad + \dot{G}_{B23}\varepsilon_2\cdot k_3\dot{G}_{B34}\varepsilon_3\cdot k_4 + \dot{G}_{B24}\varepsilon_2\cdot k_4\dot{G}_{B31}\varepsilon_3\cdot k_1 \\
&\quad + \dot{G}_{B24}\varepsilon_2\cdot k_4\dot{G}_{B32}\varepsilon_3\cdot k_2 + \dot{G}_{B24}\varepsilon_2\cdot k_4\dot{G}_{B34}\varepsilon_3\cdot k_4 \\
&\quad + \frac{1}{2}\dot{G}_{B23}\varepsilon_2\cdot\varepsilon_3\left(\dot{G}_{B21}k_2\cdot k_1 - \dot{G}_{B31}k_3\cdot k_1\right) \\
&\quad + \frac{1}{2}\dot{G}_{B23}\varepsilon_2\cdot\varepsilon_3\left(\dot{G}_{B24}k_2\cdot k_4 - \dot{G}_{B34}k_3\cdot k_4\right)\Big],
\end{aligned}
\tag{3.16}
$$

$$
\begin{aligned}
Q_4^2(23) =& \dot{G}_{B23}\dot{G}_{B32}Z_2(23) \\
&\times \Big[\dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B41}\varepsilon_4\cdot k_1 + \dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B42}\varepsilon_4\cdot k_2 \\
&\quad + \dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B43}\varepsilon_4\cdot k_3 + \dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B41}\varepsilon_4\cdot k_1 \\
&\quad + \dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B42}\varepsilon_4\cdot k_2 + \dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B43}\varepsilon_4\cdot k_3 \\
&\quad + \dot{G}_{B14}\varepsilon_1\cdot k_4\dot{G}_{B42}\varepsilon_4\cdot k_2 + \dot{G}_{B14}\varepsilon_1\cdot k_4\dot{G}_{B43}\varepsilon_4\cdot k_3 \\
&\quad + \frac{1}{2}\dot{G}_{B14}\varepsilon_1\cdot\varepsilon_4\left(\dot{G}_{B12}k_1\cdot k_2 - \dot{G}_{B42}k_4\cdot k_2\right) \\
&\quad + \frac{1}{2}\dot{G}_{B14}\varepsilon_1\cdot\varepsilon_4\left(\dot{G}_{B13}k_1\cdot k_3 - \dot{G}_{B43}k_4\cdot k_3\right)\Big],
\end{aligned}
\tag{3.17}
$$

$$
\begin{aligned}
Q_4^2(24) =& \dot{G}_{B24}\dot{G}_{B42}Z_2(24) \\
&\times \Big[\dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B31}\varepsilon_3\cdot k_1 + \dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B32}\varepsilon_3\cdot k_2 \\
&\quad + \dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B34}\varepsilon_3\cdot k_4 + \dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B32}\varepsilon_3\cdot k_2 \\
&\quad + \dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B34}\varepsilon_3\cdot k_4 + \dot{G}_{B14}\varepsilon_1\cdot k_4\dot{G}_{B31}\varepsilon_3\cdot k_1 \\
&\quad + \dot{G}_{B14}\varepsilon_1\cdot k_4\dot{G}_{B32}\varepsilon_3\cdot k_2 + \dot{G}_{B14}\varepsilon_1\cdot k_4\dot{G}_{B34}\varepsilon_3\cdot k_4 \\
&\quad + \frac{1}{2}\dot{G}_{B13}\varepsilon_1\cdot\varepsilon_3\left(\dot{G}_{B12}k_1\cdot k_2 - \dot{G}_{B32}k_3\cdot k_2\right) \\
&\quad + \frac{1}{2}\dot{G}_{B13}\varepsilon_1\cdot\varepsilon_3\left(\dot{G}_{B14}k_1\cdot k_4 - \dot{G}_{B34}k_3\cdot k_4\right)\Big],
\end{aligned}
\tag{3.18}
$$

$$
\begin{aligned}
Q_4^2(34) =& \dot{G}_{B34}\dot{G}_{B43}Z_2(34) \\
&\times \Big[\dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B23}\varepsilon_2\cdot k_3+\dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B24}\varepsilon_2\cdot k_4 \\
&+\dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B21}\varepsilon_2\cdot k_1+\dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B23}\varepsilon_2\cdot k_3 \\
&+\dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B24}\varepsilon_2\cdot k_4+\dot{G}_{B14}\varepsilon_1\cdot k_4\dot{G}_{B21}\varepsilon_2\cdot k_1 \\
&+\dot{G}_{B14}\varepsilon_1\cdot k_4\dot{G}_{B23}\varepsilon_2\cdot k_3+\dot{G}_{B14}\varepsilon_1\cdot k_4\dot{G}_{B24}\varepsilon_2\cdot k_4 \\
&+\frac{1}{2}\dot{G}_{B12}\varepsilon_1\cdot\varepsilon_2\left(\dot{G}_{B13}k_1\cdot k_3-\dot{G}_{B23}k_2\cdot k_3\right) \\
&+\frac{1}{2}\dot{G}_{B12}\varepsilon_1\cdot\varepsilon_2\left(\dot{G}_{B14}k_1\cdot k_4-\dot{G}_{B24}k_2\cdot k_4\right)\Big],
\end{aligned}
\tag{3.19}
$$

$$
Q_4^{22}(12,34)=\dot{G}_{B12}\dot{G}_{B21}Z_2(12)\dot{G}_{B34}\dot{G}_{B43}Z_2(34), \tag{3.20}
$$

$$
Q_4^{22}(13,24)=\dot{G}_{B13}\dot{G}_{B31}Z_2(13)\dot{G}_{B24}\dot{G}_{B42}Z_2(24), \tag{3.21}
$$

$$
Q_4^{22}(14,23)=\dot{G}_{B14}\dot{G}_{B41}Z_2(14)\dot{G}_{B23}\dot{G}_{B32}Z_2(23). \tag{3.22}
$$

3.1.2 Spinor $\tilde{Q}_4$

In this section, we use the "replacement rule" described in section 2.2 to obtain $\tilde{Q}_4$ from Q_4, and also we use the equations (B.2), (B.3) and (B.4) to transform any combination of G_{Fij} into some other in terms of $\dot{G}_{Bij}$. Then, we have (the sum of all the terms in this section is $\tilde{Q}_4$)

$$
\begin{aligned}
\tilde{Q}_4^4(1234) =& \Big[\left(\dot{G}_{B12}+\dot{G}_{B23}+\dot{G}_{B31}\right)\left(\dot{G}_{B34}+\dot{G}_{B41}+\dot{G}_{B13}\right) \\
&+\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B34}\dot{G}_{B41}\Big]Z_4(1234),
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
\tilde{Q}_4^4(1243) =& \Big[\left(\dot{G}_{B12}+\dot{G}_{B24}+\dot{G}_{B41}\right)\left(\dot{G}_{B43}+\dot{G}_{B31}+\dot{G}_{B14}\right) \\
&+\dot{G}_{B12}\dot{G}_{B24}\dot{G}_{B43}\dot{G}_{B31}\Big]Z_4(1243),
\end{aligned}
\tag{3.24}
$$

$$
\begin{aligned}
\tilde{Q}_4^4(1324) =& \Big[\left(\dot{G}_{B13}+\dot{G}_{B32}+\dot{G}_{B21}\right)\left(\dot{G}_{B24}+\dot{G}_{B41}+\dot{G}_{B12}\right) \\
&+\dot{G}_{B13}\dot{G}_{B32}\dot{G}_{B24}\dot{G}_{B41}\Big]Z_4(1324),
\end{aligned}
\tag{3.25}
$$

$$
\begin{aligned}
\tilde{Q}_4^3(123) =& \left(\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31}+\dot{G}_{B12}+\dot{G}_{B23}+\dot{G}_{B31}\right)Z_3(123) \\
&\times\left(\dot{G}_{B41}\varepsilon_4\cdot k_1+\dot{G}_{B42}\varepsilon_4\cdot k_2+\dot{G}_{B43}\varepsilon_4\cdot k_3\right),
\end{aligned}
\tag{3.26}
$$

$$\begin{aligned}\tilde{Q}_4^3(234) = &\left(\dot{G}_{B23}\dot{G}_{B34}\dot{G}_{B42} + \dot{G}_{B23} + \dot{G}_{B34} + \dot{G}_{B42}\right) Z_3(234) \\ &\times \left(\dot{G}_{B12}\varepsilon_1 \cdot k_2 + \dot{G}_{B13}\varepsilon_1 \cdot k_3 + \dot{G}_{B14}\varepsilon_1 \cdot k_4\right),\end{aligned} \tag{3.27}$$

$$\begin{aligned}\tilde{Q}_4^3(341) = &\left(\dot{G}_{B34}\dot{G}_{B41}\dot{G}_{B13} + \dot{G}_{B34} + \dot{G}_{B41} + \dot{G}_{B13}\right) Z_3(341) \\ &\times \left(\dot{G}_{B21}\varepsilon_2 \cdot k_1 + \dot{G}_{B23}\varepsilon_2 \cdot k_3 + \dot{G}_{B24}\varepsilon_2 \cdot k_4\right),\end{aligned} \tag{3.28}$$

$$\begin{aligned}\tilde{Q}_4^3(412) = &\left(\dot{G}_{B41}\dot{G}_{B12}\dot{G}_{B24} + \dot{G}_{B41} + \dot{G}_{B12} + \dot{G}_{B24}\right) Z_3(412) \\ &\times \left(\dot{G}_{B31}\varepsilon_3 \cdot k_1 + \dot{G}_{B32}\varepsilon_3 \cdot k_2 + \dot{G}_{B34}\varepsilon_3 \cdot k_4\right),\end{aligned} \tag{3.29}$$

$$\begin{aligned}\tilde{Q}_4^2(12) = &\left(\dot{G}_{B12}\dot{G}_{B21} + 1\right) Z_2(12) \\ &\times \Big[\dot{G}_{B31}\varepsilon_3 \cdot k_1\dot{G}_{B41}\varepsilon_4 \cdot k_1 + \dot{G}_{B31}\varepsilon_3 \cdot k_1\dot{G}_{B42}\varepsilon_4 \cdot k_2 \\ &+ \dot{G}_{B31}\varepsilon_3 \cdot k_1\dot{G}_{B43}\varepsilon_4 \cdot k_3 + \dot{G}_{B32}\varepsilon_3 \cdot k_2\dot{G}_{B41}\varepsilon_4 \cdot k_1 \\ &+ \dot{G}_{B32}\varepsilon_3 \cdot k_2\dot{G}_{B42}\varepsilon_4 \cdot k_2 + \dot{G}_{B32}\varepsilon_3 \cdot k_2\dot{G}_{B43}\varepsilon_4 \cdot k_3 \\ &+ \dot{G}_{B34}\varepsilon_3 \cdot k_4\dot{G}_{B41}\varepsilon_4 \cdot k_1 + \dot{G}_{B34}\varepsilon_3 \cdot k_4\dot{G}_{B42}\varepsilon_4 \cdot k_2 \\ &+ \frac{1}{2}\dot{G}_{B34}\varepsilon_3 \cdot \varepsilon_4\left(\dot{G}_{B31}k_3 \cdot k_1 - \dot{G}_{B41}k_4 \cdot k_1\right) \\ &+ \frac{1}{2}\dot{G}_{B34}\varepsilon_3 \cdot \varepsilon_4\left(\dot{G}_{B32}k_3 \cdot k_2 - \dot{G}_{B42}k_4 \cdot k_2\right)\Big],\end{aligned} \tag{3.30}$$

$$\begin{aligned}\tilde{Q}_4^2(13) = &\left(\dot{G}_{B13}\dot{G}_{B31} + 1\right) Z_2(13) \\ &\times \Big[\dot{G}_{B21}\varepsilon_2 \cdot k_1\dot{G}_{B41}\varepsilon_4 \cdot k_1 + \dot{G}_{B21}\varepsilon_2 \cdot k_1\dot{G}_{B42}\varepsilon_4 \cdot k_2 \\ &+ \dot{G}_{B21}\varepsilon_2 \cdot k_1\dot{G}_{B43}\varepsilon_4 \cdot k_3 + \dot{G}_{B23}\varepsilon_2 \cdot k_3\dot{G}_{B41}\varepsilon_4 \cdot k_1 \\ &+ \dot{G}_{B23}\varepsilon_2 \cdot k_3\dot{G}_{B42}\varepsilon_4 \cdot k_2 + \dot{G}_{B23}\varepsilon_2 \cdot k_3\dot{G}_{B43}\varepsilon_4 \cdot k_3 \\ &+ \dot{G}_{B24}\varepsilon_2 \cdot k_4\dot{G}_{B41}\varepsilon_4 \cdot k_1 + \dot{G}_{B24}\varepsilon_2 \cdot k_4\dot{G}_{B43}\varepsilon_4 \cdot k_3 \\ &+ \frac{1}{2}\dot{G}_{B24}\varepsilon_2 \cdot \varepsilon_4\left(\dot{G}_{B21}k_2 \cdot k_1 - \dot{G}_{B41}k_4 \cdot k_1\right) \\ &+ \frac{1}{2}\dot{G}_{B24}\varepsilon_2 \cdot \varepsilon_4\left(\dot{G}_{B23}k_2 \cdot k_3 - \dot{G}_{B43}k_4 \cdot k_3\right)\Big],\end{aligned} \tag{3.31}$$

$$
\begin{aligned}
\tilde{Q}_4^2(14) = &\left(\dot{G}_{B14}\dot{G}_{B41}+1\right) Z_2(14) \\
&\times \Big[\dot{G}_{B21}\varepsilon_2\cdot k_1\dot{G}_{B31}\varepsilon_3\cdot k_1 \;+\dot{G}_{B21}\varepsilon_2\cdot k_1\dot{G}_{B32}\varepsilon_3\cdot k_2 \\
&+\dot{G}_{B21}\varepsilon_2\cdot k_1\dot{G}_{B34}\varepsilon_3\cdot k_4+\dot{G}_{B23}\varepsilon_2\cdot k_3\dot{G}_{B31}\varepsilon_3\cdot k_1 \\
&+\dot{G}_{B23}\varepsilon_2\cdot k_3\dot{G}_{B34}\varepsilon_3\cdot k_4+\dot{G}_{B24}\varepsilon_2\cdot k_4\dot{G}_{B31}\varepsilon_3\cdot k_1 \\
&+\dot{G}_{B24}\varepsilon_2\cdot k_4\dot{G}_{B32}\varepsilon_3\cdot k_2+\dot{G}_{B24}\varepsilon_2\cdot k_4\dot{G}_{B34}\varepsilon_3\cdot k_4 \\
&+\frac{1}{2}\dot{G}_{B23}\varepsilon_2\cdot\varepsilon_3\left(\dot{G}_{B21}k_2\cdot k_1-\dot{G}_{B31}k_3\cdot k_1\right) \\
&+\frac{1}{2}\dot{G}_{B23}\varepsilon_2\cdot\varepsilon_3\left(\dot{G}_{B24}k_2\cdot k_4-\dot{G}_{B34}k_3\cdot k_4\right)\Big],
\end{aligned}
\tag{3.32}
$$

$$
\begin{aligned}
\tilde{Q}_4^2(23) = &\left(\dot{G}_{B23}\dot{G}_{B32}+1\right) Z_2(23) \\
&\times \Big[\dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B41}\varepsilon_4\cdot k_1 \;+\dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B42}\varepsilon_4\cdot k_2 \\
&+\dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B43}\varepsilon_4\cdot k_3+\dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B41}\varepsilon_4\cdot k_1 \\
&+\dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B42}\varepsilon_4\cdot k_2+\dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B43}\varepsilon_4\cdot k_3 \\
&+\dot{G}_{B14}\varepsilon_1\cdot k_4\dot{G}_{B42}\varepsilon_4\cdot k_2+\dot{G}_{B14}\varepsilon_1\cdot k_4\dot{G}_{B43}\varepsilon_4\cdot k_3 \\
&+\frac{1}{2}\dot{G}_{B14}\varepsilon_1\cdot\varepsilon_4\left(\dot{G}_{B12}k_1\cdot k_2-\dot{G}_{B42}k_4\cdot k_2\right) \\
&+\frac{1}{2}\dot{G}_{B14}\varepsilon_1\cdot\varepsilon_4\left(\dot{G}_{B13}k_1\cdot k_3-\dot{G}_{B43}k_4\cdot k_3\right)\Big],
\end{aligned}
\tag{3.33}
$$

$$
\begin{aligned}
\tilde{Q}_4^2(24) = &\left(\dot{G}_{B24}\dot{G}_{B42}+1\right) Z_2(24) \\
&\times \Big[\dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B31}\varepsilon_3\cdot k_1 \;+\dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B32}\varepsilon_3\cdot k_2 \\
&+\dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B34}\varepsilon_3\cdot k_4+\dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B32}\varepsilon_3\cdot k_2 \\
&+\dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B34}\varepsilon_3\cdot k_4+\dot{G}_{B14}\varepsilon_1\cdot k_4\dot{G}_{B31}\varepsilon_3\cdot k_1 \\
&+\dot{G}_{B14}\varepsilon_1\cdot k_4\dot{G}_{B32}\varepsilon_3\cdot k_2+\dot{G}_{B14}\varepsilon_1\cdot k_4\dot{G}_{B34}\varepsilon_3\cdot k_4 \\
&+\frac{1}{2}\dot{G}_{B13}\varepsilon_1\cdot\varepsilon_3\left(\dot{G}_{B12}k_1\cdot k_2-\dot{G}_{B32}k_3\cdot k_2\right) \\
&+\frac{1}{2}\dot{G}_{B13}\varepsilon_1\cdot\varepsilon_3\left(\dot{G}_{B14}k_1\cdot k_4-\dot{G}_{B34}k_3\cdot k_4\right)\Big],
\end{aligned}
\tag{3.34}
$$

$$\begin{aligned}\tilde{Q}_4^2(34) = &\left(\dot{G}_{B34}\dot{G}_{B43}+1\right) Z_2(34)\\ &\times\Big[\dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B23}\varepsilon_2\cdot k_3 + \dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B24}\varepsilon_2\cdot k_4\\ &+\dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B21}\varepsilon_2\cdot k_1 + \dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B23}\varepsilon_2\cdot k_3\\ &+\dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B24}\varepsilon_2\cdot k_4 + \dot{G}_{B14}\varepsilon_1\cdot k_4\dot{G}_{B21}\varepsilon_2\cdot k_1\\ &+\dot{G}_{B14}\varepsilon_1\cdot k_4\dot{G}_{B23}\varepsilon_2\cdot k_3 + \dot{G}_{B14}\varepsilon_1\cdot k_4\dot{G}_{B24}\varepsilon_2\cdot k_4\\ &+\frac{1}{2}\dot{G}_{B12}\varepsilon_1\cdot\varepsilon_2\left(\dot{G}_{B13}k_1\cdot k_3-\dot{G}_{B23}k_2\cdot k_3\right)\\ &+\frac{1}{2}\dot{G}_{B12}\varepsilon_1\cdot\varepsilon_2\left(\dot{G}_{B14}k_1\cdot k_4-\dot{G}_{B24}k_2\cdot k_4\right)\Big],\end{aligned} \tag{3.35}$$

$$\tilde{Q}_4^{22}(12,34) = \left(\dot{G}_{B12}\dot{G}_{B21}+1\right) Z_2(12)\left(\dot{G}_{B34}\dot{G}_{B43}+1\right) Z_2(34), \tag{3.36}$$

$$\tilde{Q}_4^{22}(13,24) = \left(\dot{G}_{B13}\dot{G}_{B31}+1\right) Z_2(13)\left(\dot{G}_{B24}\dot{G}_{B42}+1\right) Z_2(24), \tag{3.37}$$

$$\tilde{Q}_4^{22}(14,23) = \left(\dot{G}_{B14}\dot{G}_{B41}+1\right) Z_2(14)\left(\dot{G}_{B23}\dot{G}_{B32}+1\right) Z_2(23). \tag{3.38}$$

Notice that, in the spinor case, everithing is written in terms of $\dot{G}_{Bij}$ because we use the identities in Appendix B (equations B.2 and B.3) to transform any combination of G_{Fij} into some other in terms of $\dot{G}_{Bij}$.

3.2 One low energy photon

When we perform the integral of Q_4, $\tilde{Q}_4$ over u_4 and we recall this as $I_{(4)}$, $\tilde{I}_{(4)}$ respectively. Then $I(Q_4)$ and $\tilde{I}(\tilde{Q}_4)$ transform into

$$I(Q_4) = \int_0^1\int_0^1\int_0^1 du_1du_2du_3 I_{(4)} e^{T(G_{B12}k_1\cdot k_2+G_{B13}k_1\cdot k_3+G_{B23}k_2\cdot k_3)}, \tag{3.39}$$

$$\tilde{I}(\tilde{Q}_4) = \int_0^1\int_0^1\int_0^1 du_1du_2du_3 \tilde{I}_{(4)} e^{T(G_{B12}k_1\cdot k_2+G_{B13}k_1\cdot k_3+G_{B23}k_2\cdot k_3)}. \tag{3.40}$$

Notice that here the exponential does not contain k_4 anymore because of the linearization. (Some useful formulas that we use to perform these integrals over u_4 are enlisted in Appendix B and some details of our calculations of these expressions are in Appendices D and E.)

3.2.1 Scalar result for the one low energy photon

In this section, we write the explicit form of $I_{(4)}$. (The sum of all the terms, in this section, is $I_{(4)}$.)

$$I^4_{(4)}(1234) = \left(2G_{B12}\frac{k_2\cdot k_1}{k_2\cdot k_3} + 2G_{B23}\frac{k_2\cdot k_3}{k_2\cdot k_1} + b\right)\left(\frac{1}{3} - 2G_{B13}\right) Z_4(1234), \tag{3.41}$$

$$I^4_{(4)}(1243) = \left(2G_{B12}\frac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13}\frac{k_1\cdot k_3}{k_1\cdot k_2} + a\right)\left(\frac{1}{3} - 2G_{B23}\right) Z_4(1243), \tag{3.42}$$

$$I^4_{(4)}(1324) = \left(2G_{B13}\frac{k_1\cdot k_3}{k_2\cdot k_3} + 2G_{B23}\frac{k_2\cdot k_3}{k_1\cdot k_3} + c\right)\left(\frac{1}{3} - 2G_{B12}\right) Z_4(1324), \tag{3.43}$$

$$\begin{aligned} I^3_{(4)}(123) &= \frac{-T}{3} Z_3(123) \\ &\times \left[\left(2G_{B13}\frac{k_1\cdot k_3}{k_2\cdot k_3} + 2G_{B23}\frac{k_2\cdot k_3}{k_1\cdot k_3} + c\right)(1-4G_{B12})\, G_{B12} k_2\cdot f_4\cdot k_1\right. \\ &+ \left(2G_{B12}\frac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13}\frac{k_1\cdot k_3}{k_1\cdot k_2} + a\right)(1-4G_{B23})\, G_{B23} k_3\cdot f_4\cdot k_2 \\ &+ \left.\left(2G_{B12}\frac{k_2\cdot k_1}{k_2\cdot k_3} + 2G_{B23}\frac{k_2\cdot k_3}{k_2\cdot k_1} + b\right)(1-4G_{B13})\, G_{B13} k_1\cdot f_4\cdot k_3\right], \end{aligned} \tag{3.44}$$

$$\begin{aligned} I^3_{(4)}(234) = \left(\frac{1}{3} - 2G_{B23}\right) Z_3(234) &\left[\left(2G_{B12}\frac{k_2\cdot k_1}{k_2\cdot k_3} + 2G_{B23}\frac{k_2\cdot k_3}{k_2\cdot k_1} + b\right)\frac{r_1\cdot f_1\cdot k_2}{r_1\cdot k_1}\right. \\ &- \left.\left(2G_{B13}\frac{k_1\cdot k_3}{k_2\cdot k_3} + 2G_{B23}\frac{k_2\cdot k_3}{k_1\cdot k_3} + c\right)\frac{r_1\cdot f_1\cdot k_3}{r_1\cdot k_1}\right], \end{aligned} \tag{3.45}$$

$$\begin{aligned} I^3_{(4)}(341) = \left(\frac{1}{3} - 2G_{B13}\right) Z_3(341) &\left[\left(2G_{B12}\frac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13}\frac{k_1\cdot k_3}{k_1\cdot k_2} + a\right)\frac{r_2\cdot f_2\cdot k_1}{r_2\cdot k_2}\right. \\ &- \left.\left(2G_{B13}\frac{k_1\cdot k_3}{k_2\cdot k_3} + 2G_{B23}\frac{k_2\cdot k_3}{k_1\cdot k_3} + c\right)\frac{r_2\cdot f_2\cdot k_3}{r_2\cdot k_2}\right], \end{aligned} \tag{3.46}$$

$$I^3_{(4)}(412) = \left(\frac{1}{3} - 2G_{B12}\right) Z_3(412) \left[\left(2G_{B12}\frac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13}\frac{k_1\cdot k_3}{k_1\cdot k_2} + a\right)\frac{r_3\cdot f_3\cdot k_1}{r_3\cdot k_3} - \left(2G_{B12}\frac{k_2\cdot k_1}{k_2\cdot k_3} + 2G_{B23}\frac{k_2\cdot k_3}{k_2\cdot k_1} + b\right)\frac{r_3\cdot f_3\cdot k_2}{r_3\cdot k_3}\right]. \tag{3.47}$$

$$\begin{aligned} I^2_{(4)}(12) &= \frac{-T}{3}\left(1-4G_{B12}\right) Z_2(12) \\ &\times\Big[\left(1-4G_{B13}\right) G_{B13} k_1\cdot f_4\cdot f_3\cdot k_1 + \left(1-4G_{B23}\right) G_{B23} k_2\cdot f_4\cdot f_3\cdot k_2 \\ &+ \left(2G_{B13}\frac{k_1\cdot k_3}{k_2\cdot k_3} + 2G_{B23}\frac{k_2\cdot k_3}{k_1\cdot k_3} + c\right)\left(G_{B13} k_1\cdot f_4\cdot f_3\cdot k_2 + G_{B23} k_2\cdot f_4\cdot f_3\cdot k_1\right) \\ &- \left(2G_{B12}\frac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13}\frac{k_1\cdot k_3}{k_1\cdot k_2} + a\right) G_{B12}\frac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_1}{r_4\cdot k_3} \\ &+ \left(2G_{B12}\frac{k_2\cdot k_1}{k_2\cdot k_3} + 2G_{B23}\frac{k_2\cdot k_3}{k_2\cdot k_1} + b\right) G_{B12}\frac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_2}{r_4\cdot k_3}\Big], \end{aligned} \tag{3.48}$$

$$\begin{aligned} I^2_{(4)}(13) &= \frac{-T}{3}\left(1-4G_{B13}\right) Z_2(13) \\ &\times\Big[\left(1-4G_{B12}\right) G_{B12} k_1\cdot f_4\cdot f_2\cdot k_1 + \left(1-4G_{B23}\right) G_{B23} k_3\cdot f_4\cdot f_2\cdot k_3 \\ &+ \left(2G_{B12}\frac{k_2\cdot k_1}{k_2\cdot k_3} + 2G_{B23}\frac{k_2\cdot k_3}{k_2\cdot k_1} + b\right)\left(G_{B12} k_1\cdot f_4\cdot f_2\cdot k_3 + G_{B23} k_3\cdot f_4\cdot f_2\cdot k_1\right) \\ &- \left(2G_{B12}\frac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13}\frac{k_1\cdot k_3}{k_1\cdot k_2} + a\right) G_{B13}\frac{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_1}{r_5\cdot k_2} \\ &+ \left(2G_{B13}\frac{k_1\cdot k_3}{k_2\cdot k_3} + 2G_{B23}\frac{k_2\cdot k_3}{k_1\cdot k_3} + c\right) G_{B13}\frac{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_3}{r_5\cdot k_2}\Big], \end{aligned} \tag{3.49}$$

$$I^2_{(4)}(23) = \frac{-T}{3}\left(1-4G_{B23}\right)Z_2(23)$$
$$\times\Big[\left(1-4G_{B12}\right)G_{B12}k_2\cdot f_4\cdot f_1\cdot k_2+\left(1-4G_{B31}\right)G_{B31}k_3\cdot f_4\cdot f_1\cdot k_3$$
$$+\left(2G_{B12}\frac{k_1\cdot k_2}{k_1\cdot k_3}+2G_{B13}\frac{k_1\cdot k_3}{k_1\cdot k_2}+a\right)\left(G_{B12}k_2\cdot f_4\cdot f_1\cdot k_3+G_{B31}k_3\cdot f_4\cdot f_1\cdot k_2\right)$$
$$-\left(2G_{B21}\frac{k_2\cdot k_1}{k_2\cdot k_3}+2G_{B23}\frac{k_2\cdot k_3}{k_2\cdot k_1}+b\right)G_{B23}\frac{k_3\cdot f_4\cdot k_2 r_6\cdot f_1\cdot k_2}{r_6\cdot k_1}$$
$$+\left(2G_{B13}\frac{k_1\cdot k_3}{k_2\cdot k_3}+2G_{B23}\frac{k_2\cdot k_3}{k_1\cdot k_3}+c\right)G_{B23}\frac{k_3\cdot f_4\cdot k_2 r_6\cdot f_1\cdot k_3}{r_6\cdot k_1}\Big], \tag{3.50}$$

$$I^2_{(4)}(14) = -\frac{Z_2(14)}{3r_7\cdot k_2 r_8\cdot k_3}\left[\left(2G_{B12}\frac{k_1\cdot k_2}{k_1\cdot k_3}+2G_{B13}\frac{k_1\cdot k_3}{k_1\cdot k_2}+a\right)r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_1\right.$$
$$-\left(2G_{B13}\frac{k_1\cdot k_3}{k_2\cdot k_3}+2G_{B23}\frac{k_2\cdot k_3}{k_1\cdot k_3}+c\right)\left(r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_3+2r_7\cdot f_2\cdot f_3\cdot r_8 k_3\cdot k_1\right)$$
$$\left.-\left(2G_{B21}\frac{k_2\cdot k_1}{k_2\cdot k_3}+2G_{B23}\frac{k_2\cdot k_3}{k_2\cdot k_1}+b\right)\left(r_8\cdot f_3\cdot k_2 r_7\cdot f_2\cdot k_1-r_7\cdot f_2\cdot f_3\cdot r_8 k_2\cdot k_1\right)\right], \tag{3.51}$$

$$I^2_{(4)}(24) = -\frac{Z_2(24)}{3r_9\cdot k_1 r_{10}\cdot k_3}\left[\left(2G_{B21}\frac{k_2\cdot k_1}{k_2\cdot k_3}+2G_{B23}\frac{k_2\cdot k_3}{k_2\cdot k_1}+b\right)r_{10}\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_2\right.$$
$$-\left(2G_{B13}\frac{k_1\cdot k_3}{k_2\cdot k_3}+2G_{B23}\frac{k_2\cdot k_3}{k_1\cdot k_3}+c\right)\left(r_{10}\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_3+2r_9\cdot f_1\cdot f_3\cdot r_{10}k_3\cdot k_2\right)$$
$$\left.-\left(2G_{B12}\frac{k_1\cdot k_2}{k_1\cdot k_3}+2G_{B13}\frac{k_1\cdot k_3}{k_1\cdot k_2}+a\right)\left(r_{10}\cdot f_3\cdot k_1 r_9\cdot f_1\cdot k_2-r_9\cdot f_1\cdot f_3\cdot r_{10}k_1\cdot k_2\right)\right], \tag{3.52}$$

$$I^2_{(4)}(34) = -\frac{Z_2(34)}{3r_{11}\cdot k_1 r_{12}\cdot k_2}\left[\left(2G_{B13}\frac{k_1\cdot k_3}{k_2\cdot k_3}+2G_{B23}\frac{k_2\cdot k_3}{k_1\cdot k_3}+c\right)r_{12}\cdot f_2\cdot k_3 r_{11}\cdot f_1\cdot k_3\right.$$
$$-\left(2G_{B21}\frac{k_2\cdot k_1}{k_2\cdot k_3}+2G_{B23}\frac{k_2\cdot k_3}{k_2\cdot k_1}+b\right)\left(r_{12}\cdot f_2\cdot k_3 r_{11}\cdot f_1\cdot k_2+2r_{11}\cdot f_1\cdot f_2\cdot r_{12}k_2\cdot k_3\right)$$
$$\left.-\left(2G_{B12}\frac{k_1\cdot k_2}{k_1\cdot k_3}+2G_{B13}\frac{k_1\cdot k_3}{k_1\cdot k_2}+a\right)\left(r_{12}\cdot f_2\cdot k_1 r_{11}\cdot f_1\cdot k_3-r_{11}\cdot f_1\cdot f_2\cdot r_{12}k_1\cdot k_3\right)\right], \tag{3.53}$$

$$I^{22}_{(4)}(12,34) = \frac{1}{3}\left(1-4G_{B12}\right)Z_2(12)Z_2(34), \tag{3.54}$$

$$I^{22}_{(4)}(13,24) = \frac{1}{3}\left(1 - 4G_{B31}\right) Z_2(13) Z_2(24), \tag{3.55}$$

$$I^{22}_{(4)}(14,23) = \frac{1}{3}\left(1 - 4G_{B23}\right) Z_2(14) Z_2(23). \tag{3.56}$$

3.2.2 Spinor result for the one low energy photon

In this section, we write the explicit form of $\tilde{I}_{(4)}$. (The sum of all the terms, in this section, is $\tilde{I}_{(4)}$.)

$$\tilde{I}^4_{(4)}(1234) = Z_4(1234)\left[\left(2G_{B21}\frac{k_2\cdot k_1}{k_2\cdot k_3} + 2G_{B23}\frac{k_2\cdot k_3}{k_2\cdot k_1} + b\right)\left(\frac{1}{3} - 2G_{B13}\right)\right.$$
$$\left. + 2G_{B13}\left(\frac{k_1\cdot k_3}{k_1\cdot k_2} + \frac{k_1\cdot k_3}{k_2\cdot k_3} + 2\right) + \left(2G_{B12}\frac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B23}\frac{k_2\cdot k_3}{k_1\cdot k_3}\right) + a + c - 1\right], \tag{3.57}$$

$$\tilde{I}^4_{(4)}(1243) = Z_4(1243)\left[\left(2G_{B12}\frac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13}\frac{k_1\cdot k_3}{k_1\cdot k_2} + a\right)\left(\frac{1}{3} - 2G_{B23}\right)\right.$$
$$\left. + 2\left(G_{B23} - G_{B12} - G_{B13}\right) - \left(2G_{B12}\frac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13}\frac{k_1\cdot k_3}{k_1\cdot k_2} + a\right)\right], \tag{3.58}$$

$$\tilde{I}^4_{(4)}(1324) = Z_4(1324)\Big[\left(2G_{B13}\frac{k_1\cdot k_3}{k_2\cdot k_3} + 2G_{B23}\frac{k_2\cdot k_3}{k_1\cdot k_3} + c\right)\left(\frac{1}{3} - 2G_{B12}\right)$$
$$+ 2G_{B12}\left(\frac{k_1\cdot k_2}{k_1\cdot k_3} + \frac{k_2\cdot k_1}{k_2\cdot k_3} + 2\right) + \left(2G_{B23}\frac{k_2\cdot k_3}{k_2\cdot k_1} + 2G_{B13}\frac{k_1\cdot k_3}{k_1\cdot k_2}\right) + a + b - 1\Big], \tag{3.59}$$

$$\begin{aligned}
\tilde{I}^3_{(4)}(123) = \frac{-T}{3} Z_3(123) \,[& - (1-4G_{B12})\, G_{B12} k_2 \cdot f_4 \cdot k_1 \\
& - (1-4G_{B23})\, G_{B23} k_3 \cdot f_4 \cdot k_2 - (1-4G_{B31})\, G_{B31} k_1 \cdot f_4 \cdot k_3 \\
& + \left(2G_{B13} \frac{k_1 \cdot k_3}{k_2 \cdot k_3} + 2G_{B23} \frac{k_2 \cdot k_3}{k_1 \cdot k_3} + c \right) (1-4G_{B12})\, G_{B12} k_2 \cdot f_4 \cdot k_1 \\
& + \left(2G_{B12} \frac{k_1 \cdot k_2}{k_1 \cdot k_3} + 2G_{B13} \frac{k_1 \cdot k_3}{k_1 \cdot k_2} + a \right) (1-4G_{B23})\, G_{B23} k_3 \cdot f_4 \cdot k_2 \\
& + \left(2G_{B21} \frac{k_2 \cdot k_1}{k_2 \cdot k_3} + 2G_{B23} \frac{k_2 \cdot k_3}{k_2 \cdot k_1} + b \right) (1-4G_{B31})\, G_{B31} k_1 \cdot f_4 \cdot k_3 \\
& + \left(2G_{B21} \frac{k_2 \cdot k_1}{k_2 \cdot k_3} + 2G_{B23} \frac{k_2 \cdot k_3}{k_2 \cdot k_1} + b \right) G_{B23} k_3 \cdot f_4 \cdot k_2 \\
& + \left(2G_{B12} \frac{k_1 \cdot k_2}{k_1 \cdot k_3} + 2G_{B13} \frac{k_1 \cdot k_3}{k_1 \cdot k_2} + a \right) G_{B31} k_1 \cdot f_4 \cdot k_3 \\
& + \left(2G_{B21} \frac{k_2 \cdot k_1}{k_2 \cdot k_3} + 2G_{B23} \frac{k_2 \cdot k_3}{k_2 \cdot k_1} + b \right) G_{B12} k_2 \cdot f_4 \cdot k_1 \\
& + \left(2G_{B13} \frac{k_1 \cdot k_3}{k_2 \cdot k_3} + 2G_{B23} \frac{k_2 \cdot k_3}{k_1 \cdot k_3} + c \right) G_{B31} k_1 \cdot f_4 \cdot k_3 \\
& + \left(2G_{B12} \frac{k_1 \cdot k_2}{k_1 \cdot k_3} + 2G_{B13} \frac{k_1 \cdot k_3}{k_1 \cdot k_2} + a \right) G_{B12} k_2 \cdot f_4 \cdot k_1 \\
& + \left(2G_{B13} \frac{k_1 \cdot k_3}{k_2 \cdot k_3} + 2G_{B23} \frac{k_2 \cdot k_3}{k_1 \cdot k_3} + c \right) G_{B23} k_3 \cdot f_4 \cdot k_2 \Bigg] ,
\end{aligned} \tag{3.60}$$

$$\begin{aligned}
\tilde{I}^3_{(4)}(234) = \left(\frac{2}{3} + 2G_{B23} \right) Z_3(234) \Bigg[& \left(2G_{B13} \frac{k_1 \cdot k_3}{k_2 \cdot k_3} + 2G_{B23} \frac{k_2 \cdot k_3}{k_1 \cdot k_3} + c \right) \frac{r_1 \cdot f_1 \cdot k_3}{r_1 \cdot k_1} \\
& - \left(2G_{B21} \frac{k_2 \cdot k_1}{k_2 \cdot k_3} + 2G_{B23} \frac{k_2 \cdot k_3}{k_2 \cdot k_1} + b \right) \frac{r_1 \cdot f_1 \cdot k_2}{r_1 \cdot k_1} \Bigg] ,
\end{aligned} \tag{3.61}$$

$$\begin{aligned}
\tilde{I}^3_{(4)}(341) = \left(\frac{2}{3} + 2G_{B31} \right) Z_3(341) \Bigg[& \left(2G_{B13} \frac{k_1 \cdot k_3}{k_2 \cdot k_3} + 2G_{B23} \frac{k_2 \cdot k_3}{k_1 \cdot k_3} + c \right) \frac{r_2 \cdot f_2 \cdot k_3}{r_2 \cdot k_2} \\
& - \left(2G_{B12} \frac{k_1 \cdot k_2}{k_1 \cdot k_3} + 2G_{B13} \frac{k_1 \cdot k_3}{k_1 \cdot k_2} + a \right) \frac{r_2 \cdot f_2 \cdot k_1}{r_2 \cdot k_2} \Bigg] ,
\end{aligned} \tag{3.62}$$

$$\tilde{I}^3_{(4)}(412) = \left(\frac{2}{3} + 2G_{B12}\right) Z_3(412) \left[\left(2G_{B21}\frac{k_2\cdot k_1}{k_2\cdot k_3} + 2G_{B23}\frac{k_2\cdot k_3}{k_2\cdot k_1} + b\right)\frac{r_3\cdot f_3\cdot k_2}{r_3\cdot k_3} - \left(2G_{B12}\frac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13}\frac{k_1\cdot k_3}{k_1\cdot k_2} + a\right)\frac{r_3\cdot f_3\cdot k_1}{r_3\cdot k_3}\right]. \tag{3.63}$$

$$\begin{aligned}\tilde{I}^2_{(4)}(12) &= \frac{4}{3}TG_{B12}Z_2(12)\\ &\times\Big[\left(1-4G_{B31}\right)G_{B31}k_1\cdot f_4\cdot f_3\cdot k_1 + \left(1-4G_{B23}\right)G_{B23}k_2\cdot f_4\cdot f_3\cdot k_2\\ &+\left(2G_{B13}\frac{k_1\cdot k_3}{k_2\cdot k_3} + 2G_{B23}\frac{k_2\cdot k_3}{k_1\cdot k_3} + c\right)\left(G_{B31}k_1\cdot f_4\cdot f_3\cdot k_2 + G_{B23}k_2\cdot f_4\cdot f_3\cdot k_1\right)\\ &-\left(2G_{B12}\frac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13}\frac{k_1\cdot k_3}{k_1\cdot k_2} + a\right)G_{B12}\frac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_1}{r_4\cdot k_3}\\ &+\left(2G_{B21}\frac{k_2\cdot k_1}{k_2\cdot k_3} + 2G_{B23}\frac{k_2\cdot k_3}{k_2\cdot k_1} + b\right)G_{B12}\frac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_2}{r_4\cdot k_3}\Big],\end{aligned} \tag{3.64}$$

$$\begin{aligned}\tilde{I}^2_{(4)}(13) &= \frac{4}{3}TG_{B31}Z_2(13)\\ &\times\Big[\left(1-4G_{B12}\right)G_{B12}k_1\cdot f_4\cdot f_2\cdot k_1 + \left(1-4G_{B23}\right)G_{B23}k_3\cdot f_4\cdot f_2\cdot k_3\\ &+\left(2G_{B21}\frac{k_2\cdot k_1}{k_2\cdot k_3} + 2G_{B23}\frac{k_2\cdot k_3}{k_2\cdot k_1} + b\right)\left(G_{B12}k_1\cdot f_4\cdot f_2\cdot k_3 + G_{B23}k_3\cdot f_4\cdot f_2\cdot k_1\right)\\ &-\left(2G_{B12}\frac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13}\frac{k_1\cdot k_3}{k_1\cdot k_2} + a\right)G_{B31}\frac{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_1}{r_5\cdot k_2}\\ &+\left(2G_{B13}\frac{k_1\cdot k_3}{k_2\cdot k_3} + 2G_{B23}\frac{k_2\cdot k_3}{k_1\cdot k_3} + c\right)G_{B31}\frac{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_3}{r_5\cdot k_2}\Big],\end{aligned} \tag{3.65}$$

$$\tilde{I}^2_{(4)}(23) = \frac{4}{3} T G_{B23} Z_2(23)$$
$$\times \Big[\left(1 - 4G_{B12}\right) G_{B12} k_2 \cdot f_4 \cdot f_1 \cdot k_2 + \left(1 - 4G_{B31}\right) G_{B31} k_3 \cdot f_4 \cdot f_1 \cdot k_3$$
$$+ \left(2G_{B12} \frac{k_1 \cdot k_2}{k_1 \cdot k_3} + 2G_{B13} \frac{k_1 \cdot k_3}{k_1 \cdot k_2} + a \right) \left(G_{B12} k_2 \cdot f_4 \cdot f_1 \cdot k_3 + G_{B31} k_3 \cdot f_4 \cdot f_1 \cdot k_2 \right)$$
$$- \left(2G_{B21} \frac{k_2 \cdot k_1}{k_2 \cdot k_3} + 2G_{B23} \frac{k_2 \cdot k_3}{k_2 \cdot k_1} + b \right) G_{B23} \frac{k_3 \cdot f_4 \cdot k_2 r_6 \cdot f_1 \cdot k_2}{r_6 \cdot k_1}$$
$$+ \left(2G_{B13} \frac{k_1 \cdot k_3}{k_2 \cdot k_3} + 2G_{B23} \frac{k_2 \cdot k_3}{k_1 \cdot k_3} + c \right) G_{B23} \frac{k_3 \cdot f_4 \cdot k_2 r_6 \cdot f_1 \cdot k_3}{r_6 \cdot k_1} \Big], \tag{3.66}$$

$$\tilde{I}^2_{(4)}(14) = \frac{2Z_2(14)}{3r_7 \cdot k_2 r_8 \cdot k_3} \left[\left(2G_{B12} \frac{k_1 \cdot k_2}{k_1 \cdot k_3} + 2G_{B13} \frac{k_1 \cdot k_3}{k_1 \cdot k_2} + a \right) r_8 \cdot f_3 \cdot k_1 r_7 \cdot f_2 \cdot k_1 \right.$$
$$- \left(2G_{B13} \frac{k_1 \cdot k_3}{k_2 \cdot k_3} + 2G_{B23} \frac{k_2 \cdot k_3}{k_1 \cdot k_3} + c \right) \left(r_8 \cdot f_3 \cdot k_1 r_7 \cdot f_2 \cdot k_3 + 2 r_7 \cdot f_2 \cdot f_3 \cdot r_8 k_3 \cdot k_1 \right)$$
$$\left. - \left(2G_{B21} \frac{k_2 \cdot k_1}{k_2 \cdot k_3} + 2G_{B23} \frac{k_2 \cdot k_3}{k_2 \cdot k_1} + b \right) \left(r_8 \cdot f_3 \cdot k_2 r_7 \cdot f_2 \cdot k_1 - r_7 \cdot f_2 \cdot f_3 \cdot r_8 k_2 \cdot k_1 \right) \right], \tag{3.67}$$

$$\tilde{I}^2_{(4)}(24) = \frac{2Z_2(24)}{3r_9 \cdot k_1 r_{10} \cdot k_3} \left[\left(2G_{B21} \frac{k_2 \cdot k_1}{k_2 \cdot k_3} + 2G_{B23} \frac{k_2 \cdot k_3}{k_2 \cdot k_1} + b \right) r_{10} \cdot f_3 \cdot k_2 r_9 \cdot f_1 \cdot k_2 \right.$$
$$- \left(2G_{B13} \frac{k_1 \cdot k_3}{k_2 \cdot k_3} + 2G_{B23} \frac{k_2 \cdot k_3}{k_1 \cdot k_3} + c \right) \left(r_{10} \cdot f_3 \cdot k_2 r_9 \cdot f_1 \cdot k_3 + 2 r_9 \cdot f_1 \cdot f_3 \cdot r_{10} k_3 \cdot k_2 \right)$$
$$\left. - \left(2G_{B12} \frac{k_1 \cdot k_2}{k_1 \cdot k_3} + 2G_{B13} \frac{k_1 \cdot k_3}{k_1 \cdot k_2} + a \right) \left(r_{10} \cdot f_3 \cdot k_1 r_9 \cdot f_1 \cdot k_2 - r_9 \cdot f_1 \cdot f_3 \cdot r_{10} k_1 \cdot k_2 \right) \right], \tag{3.68}$$

$$\tilde{I}^2_{(4)}(34) = \frac{2Z_2(34)}{3r_{11} \cdot k_1 r_{12} \cdot k_2} \left[\left(2G_{B13} \frac{k_1 \cdot k_3}{k_2 \cdot k_3} + 2G_{B23} \frac{k_2 \cdot k_3}{k_1 \cdot k_3} + c \right) r_{12} \cdot f_2 \cdot k_3 r_{11} \cdot f_1 \cdot k_3 \right.$$
$$- \left(2G_{B21} \frac{k_2 \cdot k_1}{k_2 \cdot k_3} + 2G_{B23} \frac{k_2 \cdot k_3}{k_2 \cdot k_1} + b \right) \left(r_{12} \cdot f_2 \cdot k_3 r_{11} \cdot f_1 \cdot k_2 + 2 r_{11} \cdot f_1 \cdot f_2 \cdot r_{12} k_2 \cdot k_3 \right)$$
$$\left. - \left(2G_{B12} \frac{k_1 \cdot k_2}{k_1 \cdot k_3} + 2G_{B13} \frac{k_1 \cdot k_3}{k_1 \cdot k_2} + a \right) \left(r_{12} \cdot f_2 \cdot k_1 r_{11} \cdot f_1 \cdot k_3 - r_{11} \cdot f_1 \cdot f_2 \cdot r_{12} k_1 \cdot k_3 \right) \right], \tag{3.69}$$

$$\tilde{I}^{22}_{(4)}(12, 34) = \frac{8}{3} G_{B12} Z_2(12) Z_2(34), \tag{3.70}$$

$$\tilde{I}^{22}_{(4)}(13,24) = \frac{8}{3} G_{B31} Z_2(13) Z_2(24), \tag{3.71}$$

$$\tilde{I}^{22}_{(4)}(14,23) = \frac{8}{3} G_{B23} Z_2(14) Z_2(23). \tag{3.72}$$

Notice that all the previous equations are written in terms of the field strength tensor f_i and in some of these equations, it appears r_j's, as we already mention in the beginning of this chapter. Further, there are some parameters a, b and c that come from the fact that we write all in terms of G_{Bij}'s only, and are given by (see Appendix D for more details)

$$a = -\frac{1}{2}\Big[\frac{k_1 \cdot k_2 - 2}{k_1 \cdot k_3} + \frac{k_1 \cdot k_3 - 2}{k_1 \cdot k_2}\Big], \tag{3.73}$$

$$b = -\frac{1}{2}\Big[\frac{k_2 \cdot k_1 - 2}{k_2 \cdot k_3} + \frac{k_2 \cdot k_3 - 2}{k_2 \cdot k_1}\Big], \tag{3.74}$$

$$c = -\frac{1}{2}\Big[\frac{k_1 \cdot k_3 - 2}{k_2 \cdot k_3} + \frac{k_2 \cdot k_3 - 2}{k_1 \cdot k_3}\Big]. \tag{3.75}$$

3.3 Two low energy photons

Similarly, we take the result of the preceding section. Then, we take the third photon to be in the low energy limit and we integrate over u_3. Now, we have

$$I(Q_4) = \int_0^1 \int_0^1 du_1 du_2 I_{(34)} e^{TG_{B12} k_1 \cdot k_2}, \tag{3.76}$$

$$\tilde{I}(\tilde{Q}_4) = \int_0^1 \int_0^1 du_1 du_2 \tilde{I}_{(34)} e^{TG_{B12} k_1 \cdot k_2}. \tag{3.77}$$

Similarly, we write all in terms of G_{Bij}'s only and of course in terms of f_i also.

3.3.1 Scalar result for the two low energy photons

In this section, we write the explicit form of $I_{(34)}$. (The sum of all the terms, in this section, is $I_{(34)}$.)

$$I^4_{(34)}(1234) = \frac{2}{3}\left(G_{B12} - 4G^2_{B12}\right) Z_4(1234), \tag{3.78}$$

$$I^4_{(34)}(1243) = \frac{2}{3}\left(G_{B12} - 4G^2_{B12}\right) Z_4(1243), \tag{3.79}$$

$$I^4_{(34)}(1324) = \frac{1}{9}\left(1 - 12G_{B12} + 36G^2_{B21}\right) Z_4(1324), \tag{3.80}$$

$$I^3_{(34)}(123) = \frac{-T}{9}\left(G_{B12} - 10G^2_{B12} + 24G^3_{B12}\right) Z_3(123) k_2 \cdot f_4 \cdot k_1, \tag{3.81}$$

$$I^3_{(34)}(412) = \frac{-T}{9}\left(G_{B12} - 10G^2_{B12} + 24G^3_{B12}\right) Z_3(412) k_2 \cdot f_3 \cdot k_1, \tag{3.82}$$

$$\begin{aligned} I^2_{(34)}(12) = \frac{-T}{90} Z_2(12) \Big[& \left(1 - 4G_{B12}\right)\left(k_1 \cdot f_4 \cdot f_3 \cdot k_1 + k_2 \cdot f_4 \cdot f_3 \cdot k_2\right) \\ & + \left(1 - 4G_{B12} - 30G^2_{B12} + 120G^3_{B12}\right)\left(k_1 \cdot f_4 \cdot f_3 \cdot k_2 + k_2 \cdot f_4 \cdot f_3 \cdot k_1\right) \\ & + T\left(10G^2_{B12} - 80G^3_{B12} + 160G^4_{B12}\right) k_2 \cdot f_4 \cdot k_1 k_2 \cdot f_3 \cdot k_1 \Big], \end{aligned} \tag{3.83}$$

$$I^2_{(34)}(13) = \frac{-T}{9} Z_2(13)\left(G_{B12} - 4G^2_{B12}\right) k_1 \cdot f_4 \cdot f_2 \cdot k_1, \tag{3.84}$$

$$I^2_{(34)}(23) = \frac{-T}{9} Z_2(23)\left(G_{B12} - 4G^2_{B12}\right) k_2 \cdot f_4 \cdot f_1 \cdot k_2, \tag{3.85}$$

$$\begin{aligned} I^2_{(34)}(14) = \frac{Z_2(14)}{9 r_1 \cdot k_2 r_2 \cdot k_3} \Big[& T\left(G_{B12} - 4G^2_{B12}\right) k_2 \cdot k_3 r_2 \cdot f_3 \cdot k_1 r_1 \cdot f_2 \cdot k_1 \\ & + T\left(G_{B12} - 4G^2_{B12}\right) k_3 \cdot k_1 \left(r_1 \cdot f_2 \cdot f_3 \cdot r_2 k_2 \cdot k_1 - r_2 \cdot f_3 \cdot k_2 r_1 \cdot f_2 \cdot k_1\right) \\ & + \left(1 - 6G_{B12}\right)\left(r_2 \cdot f_3 \cdot k_1 r_1 \cdot f_2 \cdot k_3 + 2 r_1 \cdot f_2 \cdot f_3 \cdot r_2 k_3 \cdot k_1\right) \Big], \end{aligned} \tag{3.86}$$

$$\begin{aligned} I^2_{(34)}(24) = \frac{Z_2(24)}{9 r_3 \cdot k_1 r_4 \cdot k_3} \Big[& T\left(G_{B12} - 4G^2_{B12}\right) k_3 \cdot k_1 r_4 \cdot f_3 \cdot k_2 r_3 \cdot f_1 \cdot k_2 \\ & + T\left(G_{B12} - 4G^2_{B12}\right) k_2 \cdot k_3 \left(r_3 \cdot f_1 \cdot f_3 \cdot r_4 k_1 \cdot k_2 - r_4 \cdot f_3 \cdot k_1 r_3 \cdot f_1 \cdot k_2\right) \\ & + \left(1 - 6G_{B12}\right)\left(r_4 \cdot f_3 \cdot k_2 r_3 \cdot f_1 \cdot k_3 + 2 r_3 \cdot f_1 \cdot f_3 \cdot r_4 k_3 \cdot k_2\right) \Big], \end{aligned} \tag{3.87}$$

$$I^{22}_{(34)}(12, 34) = \frac{1}{3}\left(1 - 4G_{B12}\right) Z_2(12) Z_2(34), \tag{3.88}$$

$$I^{22}_{(34)}(13, 24) = \frac{1}{9} Z_2(13) Z_2(24), \tag{3.89}$$

$$I^{22}_{(34)}(14, 23) = \frac{1}{9} Z_2(14) Z_2(23). \tag{3.90}$$

3.3.2 Spinor result for the two low energy photons

In this section, we write the explicit form of $\tilde{I}_{(34)}$. (The sum of all the terms, in this section, is $\tilde{I}_{(34)}$.)

$$\tilde{I}^4_{(34)}(1234) = -\frac{4}{3}\left(G_{B12} - 2G^2_{B12}\right) Z_4(1234), \tag{3.91}$$

$$\tilde{I}^4_{(34)}(1243) = -\frac{4}{3}\left(G_{B12} - 2G^2_{B12}\right) Z_4(1243), \tag{3.92}$$

$$\tilde{I}^4_{(34)}(1324) = -\frac{4}{9}\left(2 - 6G_{B12} - 9G^2_{B21}\right) Z_4(1324), \tag{3.93}$$

$$\tilde{I}^3_{(34)}(123) = \frac{2T}{9}\left(G_{B12} - G^2_{B12} - 12G^3_{B12}\right) Z_3(123) k_2 \cdot f_4 \cdot k_1, \tag{3.94}$$

$$\tilde{I}^3_{(34)}(412) = \frac{2T}{9}\left(G_{B12} - G^2_{B12} - 12G^3_{B12}\right) Z_3(412) k_2 \cdot f_3 \cdot k_1, \tag{3.95}$$

$$\begin{aligned}\tilde{I}^2_{(34)}(12) =& \frac{T}{3} G_{B12} Z_2(12) \left[\frac{2}{15}\left(k_1 \cdot f_4 \cdot f_3 \cdot k_1 + k_2 \cdot f_4 \cdot f_3 \cdot k_2\right)\right. \\ &+ \frac{2}{15}\left(1 - 30G^2_{B12}\right)\left(k_1 \cdot f_4 \cdot f_3 \cdot k_2 + k_2 \cdot f_4 \cdot f_3 \cdot k_1\right) \\ &\left.+ \frac{4T}{3}\left(G^2_{B12} - 4G^3_{B12}\right) k_2 \cdot f_4 \cdot k_1 k_2 \cdot f_3 \cdot k_1\right],\end{aligned} \tag{3.96}$$

$$\tilde{I}^2_{(34)}(13) = \frac{2T}{9} Z_2(13)\left(G_{B12} - 4G^2_{B12}\right) k_1 \cdot f_4 \cdot f_2 \cdot k_1, \tag{3.97}$$

$$\tilde{I}^2_{(34)}(23) = \frac{2T}{9} Z_2(23)\left(G_{B12} - 4G^2_{B12}\right) k_2 \cdot f_4 \cdot f_1 \cdot k_2, \tag{3.98}$$

$$\begin{aligned}\tilde{I}^2_{(34)}(14) =& \frac{-2Z_2(14)}{9 r_1 \cdot k_2 r_2 \cdot k_3}\Big[T\left(G_{B12} - 4G^2_{B12}\right) k_2 \cdot k_3 r_2 \cdot f_3 \cdot k_1 r_1 \cdot f_2 \cdot k_1 \\ &+ T\left(G_{B12} - 4G^2_{B12}\right) k_3 \cdot k_1 \left(r_1 \cdot f_2 \cdot f_3 \cdot r_2 k_2 \cdot k_1 - r_2 \cdot f_3 \cdot k_2 r_1 \cdot f_2 \cdot k_1\right) \\ &+ \left(1 - 6G_{B12}\right)\left(r_2 \cdot f_3 \cdot k_1 r_1 \cdot f_2 \cdot k_3 + 2 r_1 \cdot f_2 \cdot f_3 \cdot r_2 k_3 \cdot k_1\right)\Big],\end{aligned} \tag{3.99}$$

$$\tilde{I}^{2}_{(34)}(24) = \frac{-2Z_2(24)}{9r_3\cdot k_1 r_4\cdot k_3}\Big[T\left(G_{B12}-4G^2_{B12}\right)k_3\cdot k_1 r_4\cdot f_3\cdot k_2 r_3\cdot f_1\cdot k_2 + T\left(G_{B12}-4G^2_{B12}\right)k_2\cdot k_3\left(r_3\cdot f_1\cdot f_3\cdot r_4 k_1\cdot k_2 - r_4\cdot f_3\cdot k_1 r_3\cdot f_1\cdot k_2\right) + (1-6G_{B12})\left(r_4\cdot f_3\cdot k_2 r_3\cdot f_1\cdot k_3 + 2r_3\cdot f_1\cdot f_3\cdot r_4 k_3\cdot k_2\right)\Big], \tag{3.100}$$

$$\tilde{I}^{22}_{(34)}(12,34) = \frac{8}{3}G_{B12}Z_2(12)Z_2(34), \tag{3.101}$$

$$\tilde{I}^{22}_{(34)}(13,24) = \frac{4}{9}Z_2(13)Z_2(24), \tag{3.102}$$

$$\tilde{I}^{22}_{(34)}(14,23) = \frac{4}{9}Z_2(14)Z_2(23). \tag{3.103}$$

3.4 Four low energy photons

Now, we jump to the case where we consider all photons to be in the low energy limit, because the case of three photons of low energy is equivalent as the case with four in the low energy limit. Since when we have three legs in the low energy limit, the remaining leg must be in this limit too because of momentum conservation ($k_1+k_2+k_3+k_4=0$). Then, systematically, we take the low energy limit in k_2 and we integrate over u_2 and finally, we do the same for k_1. Hence, in this case $I(Q_4)$ and $\tilde{I}(\tilde{Q}_4)$ are simply

$$I(Q_4) = I_{(1234)}, \tag{3.104}$$

$$\tilde{I}(\tilde{Q}_4) = \tilde{I}_{(1234)}. \tag{3.105}$$

3.4.1 Scalar result for the four low energy photons

In this section, we write the explicit form of $I_{(1234)}$. (The sum of all the terms, in this section, is $I_{(1234)}$.)

$$I^{4}_{(1234)}(1234) = \frac{1}{45}Z_4(1234), \tag{3.106}$$

$$I^{4}_{(1234)}(1243) = \frac{1}{45}Z_4(1243), \tag{3.107}$$

$$I^4_{(1234)}(1324) = \frac{1}{45} Z_4(1324), \tag{3.108}$$

$$I^{22}_{(1234)}(12,34) = \frac{1}{9} Z_2(12) Z_2(34), \tag{3.109}$$

$$I^{22}_{(1234)}(13,24) = \frac{1}{9} Z_2(13) Z_2(24), \tag{3.110}$$

$$I^{22}_{(1234)}(14,23) = \frac{1}{9} Z_2(14) Z_2(23). \tag{3.111}$$

3.4.2 Spinor result for the four low energy photons

In this section, we write the explicit form of $\tilde{I}_{(1234)}$. (The sum of all the terms, in this section, is $\tilde{I}_{(1234)}$.)

$$\tilde{I}^4_{(1234)}(1234) = \frac{-14}{45} Z_4(1234), \tag{3.112}$$

$$\tilde{I}^4_{(1234)}(1243) = \frac{-14}{45} Z_4(1243), \tag{3.113}$$

$$\tilde{I}^4_{(1234)}(1324) = \frac{-14}{45} Z_4(1324), \tag{3.114}$$

$$\tilde{I}^{22}_{(1234)}(12,34) = \frac{4}{9} Z_2(12) Z_2(34), \tag{3.115}$$

$$\tilde{I}^{22}_{(1234)}(13,24) = \frac{4}{9} Z_2(13) Z_2(24), \tag{3.116}$$

$$\tilde{I}^{22}_{(1234)}(14,23) = \frac{4}{9} Z_2(14) Z_2(23). \tag{3.117}$$

In the next chapters, we will solve the integrals that remains and we will obtain the amplitude in explicit form for each of those cases, except for the case where all photons have arbitrary momentum.

Chapter 4

Low energy limit in one external leg

In the previous chapter, we performed the integral over u_4 with the assumption that the corresponding photon was in the low energy limit (section 3.2). We are left with the following integrals

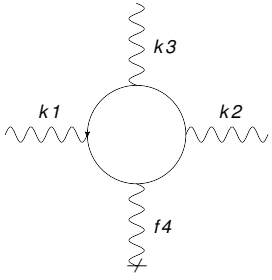

Figure 4.1: *Feynman diagram for the four-photon amplitude with the fourth photon in the low energy limit.*

$$I^l_{T[123;4)} = \int_0^1 \int_0^1 \int_0^1 du_1 du_2 du_3 \int_0^\infty \frac{dT}{T} T^{4-\frac{D}{2}} e^{-T(m^2 - \frac{1}{2}\sum_{i,j}^3 G_{Bij} k_i \cdot k_j)} I^l_{(4)}, \tag{4.1}$$

that we need to compute in order to obtain the scalar amplitude. For the spinor amplitude, we have a similar expression for $\tilde{I}^l_{T[123;4)}$. In these

expressions we use the super-index $l = 4, 3, 2, 22$ to make reference to all the equations in section 3.2, the subindex "T" tell us that the integral over the proper time is included,. In fact, we already perform this integral. Moreover, the subindex $[123; 4)$ tell us that the fourth integral is performed with the assumption that the corresponding photon is in the low energy limit and the other three integrals are performed without any condition, since we use the notation "$(\ldots)$" for low energy integration and "$[\ldots]$" for integration without conditions.

Notice that the most general integral that we want to solve looks like:

$$\begin{aligned}
&\int_0^1\int_0^1\int_0^1 du_1du_2du_3\int_0^\infty \frac{dT}{T}T^{n-\frac{D}{2}}e^{-T(m^2-\frac{1}{2}\sum G_{Brs}k_r\cdot k_s)}G^l_{B12}G^i_{B13}G^j_{B23}\\
&=\Gamma\left(n-\frac{D}{2}\right)\int_0^1\int_0^1\int_0^1\frac{du_1du_2du_3G^l_{B12}G^i_{B13}G^j_{B23}}{(m^2-G_{B12}k_1\cdot k_2-G_{B13}k_1\cdot k_3-G_{B23}k_2\cdot k_3)^{n-\frac{D}{2}}}\\
&=\frac{\Gamma\left(n-\frac{D}{2}\right)}{m^{2n-D}}\int_0^1\int_0^1\int_0^1\frac{du_1du_2du_3G^l_{B12}G^i_{B13}G^j_{B23}}{(1-G_{B12}x-G_{B13}y-G_{B23}z)^{n-\frac{D}{2}}}\equiv Y_{nlij},
\end{aligned} \tag{4.2}$$

where $x = k_1\cdot k_2/m^2$, $y = k_1\cdot k_3/m^2$, $z = k_2\cdot k_3/m^2$ and Y_{nlij} can be rewritten as

$$\begin{aligned}
Y_{nlij} &= \frac{\Gamma\left(n-\frac{D}{2}\right)}{m^{2n-D}(n-l-\frac{D}{2})_l(n-l-i-\frac{D}{2})_i(n-l-i-j-\frac{D}{2})_j}\\
&\times\frac{\partial^l}{\partial x^l}\frac{\partial^i}{\partial y^i}\frac{\partial^j}{\partial z^j}\int_0^1\int_0^1\int_0^1\frac{du_1du_2du_3}{(1-G_{B12}x-G_{B13}y-G_{B23}z)^{n-l-i-j-\frac{D}{2}}}\\
&\equiv\partial^l_x\partial^i_y\partial^j_z K_n,
\end{aligned} \tag{4.3}$$

in which $(a)_n$ is the Pochhammer symbol, defined as

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1),\quad (a)_0 = 1, \tag{4.4}$$

for general a and nonnegative integer n. Now, we define

$$K_n \equiv \frac{\Gamma\left(n-\frac{D}{2}\right)}{m^{2n-D}}\int_0^1\int_0^1\int_0^1\frac{du_1du_2du_3}{(1-G_{B12}x-G_{B13}y-G_{B23}z)^{n-\frac{D}{2}}}, \tag{4.5}$$

and the operator "∂" acting on K_n such as

$$\partial_x^l K_n \equiv \frac{\Gamma\left(n-\frac{D}{2}\right)}{m^{2n-D}(n-l-\frac{D}{2})_l}\frac{\partial^l}{\partial x^l}\int_0^1\int_0^1\int_0^1 \frac{du_1 du_2 du_3}{(1-G_{B12}x-G_{B13}y-G_{B23}z)^{n-l-\frac{D}{2}}}, \tag{4.6}$$

similarly for y and z. This is the equivalent of "tensor reduction" in this formalism. It is important to mention that the analytic result for the triple integral in (4.5) is already known [41]. Therefore, we can now write all in section 3.2 in terms of the three point function K_n and its derivatives.

4.1 Explicit result for scalar case

In this section, we write the explicit result for $I_{T[123;4)}$, in terms of ∂_x, ∂_y, ∂_z and K_n. (The sum of all the "I" terms, in this section, is $I_{T[123;4)}$.)

$$I^4_{T[123;4)}(1234) = \left(2\partial_x\frac{k_2\cdot k_1}{k_2\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_2\cdot k_1}+b\right)\left(\frac{1}{3}-2\partial_y\right)K_4 Z_4(1234) \tag{4.7}$$

$$I^4_{T[123;4)}(1243) = \left(2\partial_x\frac{k_1\cdot k_2}{k_1\cdot k_3}+2\partial_y\frac{k_1\cdot k_3}{k_1\cdot k_2}+a\right)\left(\frac{1}{3}-2\partial_z\right)K_4 Z_4(1243) \tag{4.8}$$

$$I^4_{T[123;4)}(1324) = \left(2\partial_y\frac{k_1\cdot k_3}{k_2\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_1\cdot k_3}+c\right)\left(\frac{1}{3}-2\partial_x\right)K_4 Z_4(1324), \tag{4.9}$$

$$\begin{aligned} I^3_{T[123;4)}(123) =& \frac{-1}{3}Z_3(123)\left[\left(2\partial_y\frac{k_1\cdot k_3}{k_2\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_1\cdot k_3}+c\right)\left(\partial_x-4\partial_x^2\right)k_2\cdot f_4\cdot k_1\right.\\ &+\left(2\partial_x\frac{k_1\cdot k_2}{k_1\cdot k_3}+2\partial_y\frac{k_1\cdot k_3}{k_1\cdot k_2}+a\right)\left(\partial_z-4\partial_z^2\right)k_3\cdot f_4\cdot k_2\\ &+\left.\left(2\partial_x\frac{k_2\cdot k_1}{k_2\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_2\cdot k_1}+b\right)\left(\partial_y-4\partial_y^2\right)k_1\cdot f_4\cdot k_3\right]K_5, \end{aligned} \tag{4.10}$$

$$\begin{aligned} I^3_{T[123;4)}(234) =& \left(\frac{1}{3}-2\partial_z\right)Z_3(234)\left[\left(2\partial_x\frac{k_2\cdot k_1}{k_2\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_2\cdot k_1}+b\right)\frac{r_1\cdot f_1\cdot k_2}{r_1\cdot k_1}\right.\\ &-\left.\left(2\partial_y\frac{k_1\cdot k_3}{k_2\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_1\cdot k_3}+c\right)\frac{r_1\cdot f_1\cdot k_3}{r_1\cdot k_1}\right]K_4, \end{aligned} \tag{4.11}$$

$$\begin{aligned} I^3_{T[123;4)}(341) =& \left(\frac{1}{3}-2\partial_y\right)Z_3(341)\left[\left(2\partial_x\frac{k_1\cdot k_2}{k_1\cdot k_3}+2\partial_y\frac{k_1\cdot k_3}{k_1\cdot k_2}+a\right)\frac{r_2\cdot f_2\cdot k_1}{r_2\cdot k_2}\right.\\ &-\left.\left(2\partial_y\frac{k_1\cdot k_3}{k_2\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_1\cdot k_3}+c\right)\frac{r_2\cdot f_2\cdot k_3}{r_2\cdot k_2}\right]K_4, \end{aligned} \tag{4.12}$$

$$\begin{aligned} I^3_{T[123;4)}(412) =& \left(\frac{1}{3}-2\partial_x\right)Z_3(412)\left[\left(2\partial_x\frac{k_1\cdot k_2}{k_1\cdot k_3}+2\partial_y\frac{k_1\cdot k_3}{k_1\cdot k_2}+a\right)\frac{r_3\cdot f_3\cdot k_1}{r_3\cdot k_3}\right.\\ &-\left.\left(2\partial_x\frac{k_2\cdot k_1}{k_2\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_2\cdot k_1}+b\right)\frac{r_3\cdot f_3\cdot k_2}{r_3\cdot k_3}\right]K_4, \end{aligned} \tag{4.13}$$

$$
\begin{aligned}
I^2_{T[123;4)}(12) = {} & \frac{-1}{3}\left(1-4\partial_x\right) Z_2(12) \\
& \times \Big\{ \left(\partial_y - 4\partial_y^2\right) k_1\cdot f_4\cdot f_3\cdot k_1 + \left(\partial_z - 4\partial_z^2\right) k_2\cdot f_4\cdot f_3\cdot k_2 \\
& + \left(2\partial_y \frac{k_1\cdot k_3}{k_2\cdot k_3} + 2\partial_z \frac{k_2\cdot k_3}{k_1\cdot k_3} + c\right)\left[\partial_y k_1\cdot f_4\cdot f_3\cdot k_2 + \partial_z k_2\cdot f_4\cdot f_3\cdot k_1\right] \\
& - \left(2\partial_x \frac{k_1\cdot k_2}{k_1\cdot k_3} + 2\partial_y \frac{k_1\cdot k_3}{k_1\cdot k_2} + a\right)\partial_x \frac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_1}{r_4\cdot k_3} \\
& + \left(2\partial_x \frac{k_2\cdot k_1}{k_2\cdot k_3} + 2\partial_z \frac{k_2\cdot k_3}{k_2\cdot k_1} + b\right)\partial_x \frac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_2}{r_4\cdot k_3}\Big\} K_5,
\end{aligned}
\tag{4.14}
$$

$$
\begin{aligned}
I^2_{T[123;4)}(13) = {} & \frac{-1}{3}\left(1-4\partial_y\right) Z_2(13) \\
& \times \Big\{ \left(\partial_x - 4\partial_x^2\right) k_1\cdot f_4\cdot f_2\cdot k_1 + \left(\partial_z - 4\partial_z^2\right) k_3\cdot f_4\cdot f_2\cdot k_3 \\
& + \left(2\partial_x \frac{k_2\cdot k_1}{k_2\cdot k_3} + 2\partial_z \frac{k_2\cdot k_3}{k_2\cdot k_1} + b\right)\left[\partial_x k_1\cdot f_4\cdot f_2\cdot k_3 + \partial_z k_3\cdot f_4\cdot f_2\cdot k_1\right] \\
& - \left(2\partial_x \frac{k_1\cdot k_2}{k_1\cdot k_3} + 2\partial_y \frac{k_1\cdot k_3}{k_1\cdot k_2} + a\right)\partial_y \frac{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_1}{r_5\cdot k_2} \\
& + \left(2\partial_y \frac{k_1\cdot k_3}{k_2\cdot k_3} + 2\partial_z \frac{k_2\cdot k_3}{k_1\cdot k_3} + c\right)\partial_y \frac{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_3}{r_5\cdot k_2}\Big\} K_5,
\end{aligned}
\tag{4.15}
$$

$$
\begin{aligned}
I^2_{T[123;4)}(23) = {} & \frac{-T}{3}\left(1-4\partial_z\right) Z_2(23) \\
& \times \Big\{ \left(\partial_x - 4\partial_x^2\right) k_2\cdot f_4\cdot f_1\cdot k_2 + \left(\partial_y - 4\partial_y^2\right) k_3\cdot f_4\cdot f_1\cdot k_3 \\
& + \left(2\partial_x \frac{k_1\cdot k_2}{k_1\cdot k_3} + 2\partial_y \frac{k_1\cdot k_3}{k_1\cdot k_2} + a\right)\left[\partial_x k_2\cdot f_4\cdot f_1\cdot k_3 + \partial_y k_3\cdot f_4\cdot f_1\cdot k_2\right] \\
& - \left(2\partial_x \frac{k_2\cdot k_1}{k_2\cdot k_3} + 2\partial_z \frac{k_2\cdot k_3}{k_2\cdot k_1} + b\right)\partial_z \frac{k_3\cdot f_4\cdot k_2 r_6\cdot f_1\cdot k_2}{r_6\cdot k_1} \\
& + \left(2\partial_y \frac{k_1\cdot k_3}{k_2\cdot k_3} + 2\partial_z \frac{k_2\cdot k_3}{k_1\cdot k_3} + c\right)\partial_z \frac{k_3\cdot f_4\cdot k_2 r_6\cdot f_1\cdot k_3}{r_6\cdot k_1}\Big\} K_5,
\end{aligned}
\tag{4.16}
$$

$$
\begin{aligned}
I^2_{T[123;4)}(14) = {} & -\frac{Z_2(14)}{3 r_7\cdot k_2 r_8\cdot k_3}\left[\left(2\partial_x \frac{k_1\cdot k_2}{k_1\cdot k_3} + 2\partial_y \frac{k_1\cdot k_3}{k_1\cdot k_2} + a\right) r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_1\right. \\
& - \left(2\partial_y \frac{k_1\cdot k_3}{k_2\cdot k_3} + 2\partial_z \frac{k_2\cdot k_3}{k_1\cdot k_3} + c\right)\left(r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_3 + 2 r_7\cdot f_2\cdot f_3\cdot r_8 k_3\cdot k_1\right) \\
& \left. - \left(2\partial_x \frac{k_2\cdot k_1}{k_2\cdot k_3} + 2\partial_z \frac{k_2\cdot k_3}{k_2\cdot k_1} + b\right)\left(r_8\cdot f_3\cdot k_2 r_7\cdot f_2\cdot k_1 - r_7\cdot f_2\cdot f_3\cdot r_8 k_2\cdot k_1\right)\right] K_4,
\end{aligned}
\tag{4.17}
$$

$$I^2_{T[123;4)}(24) = -\frac{Z_2(24)}{3r_9\cdot k_1 r_{10}\cdot k_3}\left[\left(2\partial_x\frac{k_2\cdot k_1}{k_2\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_2\cdot k_1}+b\right)r_{10}\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_2\right.$$
$$-\left(2\partial_y\frac{k_1\cdot k_3}{k_2\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_1\cdot k_3}+c\right)(r_{10}\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_3+2r_9\cdot f_1\cdot f_3\cdot r_{10}k_3\cdot k_2)$$
$$\left.-\left(2\partial_x\frac{k_1\cdot k_2}{k_1\cdot k_3}+2\partial_y\frac{k_1\cdot k_3}{k_1\cdot k_2}+a\right)(r_{10}\cdot f_3\cdot k_1 r_9\cdot f_1\cdot k_2-r_9\cdot f_1\cdot f_3\cdot r_{10}k_1\cdot k_2)\right]K_4, \tag{4.18}$$

$$I^2_{T[123;4)}(34) = -\frac{Z_2(34)}{3r_{11}\cdot k_1 r_{12}\cdot k_2}\left[\left(2\partial_y\frac{k_1\cdot k_3}{k_2\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_1\cdot k_3}+c\right)r_{12}\cdot f_2\cdot k_3 r_{11}\cdot f_1\cdot k_3\right.$$
$$-\left(2\partial_x\frac{k_2\cdot k_1}{k_2\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_2\cdot k_1}+b\right)(r_{12}\cdot f_2\cdot k_3 r_{11}\cdot f_1\cdot k_2+2r_{11}\cdot f_1\cdot f_2\cdot r_{12}k_2\cdot k_3)$$
$$\left.-\left(2\partial_x\frac{k_1\cdot k_2}{k_1\cdot k_3}+2\partial_y\frac{k_1\cdot k_3}{k_1\cdot k_2}+a\right)(r_{12}\cdot f_2\cdot k_1 r_{11}\cdot f_1\cdot k_3-r_{11}\cdot f_1\cdot f_2\cdot r_{12}k_1\cdot k_3)\right]K_4, \tag{4.19}$$

$$I^{22}_{T[123;4)}(12,34) = \frac{1}{3}\left(1-4\partial_x\right)K_4 Z_2(12)Z_2(34), \tag{4.20}$$

$$I^{22}_{T[123;4)}(13,24) = \frac{1}{3}\left(1-4\partial_y\right)K_4 Z_2(13)Z_2(24), \tag{4.21}$$

$$I^{22}_{T[123;4)}(14,23) = \frac{1}{3}\left(1-4\partial_z\right)K_4 Z_2(14)Z_2(23). \tag{4.22}$$

Then the amplitude for this case looks like

$$\Gamma_{scal}[k_1,\varepsilon_1;...;k_4,\varepsilon_4] = \frac{e^4}{(4\pi)^{\frac{D}{2}}}I_{T[123;4)}. \tag{4.23}$$

4.2 Explicit result for spinor case

In this section, we write the explicit result for $\tilde{I}_{T[123;4)}$, in terms of ∂_x, ∂_y, ∂_z and K_n. (The sum of all the "$\tilde{I}$" terms, in this section, is $\tilde{I}_{T[123;4)}$.)

$$\tilde{I}^4_{T[123;4)}(1234) = Z_4(1234)\left[\left(2\partial_x\frac{k_2\cdot k_1}{k_2\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_2\cdot k_1}+b\right)\left(\frac{1}{3}-2\partial_y\right)\right.$$
$$\left.+\left(2\partial_x\frac{k_1\cdot k_2}{k_1\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_1\cdot k_3}\right)+2\partial_y\left(\frac{k_1\cdot k_3}{k_1\cdot k_2}+\frac{k_1\cdot k_3}{k_2\cdot k_3}+2\right)+a+c-1\right]K_4 \tag{4.24}$$

$$\tilde{I}^4_{T[123;4)}(1243) = Z_4(1243)\left[\left(2\partial_x\frac{k_1\cdot k_2}{k_1\cdot k_3}+2\partial_y\frac{k_1\cdot k_3}{k_1\cdot k_2}+a\right)\left(\frac{1}{3}-2\partial_z\right)\right.$$
$$\left.+\,2\left(\partial_z-\partial_x-\partial_y\right)-\left(2\partial_x\frac{k_1\cdot k_2}{k_1\cdot k_3}+2\partial_y\frac{k_1\cdot k_3}{k_1\cdot k_2}+a\right)\right]K_4 \tag{4.25}$$

$$\begin{aligned}\tilde{I}^4_{T[123;4)}(1324) = Z_4(1324)\Bigg[&\left(2\partial_y\frac{k_1\cdot k_3}{k_2\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_1\cdot k_3}+c\right)\left(\frac{1}{3}-2\partial_x\right)\\ &+2\partial_x\left(\frac{k_1\cdot k_2}{k_1\cdot k_3}+\frac{k_2\cdot k_1}{k_2\cdot k_3}+2\right)+\left(2\partial_y\frac{k_1\cdot k_3}{k_1\cdot k_2}+2\partial_z\frac{k_2\cdot k_3}{k_2\cdot k_1}\right)+a+b-1\Bigg]K_4,\end{aligned} \tag{4.26}$$

$$\begin{aligned}\tilde{I}^3_{T[123;4)}(123) = \frac{-1}{3}Z_3(123)\Big[&-\left(\partial_x-4\partial_x^2\right)k_2\cdot f_4\cdot k_1\\ &-\left(\partial_z-4\partial_z^2\right)k_3\cdot f_4\cdot k_2-\left(\partial_y-4\partial_y^2\right)k_1\cdot f_4\cdot k_3\\ &+\left(2\partial_y\frac{k_1\cdot k_3}{k_2\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_1\cdot k_3}+c\right)\left(\partial_x-4\partial_x^2\right)k_2\cdot f_4\cdot k_1\\ &+\left(2\partial_x\frac{k_1\cdot k_2}{k_1\cdot k_3}+2\partial_y\frac{k_1\cdot k_3}{k_1\cdot k_2}+a\right)\left(\partial_z-4\partial_z^2\right)k_3\cdot f_4\cdot k_2\\ &+\left(2\partial_x\frac{k_2\cdot k_1}{k_2\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_2\cdot k_1}+b\right)\left(\partial_y-4\partial_y^2\right)k_1\cdot f_4\cdot k_3\\ &+\left(2\partial_x\frac{k_2\cdot k_1}{k_2\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_2\cdot k_1}+b\right)\partial_z k_3\cdot f_4\cdot k_2\\ &+\left(2\partial_x\frac{k_1\cdot k_2}{k_1\cdot k_3}+2\partial_y\frac{k_1\cdot k_3}{k_1\cdot k_2}+a\right)\partial_y k_1\cdot f_4\cdot k_3\\ &+\left(2\partial_x\frac{k_2\cdot k_1}{k_2\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_2\cdot k_1}+b\right)\partial_x k_2\cdot f_4\cdot k_1\\ &+\left(2\partial_y\frac{k_1\cdot k_3}{k_2\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_1\cdot k_3}+c\right)\partial_y k_1\cdot f_4\cdot k_3\\ &+\left(2\partial_x\frac{k_1\cdot k_2}{k_1\cdot k_3}+2\partial_y\frac{k_1\cdot k_3}{k_1\cdot k_2}+a\right)\partial_x k_2\cdot f_4\cdot k_1\\ &+\left(2\partial_y\frac{k_1\cdot k_3}{k_2\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_1\cdot k_3}+c\right)\partial_z k_3\cdot f_4\cdot k_2\Big]K_5,\end{aligned} \tag{4.27}$$

$$\begin{aligned}\tilde{I}^3_{T[123;4)}(234) = \left(\frac{2}{3}+2\partial_z\right)Z_3(234)\Bigg[&\left(2\partial_y\frac{k_1\cdot k_3}{k_2\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_1\cdot k_3}+c\right)\frac{r_1\cdot f_1\cdot k_3}{r_1\cdot k_1}\\ &-\left(2\partial_x\frac{k_2\cdot k_1}{k_2\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_2\cdot k_1}+b\right)\frac{r_1\cdot f_1\cdot k_2}{r_1\cdot k_1}\Bigg]K_4,\end{aligned} \tag{4.28}$$

$$\begin{aligned}\tilde{I}^3_{T[123;4)}(341) = \left(\frac{2}{3}+2\partial_y\right)Z_3(341)\Bigg[&\left(2\partial_y\frac{k_1\cdot k_3}{k_2\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_1\cdot k_3}+c\right)\frac{r_2\cdot f_2\cdot k_3}{r_2\cdot k_2}\\ &-\left(2\partial_x\frac{k_1\cdot k_2}{k_1\cdot k_3}+2\partial_y\frac{k_1\cdot k_3}{k_1\cdot k_2}+a\right)\frac{r_2\cdot f_2\cdot k_1}{r_2\cdot k_2}\Bigg]K_4,\end{aligned} \tag{4.29}$$

$$\begin{aligned}\tilde{I}^3_{T[123;4)}(412) = \left(\frac{2}{3}+2\partial_x\right)Z_3(412)\Bigg[&\left(2\partial_x\frac{k_2\cdot k_1}{k_2\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_2\cdot k_1}+b\right)\frac{r_3\cdot f_3\cdot k_2}{r_3\cdot k_3}\\ &-\left(2\partial_x\frac{k_1\cdot k_2}{k_1\cdot k_3}+2\partial_y\frac{k_1\cdot k_3}{k_1\cdot k_2}+a\right)\frac{r_3\cdot f_3\cdot k_1}{r_3\cdot k_3}\Bigg]K_4,\end{aligned} \tag{4.30}$$

$$\begin{aligned}\tilde{I}^2_{T[123;4)}(12) = \frac{4}{3}\partial_x Z_2(12)\Big\{ &\left(\partial_y - 4\partial_y^2\right) k_1\cdot f_4\cdot f_3\cdot k_1 + \left(\partial_z - 4\partial_z^2\right) k_2\cdot f_4\cdot f_3\cdot k_2\\ &+\left(2\partial_y\frac{k_1\cdot k_3}{k_2\cdot k_3} + 2\partial_z\frac{k_2\cdot k_3}{k_1\cdot k_3} + c\right)\left[\partial_y k_1\cdot f_4\cdot f_3\cdot k_2 + \partial_z k_2\cdot f_4\cdot f_3\cdot k_1\right]\\ &-\left(2\partial_x\frac{k_1\cdot k_2}{k_1\cdot k_3} + 2\partial_y\frac{k_1\cdot k_3}{k_1\cdot k_2} + a\right)\partial_x\frac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_1}{r_4\cdot k_3}\\ &+\left(2\partial_x\frac{k_2\cdot k_1}{k_2\cdot k_3} + 2\partial_z\frac{k_2\cdot k_3}{k_2\cdot k_1} + b\right)\partial_x\frac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_2}{r_4\cdot k_3}\Big\} K_5,\end{aligned} \tag{4.31}$$

$$\begin{aligned}\tilde{I}^2_{T[123;4)}(13) = \frac{4}{3}\partial_y Z_2(13)\Big\{ &\left(\partial_x - 4\partial_x^2\right) k_1\cdot f_4\cdot f_2\cdot k_1 + \left(\partial_z - 4\partial_z^2\right) k_3\cdot f_4\cdot f_2\cdot k_3\\ &+\left(2\partial_x\frac{k_2\cdot k_1}{k_2\cdot k_3} + 2\partial_z\frac{k_2\cdot k_3}{k_2\cdot k_1} + b\right)\left[\partial_x k_1\cdot f_4\cdot f_2\cdot k_3 + \partial_z k_3\cdot f_4\cdot f_2\cdot k_1\right]\\ &-\left(2\partial_x\frac{k_1\cdot k_2}{k_1\cdot k_3} + 2\partial_y\frac{k_1\cdot k_3}{k_1\cdot k_2} + a\right)\partial_y\frac{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_1}{r_5\cdot k_2}\\ &+\left(2\partial_y\frac{k_1\cdot k_3}{k_2\cdot k_3} + 2\partial_z\frac{k_2\cdot k_3}{k_1\cdot k_3} + c\right)\partial_y\frac{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_3}{r_5\cdot k_2}\Big\} K_5,\end{aligned} \tag{4.32}$$

$$\begin{aligned}\tilde{I}^2_{T[123;4)}(23) = \frac{4}{3}\partial_z Z_2(23)\Big\{ &\left(\partial_x - 4\partial_x^2\right) k_2\cdot f_4\cdot f_1\cdot k_2 + \left(\partial_y - 4\partial_y^2\right) k_3\cdot f_4\cdot f_1\cdot k_3\\ &+\left(2\partial_x\frac{k_1\cdot k_2}{k_1\cdot k_3} + 2\partial_y\frac{k_1\cdot k_3}{k_1\cdot k_2} + a\right)\left[\partial_x k_2\cdot f_4\cdot f_1\cdot k_3 + \partial_y k_3\cdot f_4\cdot f_1\cdot k_2\right]\\ &-\left(2\partial_x\frac{k_2\cdot k_1}{k_2\cdot k_3} + 2\partial_z\frac{k_2\cdot k_3}{k_2\cdot k_1} + b\right)\partial_z\frac{k_3\cdot f_4\cdot k_2 r_6\cdot f_1\cdot k_2}{r_6\cdot k_1}\\ &+\left(2\partial_y\frac{k_1\cdot k_3}{k_2\cdot k_3} + 2\partial_z\frac{k_2\cdot k_3}{k_1\cdot k_3} + c\right)\partial_z\frac{k_3\cdot f_4\cdot k_2 r_6\cdot f_1\cdot k_3}{r_6\cdot k_1}\Big\} K_5,\end{aligned} \tag{4.33}$$

$$\begin{aligned}\tilde{I}^2_{T[123;4)}(14) = \frac{2Z_2(14)}{3r_7\cdot k_2 r_8\cdot k_3}\Bigg[&\left(2\partial_x\frac{k_1\cdot k_2}{k_1\cdot k_3} + 2\partial_y\frac{k_1\cdot k_3}{k_1\cdot k_2} + a\right) r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_1\\ &-\left(2\partial_y\frac{k_1\cdot k_3}{k_2\cdot k_3} + 2\partial_z\frac{k_2\cdot k_3}{k_1\cdot k_3} + c\right)\left(r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_3 + 2r_7\cdot f_2\cdot f_3\cdot r_8 k_3\cdot k_1\right)\\ &-\left(2\partial_x\frac{k_2\cdot k_1}{k_2\cdot k_3} + 2\partial_z\frac{k_2\cdot k_3}{k_2\cdot k_1} + b\right)\left(r_8\cdot f_3\cdot k_2 r_7\cdot f_2\cdot k_1 - r_7\cdot f_2\cdot f_3\cdot r_8 k_2\cdot k_1\right)\Bigg] K_4,\end{aligned} \tag{4.34}$$

$$\tilde{I}^2_{T[123;4)}(24) = \frac{2Z_2(24)}{3r_9\cdot k_1 r_{10}\cdot k_3}\left[\left(2\partial_x\frac{k_2\cdot k_1}{k_2\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_2\cdot k_1}+b\right)r_{10}\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_2\right.$$
$$-\left(2\partial_y\frac{k_1\cdot k_3}{k_2\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_1\cdot k_3}+c\right)(r_{10}\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_3+2r_9\cdot f_1\cdot f_3\cdot r_{10}k_3\cdot k_2)$$
$$\left.-\left(2\partial_x\frac{k_1\cdot k_2}{k_1\cdot k_3}+2\partial_y\frac{k_1\cdot k_3}{k_1\cdot k_2}+a\right)(r_{10}\cdot f_3\cdot k_1 r_9\cdot f_1\cdot k_2-r_9\cdot f_1\cdot f_3\cdot r_{10}k_1\cdot k_2)\right]K_4, \tag{4.35}$$

$$\tilde{I}^2_{T[123;4)}(34) = \frac{2Z_2(34)}{3r_{11}\cdot k_1 r_{12}\cdot k_2}\left[\left(2\partial_y\frac{k_1\cdot k_3}{k_2\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_1\cdot k_3}+c\right)r_{12}\cdot f_2\cdot k_3 r_{11}\cdot f_1\cdot k_3\right.$$
$$-\left(2\partial_x\frac{k_2\cdot k_1}{k_2\cdot k_3}+2\partial_z\frac{k_2\cdot k_3}{k_2\cdot k_1}+b\right)(r_{12}\cdot f_2\cdot k_3 r_{11}\cdot f_1\cdot k_2+2r_{11}\cdot f_1\cdot f_2\cdot r_{12}k_2\cdot k_3)$$
$$\left.-\left(2\partial_x\frac{k_1\cdot k_2}{k_1\cdot k_3}+2\partial_y\frac{k_1\cdot k_3}{k_1\cdot k_2}+a\right)(r_{12}\cdot f_2\cdot k_1 r_{11}\cdot f_1\cdot k_3-r_{11}\cdot f_1\cdot f_2\cdot r_{12}k_1\cdot k_3)\right]K_4, \tag{4.36}$$

$$\tilde{I}^{22}_{T[123;4)}(12,34) = \frac{8}{3}\partial_x K_4 Z_2(12)Z_2(34), \tag{4.37}$$

$$\tilde{I}^{22}_{T[123;4)}(13,24) = \frac{8}{3}\partial_y K_4 Z_2(13)Z_2(24), \tag{4.38}$$

$$\tilde{I}^{22}_{T[123;4)}(14,23) = \frac{8}{3}\partial_z K_4 Z_2(14)Z_2(23). \tag{4.39}$$

Similarly, the amplitude in this case is

$$\Gamma_{scal}[k_1,\varepsilon_1;...;k_4,\varepsilon_4] = \frac{-2e^4}{(4\pi)^{\frac{D}{2}}}\tilde{I}_{T[123;4)}, \tag{4.40}$$

4.3 How does K_n look like in the literature?

As we already mentioned, the exact result for the triple integral in K_n is given in [41] by J. Blümlein, K. G. Phan and T. Riemann. Hence, let's see how is it related the notation used in [41] against ours. For our particular case, they have

$$J_3 = J(d;\{p_i p_j\},\{m_i^2\}) = \int \frac{d^dk}{i\pi^{\frac{d}{2}}}\frac{1}{D_1D_2D_3}, \tag{4.41}$$

with inverse propagators $D_i = (k+q_i)^2 - m_i^2 + i\varepsilon$, momentum conservation $\sum_i^3 p_i = 0$ and all momenta to be incoming. Hence, we need to write J_3 in a similar way as (4.5). In the following, there are enlisted the steps to do this.

- First, notice that they use a different convention of the metric. They use $(+---)$, while we use $(-+++)$. So, if we change their convention of the metric to ours, we get $D_i = -(k+q_i)^2 - m_i^2 + i\varepsilon$.
- Second, J_3 is in the Minkowski space, so we change to Euclidean. This removes the factor of "i" in the integral.
- Third, we change p_i by k_i and k by l, just notation.
- Fourth, we choose $q_1 = 0$, $q_2 = -k_1$ and $q_3 = k_2$. Then, J_3 can be written as

$$J_3 = \int \frac{d^d l}{\pi^{\frac{d}{2}}} \frac{1}{(l^2+m^2)[(l-k_1)^2+m^2][(l+k_2)^2+m^2]}. \tag{4.42}$$

In Srednicki [42] it is shown, in section 16, that this integral can be written as

$$J_3 = \Gamma(3-\frac{d}{2}) \int_0^1 \int_0^1 \int_0^1 dx_1 dx_2 dx_3 \delta(x_1+x_2+x_3-1) \frac{1}{D_*^{3-\frac{d}{2}}}, \tag{4.43}$$

where

$$D_* = m^2 + x_1 x_3 k_1^2 + x_2 x_3 k_2^2 + x_1 x_2 k_3^2. \tag{4.44}$$

Using that $k_1 + k_2 + k_3 = 0$ and $x_1 + x_2 + x_3 = 1$, we can rewrite D_* as

$$D_* = m^2 - x_3(1-x_3)k_1 \cdot k_2 - x_1(1-x_1)k_1 \cdot k_3 - x_2(1-x_2)k_2 \cdot k_3. \tag{4.45}$$

If we fix an order and we use the change of variables

$$\begin{aligned} x_1 &= 1 - u_3, \\ x_2 &= u_3 - u_2, \\ x_3 &= u_2, \end{aligned} \tag{4.46}$$

we can write our Green functions as

$$\begin{aligned} G_{B12} &= x_3(1-x_3), \\ G_{B13} &= x_1(1-x_1), \\ G_{B23} &= x_2(1-x_2), \end{aligned} \tag{4.47}$$

here we have made use of the freedom to choose the zero somewhere on the "worldloop" for setting $u_1 = 0$. And now we can write D_* as

$$D_* = m^2 - G_{B12}k_1 \cdot k_2 - G_{B13}k_1 \cdot k_3 - G_{B23}k_2 \cdot k_3, \tag{4.48}$$

that is exactly what we have in the denominator of (4.5) and this tells us that J_3 and K_n are related by

$$K_n = J_3(D + 6 - 2n; \{k_i k_j\}, \{m^2\}). \tag{4.49}$$

Chapter 5

Low energy limit in two external legs

In section 3.3, we performed the integrals over u_4 and u_3 under the assumption that the corresponding photons were in the low energy limit and this leaves us with the following integrals

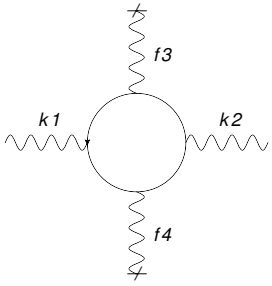

Figure 5.1: *Feynman diagram for the four-photon amplitude with the third and fourth photons in the low energy limit.*

$$I^i_{T[12;34)} = \int_0^1 \int_0^1 du_1 du_2 \int_0^\infty \frac{dT}{T} T^{4-\frac{D}{2}} e^{-T(m^2 - G_{B12} k_1 \cdot k_2)} I^i_{(34)}, \qquad (5.1)$$

for the scalar amplitude and equivalently $\tilde{I}^l_{T[12;34)}$ for the spinor amplitude. Similarly, we include the integral over T and the integrals over u_1 and

u_2 must be performed with arbitrary momenta. Because of the translation invariance in the loop we can fix the zero to be at the location of the second vertex operator, $u_2 = 0$ and $u_1 = u$. Hence, (5.1) transforms into

$$I^i_{T[12;34)} = \int_0^1 du \int_0^\infty \frac{dT}{T} T^{4-\frac{D}{2}} e^{-T(m^2 - G_B k_1\cdot k_2)} I^i_{(34)}, \tag{5.2}$$

(similarly for $\tilde{I}^l_{T[12;34)}$) where $G_B(u_1 = u, u_2 = 0) = u(1-u) = G_B$. Notice that the most general integral that we want to solve looks like

$$\begin{aligned} &\int_0^1 du \int_0^\infty \frac{dT}{T} T^{n-\frac{D}{2}} e^{-T(m^2 - G_B k_1\cdot k_2)} G_B^l \\ &= \Gamma\left(n - \frac{D}{2}\right) \int_0^1 du \frac{G_B^l}{(m^2 - G_B k_1\cdot k_2)^{n-\frac{D}{2}}} \equiv Y_{nl}, \end{aligned} \tag{5.3}$$

and this can be rewritten as

$$\begin{aligned} Y_{nl} &= \frac{\Gamma\left(n-\frac{D}{2}\right)}{m^{2n-D}} \int_0^1 du \frac{G_B^l}{(1-G_B x)^{n-\frac{D}{2}}} \\ &= \frac{\Gamma\left(n-\frac{D}{2}\right)}{m^{2n-D}(n-l-\frac{D}{2})_l} \frac{d^l}{dx^l} \int_0^1 du \frac{1}{(1-G_B x)^{n-l-\frac{D}{2}}}, \end{aligned} \tag{5.4}$$

with $x = k_1\cdot k_2/m^2$, $(a)_n$ the Pochhammer symbol, and we know that

$$\int_0^1 du \frac{1}{(1-G_B x)^\lambda} = {}_2F_1\left(1, \lambda; \frac{3}{2}; \frac{x}{4}\right), \tag{5.5}$$

where ${}_2F_1$ is an hypergeometric function whose series expansion is (see Appendix F)

$${}_2F_1(a,b;c;x) = \sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}. \tag{5.6}$$

Therefore, in the next two sections we write the solutions to our integrals in terms of what we call Y_{nl}. (In Appendix G there is an explicit form of some Y_{nl}'s, which are required in this chapter.)

5.1 Explicit result for scalar case

In this section, we write the explicit result for $I_{T[12;34)}$, in terms of Y_{nl}. (The sum of all the "I" terms, in this section, is $I_{T[12;34)}$.)

$$I^4_{T[12;34)}(1234) = \frac{2}{3}\left(Y_{41} - 4Y_{42}\right) Z_4(1234), \tag{5.7}$$

$$I^4_{T[12;34)}(1243) = \frac{2}{3}\left(Y_{41} - 4Y_{42}\right) Z_4(1243), \tag{5.8}$$

$$I^4_{T[12;34)}(1324) = \frac{1}{9}\left(Y_{40} - 12Y_{41} + 36Y_{42}\right) Z_4(1324), \tag{5.9}$$

$$I^3_{T[12;34)}(123) = \frac{-1}{9}\left(Y_{51} - 10Y_{52} + 24Y_{53}\right) Z_3(123) k_2 \cdot f_4 \cdot k_1, \tag{5.10}$$

$$I^3_{T[12;34)}(412) = \frac{-1}{9}\left(Y_{51} - 10Y_{52} + 24Y_{53}\right) Z_3(412) k_2 \cdot f_3 \cdot k_1, \tag{5.11}$$

$$\begin{aligned} I^2_{T[12;34)}(12) = \frac{-1}{90} Z_2(12) \big[& \left(Y_{50} - 4Y_{51}\right)\left(k_1 \cdot f_4 \cdot f_3 \cdot k_1 + k_2 \cdot f_4 \cdot f_3 \cdot k_2\right) \\ & + \left(Y_{50} - 4Y_{51} - 30Y_{52} + 120Y_{53}\right)\left(k_1 \cdot f_4 \cdot f_3 \cdot k_2 + k_2 \cdot f_4 \cdot f_3 \cdot k_1\right) \\ & + \left(10Y_{62} - 80Y_{63} + 160Y_{64}\right) k_2 \cdot f_4 \cdot k_1 k_2 \cdot f_3 \cdot k_1 \big], \end{aligned} \tag{5.12}$$

$$I^2_{T[12;34)}(13) = \frac{-1}{9} Z_2(13)\left(Y_{51} - 4Y_{52}\right) k_1 \cdot f_4 \cdot f_2 \cdot k_1, \tag{5.13}$$

$$I^2_{T[12;34)}(23) = \frac{-1}{9} Z_2(23)\left(Y_{51} - 4Y_{52}\right) k_2 \cdot f_4 \cdot f_1 \cdot k_2, \tag{5.14}$$

$$\begin{aligned} I^2_{T[12;34)}(14) = \frac{Z_2(14)}{9 r_1 \cdot k_2 r_2 \cdot k_3} \big[& \left(Y_{51} - 4Y_{52}\right) k_2 \cdot k_3 r_2 \cdot f_3 \cdot k_1 r_1 \cdot f_2 \cdot k_1 \\ & + \left(Y_{51} - 4Y_{52}\right) k_3 \cdot k_1 \left(r_1 \cdot f_2 \cdot f_3 \cdot r_2 k_2 \cdot k_1 - r_2 \cdot f_3 \cdot k_2 r_1 \cdot f_2 \cdot k_1\right) \\ & + \left(Y_{40} - 6Y_{41}\right)\left(r_2 \cdot f_3 \cdot k_1 r_1 \cdot f_2 \cdot k_3 + 2 r_1 \cdot f_2 \cdot f_3 \cdot r_2 k_3 \cdot k_1\right) \big], \end{aligned} \tag{5.15}$$

$$\begin{aligned} I^2_{T[12;34)}(24) = \frac{Z_2(24)}{9 r_3 \cdot k_1 r_4 \cdot k_3} \big[& \left(Y_{51} - 4Y_{52}\right) k_3 \cdot k_1 r_4 \cdot f_3 \cdot k_2 r_3 \cdot f_1 \cdot k_2 \\ & + \left(Y_{51} - 4Y_{52}\right) k_2 \cdot k_3 \left(r_3 \cdot f_1 \cdot f_3 \cdot r_4 k_1 \cdot k_2 - r_4 \cdot f_3 \cdot k_1 r_3 \cdot f_1 \cdot k_2\right) \\ & + \left(Y_{40} - 6Y_{41}\right)\left(r_4 \cdot f_3 \cdot k_2 r_3 \cdot f_1 \cdot k_3 + 2 r_3 \cdot f_1 \cdot f_3 \cdot r_4 k_3 \cdot k_2\right) \big], \end{aligned} \tag{5.16}$$

$$I^{22}_{T[12;34)}(12,34) = \frac{1}{3}\left(Y_{40} - 4Y_{41}\right) Z_2(12) Z_2(34), \tag{5.17}$$

$$I^{22}_{T[12;34)}(13,24) = \frac{1}{9} Y_{40} Z_2(13) Z_2(24), \tag{5.18}$$

$$I^{22}_{T[12;34)}(14,23) = \frac{1}{9} Y_{40} Z_2(14) Z_2(23). \tag{5.19}$$

Then, the amplitude in this case is

$$\Gamma_{scal}[k_1,\varepsilon_1;...;k_4,\varepsilon_4] = \frac{e^4}{(4\pi)^{\frac{D}{2}}} I_{T[12;34)}. \tag{5.20}$$

5.2 Explicit result for spinor case

In this section, we write the explicit result for $\tilde{I}_{T[12;34)}$, in terms of Y_{nl}. (The sum of all the "$\tilde{I}$" terms, in this section, is $\tilde{I}_{T[12;34)}$.)

$$\tilde{I}^4_{T[12;34)}(1234) = -\frac{4}{3}\left(Y_{41} - 2Y_{42}\right) Z_4(1234), \tag{5.21}$$

$$\tilde{I}^4_{T[12;34)}(1243) = -\frac{4}{3}\left(Y_{41} - 2Y_{42}\right) Z_4(1243), \tag{5.22}$$

$$\tilde{I}^4_{T[12;34)}(1324) = -\frac{4}{9}\left(2Y_{40} - 6Y_{41} - 9Y_{42}\right) Z_4(1324), \tag{5.23}$$

$$\tilde{I}^3_{T[12;34)}(123) = \frac{2}{9}\left(Y_{51} - Y_{52} - 12Y_{53}\right) Z_3(123) k_2 \cdot f_4 \cdot k_1, \tag{5.24}$$

$$\tilde{I}^3_{T[12;34)}(412) = \frac{2}{9}\left(Y_{51} - Y_{52} - 12Y_{53}\right) Z_3(412) k_2 \cdot f_3 \cdot k_1, \tag{5.25}$$

$$\begin{aligned}\tilde{I}^2_{T[12;34)}(12) =& \frac{1}{3} Z_2(12) \left[\frac{2}{15} Y_{50}\left(k_1 \cdot f_4 \cdot f_3 \cdot k_1 + k_2 \cdot f_4 \cdot f_3 \cdot k_2\right)\right. \\ &+ \frac{2}{15}\left(Y_{51} - 30Y_{53}\right)\left(k_1 \cdot f_4 \cdot f_3 \cdot k_2 + k_2 \cdot f_4 \cdot f_3 \cdot k_1\right) \\ &\left.+ \frac{4}{3}\left(Y_{63} - 4Y_{64}\right) k_2 \cdot f_4 \cdot k_1 k_2 \cdot f_3 \cdot k_1\right],\end{aligned} \tag{5.26}$$

$$\tilde{I}^2_{T[12;34)}(13) = \frac{2}{9} Z_2(13)\left(Y_{51} - 4Y_{52}\right) k_1 \cdot f_4 \cdot f_2 \cdot k_1, \tag{5.27}$$

$$\tilde{I}^2_{T[12;34)}(23) = \frac{2}{9} Z_2(23)\left(Y_{51} - 4Y_{52}\right) k_2 \cdot f_4 \cdot f_1 \cdot k_2, \tag{5.28}$$

$$\begin{aligned}\tilde{I}^2_{T[12;34)}(14) =& \frac{-2Z_2(14)}{9 r_1 \cdot k_2 r_2 \cdot k_3}\left[\left(Y_{51} - 4Y_{52}\right) k_2 \cdot k_3 r_2 \cdot f_3 \cdot k_1 r_1 \cdot f_2 \cdot k_1\right. \\ &+ \left(Y_{51} - 4Y_{52}\right) k_3 \cdot k_1 \left(r_1 \cdot f_2 \cdot f_3 \cdot r_2 k_2 \cdot k_1 - r_2 \cdot f_3 \cdot k_2 r_1 \cdot f_2 \cdot k_1\right) \\ &\left.+ \left(Y_{40} - 6Y_{41}\right)\left(r_2 \cdot f_3 \cdot k_1 r_1 \cdot f_2 \cdot k_3 + 2 r_1 \cdot f_2 \cdot f_3 \cdot r_2 k_3 \cdot k_1\right)\right],\end{aligned} \tag{5.29}$$

$$\begin{aligned}\tilde{I}^{2}_{T[12;34)}(24) = \frac{-2Z_2(24)}{9r_3\cdot k_1 r_4\cdot k_3}\,&[(Y_{51}-4Y_{52})\,k_3\cdot k_1 r_4\cdot f_3\cdot k_2 r_3\cdot f_1\cdot k_2\\ &+(Y_{51}-4Y_{52})\,k_2\cdot k_3\,(r_3\cdot f_1\cdot f_3\cdot r_4 k_1\cdot k_2 - r_4\cdot f_3\cdot k_1 r_3\cdot f_1\cdot k_2)\\ &+(Y_{40}-6Y_{41})\,(r_4\cdot f_3\cdot k_2 r_3\cdot f_1\cdot k_3 + 2r_3\cdot f_1\cdot f_3\cdot r_4 k_3\cdot k_2)]\,,\end{aligned} \tag{5.30}$$

$$\tilde{I}^{22}_{T[12;34)}(12,34) = \frac{8}{3}Y_{41}Z_2(12)Z_2(34), \tag{5.31}$$

$$\tilde{I}^{22}_{T[12;34)}(13,24) = \frac{4}{9}Y_{40}Z_2(13)Z_2(24), \tag{5.32}$$

$$\tilde{I}^{22}_{T[12;34)}(14,23) = \frac{4}{9}Y_{40}Z_2(14)Z_2(23). \tag{5.33}$$

Similarly, the amplitude in this case is

$$\Gamma_{scal}[k_1,\varepsilon_1;...;k_4,\varepsilon_4] = \frac{-2e^4}{(4\pi)^{\frac{D}{2}}}\tilde{I}_{T[12;34)}. \tag{5.34}$$

5.3 Verification of results

In this section, we test our previous results by comparing them with known results.

5.3.1 Propagator in an external field

In order to verify our results, we will compare them with those obtained in [26, 40] for the photon propagator in presence of a constant electromagnetic field. This propagator in terms of Feynman diagrams is shown in Figure 5.2, which comparing with Figure 5.1 we can see that the four photon amplitude with two photons in the low energy limit is equal to the second term in the expansion of the propagator in Figure 5.2 when we fix $k_1 = -k_2 = k$.

In [26, 40], an integral representation for the dimensionally regularized scalar/spinor QED vacuum polarization tensors is obtained as

$$\Pi^{\mu\nu}_{scal}(k) = \frac{-e^2}{(4\pi)^{\frac{D}{2}}}\int_0^\infty \frac{dT}{T}T^{2-\frac{D}{2}}e^{-m^2T}\det{}^{-1/2}\left[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\right]\int_0^1 du_1 I^{\mu\nu}_{scal}e^{-Tk\cdot\Phi_{12}\cdot k}, \tag{5.35}$$

$$\Pi^{\mu\nu}_{spin}(k) = \frac{2e^2}{(4\pi)^{\frac{D}{2}}}\int_0^\infty \frac{dT}{T}T^{2-\frac{D}{2}}e^{-m^2T}\det{}^{-1/2}\left[\frac{\tan(\mathcal{Z})}{\mathcal{Z}}\right]\int_0^1 du_1 I^{\mu\nu}_{spin}e^{-Tk\cdot\Phi_{12}\cdot k}, \tag{5.36}$$

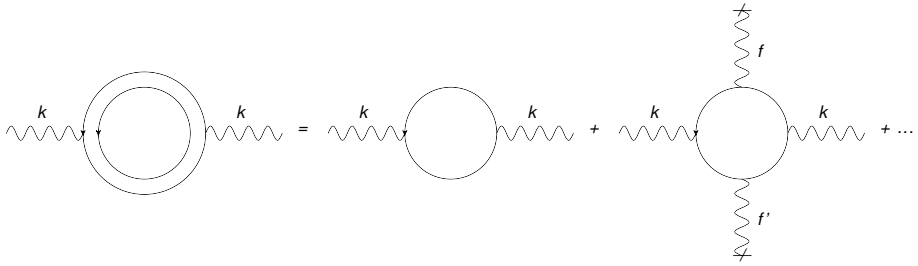

Figure 5.2: *Feynman diagram for photon propagator in an external field. (left photon ingoing and right outgoing.)*

where $\mathcal{Z} = eTF$, with $F = i(f_3 + f_4)$ the field strength tensor,

$$I^{\mu\nu}_{spin} = \dot{\mathcal{G}}^{\mu\nu}_{B12} k \cdot \dot{\mathcal{G}}_{B12} \cdot k - \mathcal{G}^{\mu\nu}_{F12} k \cdot \mathcal{G}_{F12} \cdot k - \left[\left(\dot{\mathcal{G}}_{B11} - \mathcal{G}_{F11} - \dot{\mathcal{G}}_{B12} \right)^{\mu\lambda} \left(\dot{\mathcal{G}}_{B21} - \dot{\mathcal{G}}_{B22} + \mathcal{G}_{F22} \right)^{\nu\kappa} + \mathcal{G}^{\mu\lambda}_{F12} \mathcal{G}^{\nu\kappa}_{F12} \right] k^\kappa k^\lambda. \tag{5.37}$$

Here $\mathcal{G}_{B12}$, $\mathcal{G}_{F12}$ are the generalized Green's functions, and the first few terms of $\dot{\mathcal{G}}_{B12}$, $\mathcal{G}_{F12}$ in the expansion in F are

$$\dot{\mathcal{G}}_{B12} = \dot{G}_{B12} + 2i \left(G_{B12} - \frac{T}{6} \right) eF + \frac{2}{3} \dot{G}_{B12} G_{B12} T (eF)^2 + O(F^3), \tag{5.38}$$

$$\mathcal{G}_{F12} = G_{F12} - i G_{F12} \dot{G}_{B12} T e F + 2 G_{F12} G_{B12} T (eF)^2 + O(F^3). \tag{5.39}$$

In our particular case,

$$Tk \cdot \Phi_{12} \cdot k = G_{B12} k^2 + \frac{1}{3T} G^2_{B12} k \cdot \mathcal{Z}^2 \cdot k, \tag{5.40}$$

and $I^{\mu\nu}_{scal}$ is obtained from $I^{\mu\nu}_{spin}$ by deleting all quantities carrying a subscript "F"; also here it is rescaled to the unit circle and set $u_2 = 0$.

We are interested in the expansion of (5.35) and (5.36) in F up to $O(F^2)$ because this is exactly what it is shown in Figure 5.2. Hence, after some mathematical manipulations of the previous equations and taking the product of the polarization tensor with the corresponding polarization

vectors, we obtain

$$\varepsilon_{1\mu}\varepsilon_{2\nu}\Pi^{\mu\nu}_{scal}(k) = \tfrac{e^4}{(4\pi)^{\frac{D}{2}}}\int_0^\infty \tfrac{dT}{T}T^{4-\frac{D}{2}}e^{-m^2T}\int_0^1 du_1 e^{-TG_{B12}k^2}$$
$$\times\Big\{\dot{G}^2_{B12}(\varepsilon_1\cdot\varepsilon_2 k^2-\varepsilon_1\cdot k\varepsilon_2\cdot k)\Big[-(eT)^{-2}-\tfrac{T}{24}\left(1-\dot{G}^2_{B12}\right)^2 k\cdot f_3\cdot f_4\cdot k$$
$$+\tfrac{1}{6}\mathrm{tr}(f_3f_4)\Big]+\tfrac{1}{6}\left(\dot{G}^2_{B12}-\dot{G}^4_{B12}\right)\Big[2\varepsilon_1\cdot\varepsilon_2 k\cdot f_3\cdot f_4\cdot k+k^2\varepsilon_1\cdot f_3\cdot f_4\cdot\varepsilon_2$$
$$-\varepsilon_2\cdot k(\varepsilon_1\cdot f_3\cdot f_4\cdot k+\varepsilon_1\cdot f_4\cdot f_3\cdot k)-\varepsilon_1\cdot k(\varepsilon_2\cdot f_3\cdot f_4\cdot k+\varepsilon_2\cdot f_4\cdot f_3\cdot k)$$
$$+k^2\varepsilon_1\cdot f_4\cdot f_3\varepsilon_2\Big]-\tfrac{1}{4}\left(1-\dot{G}^2_{B12}\right)^2(\varepsilon_1\cdot f_3\cdot k\varepsilon_2\cdot f_4\cdot k+\varepsilon_1\cdot f_4\cdot k\varepsilon_2\cdot f_3\cdot k)\Big\}, \tag{5.41}$$

$$\varepsilon_{1\mu}\varepsilon_{2\nu}\Pi^{\mu\nu}_{spin}(k) = \tfrac{-2e^4}{(4\pi)^{\frac{D}{2}}}\int_0^\infty \tfrac{dT}{T}T^{4-\frac{D}{2}}e^{-m^2T}\int_0^1 du_1 e^{-TG_{B12}k^2}$$
$$\times\Big\{\left(\dot{G}^2_{B12}-1\right)(\varepsilon_1\cdot\varepsilon_2 k^2-\varepsilon_1\cdot k\varepsilon_2\cdot k)\Big[-(eT)^{-2}-\tfrac{T}{24}\left(1-\dot{G}^2_{B12}\right)^2 k\cdot f_3\cdot f_4\cdot k$$
$$-\tfrac{1}{3}\mathrm{tr}(f_3f_4)\Big]+\tfrac{1}{6}\left(1-\dot{G}^2_{B12}\right)\left(\dot{G}^2_{B12}-3\right)\Big[2\varepsilon_1\cdot\varepsilon_2 k\cdot f_3\cdot f_4\cdot k+k^2\varepsilon_1\cdot f_3\cdot f_4\cdot\varepsilon_2$$
$$-\varepsilon_2\cdot k(\varepsilon_1\cdot f_3\cdot f_4\cdot k+\varepsilon_1\cdot f_4\cdot f_3\cdot k)-\varepsilon_1\cdot k(\varepsilon_2\cdot f_3\cdot f_4\cdot k+\varepsilon_2\cdot f_4\cdot f_3\cdot k)$$
$$+k^2\varepsilon_1\cdot f_4\cdot f_3\varepsilon_2\Big]-\tfrac{1}{4}\left(1-\dot{G}^2_{B12}\right)^2(\varepsilon_1\cdot f_3\cdot k\varepsilon_2\cdot f_4\cdot k+\varepsilon_1\cdot f_4\cdot k\varepsilon_2\cdot f_3\cdot k)\Big\}. \tag{5.42}$$

Notice that the previous expressions are written in terms of $\dot{G}_{B12}$. Thus, we take the result presented in sections 5.1 and 5.2 and with the help of (B.1) we write this in terms of $\dot{G}_{B12}$. Then, we fix $k_1=-k_2=k$ and we compute the corresponding expressions obtaining

Scalar result

$$I^4_{(34)}(1234) = \frac{1}{6}\left(\dot{G}^2_{B12}-\dot{G}^4_{B12}\right)Z_4(1234), \tag{5.43}$$

$$I^4_{(34)}(1243) = \frac{1}{6}\left(\dot{G}^2_{B12}-\dot{G}^4_{B12}\right)Z_4(1243), \tag{5.44}$$

$$I^4_{(34)}(1324) = \frac{1}{6^2}\left(1-6\dot{G}^2_{B12}+9\dot{G}^4_{B21}\right)Z_4(1324), \tag{5.45}$$

$$I^2_{(34)}(12) = \tfrac{-T}{24}\dot{G}^2_{B12}\left(1-2\dot{G}^2_{B12}+\dot{G}^4_{B12}\right)Z_2(12)k\cdot f_4\cdot f_3\cdot k, \tag{5.46}$$

$$I^2_{(34)}(14) = -\tfrac{1}{9}\left(1-3\dot{G}^2_{B12}\right)Z_2(14)\varepsilon_2\cdot f_3\cdot k, \tag{5.47}$$

$$I^2_{(34)}(24) = \tfrac{1}{9}\left(1-3\dot{G}^2_{B12}\right)Z_2(24)\varepsilon_1\cdot f_3\cdot k, \tag{5.48}$$

$$I^{22}_{(34)}(12,34) = \frac{1}{3}\dot{G}^2_{B12} Z_2(12) Z_2(34), \tag{5.49}$$

$$I^{22}_{(34)}(13,24) = \frac{1}{9} Z_2(13) Z_2(24), \tag{5.50}$$

$$I^{22}_{(34)}(14,23) = \frac{1}{9} Z_2(14) Z_2(23). \tag{5.51}$$

Spinor result

$$\tilde{I}^4_{(34)}(1234) = \frac{1}{6}\left(4\dot{G}^2_{B12} - \dot{G}^4_{B12} - 3\right) Z_4(1234), \tag{5.52}$$

$$\tilde{I}^4_{(34)}(1243) = \frac{1}{6}\left(4\dot{G}^2_{B12} - \dot{G}^4_{B12} - 3\right) Z_4(1243), \tag{5.53}$$

$$\tilde{I}^4_{(34)}(1324) = \frac{1}{6^2}\left(1 - 42\dot{G}^2_{B12} + 9\dot{G}^4_{B21}\right) Z_4(1324), \tag{5.54}$$

$$\tilde{I}^2_{(34)}(12) = \tfrac{T}{24}\left(1 - \dot{G}^2_{B12}\right)\left(1 - 2\dot{G}^2_{B12} + \dot{G}^4_{B12}\right) Z_2(12) k \cdot f_4 \cdot f_3 \cdot k, \tag{5.55}$$

$$\tilde{I}^2_{(34)}(14) = \tfrac{2}{9}\left(1 - 3\dot{G}^2_{B12}\right) Z_2(14) \varepsilon_2 \cdot f_3 \cdot k, \tag{5.56}$$

$$\tilde{I}^2_{(34)}(24) = -\tfrac{2}{9}\left(1 - 3\dot{G}^2_{B12}\right) Z_2(24) \varepsilon_1 \cdot f_3 \cdot k, \tag{5.57}$$

$$\tilde{I}^{22}_{(34)}(12,34) = \frac{2}{3}\left(1 - \dot{G}^2_{B12}\right) Z_2(12) Z_2(34), \tag{5.58}$$

$$\tilde{I}^{22}_{(34)}(13,24) = \frac{4}{9} Z_2(13) Z_2(24), \tag{5.59}$$

$$\tilde{I}^{22}_{(34)}(14,23) = \frac{4}{9} Z_2(14) Z_2(23). \tag{5.60}$$

Now, we need to write the Lorentz cycles "Z_n" in terms of ε_1, ε_2, k, f_3 and f_4 in the scalar/spinor result, then we put similar terms together and we obtain exactly the terms that appear in (5.41), (5.42) respectively, with the only exception of the term "$(eT)^{-2}$", because this corresponds to the vacuum polarization diagram in Figure 5.2.

5.3.2 Two-loop QED β function

In sections 5.1 and 5.2 we presented the off-shell four-photon amplitude with two external legs in the low energy limit for scalar and spinor QED respectively. Hence, as the amplitude is off-shell, from these expressions we can compute the two-loop vacuum polarization amplitude (Figure 5.3) by performing the following replacement

$$k_1 \to k, \quad k_2 \to -k \quad \text{and} \quad \varepsilon_1^\mu \varepsilon_2^\nu \to \frac{\delta^{\mu\mu}}{k^2}, \tag{5.61}$$

in Feynman gauge, and computing the integral over the momentum k. In the previous section we already performed the first two replacements ($k_1 = -k_2 = k$). Then, we take the part that contains the information on the momenta and polarizations in equations (5.52)-(5.60), and we perform the third replacement ($\varepsilon_1^\mu \varepsilon_2^\nu \to \frac{\delta^{\mu\mu}}{k^2}$). Thus, in terms of f_3, f_4 and k, we have the following relations

$$\begin{aligned}
&k^2 Z_4(1234) = (D-2)k\cdot f_3\cdot f_4\cdot k + k^2 \mathrm{tr}(f_3 f_4),\\
&k^2 Z_4(1243) = (D-2)k\cdot f_3\cdot f_4\cdot k + k^2 \mathrm{tr}(f_3 f_4),\\
&k^2 Z_4(1324) = 2k\cdot f_3\cdot f_4\cdot k,\\
&k^2 Z_2(12) k\cdot f_4\cdot f_3\cdot k = (D-1)k^2 k\cdot f_4\cdot f_3\cdot k,\\
&k^2 Z_2(14)\varepsilon_2\cdot f_3\cdot k = -k\cdot f_3\cdot f_4\cdot k,\\
&k^2 Z_2(24)\varepsilon_1\cdot f_3\cdot k = k\cdot f_3\cdot f_4\cdot k,\\
&k^2 Z_2(12) Z_2(34) = (D-1)k^2 Z_2(34),\\
&k^2 Z_2(13) Z_2(24) = k\cdot f_3\cdot f_4\cdot k,\\
&k^2 Z_2(14) Z_2(23) = k\cdot f_3\cdot f_4\cdot k.
\end{aligned} \tag{5.62}$$

Now, we use the fact that $k\cdot f_3\cdot f_4\cdot k = \frac{1}{D}k^2\mathrm{tr}(f_3 f_4)$ to rewrite relations above as

$$\begin{aligned}
&Z_4(1234) = \tfrac{2D-2}{D}\mathrm{tr}(f_3 f_4),\\
&Z_4(1243) = \tfrac{2D-2}{D}\mathrm{tr}(f_3 f_4),\\
&Z_4(1324) = \tfrac{2}{D}\mathrm{tr}(f_3 f_4),\\
&Z_2(12) k\cdot f_4\cdot f_3\cdot k = \tfrac{D-1}{D}k^2\mathrm{tr}(f_3 f_4),\\
&Z_2(14)\varepsilon_2\cdot f_3\cdot k = -\tfrac{1}{D}\mathrm{tr}(f_3 f_4),\\
&Z_2(24)\varepsilon_1\cdot f_3\cdot k = \tfrac{1}{D}\mathrm{tr}(f_3 f_4),\\
&Z_2(12) Z_2(34) = \tfrac{1}{2}(D-1)\mathrm{tr}(f_3 f_4),\\
&Z_2(13) Z_2(24) = \tfrac{1}{D}\mathrm{tr}(f_3 f_4),\\
&Z_2(14) Z_2(23) = \tfrac{1}{D}\mathrm{tr}(f_3 f_4).
\end{aligned} \tag{5.63}$$

Notice that the result presented in equations (5.52)-(5.60) is simpler than the presented in sections 5.1 and 5.2 because of the replace $k_1 = -k_2 = k$ already done in (5.52)-(5.60). Then we take this result and with the help of (B.1) we write it in terms of G_{B12} for each case, scalar and spinor. Furthermore, we compute the two-loop scalar/spinor amplitude and the corresponding β-function correction for each case.

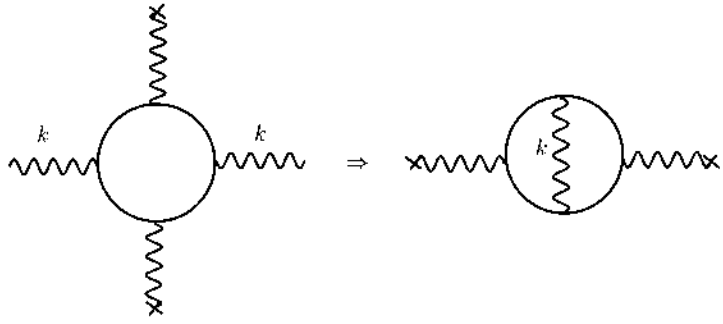

Figure 5.3: *One-loop four-photon amplitude transforms into the two-loop vacuum polarization amplitude.*

Scalar result

We rewrite (5.43)-(5.51) in terms of G_{B12}

$$I^4_{(34)}(1234) = \frac{2}{3}\left(G_{B12} - 4G^2_{B12}\right) Z_4(1234), \tag{5.64}$$

$$I^4_{(34)}(1243) = \frac{2}{3}\left(G_{B12} - 4G^2_{B12}\right) Z_4(1243), \tag{5.65}$$

$$I^4_{(34)}(1324) = \frac{1}{9}\left(1 - 12G_{B12} + 36G^2_{B21}\right) Z_4(1324), \tag{5.66}$$

$$I^2_{(34)}(12) = \tfrac{-2T}{3}\left(G^2_{B12} - 4G^3_{B12}\right) Z_2(12) k \cdot f_4 \cdot f_3 \cdot k, \tag{5.67}$$

$$I^2_{(34)}(14) = \tfrac{2}{9}\left(1 - 6G_{B12}\right) Z_2(14) \varepsilon_2 \cdot f_3 \cdot k, \tag{5.68}$$

$$I^2_{(34)}(24) = -\tfrac{2}{9}\left(1 - 6G_{B12}\right) Z_2(24) \varepsilon_1 \cdot f_3 \cdot k, \tag{5.69}$$

$$I^{22}_{(34)}(12,34) = \frac{1}{3}\left(1 - 4G_{B12}\right) Z_2(12) Z_2(34), \tag{5.70}$$

$$I^{22}_{(34)}(13,24) = \frac{1}{9} Z_2(13) Z_2(24), \tag{5.71}$$

$$I^{22}_{(34)}(14,23) = \frac{1}{9} Z_2(14) Z_2(23). \tag{5.72}$$

Now, with the help of (5.4) and (5.63) we write all in terms of Y_{nl} and $\mathrm{tr}(f_3 f_4)$

$$I^{4}_{(34)}(1234) = \frac{4(D-1)}{3D}\left(Y_{41} - 4Y_{42}\right) \mathrm{tr}(f_3 f_4), \tag{5.73}$$

$$I^{4}_{(34)}(1243) = \frac{4(D-1)}{3D}\left(Y_{41} - 4Y_{42}\right) \mathrm{tr}(f_3 f_4), \tag{5.74}$$

$$I^{4}_{(34)}(1324) = \frac{2}{9D}\left(Y_{40} - 12Y_{41} + 36Y_{42}\right) \mathrm{tr}(f_3 f_4), \tag{5.75}$$

$$I^{2}_{(34)}(12) = -\tfrac{2(D-1)}{3D}\left(Y_{52} - 4Y_{53}\right) k^2 \mathrm{tr}(f_3 f_4), \tag{5.76}$$

$$I^{2}_{(34)}(14) = -\tfrac{2}{9D}\left(Y_{40} - 6Y_{41}\right) \mathrm{tr}(f_3 f_4), \tag{5.77}$$

$$I^{2}_{(34)}(24) = -\tfrac{2}{9D}\left(Y_{40} - 6Y_{41}\right) \mathrm{tr}(f_3 f_4), \tag{5.78}$$

$$I^{22}_{(34)}(12,34) = \frac{D-1}{6}\left(Y_{40} - 4Y_{41}\right) \mathrm{tr}(f_3 f_4), \tag{5.79}$$

$$I^{22}_{(34)}(13,24) = \frac{1}{9D} Y_{40} \mathrm{tr}(f_3 f_4), \tag{5.80}$$

$$I^{22}_{(34)}(14,23) = \frac{1}{9D} Y_{40} \mathrm{tr}(f_3 f_4). \tag{5.81}$$

Finally, summing all these terms and taking the integral over k (the factor $1/(2\pi)^D$ comes from the convention of this integral), we have the two-loop amplitude for scalar QED (after sewing)

$$\begin{aligned}\Gamma^{(2)}_{scal} =& \frac{e^4}{(4\pi)^{\frac{D}{2}}} \int \frac{d^D k}{(2\pi)^D} \Big[\frac{D-1}{6} Y_{40} - \frac{2}{3D}(D-4)(D-1) Y_{41} \\ & - \frac{8}{3D}(4D-7) Y_{42} - \frac{2(D-1)}{3D}\left(Y_{52} - 4Y_{53}\right) k^2 \Big] \mathrm{tr}(f_3 f_4).\end{aligned} \tag{5.82}$$

Spinor result

We rewrite (5.52)-(5.60) in terms of G_{B12}

$$\tilde{I}^4_{(34)}(1234) = -\frac{4}{3}\left(G_{B12} - 2G^2_{B12}\right) Z_4(1234), \tag{5.83}$$

$$\tilde{I}^4_{(34)}(1243) = -\frac{4}{3}\left(G_{B12} - 2G^2_{B12}\right) Z_4(1243), \tag{5.84}$$

$$\tilde{I}^4_{(34)}(1324) = -\frac{4}{9}\left(2 - 6G_{B12} - 9G^2_{B12}\right) Z_4(1324), \tag{5.85}$$

$$\tilde{I}^2_{(34)}(12) = \tfrac{8T}{3} G^3_{B12} Z_2(12) k \cdot f_4 \cdot f_3 \cdot k, \tag{5.86}$$

$$\tilde{I}^2_{(34)}(14) = -\tfrac{4}{9}\left(1 - 6G_{B12}\right) Z_2(14) \varepsilon_2 \cdot f_3 \cdot k, \tag{5.87}$$

$$\tilde{I}^2_{(34)}(24) = \tfrac{4}{9}\left(1 - 6G_{B12}\right) Z_2(24) \varepsilon_1 \cdot f_3 \cdot k, \tag{5.88}$$

$$\tilde{I}^{22}_{(34)}(12,34) = \frac{8}{3} G_{B12} Z_2(12) Z_2(34), \tag{5.89}$$

$$\tilde{I}^{22}_{(34)}(13,24) = \frac{4}{9} Z_2(13) Z_2(24), \tag{5.90}$$

$$\tilde{I}^{22}_{(34)}(14,23) = \frac{4}{9} Z_2(14) Z_2(23). \tag{5.91}$$

Now, with the help of (5.4) and (5.63) we write all in terms of Y_{nl} and $\mathrm{tr}(f_3 f_4)$

$$\tilde{I}^4_{(34)}(1234) = -\frac{8(D-1)}{3D}\left(Y_{41} - 2Y_{42}\right) \mathrm{tr}(f_3 f_4), \tag{5.92}$$

$$\tilde{I}^4_{(34)}(1243) = -\frac{8(D-1)}{3D}\left(Y_{41} - 2Y_{42}\right) \mathrm{tr}(f_3 f_4), \tag{5.93}$$

$$\tilde{I}^4_{(34)}(1324) = -\frac{8}{9D}\left(2Y_{40} - 6Y_{41} - 9Y_{42}\right) \mathrm{tr}(f_3 f_4), \tag{5.94}$$

$$\tilde{I}^2_{(34)}(12) = \tfrac{8(D-1)}{3D} Y_{53} k^2 \mathrm{tr}(f_3 f_4), \tag{5.95}$$

$$\tilde{I}^2_{(34)}(14) = \tfrac{4}{9D}\left(Y_{40} - 6Y_{41}\right) \mathrm{tr}(f_3 f_4), \tag{5.96}$$

$$\tilde{I}^2_{(34)}(24) = \tfrac{4}{9D}\left(Y_{40} - 6Y_{41}\right) \mathrm{tr}(f_3 f_4), \tag{5.97}$$

$$\tilde{I}^{22}_{(34)}(12,34) = \frac{4(D-1)}{3} Y_{41} \mathrm{tr}(f_3 f_4), \tag{5.98}$$

$$\tilde{I}^{22}_{(34)}(13,24) = \frac{4}{9D} Y_{40} \mathrm{tr}(f_3 f_4), \tag{5.99}$$

$$\tilde{I}^{22}_{(34)}(14,23) = \frac{4}{9D} Y_{40} \mathrm{tr}(f_3 f_4). \tag{5.100}$$

Finally, summing all these terms and taking the integral over k, we have the two-loop amplitude for spinor QED (after sewing)

$$\begin{aligned} \Gamma^{(2)}_{spin} =& \frac{-2e^4}{(4\pi)^{\frac{D}{2}}} \int \frac{d^D k}{(2\pi)^D} \Big[\frac{4}{3D}(D-4)(D-1) Y_{41} \\ &+ \frac{8}{3D}(2D+1) Y_{42} + \frac{8(D-1)}{3D} Y_{53} k^2 \Big] \mathrm{tr}(f_3 f_4). \end{aligned} \tag{5.101}$$

Notice that, in order to calculate this β-function, we combine the worldline formalism with Feynman rules, since the off-shell four-photon amplitude with two external legs in the low energy limit was obtained using the worldline formalism and the procedure that we followed to obtain the two-loop amplitude comes from Feynman rules.

Result of β function

Now, we compute the integrals in (5.82) and (5.101) in Mathematica. Then, we fix the dimension to be $D = 4 - 2\epsilon$, (for $\epsilon > 0$ & $\epsilon \ll 1$) and we expand around $\epsilon \approx 0$, obtaining

$$\Gamma^{(2)}_{scal} \approx -\frac{e^4}{(4\pi)^4 \epsilon} \mathrm{tr}(f_3 f_4) + O(\epsilon^0), \tag{5.102}$$

$$\Gamma^{(2)}_{spin} \approx -\frac{6e^4}{(4\pi)^4 \epsilon} \mathrm{tr}(f_3 f_4) + O(\epsilon^0). \tag{5.103}$$

These two expressions are in agreement with those obtained from the two-loop with the worldline formalism in [40]. Hence, the beta functions are

$$\beta^{(2)}_{scal}(\alpha) = \beta^{(2)}_{spin}(\alpha) = \frac{\alpha^3}{2\pi^2}, \tag{5.104}$$

which is exactly the same result that was obtained in [19] with the standard Feynman procedure and in [40] with the worldline formalism in a direct two-loop calculation.

Chapter 6

Low energy limit in four external legs

In Chapter 3, we presented the integral result for the four photon amplitude with all its legs in the low energy limit (section 3.4). In fact, in these expressions the only integral that remains is

$$\int_0^\infty \frac{dT}{T} T^{4-\frac{D}{2}} e^{-m^2 T} = \frac{\Gamma\left(4-\frac{D}{2}\right)}{(m^2)^{4-\frac{D}{2}}}. \tag{6.1}$$

Therefore, in the next section we present the result for these amplitudes.

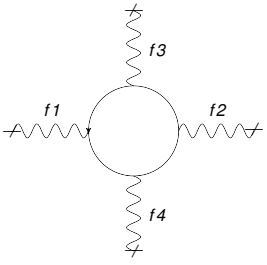

Figure 6.1: *Feynman diagram for the four-photon amplitude with all photons in the low energy limit.*

6.1 Explicit result for scalar case

Now, we write the explicit result for $I(Q_4)$

$$\begin{aligned} I(Q_4) =& \frac{1}{45}[Z_4(1234) + Z_4(1243) + Z_4(1324)] \\ &+ \frac{1}{9}[Z_2(12)Z_2(34) + Z_2(13)Z_2(24) + Z_2(14)Z_2(23)]. \end{aligned} \tag{6.2}$$

Then, from (3.1), the amplitude is

$$\Gamma_{scal}[k_1, \varepsilon_1; ...; k_4, \varepsilon_4] = \frac{e^4}{(4\pi)^{\frac{D}{2}}} m^{-8+D} \Gamma\left(4 - \frac{D}{2}\right) I(Q_4). \tag{6.3}$$

6.2 Explicit result for spinor case

Now, we write the explicit result for $\tilde{I}(\tilde{Q}_4)$

$$\begin{aligned} \tilde{I}(\tilde{Q}_4) =& \frac{-14}{45} [Z_4(1234) + Z_4(1243) + Z_4(1324)] \\ &+ \frac{4}{9} [Z_2(12)Z_2(34) + Z_2(13)Z_2(24) + Z_2(14)Z_2(23)] . \end{aligned} \tag{6.4}$$

Then, from (3.3), the amplitude is

$$\Gamma_{spin}[k_1, \varepsilon_1; ...; k_4, \varepsilon_4] = \frac{-2e^4}{(4\pi)^{\frac{D}{2}}} m^{-8+D} \Gamma\left(4 - \frac{D}{2}\right) \tilde{I}(\tilde{Q}_4). \tag{6.5}$$

It is worthwhile to point out that it is also possible to compute these amplitudes directly from the Euler-Heisenberg effective Lagrangian, [13, 16, 15].

Chapter 7

Conclusions

The worldline formalism is totally equivalent to the standard "a la Feynman"; it is often possible to find the result in an easier way.

We obtain the analytic result for the one-loop four-photon amplitude off-shell for one, two and four external photon lines in the low energy limit in D dimension.

We obtain the result for the off-shell four-photon amplitude with one external photon line in the low energy limit in terms of derivatives of known integrals, J_3 from [41]. This is a new result.

We obtain the explicit result for the off-shell four-photon amplitude with two external photon lines in the low energy limit in terms of the hypergeometric function ${}_2F_1$. This is more general than what had been obtained by previous authors, because, for us the only condition on the momenta is $\sum_{i=1}^{4} k_i = 0$, while other authors also fix $k_1 = -k_2$.

It was possible to verify some of our results by comparing them with some particular cases in the literature.

The techniques developed here for the calculation of the four-photon amplitudes are also of relevance for other amplitudes with four gauge bosons, such as, the four-gluon amplitudes and amplitudes involving W or Z bosons.

We do not need to calculate any trace of gamma matrices. This is possible because of the "replacement rule".

We can write the effective action or the amplitude in terms of the field strength tensor making the gauge invariance manifest.

We do not need to compute any momentum integral over the loop.

We do not need to fix any order to perform the integrals (for QED).

In the specific case of the four-photon amplitudes, the Feynman diagram calculation suffers from spurious UV divergences that in the worldline formalism are eliminated by the standard integration-by-parts procedure.

Future work

Compute the exact result for the one-loop four-photon amplitude off-shell with arbitrary momentum in each external photon line.

Use our approach to calculate some particular multi-loop corrections.

Use our approach to recalculate the three-loop correction to the "$g-2$" factor (see Figure 7.1 and [44]).

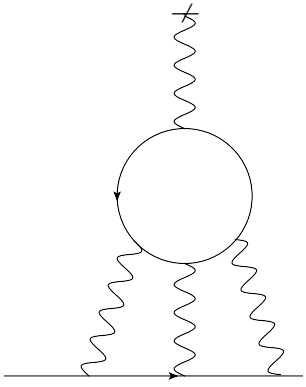

Figure 7.1: *Feynman diagram for the three loop correction to the $g-2$ factor with one external photon line in the low energy limit.*

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Appendix A

Conventions

In all our calculations we use natural units ($\hbar = c = 1$) and our metric convention for Minkowski space is $(-+++)$ and at the path integral level we work in the Euclidean space with a positive metric convention $(++++)$. The Roman indexes run from 1 to n ($i = 1, 2, 3, ...$), and Greek indexes run from 0 to n ($\mu = 0, 1, 2, ...$), with n a positive integer.

Appendix B

Some useful formulas and integrals

B.1 Formulas

In general, we have the following relations

$$\dot{G}^2_{Bij} = 1 - 4G_{Bij}, \tag{B.1}$$

$$\dot{G}^2_{Fij} = 1, \tag{B.2}$$

$$\dot{G}_{B12} + \dot{G}_{B23} + \dot{G}_{B31} = -G_{F12}G_{F23}G_{F31}, \tag{B.3}$$

$$G_{F12}G_{F23}G_{F34}G_{F41} = -\left(\dot{G}_{B12} + \dot{G}_{B23} + \dot{G}_{B31}\right)\left(\dot{G}_{B34} + \dot{G}_{B41} + \dot{G}_{B13}\right), \tag{B.4}$$

$$k_i \cdot f_l \cdot k_j = k_i \cdot k_l \varepsilon_l \cdot k_j - k_i \cdot \varepsilon_l k_l \cdot k_j, \tag{B.5}$$

$$\begin{aligned} r_l \cdot f_i \cdot f_j \cdot r_n = {} & r_l \cdot k_i \varepsilon_i \cdot k_j \varepsilon_j \cdot r_n - r_l \cdot k_i \varepsilon_i \cdot \varepsilon_j k_j \cdot r_n \\ & - r_l \cdot \varepsilon_i k_i \cdot k_j \varepsilon_j \cdot r_n + r_l \cdot \varepsilon_i k_i \cdot \varepsilon_j k_j \cdot r_n. \end{aligned} \tag{B.6}$$

B.2 Integrals

In general, we have the following integrals

$$\int_0^1 du_1 G_{B12} = \frac{1}{6}, \tag{B.7}$$

$$\int_0^1 du_1 \dot{G}_{B12} = 0, \tag{B.8}$$

$$\int_0^1 du_3 G_{B13} G_{B32} = -\frac{1}{6} G_{B12}^2 + \frac{1}{30}, \tag{B.9}$$

$$\int_0^1 du_3 \dot{G}_{B13} G_{B32} = -\frac{1}{3} \dot{G}_{B12} G_{B12}, \tag{B.10}$$

$$\int_0^1 du_3 \dot{G}_{B13}^k \dot{G}_{B32}^l = \frac{k!l!}{2} \sum_{m=0}^{l} \frac{(1-(-1)^{k+l-m+1}) \dot{G}_{B12}^m - (1-(-1)^m) \dot{G}_{B12}^{k+l-m+1}}{m!(k+l-m+1)!}, \tag{B.11}$$

$$\int_0^1 du_4 \dot{G}_{B41} \dot{G}_{B42} \dot{G}_{B43} = -\frac{1}{6} (\dot{G}_{B12} - \dot{G}_{B23})(\dot{G}_{B23} - \dot{G}_{B31})(\dot{G}_{B31} - \dot{G}_{B12}). \tag{B.12}$$

Some particular cases of (B.11) are

$$\int_0^1 du_3 \dot{G}_{B13}^2 = \frac{1}{3}, \tag{B.13}$$

$$\int_0^1 du_3 \dot{G}_{B13}^4 = \frac{1}{5}, \tag{B.14}$$

$$\int_0^1 du_3 \dot{G}_{B13} \dot{G}_{B32} = \frac{1}{6} - \frac{1}{2} \dot{G}_{B12}^2 = 2G_{B12} - \frac{1}{3}, \tag{B.15}$$

$$\int_0^1 du_3 \dot{G}_{B13} \dot{G}_{B32}^2 = \int_0^1 du_3 \dot{G}_{B13}^2 \dot{G}_{B32} = \frac{1}{3} (\dot{G}_{B12} - \dot{G}_{B12}^3), \tag{B.16}$$

$$\int_0^1 du_3 \dot{G}_{B13}^3 \dot{G}_{B32} = \int_0^1 du_3 \dot{G}_{B13} \dot{G}_{B32}^3 = \frac{3!}{5!} - \frac{\dot{G}_{B12}^4}{4}, \tag{B.17}$$

$$\int_0^1 du_3 \dot{G}_{B13}^2 \dot{G}_{B32}^2 = \frac{4}{5!} + \frac{2\dot{G}_{B12}^2}{3!} - \frac{\dot{G}_{B12}^4}{3!}, \tag{B.18}$$

$$\int_0^1 du_3 \dot{G}_{B13}^4 \dot{G}_{B32} = \int_0^1 du_3 \dot{G}_{B13} \dot{G}_{B32}^4 = \frac{1}{5} (\dot{G}_{B12} - \dot{G}_{B12}^5), \tag{B.19}$$

$$\int_0^1 du_3 \dot{G}_{B13}^3 \dot{G}_{B32}^2 = \int_0^1 du_3 \dot{G}_{B13}^2 \dot{G}_{B32}^3 = \frac{1}{10} (\dot{G}_{B12} - \dot{G}_{B12}^5). \tag{B.20}$$

In the case in which we have one leg in the low energy limit, we will need

$$\int_0^1 du_4 \dot{G}_{B4j}\, e^{(\cdot 4)}\Big|_{lin\ k_4} = \frac{T}{3}\sum_{i=1}^{3} \dot{G}_{Bji} G_{Bij} k_i \cdot k_4, \tag{B.21}$$

where

$$e^{(\cdot 4)} = e^{T(G_{B14} k_1\cdot k_4 + G_{B24} k_2\cdot k_4 + G_{B34} k_3\cdot k_4)}. \tag{B.22}$$

Similarly, when we have two legs in the low energy limit, we will need

$$\int_0^1 du_3\, e^{(\cdot 3)}\Big|_{lin\ k_3} = \frac{1}{6}\left(k_2\cdot k_3 + k_3\cdot k_1\right), \tag{B.23}$$

where

$$e^{(\cdot 3)} = e^{G_{B23} k_2\cdot k_3 + G_{B31} k_3\cdot k_1}. \tag{B.24}$$

Further,

$$\int_0^1 du_3 \dot{G}_{B31}\, e^{(\cdot 3)}\Big|_{lin\ k_3} = \frac{1}{12}(\dot{G}_{B12} - \dot{G}^3_{B12}) k_2\cdot k_3, \tag{B.25}$$

$$\int_0^1 du_3 \dot{G}_{B23}\, e^{(\cdot 3)}\Big|_{lin\ k_3} = \frac{1}{12}(\dot{G}_{B12} - \dot{G}^3_{B12}) k_3\cdot k_1, \tag{B.26}$$

$$\int_0^1 du_3 \dot{G}^2_{B31}\, e^{(\cdot 3)}\Big|_{lin\ k_3} = \frac{1}{4}\left[\left(\frac{3}{10} - \frac{2\dot{G}^2_{B12}}{3!} + \frac{\dot{G}^4_{B12}}{3!}\right) k_2\cdot k_3 + \frac{2}{15} k_3\cdot k_1\right], \tag{B.27}$$

$$\int_0^1 du_3 \dot{G}^2_{B23}\, e^{(\cdot 3)}\Big|_{lin\ k_3} = \frac{1}{4}\left[\frac{2}{15} k_2\cdot k_3 + \left(\frac{3}{10} - \frac{2\dot{G}^2_{B12}}{3!} + \frac{\dot{G}^4_{B12}}{3!}\right) k_3\cdot k_1\right], \tag{B.28}$$

$$\int_0^1 du_3 \dot{G}_{B23}\dot{G}_{B31}\, e^{(\cdot 3)}\Big|_{lin\ k_3} = \frac{1}{4}\left(\frac{14}{5!} - \frac{\dot{G}^2_{B12}}{2} + \frac{\dot{G}^4_{B12}}{4}\right)\left(k_2\cdot k_3 + k_3\cdot k_1\right), \tag{B.29}$$

$$\begin{aligned}\int_0^1 du_3 \dot{G}^2_{B23}\dot{G}_{B31}\, e^{(\cdot 3)}\Big|_{lin\ k_3} =& \frac{1}{4}\left[\left(\frac{-2}{15}\dot{G}_{B12} + \frac{1}{3}\dot{G}^3_{B12} + \frac{1}{5}\dot{G}^5_{B12}\right) k_2\cdot k_3\right.\\ &\left. + \left(\frac{-7}{30}\dot{G}_{B12} + \frac{1}{3}\dot{G}^3_{B12} + \frac{1}{10}\dot{G}^5_{B12}\right) k_3\cdot k_1\right],\end{aligned} \tag{B.30}$$

$$\int_0^1 du_3 \dot{G}_{B23} \dot{G}_{B31}^2 \, e^{(\cdot 3)}\Big|_{lin\ k_3} = \frac{1}{4}\left[\left(\frac{-7}{30}\dot{G}_{B12} + \frac{1}{3}\dot{G}_{B12}^3 + \frac{1}{10}\dot{G}_{B12}^5\right) k_2 \cdot k_3 + \left(\frac{-2}{15}\dot{G}_{B12} + \frac{1}{3}\dot{G}_{B12}^3 + \frac{1}{5}\dot{G}_{B12}^5\right) k_3 \cdot k_1\right]. \tag{B.31}$$

Appendix C

Explicit derivation of Bern-Kosower master formula

In order to construct the Bern-Kosower master formula, we start from the effective action (2.1)

$$\Gamma[A] = \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int \mathcal{D}x \; e^{-\int_0^T d\tau \left(\frac{\dot{x}^2}{4} + ie\dot{x}\cdot A(x)\right)}, \tag{C.1}$$

and we choose the background field to be a sum of plane waves

$$A_\mu(x) = \sum_{i=1}^{N} \varepsilon_{i\mu} \; e^{ik_i \cdot x}. \tag{C.2}$$

Therefore, we replace (C.2) into (C.1), we expand the exponential of $A(x)$ and we only take the term completely mixed in ε_i, i.e. the term that contains every polarization vector at linear order; this looks like

$$\begin{aligned} e^{-\int_0^T d\tau \left(\frac{\dot{x}^2}{4} + ie\dot{x}\cdot A(x)\right)} &= e^{-\int_0^T d\tau \frac{\dot{x}^2}{4}} \int_0^T d\tau_1 (-ie)\varepsilon_1 \cdot \dot{x} \; e^{ik_1 \cdot x} \\ &\times \int_0^T d\tau_2 (-ie)\varepsilon_2 \cdot \dot{x} \; e^{ik_2 \cdot x} \cdots \int_0^T d\tau_N (-ie)\varepsilon_N \cdot \dot{x} \; e^{ik_N \cdot x} \\ &= (-ie)^N \int_0^T d\tau_1 \cdots \int_0^T d\tau_N \; e^{-\int_0^T d\tau \frac{\dot{x}^2}{4} + \sum_{i=1}^N (\varepsilon_i \cdot \dot{x} + ik_i \cdot x)} \Big|_{lin \; \varepsilon_i}. \end{aligned} \tag{C.3}$$

Since, these are the only terms that contribute to the one loop amplitude and also we use the trick of (2.3). Moreover, notice that

$$\sum_{i=1}^{N}\left(\varepsilon_i\cdot\dot{x}(\tau_i)+ik_i\cdot x(\tau_i)\right)=\int_0^T d\tau\sum_{i=1}^{N}\left(-\varepsilon_i\cdot x(\tau)\dot{\delta}(\tau-\tau_i)+ik_i\cdot x(\tau)\delta(\tau-\tau_i)\right)$$
$$=\int_0^T d\tau j(\tau)\cdot x(\tau), \tag{C.4}$$

where we have defined

$$j(\tau)\equiv\sum_{i=1}^{N}\left(i\delta(\tau-\tau_i)k_i-\dot{\delta}(\tau-\tau_i)\varepsilon_i\right). \tag{C.5}$$

Then, putting all this together we are able to write (C.1) as:

$$\Gamma[A]=(-ie)^N\int_0^\infty\frac{dT}{T}e^{-m^2T}\int\mathcal{D}x\int_0^T d\tau_1\cdots\int_0^T d\tau_N e^{-\int_0^T d\tau\left(\frac{\dot{x}^2}{4}-j\cdot x\right)}\Big|_{lin\ \varepsilon_i}. \tag{C.6}$$

Now, the integral over "x" has the form of a Gaussian integral in D dimensions. Hence, we look at the free coordinate path integral at fixed proper time T

$$\int\mathcal{D}x\ e^{-\int_0^T d\tau\frac{\dot{x}^2}{4}}=(4\pi T)^{D/2}, \tag{C.7}$$

and then we write the amplitude in terms of it

$$\Gamma[A]=(-ie)^N\int_0^\infty\frac{dT}{T}(4\pi T)^{-D/2}e^{-m^2T}\int_0^T d\tau_1\cdots\int_0^T d\tau_N$$
$$\times\left.\frac{\int\mathcal{D}x\ e^{-\int_0^T d\tau\left(\frac{\dot{x}^2}{4}-j\cdot x\right)}}{\int\mathcal{D}x\ e^{-\int_0^T d\tau\frac{\dot{x}^2}{4}}}\right|_{lin\ \varepsilon_i}. \tag{C.8}$$

Now, in order to perform the Gaussian integral, we need first to separate off the integration over the loop center of mass x_0 defined as

$$x_0^\mu\equiv\frac{1}{T}\int_0^T d\tau x^\mu(\tau). \tag{C.9}$$

Thus, reducing the path integral to an integral over the relative coordinate q:

$$x^\mu(\tau) = x_0^\mu + q^\mu(\tau), \tag{C.10}$$

where we can see from (C.9) and (C.10) that q satisfies

$$\int_0^T d\tau q(\tau) = 0. \tag{C.11}$$

Furthermore, using (C.10) in (C.8) we can see that

$$\int \mathcal{D}x\ e^{-\int_0^T d\tau\left(\frac{\dot{x}^2}{4} - j\cdot x\right)} = \int d^D x_0 \int \mathcal{D}q\ e^{-\int_0^T d\tau\left(\frac{\dot{q}^2}{4} - j\cdot q\right)} e^{ix_0\cdot\sum_{i=1}^N k_i}, \tag{C.12}$$

look carefully (C.4) to see that

$$\int_0^T d\tau j(\tau)\cdot x(\tau) = \int_0^T d\tau j(\tau)\cdot q(\tau) + ix_0\cdot\sum_{i=1}^N k_i, \tag{C.13}$$

and of course

$$\int \mathcal{D}x\ e^{-\int_0^T d\tau\frac{\dot{x}^2}{4}} = \int \mathcal{D}q\ e^{-\int_0^T d\tau\frac{\dot{q}^2}{4}}. \tag{C.14}$$

Then, this transforms (C.8) into

$$\begin{aligned}\Gamma[A] =& (-ie)^N \int_0^\infty \frac{dT}{T}(4\pi T)^{-D/2} e^{-m^2T} \int_0^T d\tau_1 \cdots \int_0^T d\tau_N \\ &\times \int d^D x_0 \frac{\int \mathcal{D}q\ e^{-\int_0^T d\tau\left(\frac{\dot{q}^2}{4} - j\cdot q\right)}}{\int \mathcal{D}q\ e^{-\int_0^T d\tau\frac{\dot{q}^2}{4}}} e^{ix_0\cdot\sum_{i=1}^N k_i}\Bigg|_{lin\ \varepsilon_i},\end{aligned} \tag{C.15}$$

where

$$\int d^D x_0 e^{ix_0\cdot\sum_{i=1}^N k_i} = (2\pi)^D \delta\left(\sum_{i=1}^N k_i\right), \tag{C.16}$$

$$\begin{aligned}\frac{\int \mathcal{D}q\ e^{-\int_0^T d\tau\left(\frac{\dot{q}^2}{4} - j\cdot q\right)}}{\int \mathcal{D}q\ e^{-\int_0^T d\tau\frac{\dot{q}^2}{4}}} &= e^{\frac{1}{4}\int_0^T d\tau\int_0^T d\tau' j(\tau)\left(\frac{-1}{4}\frac{d^2}{d\tau^2}\right)^{-1} j(\tau')} \\ &= e^{\frac{-1}{2}\int_0^T d\tau\int_0^T d\tau' j(\tau) G_B(\tau-\tau') j(\tau')}.\end{aligned} \tag{C.17}$$

Here, G_B is the Green function of the operator $\left(\frac{-1}{4}\frac{d^2}{d\tau^2}\right)$,

$$\langle\tau|\left(\frac{-1}{4}\frac{d^2}{d\tau^2}\right)^{-1}|\tau'\rangle = -2G_B(\tau-\tau'), \tag{C.18}$$

$$G_B(\tau-\tau') = |\tau-\tau'| - \frac{(\tau-\tau')^2}{T} - \frac{T}{6}. \tag{C.19}$$

Further, we notice that

$$\begin{aligned}&\frac{-1}{2}\int_0^T d\tau \int_0^T d\tau' j(\tau)G_B(\tau-\tau')j(\tau')\\ &= \sum_{i,j=1}^{N}\left(\frac{1}{2}G_{Bij}k_i\cdot k_j - i\dot{G}_{Bij}\varepsilon_i\cdot k_j + \frac{1}{2}\ddot{G}_{Bij}\varepsilon_i\cdot\varepsilon_j\right),\end{aligned} \tag{C.20}$$

where $G_{Bij} = G_B(\tau_i-\tau_j)$ and dots are the derivatives respect to the first parameter τ_i

$$\begin{aligned}\dot{G}_B(\tau_i-\tau_j) &= sign(\tau_i-\tau_j) - 2\frac{(\tau_i-\tau_j)}{T},\\ \ddot{G}_B(\tau_i-\tau_j) &= 2\delta(\tau_i-\tau_j) - \frac{2}{T}.\end{aligned} \tag{C.21}$$

Hence, we can write (C.15) as

$$\begin{aligned}\Gamma[A] =&(-ie)^N\int_0^\infty\frac{dT}{T}(4\pi T)^{-D/2}e^{-m^2T}\int_0^T d\tau_1\cdots\int_0^T d\tau_N\\ &\times(2\pi)^D\delta\left(\sum_{i=1}^N k_i\right)e^{\sum_{i,j=1}^N\left(\frac{1}{2}G_{Bij}k_i\cdot k_j - i\dot{G}_{Bij}\varepsilon_i\cdot k_j+\frac{1}{2}\ddot{G}_{Bij}\varepsilon_i\cdot\varepsilon_j\right)}\Bigg|_{lin\ \varepsilon_i},\end{aligned} \tag{C.22}$$

or

$$\begin{aligned}\Gamma[k_1,\varepsilon_1;...;k_N,\varepsilon_N] &= (-ie)^N(2\pi)^D\delta\left(\sum_{i=1}^N k_i\right)\int_0^\infty\frac{dT}{T}(4\pi T)^{-D/2}e^{-m^2T}\\ &\times\Pi_{k=1}^N\int_0^T d\tau_k exp\left[\sum_{i,j=1}^N\left(\frac{1}{2}G_{Bij}k_i\cdot k_j - i\dot{G}_{Bij}\varepsilon_i\cdot k_j+\frac{1}{2}\ddot{G}_{Bij}\varepsilon_i\cdot\varepsilon_j\right)\right]\Bigg|_{lin\ \varepsilon_i},\end{aligned} \tag{C.23}$$

that is exactly the Bern-Kosower master formula.

Appendix D

Explicit calculation of low energy integrals (scalar case)

In this appendix we present explicitly all the calculations needed to obtain all the formulas presented in Chapter 3, in the scalar cases.

D.1 One low energy photon

In this section we will perform the integral over u_4 of (3.2) with the assumption that the corresponding photon is in the low energy limit. This means that we only consider the linear terms in k_4. Then we define

$$I_{(4)} = \int_0^1 du_4 Q_4 e^{(\cdot 4)} = I^4_{(4)} + I^3_{(4)} + I^2_{(4)} + I^{22}_{(4)}, \tag{D.1}$$

with

$$e^{(\cdot 4)} = e^{T(G_{B14} k_1 \cdot k_4 + G_{B24} k_2 \cdot k_4 + G_{B34} k_3 \cdot k_4)}, \tag{D.2}$$

and

$$I^i_{(4)} = \int_0^1 du_4 Q^i_4 e^{(\cdot 4)}, \quad i = 4, 3, 2, 22. \tag{D.3}$$

D.1.1 Integral of Q_4^4

Since Q_4^4 is linear in k_4, we can disregard the exponential. Then with the help of (B.15) we can integrate each term of Q_4^4 in (3.7) and we obtain

$$\begin{aligned} I^4_{(4)}(1234) &= \dot{G}_{B12}\dot{G}_{B23}\left(\tfrac{1}{6}-\tfrac{1}{2}\dot{G}^2_{B31}\right)Z_4(1234),\\ I^4_{(4)}(1243) &= \dot{G}_{B12}\dot{G}_{B31}\left(\tfrac{1}{6}-\tfrac{1}{2}\dot{G}^2_{B23}\right)Z_4(1243),\\ I^4_{(4)}(1324) &= \dot{G}_{B23}\dot{G}_{B31}\left(\tfrac{1}{6}-\tfrac{1}{2}\dot{G}^2_{B12}\right)Z_4(1324). \end{aligned} \tag{D.4}$$

D.1.2 Integral of Q_4^3

First we cancel all the terms $O(k_4^2)$ in (3.10)-(3.13)

$$\begin{aligned} Q_4^3(123) = \;& \dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31}Z_3(123)\left(\dot{G}_{B41}\varepsilon_4\cdot k_1+\dot{G}_{B42}\varepsilon_4\cdot k_2\right)\\ &+\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31}Z_3(123)\dot{G}_{B43}\varepsilon_4\cdot k_3, \end{aligned} \tag{D.5}$$

$$\begin{aligned} Q_4^3(234) = \;& \dot{G}_{B23}\dot{G}_{B34}\dot{G}_{B42}Z_3(234)\left(\dot{G}_{B12}\varepsilon_1\cdot k_2+\dot{G}_{B13}\varepsilon_1\cdot k_3\right)\\ &+\dot{G}_{B23}\dot{G}_{B34}\dot{G}_{B42}Z_3(234)\dot{G}_{B14}\cancel{\varepsilon_1\cdot k_4}, \end{aligned} \tag{D.6}$$

$$\begin{aligned} Q_4^3(341) = \;& \dot{G}_{B34}\dot{G}_{B41}\dot{G}_{B13}Z_3(341)\left(\dot{G}_{B21}\varepsilon_2\cdot k_1+\dot{G}_{B23}\varepsilon_2\cdot k_3\right)\\ &+\dot{G}_{B34}\dot{G}_{B41}\dot{G}_{B13}Z_3(341)\dot{G}_{B24}\cancel{\varepsilon_2\cdot k_4}, \end{aligned} \tag{D.7}$$

$$\begin{aligned} Q_4^3(412) = \;& \dot{G}_{B41}\dot{G}_{B12}\dot{G}_{B24}Z_3(412)\left(\dot{G}_{B31}\varepsilon_3\cdot k_1+\dot{G}_{B32}\varepsilon_3\cdot k_2\right)\\ &+\dot{G}_{B41}\dot{G}_{B12}\dot{G}_{B24}Z_3(412)\dot{G}_{B34}\cancel{\varepsilon_3\cdot k_4}. \end{aligned} \tag{D.8}$$

Now this equations without $O(k_4^2)$ terms and with the help of (B.5) can be rewritten as

$$Q_4^3(123) = \dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31}Z_3(123)\left[\dot{G}_{B41}\varepsilon_4\cdot k_1+\dot{G}_{B42}\varepsilon_4\cdot k_2+\dot{G}_{B43}\varepsilon_4\cdot k_3\right], \tag{D.9}$$

$$\begin{aligned} Q_4^3(234) = \;& \dot{G}_{B23}\dot{G}_{B34}\dot{G}_{B42}Z_3(234)\left(\dot{G}_{B12}\tfrac{r_1\cdot f_1\cdot k_2}{r_1\cdot k_1}+\dot{G}_{B12}\tfrac{r_1\cdot\varepsilon_1 k_1\cdot k_2}{r_1\cdot k_1}\right)\\ &+\dot{G}_{B23}\dot{G}_{B34}\dot{G}_{B42}Z_3(234)\left(\dot{G}_{B13}\tfrac{r_1\cdot f_1\cdot k_3}{r_1\cdot k_1}+\dot{G}_{B13}\tfrac{r_1\cdot\varepsilon_1 k_1\cdot k_3}{r_1\cdot k_1}\right), \end{aligned} \tag{D.10}$$

$$Q_4^3(341) = \dot{G}_{B34}\dot{G}_{B41}\dot{G}_{B13}Z_3(341)\left(\dot{G}_{B21}\frac{r_2\cdot f_2\cdot k_1}{r_2\cdot k_2} + \dot{G}_{B21}\frac{r_2\cdot\varepsilon_2 k_2\cdot k_1}{r_2\cdot k_2}\right) + \dot{G}_{B34}\dot{G}_{B41}\dot{G}_{B13}Z_3(341)\left(\dot{G}_{B23}\frac{r_2\cdot f_2\cdot k_3}{r_2\cdot k_2} + \dot{G}_{B23}\frac{r_2\cdot\varepsilon_2 k_2\cdot k_3}{r_2\cdot k_2}\right), \tag{D.11}$$

$$Q_4^3(412) = \dot{G}_{B41}\dot{G}_{B12}\dot{G}_{B24}Z_3(412)\left(\dot{G}_{B31}\frac{r_3\cdot f_3\cdot k_1}{r_3\cdot k_3} + \dot{G}_{B31}\frac{r_3\cdot\varepsilon_3 k_3\cdot k_1}{r_3\cdot k_3}\right) + \dot{G}_{B41}\dot{G}_{B12}\dot{G}_{B24}Z_3(412)\left(\dot{G}_{B32}\frac{r_3\cdot f_3\cdot k_2}{r_3\cdot k_3} + \dot{G}_{B32}\frac{r_3\cdot\varepsilon_3 k_3\cdot k_2}{r_3\cdot k_3}\right). \tag{D.12}$$

Here, we introduce the vectors r_1, r_2 and r_3 with the only conditions that $r_1 \cdot k_1 \neq 0$, $r_2 \cdot k_2 \neq 0$ and $r_3 \cdot k_3 \neq 0$ and we use this because with this trick we can write these equations in terms of f_i the field strength tensor. Moreover, notice that $I^3_{(4)}(123)$ must be calculated using (B.21) because $Q_4^3(123)$ is zero order in k_4, (here an in the following we will use $T = 1$, but this extra factors of T must be restored and as we can see there are extra factors of T only when we take the linear term of th exponential)

$$I^3_{(4)}(123) = \dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31}Z_3(123)\left[\tfrac{1}{3}\textstyle\sum_{i=1}^3 \dot{G}_{B1i}G_{Bi1}k_i\cdot k_4\varepsilon_4\cdot k_1 + \tfrac{1}{3}\sum_{i=1}^3 \dot{G}_{B2i}G_{Bi2}k_i\cdot k_4\varepsilon_4\cdot k_2 + \tfrac{1}{3}\sum_{i=1}^3 \dot{G}_{B3i}G_{Bi3}k_i\cdot k_4\varepsilon_4\cdot k_3\right], \tag{D.13}$$

we expand the sums

$$I^3_{(4)}(123) = \tfrac{1}{3}\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31}Z_3(123)\left[\dot{G}_{B12}G_{B21}k_2\cdot k_4\varepsilon_4\cdot k_1 + \dot{G}_{B13}G_{B31}k_3\cdot k_4\varepsilon_4\cdot k_1 + \dot{G}_{B21}G_{B12}k_1\cdot k_4\varepsilon_4\cdot k_2 + \dot{G}_{B23}G_{B32}k_3\cdot k_4\varepsilon_4\cdot k_2 + \dot{G}_{B31}G_{B13}k_1\cdot k_4\varepsilon_4\cdot k_3 + \dot{G}_{B32}G_{B23}k_2\cdot k_4\varepsilon_4\cdot k_3\right], \tag{D.14}$$

we factorize

$$I^3_{(4)}(123) = \tfrac{1}{3}\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31}Z_3(123)\left[\dot{G}_{B12}G_{B21}(k_2\cdot k_4\varepsilon_4\cdot k_1 - k_1\cdot k_4\varepsilon_4\cdot k_2) + \dot{G}_{B23}G_{B32}(k_3\cdot k_4\varepsilon_4\cdot k_2 - k_2\cdot k_4\varepsilon_4\cdot k_3) + \dot{G}_{B31}G_{B13}(k_1\cdot k_4\varepsilon_4\cdot k_3 - k_3\cdot k_4\varepsilon_4\cdot k_1)\right], \tag{D.15}$$

finally we use (B.5) to write this as

$$I^3_{(4)}(123) = \tfrac{1}{3}\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31}Z_3(123)\left[\dot{G}_{B12}G_{B21}k_2\cdot f_4\cdot k_1 + \dot{G}_{B23}G_{B32}k_3\cdot f_4\cdot k_2 + \dot{G}_{B31}G_{B13}k_1\cdot f_4\cdot k_3\right]. \tag{D.16}$$

While in (D.10)-(D.12) we cancel the marked terms because these form a total derivative, in each case, that does not contribute to the integral and we use (B.11) and (B.12) to integrate the remaining terms

$$I^3_{(4)}(234) = \dot{G}_{B23}\left(2G_{B32} - \frac{1}{3}\right) Z_3(234)\left(\dot{G}_{B12}\frac{r_1\cdot f_1\cdot k_2}{r_1\cdot k_1} + \dot{G}_{B13}\frac{r_1\cdot f_1\cdot k_3}{r_1\cdot k_1}\right), \tag{D.17}$$

$$I^3_{(4)}(341) = \left(2G_{B31} - \frac{1}{3}\right)\dot{G}_{B13} Z_3(341)\left(\dot{G}_{B21}\frac{r_2\cdot f_2\cdot k_1}{r_2\cdot k_2} + \dot{G}_{B23}\frac{r_2\cdot f_2\cdot k_3}{r_2\cdot k_2}\right), \tag{D.18}$$

$$I^3_{(4)}(412) = \left(2G_{B12} - \frac{1}{3}\right)\dot{G}_{B12} Z_3(412)\left(\dot{G}_{B31}\frac{r_3\cdot f_3\cdot k_1}{r_3\cdot k_3} + \dot{G}_{B32}\frac{r_3\cdot f_3\cdot k_2}{r_3\cdot k_3}\right). \tag{D.19}$$

Now we use (B.1) to write all in terms of $\dot{G}$'s (also we arrange the subscripts)

Result of $I^3_{(4)}$

$$\begin{aligned} I^3_{(4)}(123) = {} & \tfrac{1}{12}\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31} Z_3(123)\Big[\dot{G}_{B12}\left(1-\dot{G}^2_{B12}\right) k_2\cdot f_4\cdot k_1 \\ & + \dot{G}_{B23}\left(1-\dot{G}^2_{B23}\right) k_3\cdot f_4\cdot k_2 + \dot{G}_{B31}\left(1-\dot{G}^2_{B31}\right) k_1\cdot f_4\cdot k_3\Big], \end{aligned} \tag{D.20}$$

$$I^3_{(4)}(234) = \frac{1}{2}\dot{G}_{B23}\left(\frac{1}{3} - \dot{G}^2_{B23}\right) Z_3(234)\left(\dot{G}_{B12}\frac{r_1\cdot f_1\cdot k_2}{r_1\cdot k_1} - \dot{G}_{B31}\frac{r_1\cdot f_1\cdot k_3}{r_1\cdot k_1}\right), \tag{D.21}$$

$$I^3_{(4)}(341) = \frac{1}{2}\left(\frac{1}{3} - \dot{G}^2_{B31}\right)\dot{G}_{B31} Z_3(341)\left(\dot{G}_{B12}\frac{r_2\cdot f_2\cdot k_1}{r_2\cdot k_2} - \dot{G}_{B23}\frac{r_2\cdot f_2\cdot k_3}{r_2\cdot k_2}\right), \tag{D.22}$$

$$I^3_{(4)}(412) = \frac{1}{2}\left(\frac{1}{3} - \dot{G}^2_{B12}\right)\dot{G}_{B12} Z_3(412)\left(\dot{G}_{B31}\frac{r_3\cdot f_3\cdot k_1}{r_3\cdot k_3} - \dot{G}_{B23}\frac{r_3\cdot f_3\cdot k_2}{r_3\cdot k_3}\right). \tag{D.23}$$

D.1.3 Integral of Q_4^2

First we cancel all the terms $O(k_4^2)$ in (3.14)-(3.19)

$$
\begin{aligned}
Q_4^2(12) = \dot{G}_{B12}\dot{G}_{B21}Z_2(12)\Big[&\dot{G}_{B31}\varepsilon_3\cdot k_1\dot{G}_{B41}\varepsilon_4\cdot k_1 + \dot{G}_{B31}\varepsilon_3\cdot k_1\dot{G}_{B42}\varepsilon_4\cdot k_2\\
&+\dot{G}_{B31}\varepsilon_3\cdot k_1\dot{G}_{B43}\varepsilon_4\cdot k_3 + \dot{G}_{B32}\varepsilon_3\cdot k_2\dot{G}_{B41}\varepsilon_4\cdot k_1\\
&+\dot{G}_{B32}\varepsilon_3\cdot k_2\dot{G}_{B42}\varepsilon_4\cdot k_2 + \dot{G}_{B32}\varepsilon_3\cdot k_2\dot{G}_{B43}\varepsilon_4\cdot k_3\\
&+\dot{G}_{B34}\varepsilon_3\cdot k_4\dot{G}_{B41}\varepsilon_4\cdot k_1 + \dot{G}_{B34}\varepsilon_3\cdot k_4\dot{G}_{B42}\varepsilon_4\cdot k_2\\
&+\tfrac{1}{2}\dot{G}_{B34}\varepsilon_3\cdot\varepsilon_4\left(\dot{G}_{B31}k_3\cdot k_1 - \dot{G}_{B41}k_4\cdot k_1\right)\\
&+\tfrac{1}{2}\dot{G}_{B34}\varepsilon_3\cdot\varepsilon_4\left(\dot{G}_{B32}k_3\cdot k_2 - \dot{G}_{B42}k_4\cdot k_2\right)\Big],
\end{aligned}
\tag{D.24}
$$

$$
\begin{aligned}
Q_4^2(13) = \dot{G}_{B13}\dot{G}_{B31}Z_2(13)\Big[&\dot{G}_{B21}\varepsilon_2\cdot k_1\dot{G}_{B41}\varepsilon_4\cdot k_1 + \dot{G}_{B21}\varepsilon_2\cdot k_1\dot{G}_{B42}\varepsilon_4\cdot k_2\\
&+\dot{G}_{B21}\varepsilon_2\cdot k_1\dot{G}_{B43}\varepsilon_4\cdot k_3 + \dot{G}_{B23}\varepsilon_2\cdot k_3\dot{G}_{B41}\varepsilon_4\cdot k_1\\
&+\dot{G}_{B23}\varepsilon_2\cdot k_3\dot{G}_{B42}\varepsilon_4\cdot k_2 + \dot{G}_{B23}\varepsilon_2\cdot k_3\dot{G}_{B43}\varepsilon_4\cdot k_3\\
&+\dot{G}_{B24}\varepsilon_2\cdot k_4\dot{G}_{B41}\varepsilon_4\cdot k_1 + \dot{G}_{B24}\varepsilon_2\cdot k_4\dot{G}_{B43}\varepsilon_4\cdot k_3\\
&+\tfrac{1}{2}\dot{G}_{B24}\varepsilon_2\cdot\varepsilon_4\left(\dot{G}_{B21}k_2\cdot k_1 - \dot{G}_{B41}k_4\cdot k_1\right)\\
&+\tfrac{1}{2}\dot{G}_{B24}\varepsilon_2\cdot\varepsilon_4\left(\dot{G}_{B23}k_2\cdot k_3 - \dot{G}_{B43}k_4\cdot k_3\right)\Big],
\end{aligned}
\tag{D.25}
$$

$$
\begin{aligned}
Q_4^2(14) = \dot{G}_{B14}\dot{G}_{B41}Z_2(14)\Big[&\dot{G}_{B21}\varepsilon_2\cdot k_1\dot{G}_{B31}\varepsilon_3\cdot k_1 + \dot{G}_{B21}\varepsilon_2\cdot k_1\dot{G}_{B32}\varepsilon_3\cdot k_2\\
&+\dot{G}_{B21}\varepsilon_2\cdot k_1\dot{G}_{B34}\cancel{\varepsilon_3\cdot k_4} + \dot{G}_{B23}\varepsilon_2\cdot k_3\dot{G}_{B31}\varepsilon_3\cdot k_1\\
&+\dot{G}_{B23}\varepsilon_2\cdot k_3\dot{G}_{B34}\cancel{\varepsilon_3\cdot k_4} + \dot{G}_{B24}\cancel{\varepsilon_2\cdot k_4}\dot{G}_{B31}\varepsilon_3\cdot k_1\\
&+\dot{G}_{B24}\cancel{\varepsilon_2\cdot k_4}\dot{G}_{B32}\varepsilon_3\cdot k_2 + \dot{G}_{B24}\cancel{\varepsilon_2\cdot k_4}\dot{G}_{B34}\cancel{\varepsilon_3\cdot k_4}\\
&+\tfrac{1}{2}\dot{G}_{B23}\varepsilon_2\cdot\varepsilon_3\left(\dot{G}_{B21}k_2\cdot k_1 - \dot{G}_{B31}k_3\cdot k_1\right)\\
&+\tfrac{1}{2}\dot{G}_{B23}\varepsilon_2\cdot\varepsilon_3\left(\dot{G}_{B24}\cancel{k_2\cdot k_4} - \dot{G}_{B34}\cancel{k_3\cdot k_4}\right)\Big],
\end{aligned}
\tag{D.26}
$$

$$
\begin{aligned}
Q_4^2(23) = \dot{G}_{B23}\dot{G}_{B32}Z_2(23)\Big[&\dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B41}\varepsilon_4\cdot k_1 + \dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B42}\varepsilon_4\cdot k_2\\
&+\dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B43}\varepsilon_4\cdot k_3 + \dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B41}\varepsilon_4\cdot k_1\\
&+\dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B42}\varepsilon_4\cdot k_2 + \dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B43}\varepsilon_4\cdot k_3\\
&+\dot{G}_{B14}\varepsilon_1\cdot k_4\dot{G}_{B42}\varepsilon_4\cdot k_2 + \dot{G}_{B14}\varepsilon_1\cdot k_4\dot{G}_{B43}\varepsilon_4\cdot k_3\\
&+\tfrac{1}{2}\dot{G}_{B14}\varepsilon_1\cdot\varepsilon_4\left(\dot{G}_{B12}k_1\cdot k_2 - \dot{G}_{B42}k_4\cdot k_2\right)\\
&+\tfrac{1}{2}\dot{G}_{B14}\varepsilon_1\cdot\varepsilon_4\left(\dot{G}_{B13}k_1\cdot k_3 - \dot{G}_{B43}k_4\cdot k_3\right)\Big],
\end{aligned}
\tag{D.27}
$$

$$
\begin{aligned}
Q_4^2(24) = \dot{G}_{B24}\dot{G}_{B42}Z_2(24)\Big[&\dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B31}\varepsilon_3\cdot k_1 + \dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B32}\varepsilon_3\cdot k_2\\
&+\dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B34}\cancel{\varepsilon_3\cdot k_4} + \dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B32}\varepsilon_3\cdot k_2\\
&+\dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B34}\cancel{\varepsilon_3\cdot k_4} + \dot{G}_{B14}\cancel{\varepsilon_1\cdot k_4}\dot{G}_{B31}\varepsilon_3\cdot k_1\\
&+\dot{G}_{B14}\cancel{\varepsilon_1\cdot k_4}\dot{G}_{B32}\varepsilon_3\cdot k_2 + \dot{G}_{B14}\cancel{\varepsilon_1\cdot k_4}\dot{G}_{B34}\cancel{\varepsilon_3\cdot k_4}\\
&+\tfrac{1}{2}\dot{G}_{B13}\varepsilon_1\cdot\varepsilon_3\left(\dot{G}_{B12}k_1\cdot k_2 - \dot{G}_{B32}k_3\cdot k_2\right)\\
&+\tfrac{1}{2}\dot{G}_{B13}\varepsilon_1\cdot\varepsilon_3\left(\dot{G}_{B14}\cancel{k_1\cdot k_4} - \dot{G}_{B34}\cancel{k_3\cdot k_4}\right)\Big],
\end{aligned}
\tag{D.28}
$$

$$
\begin{aligned}
Q_4^2(34) = \dot{G}_{B34}\dot{G}_{B43}Z_2(34)\Big[&\dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B23}\varepsilon_2\cdot k_3 + \dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B24}\cancel{\varepsilon_2\cdot k_4}\\
&+\dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B21}\varepsilon_2\cdot k_1 + \dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B23}\varepsilon_2\cdot k_3\\
&+\dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B24}\cancel{\varepsilon_2\cdot k_4} + \dot{G}_{B14}\cancel{\varepsilon_1\cdot k_4}\dot{G}_{B21}\varepsilon_2\cdot k_1\\
&+\dot{G}_{B14}\cancel{\varepsilon_1\cdot k_4}\dot{G}_{B23}\varepsilon_2\cdot k_3 + \dot{G}_{B14}\cancel{\varepsilon_1\cdot k_4}\dot{G}_{B24}\cancel{\varepsilon_2\cdot k_4}\\
&+\tfrac{1}{2}\dot{G}_{B12}\varepsilon_1\cdot\varepsilon_2\left(\dot{G}_{B13}k_1\cdot k_3 - \dot{G}_{B23}k_2\cdot k_3\right)\\
&+\tfrac{1}{2}\dot{G}_{B12}\varepsilon_1\cdot\varepsilon_2\left(\dot{G}_{B14}\cancel{k_1\cdot k_4} - \dot{G}_{B24}\cancel{k_2\cdot k_4}\right)\Big].
\end{aligned}
\tag{D.29}
$$

Then, after a factorization respect to $\dot{G}_{Bij}$, $I^2_{(4)}$ looks like

$$
\begin{aligned}
I^2_{(4)}(12) = \dot{G}_{B12}\dot{G}_{B21}Z_2(12)\int_0^1\Big[&\dot{G}_{B31}\dot{G}_{B43}(\varepsilon_3\cdot k_1\varepsilon_4\cdot k_3 - \tfrac{1}{2}\varepsilon_3\cdot\varepsilon_4 k_3\cdot k_1)\\
&+\dot{G}_{B32}\dot{G}_{B43}(\varepsilon_3\cdot k_2\varepsilon_4\cdot k_3 - \tfrac{1}{2}\varepsilon_3\cdot\varepsilon_4 k_3\cdot k_2)\\
&+\dot{G}_{B41}(\dot{G}_{B31}\varepsilon_3\cdot k_1\varepsilon_4\cdot k_1 + \dot{G}_{B32}\varepsilon_3\cdot k_2\varepsilon_4\cdot k_1)\\
&+\dot{G}_{B42}(\dot{G}_{B31}\varepsilon_3\cdot k_1\varepsilon_4\cdot k_2 + \dot{G}_{B32}\varepsilon_3\cdot k_2\varepsilon_4\cdot k_2)\Big]\, e^{(i4)}du_4\\
+\dot{G}_{B12}\dot{G}_{B21}Z_2(12)\int_0^1\Big[&\dot{G}_{B34}\dot{G}_{B41}(\varepsilon_3\cdot k_4\varepsilon_4\cdot k_1 - \tfrac{1}{2}\varepsilon_3\cdot\varepsilon_4 k_4\cdot k_1)\\
+\dot{G}_{B34}\dot{G}_{B42}&(\varepsilon_3\cdot k_4\varepsilon_4\cdot k_2 - \tfrac{1}{2}\varepsilon_3\cdot\varepsilon_4 k_4\cdot k_2)\Big]\, du_4,
\end{aligned}
\tag{D.30}
$$

$$
\begin{aligned}
I^2_{(4)}(13) = \dot{G}_{B13}\dot{G}_{B31}Z_2(13)\int_0^1 \Big[&\dot{G}_{B21}\dot{G}_{B42}(\varepsilon_2\cdot k_1\varepsilon_4\cdot k_2 - \tfrac{1}{2}\varepsilon_2\cdot\varepsilon_4 k_2\cdot k_1)\\
&+\dot{G}_{B23}\dot{G}_{B42}(\varepsilon_2\cdot k_3\varepsilon_4\cdot k_2 - \tfrac{1}{2}\varepsilon_2\cdot\varepsilon_4 k_2\cdot k_3)\\
&+\dot{G}_{B41}(\dot{G}_{B21}\varepsilon_2\cdot k_1\varepsilon_4\cdot k_1 + \dot{G}_{B23}\varepsilon_2\cdot k_3\varepsilon_4\cdot k_1)\\
&+\dot{G}_{B43}(\dot{G}_{B21}\varepsilon_2\cdot k_1\varepsilon_4\cdot k_3 + \dot{G}_{B23}\varepsilon_2\cdot k_3\varepsilon_4\cdot k_3)\Big]\, e^{(i4)}du_4\\
&+\dot{G}_{B13}\dot{G}_{B31}Z_2(13)\int_0^1 \Big[\dot{G}_{B24}\dot{G}_{B41}(\varepsilon_2\cdot k_4\varepsilon_4\cdot k_1 - \tfrac{1}{2}\varepsilon_2\cdot\varepsilon_4 k_4\cdot k_1)\\
&+\dot{G}_{B24}\dot{G}_{B43}(\varepsilon_2\cdot k_4\varepsilon_4\cdot k_3 - \tfrac{1}{2}\varepsilon_2\cdot\varepsilon_4 k_4\cdot k_3)\Big]\, du_4,
\end{aligned}
\tag{D.31}
$$

$$
\begin{aligned}
I^2_{(4)}(23) = \dot{G}_{B23}\dot{G}_{B32}Z_2(23)\int_0^1 \Big[&\dot{G}_{B12}\dot{G}_{B41}(\varepsilon_1\cdot k_2\varepsilon_4\cdot k_1 - \tfrac{1}{2}\varepsilon_1\cdot\varepsilon_4 k_1\cdot k_2)\\
&+\dot{G}_{B13}\dot{G}_{B41}(\varepsilon_1\cdot k_3\varepsilon_4\cdot k_1 - \tfrac{1}{2}\varepsilon_1\cdot\varepsilon_4 k_1\cdot k_3)\\
&+\dot{G}_{B42}(\dot{G}_{B12}\varepsilon_1\cdot k_2\varepsilon_4\cdot k_2 + \dot{G}_{B13}\varepsilon_1\cdot k_3\varepsilon_4\cdot k_2)\\
&+\dot{G}_{B43}(\dot{G}_{B12}\varepsilon_1\cdot k_2\varepsilon_4\cdot k_3 + \dot{G}_{B13}\varepsilon_1\cdot k_3\varepsilon_4\cdot k_3)\Big]\, e^{(i4)}du_4\\
&+\dot{G}_{B23}\dot{G}_{B32}Z_2(23)\int_0^1 \Big[\dot{G}_{B14}\dot{G}_{B42}(\varepsilon_1\cdot k_4\varepsilon_4\cdot k_2 - \tfrac{1}{2}\varepsilon_1\cdot\varepsilon_4 k_4\cdot k_2)\\
&+\dot{G}_{B14}\dot{G}_{B43}(\varepsilon_1\cdot k_4\varepsilon_4\cdot k_3 - \tfrac{1}{2}\varepsilon_1\cdot\varepsilon_4 k_4\cdot k_3)\Big]\, du_4,
\end{aligned}
\tag{D.32}
$$

$$
\begin{aligned}
I^2_{(4)}(14) = \int_0^1 \dot{G}_{B14}\dot{G}_{B41}Z_2(14)du_4\Big[&\dot{G}_{B21}\dot{G}_{B31}\varepsilon_2\cdot k_1\varepsilon_3\cdot k_1\\
&+\dot{G}_{B23}\dot{G}_{B31}(\varepsilon_2\cdot k_3\varepsilon_3\cdot k_1 - \tfrac{1}{2}\varepsilon_2\cdot\varepsilon_3 k_3\cdot k_1)\\
&+\dot{G}_{B21}\dot{G}_{B32}(\varepsilon_2\cdot k_1\varepsilon_3\cdot k_2 - \tfrac{1}{2}\varepsilon_2\cdot\varepsilon_3 k_2\cdot k_1)\Big],
\end{aligned}
\tag{D.33}
$$

$$
\begin{aligned}
I^2_{(4)}(24) = \int_0^1 \dot{G}_{B24}\dot{G}_{B42}Z_2(24)du_4\Big[&\dot{G}_{B12}\dot{G}_{B32}\varepsilon_1\cdot k_2\varepsilon_3\cdot k_2\\
&+\dot{G}_{B13}\dot{G}_{B32}(\varepsilon_1\cdot k_3\varepsilon_3\cdot k_2 - \tfrac{1}{2}\varepsilon_1\cdot\varepsilon_3 k_3\cdot k_2)\\
&+\dot{G}_{B12}\dot{G}_{B31}(\varepsilon_1\cdot k_2\varepsilon_3\cdot k_1 - \tfrac{1}{2}\varepsilon_1\cdot\varepsilon_3 k_1\cdot k_2)\Big],
\end{aligned}
\tag{D.34}
$$

$$
\begin{aligned}
I^2_{(4)}(34) = \int_0^1 \dot{G}_{B34}\dot{G}_{B43}Z_2(34)du_4\Big[&\dot{G}_{B13}\dot{G}_{B23}\varepsilon_1\cdot k_3\varepsilon_2\cdot k_3\\
&+\dot{G}_{B13}\dot{G}_{B21}(\varepsilon_1\cdot k_3\varepsilon_2\cdot k_1 - \tfrac{1}{2}\varepsilon_1\cdot\varepsilon_2 k_1\cdot k_3)\\
&+\dot{G}_{B12}\dot{G}_{B23}(\varepsilon_1\cdot k_2\varepsilon_2\cdot k_3 - \tfrac{1}{2}\varepsilon_1\cdot\varepsilon_2 k_2\cdot k_3)\Big].
\end{aligned}
\tag{D.35}
$$

We use (B.15) and (B.21) to integrate $I^2_{(4)}(12)$, $I^2_{(4)}(13)$ and $I^2_{(4)}(23)$ (again we take $T = 1$). While we only use (B.15) to integrate $I^2_{(4)}(14)$, $I^2_{(4)}(24)$

and $I^2_{(4)}(34)$, as result we have

$$
\begin{aligned}
I^2_{(4)}(12) = \dot{G}_{B12}\dot{G}_{B21}Z_2(12)\Big[& \\
&+\tfrac{1}{3}\dot{G}_{B31}\textstyle\sum_i \dot{G}_{B3i}G_{Bi3}k_i\cdot k_4(\varepsilon_3\cdot k_1\varepsilon_4\cdot k_3-\tfrac{1}{2}\varepsilon_3\cdot\varepsilon_4 k_3\cdot k_1)\\
&+\tfrac{1}{3}\dot{G}_{B32}\textstyle\sum_i \dot{G}_{B3i}G_{Bi3}k_i\cdot k_4(\varepsilon_3\cdot k_2\varepsilon_4\cdot k_3-\tfrac{1}{2}\varepsilon_3\cdot\varepsilon_4 k_3\cdot k_2)\\
&+\tfrac{1}{3}\textstyle\sum_i \dot{G}_{B1i}G_{Bi1}k_i\cdot k_4(\dot{G}_{B31}\varepsilon_3\cdot k_1\varepsilon_4\cdot k_1+\dot{G}_{B32}\varepsilon_3\cdot k_2\varepsilon_4\cdot k_1)\\
&+\tfrac{1}{3}\textstyle\sum_i \dot{G}_{B2i}G_{Bi2}k_i\cdot k_4(\dot{G}_{B31}\varepsilon_3\cdot k_1\varepsilon_4\cdot k_2+\dot{G}_{B32}\varepsilon_3\cdot k_2\varepsilon_4\cdot k_2)\Big]\\
&+\dot{G}_{B12}\dot{G}_{B21}Z_2(12)\left[(2G_{B31}-\tfrac{1}{3})(\varepsilon_3\cdot k_4\varepsilon_4\cdot k_1-\tfrac{1}{2}\varepsilon_3\cdot\varepsilon_4 k_4\cdot k_1)\right.\\
&\left.+(2G_{B32}-\tfrac{1}{3})(\varepsilon_3\cdot k_4\varepsilon_4\cdot k_2-\tfrac{1}{2}\varepsilon_3\cdot\varepsilon_4 k_4\cdot k_2)\right],
\end{aligned}
\tag{D.36}
$$

$$
\begin{aligned}
I^2_{(4)}(13) = \dot{G}_{B13}\dot{G}_{B31}Z_2(13)\Big[& \\
&+\tfrac{1}{3}\dot{G}_{B21}\textstyle\sum_i \dot{G}_{B2i}G_{Bi2}k_i\cdot k_4(\varepsilon_2\cdot k_1\varepsilon_4\cdot k_2-\tfrac{1}{2}\varepsilon_2\cdot\varepsilon_4 k_2\cdot k_1)\\
&+\tfrac{1}{3}\dot{G}_{B23}\textstyle\sum_i \dot{G}_{B2i}G_{Bi2}k_i\cdot k_4(\varepsilon_2\cdot k_3\varepsilon_4\cdot k_2-\tfrac{1}{2}\varepsilon_2\cdot\varepsilon_4 k_2\cdot k_3)\\
&+\tfrac{1}{3}\textstyle\sum_i \dot{G}_{B1i}G_{Bi1}k_i\cdot k_4(\dot{G}_{B21}\varepsilon_2\cdot k_1\varepsilon_4\cdot k_1+\dot{G}_{B23}\varepsilon_2\cdot k_3\varepsilon_4\cdot k_1)\\
&+\tfrac{1}{3}\textstyle\sum_i \dot{G}_{B3i}G_{Bi3}k_i\cdot k_4(\dot{G}_{B21}\varepsilon_2\cdot k_1\varepsilon_4\cdot k_3+\dot{G}_{B23}\varepsilon_2\cdot k_3\varepsilon_4\cdot k_3)\Big]\\
&+\dot{G}_{B13}\dot{G}_{B31}Z_2(13)\left[(2G_{B21}-\tfrac{1}{3})(\varepsilon_2\cdot k_4\varepsilon_4\cdot k_1-\tfrac{1}{2}\varepsilon_2\cdot\varepsilon_4 k_4\cdot k_1)\right.\\
&\left.+(2G_{B23}-\tfrac{1}{3})(\varepsilon_2\cdot k_4\varepsilon_4\cdot k_3-\tfrac{1}{2}\varepsilon_2\cdot\varepsilon_4 k_4\cdot k_3)\right],
\end{aligned}
\tag{D.37}
$$

$$
\begin{aligned}
I^2_{(4)}(23) = \dot{G}_{B23}\dot{G}_{B32}Z_2(23)\Big[& \\
&+\tfrac{1}{3}\dot{G}_{B12}\textstyle\sum_i \dot{G}_{B1i}G_{Bi1}k_i\cdot k_4(\varepsilon_1\cdot k_2\varepsilon_4\cdot k_1-\tfrac{1}{2}\varepsilon_1\cdot\varepsilon_4 k_1\cdot k_2)\\
&+\tfrac{1}{3}\dot{G}_{B13}\textstyle\sum_i \dot{G}_{B1i}G_{Bi1}k_i\cdot k_4(\varepsilon_1\cdot k_3\varepsilon_4\cdot k_1-\tfrac{1}{2}\varepsilon_1\cdot\varepsilon_4 k_1\cdot k_3)\\
&+\tfrac{1}{3}\textstyle\sum_i \dot{G}_{B2i}G_{Bi2}k_i\cdot k_4(\dot{G}_{B12}\varepsilon_1\cdot k_2\varepsilon_4\cdot k_2+\dot{G}_{B13}\varepsilon_1\cdot k_3\varepsilon_4\cdot k_2)\\
&+\tfrac{1}{3}\textstyle\sum_i \dot{G}_{B3i}G_{Bi3}k_i\cdot k_4(\dot{G}_{B12}\varepsilon_1\cdot k_2\varepsilon_4\cdot k_3+\dot{G}_{B13}\varepsilon_1\cdot k_3\varepsilon_4\cdot k_3)\Big]\\
&+\dot{G}_{B23}\dot{G}_{B32}Z_2(23)\left[(2G_{B12}-\tfrac{1}{3})(\varepsilon_1\cdot k_4\varepsilon_4\cdot k_2-\tfrac{1}{2}\varepsilon_1\cdot\varepsilon_4 k_4\cdot k_2)\right.\\
&\left.+(2G_{B13}-\tfrac{1}{3})(\varepsilon_1\cdot k_4\varepsilon_4\cdot k_3-\tfrac{1}{2}\varepsilon_1\cdot\varepsilon_4 k_4\cdot k_3)\right],
\end{aligned}
\tag{D.38}
$$

$$
\begin{aligned}
I^2_{(4)}(14) = -\tfrac{1}{3}Z_2(14)\Big[& \dot{G}_{B21}\dot{G}_{B31}\varepsilon_2\cdot k_1\varepsilon_3\cdot k_1\\
&+\dot{G}_{B23}\dot{G}_{B31}(\varepsilon_2\cdot k_3\varepsilon_3\cdot k_1-\tfrac{1}{2}\varepsilon_2\cdot\varepsilon_3 k_3\cdot k_1)\\
&+\dot{G}_{B21}\dot{G}_{B32}(\varepsilon_2\cdot k_1\varepsilon_3\cdot k_2-\tfrac{1}{2}\varepsilon_2\cdot\varepsilon_3 k_2\cdot k_1)\Big],
\end{aligned}
\tag{D.39}
$$

$$
\begin{aligned}
I^2_{(4)}(24) = -\tfrac{1}{3} Z_2(24) \Big[& \dot{G}_{B12}\dot{G}_{B32}\varepsilon_1 \cdot k_2 \varepsilon_3 \cdot k_2 \\
& + \dot{G}_{B13}\dot{G}_{B32}(\varepsilon_1 \cdot k_3 \varepsilon_3 \cdot k_2 - \tfrac{1}{2}\varepsilon_1 \cdot \varepsilon_3 k_3 \cdot k_2) \\
& + \dot{G}_{B12}\dot{G}_{B31}(\varepsilon_1 \cdot k_2 \varepsilon_3 \cdot k_1 - \tfrac{1}{2}\varepsilon_1 \cdot \varepsilon_3 k_1 \cdot k_2) \Big] ,
\end{aligned}
\tag{D.40}
$$

$$
\begin{aligned}
I^2_{(4)}(34) = -\tfrac{1}{3} Z_2(34) \Big[& \dot{G}_{B13}\dot{G}_{B23}\varepsilon_1 \cdot k_3 \varepsilon_2 \cdot k_3 \\
& + \dot{G}_{B13}\dot{G}_{B21}(\varepsilon_1 \cdot k_3 \varepsilon_2 \cdot k_1 - \tfrac{1}{2}\varepsilon_1 \cdot \varepsilon_2 k_1 \cdot k_3) \\
& + \dot{G}_{B12}\dot{G}_{B23}(\varepsilon_1 \cdot k_2 \varepsilon_2 \cdot k_3 - \tfrac{1}{2}\varepsilon_1 \cdot \varepsilon_2 k_2 \cdot k_3) \Big] .
\end{aligned}
\tag{D.41}
$$

Working with $I^2_{(4)}(12)$, $I^2_{(4)}(13)$ and $I^2_{(4)}(23)$

For now, we will work with $I^2_{(4)}(12)$, $I^2_{(4)}(13)$ and $I^2_{(4)}(23)$ only. Our purpose is to write this expressions in terms of the field strength tensor (2.12). First we expand the sums

$$
\begin{aligned}
I^2_{(4)}(12) = & \tfrac{1}{3}\dot{G}_{B12}\dot{G}_{B21} Z_2(12) \Big[\\
& \dot{G}_{B31}(\dot{G}_{B31} G_{B13} k_1 \cdot k_4 + \dot{G}_{B32} G_{B23} k_2 \cdot k_4)(\varepsilon_3 \cdot k_1 \varepsilon_4 \cdot k_3 - \tfrac{1}{2}\varepsilon_3 \cdot \varepsilon_4 k_3 \cdot k_1) \\
& + \dot{G}_{B32}(\dot{G}_{B31} G_{B13} k_1 \cdot k_4 + \dot{G}_{B32} G_{B23} k_2 \cdot k_4)(\varepsilon_3 \cdot k_2 \varepsilon_4 \cdot k_3 - \tfrac{1}{2}\varepsilon_3 \cdot \varepsilon_4 k_3 \cdot k_2) \\
& + (\dot{G}_{B12} G_{B21} k_2 \cdot k_4 + \dot{G}_{B13} G_{B31} k_3 \cdot k_4)(\dot{G}_{B31}\varepsilon_3 \cdot k_1 \varepsilon_4 \cdot k_1 + \dot{G}_{B32}\varepsilon_3 \cdot k_2 \varepsilon_4 \cdot k_1) \\
& + (\dot{G}_{B21} G_{B12} k_1 \cdot k_4 + \dot{G}_{B23} G_{B32} k_3 \cdot k_4)(\dot{G}_{B31}\varepsilon_3 \cdot k_1 \varepsilon_4 \cdot k_2 + \dot{G}_{B32}\varepsilon_3 \cdot k_2 \varepsilon_4 \cdot k_2) \Big] \\
& + \dot{G}_{B12}\dot{G}_{B21} Z_2(12) \big[(2G_{B31} - \tfrac{1}{3})(\varepsilon_3 \cdot k_4 \varepsilon_4 \cdot k_1 - \tfrac{1}{2}\varepsilon_3 \cdot \varepsilon_4 k_4 \cdot k_1) \\
& + (2G_{B32} - \tfrac{1}{3})(\varepsilon_3 \cdot k_4 \varepsilon_4 \cdot k_2 - \tfrac{1}{2}\varepsilon_3 \cdot \varepsilon_4 k_4 \cdot k_2) \big] ,
\end{aligned}
\tag{D.42}
$$

$$
\begin{aligned}
I^2_{(4)}(13) = & \tfrac{1}{3}\dot{G}_{B13}\dot{G}_{B31} Z_2(13) \Big[\\
& \dot{G}_{B21}(\dot{G}_{B21} G_{B12} k_1 \cdot k_4 + \dot{G}_{B23} G_{B32} k_3 \cdot k_4)(\varepsilon_2 \cdot k_1 \varepsilon_4 \cdot k_2 - \tfrac{1}{2}\varepsilon_2 \cdot \varepsilon_4 k_2 \cdot k_1) \\
& + \dot{G}_{B23}(\dot{G}_{B21} G_{B12} k_1 \cdot k_4 + \dot{G}_{B23} G_{B32} k_3 \cdot k_4)(\varepsilon_2 \cdot k_3 \varepsilon_4 \cdot k_2 - \tfrac{1}{2}\varepsilon_2 \cdot \varepsilon_4 k_2 \cdot k_3) \\
& + (\dot{G}_{B12} G_{B21} k_2 \cdot k_4 + \dot{G}_{B13} G_{B31} k_3 \cdot k_4)(\dot{G}_{B21}\varepsilon_2 \cdot k_1 \varepsilon_4 \cdot k_1 + \dot{G}_{B23}\varepsilon_2 \cdot k_3 \varepsilon_4 \cdot k_1) \\
& + (\dot{G}_{B31} G_{B13} k_1 \cdot k_4 + \dot{G}_{B32} G_{B23} k_2 \cdot k_4)(\dot{G}_{B21}\varepsilon_2 \cdot k_1 \varepsilon_4 \cdot k_3 + \dot{G}_{B23}\varepsilon_2 \cdot k_3 \varepsilon_4 \cdot k_3) \Big] \\
& + \dot{G}_{B13}\dot{G}_{B31} Z_2(13) \big[(2G_{B21} - \tfrac{1}{3})(\varepsilon_2 \cdot k_4 \varepsilon_4 \cdot k_1 - \tfrac{1}{2}\varepsilon_2 \cdot \varepsilon_4 k_4 \cdot k_1) \\
& + (2G_{B23} - \tfrac{1}{3})(\varepsilon_2 \cdot k_4 \varepsilon_4 \cdot k_3 - \tfrac{1}{2}\varepsilon_2 \cdot \varepsilon_4 k_4 \cdot k_3) \big] ,
\end{aligned}
\tag{D.43}
$$

$$
\begin{aligned}
I^2_{(4)}(23) = \tfrac{1}{3}\dot{G}_{B23}\dot{G}_{B32}Z_2(23)\Big[& \\
& \dot{G}_{B12}(\dot{G}_{B12}G_{B21}k_2\cdot k_4 + \dot{G}_{B13}G_{B31}k_3\cdot k_4)(\varepsilon_1\cdot k_2\varepsilon_4\cdot k_1 - \tfrac{1}{2}\varepsilon_1\cdot\varepsilon_4 k_1\cdot k_2) \\
& +\dot{G}_{B13}(\dot{G}_{B12}G_{B21}k_2\cdot k_4 + \dot{G}_{B13}G_{B31}k_3\cdot k_4)(\varepsilon_1\cdot k_3\varepsilon_4\cdot k_1 - \tfrac{1}{2}\varepsilon_1\cdot\varepsilon_4 k_1\cdot k_3) \\
& +(\dot{G}_{B21}G_{B12}k_1\cdot k_4 + \dot{G}_{B23}G_{B32}k_3\cdot k_4)(\dot{G}_{B12}\varepsilon_1\cdot k_2\varepsilon_4\cdot k_2 + \dot{G}_{B13}\varepsilon_1\cdot k_3\varepsilon_4\cdot k_2) \\
& +(\dot{G}_{B31}G_{B13}k_1\cdot k_4 + \dot{G}_{B32}G_{B23}k_2\cdot k_4)(\dot{G}_{B12}\varepsilon_1\cdot k_2\varepsilon_4\cdot k_3 + \dot{G}_{B13}\varepsilon_1\cdot k_3\varepsilon_4\cdot k_3)\Big] \\
& +\dot{G}_{B23}\dot{G}_{B32}Z_2(23)\left[(2G_{B12} - \tfrac{1}{3})(\varepsilon_1\cdot k_4\varepsilon_4\cdot k_2 - \tfrac{1}{2}\varepsilon_1\cdot\varepsilon_4 k_4\cdot k_2)\right. \\
& \left.+ (2G_{B13} - \tfrac{1}{3})(\varepsilon_1\cdot k_4\varepsilon_4\cdot k_3 - \tfrac{1}{2}\varepsilon_1\cdot\varepsilon_4 k_4\cdot k_3)\right].
\end{aligned}
\tag{D.44}
$$

Then, we factorize respect to G_{Bij}'s and $\dot{G}_{Bij}$'s

$$
\begin{aligned}
I^2_{(4)}(12) = \tfrac{1}{3}\dot{G}_{B12}\dot{G}_{B21}Z_2(12)\Big[& \\
& -\tfrac{1}{2}\varepsilon_3\cdot\varepsilon_4 k_3\cdot k_1\dot{G}_{B31}(\dot{G}_{B31}G_{B13}k_1\cdot k_4 + \dot{G}_{B32}G_{B23}k_2\cdot k_4) \\
& -\tfrac{1}{2}\varepsilon_3\cdot\varepsilon_4 k_3\cdot k_2\dot{G}_{B32}(\dot{G}_{B31}G_{B13}k_1\cdot k_4 + \dot{G}_{B32}G_{B23}k_2\cdot k_4) \\
& +\dot{G}_{B31}\dot{G}_{B31}G_{B13}\varepsilon_3\cdot k_1(k_1\cdot k_4\varepsilon_4\cdot k_3 - k_3\cdot k_4\varepsilon_4\cdot k_1) \\
& +\dot{G}_{B32}\dot{G}_{B31}G_{B13}\varepsilon_3\cdot k_2(k_1\cdot k_4\varepsilon_4\cdot k_3 - k_3\cdot k_4\varepsilon_4\cdot k_1) \\
& +\dot{G}_{B31}\dot{G}_{B32}G_{B23}\varepsilon_3\cdot k_1(k_2\cdot k_4\varepsilon_4\cdot k_3 - k_3\cdot k_4\varepsilon_4\cdot k_2) \\
& +\dot{G}_{B32}\dot{G}_{B32}G_{B23}\varepsilon_3\cdot k_2(k_2\cdot k_4\varepsilon_4\cdot k_3 - k_3\cdot k_4\varepsilon_4\cdot k_2) \\
& +\dot{G}_{B12}\dot{G}_{B31}G_{B21}\varepsilon_3\cdot k_1(k_2\cdot k_4\varepsilon_4\cdot k_1 - k_1\cdot k_4\varepsilon_4\cdot k_2) \\
& +\dot{G}_{B12}\dot{G}_{B32}G_{B21}\varepsilon_3\cdot k_2(k_2\cdot k_4\varepsilon_4\cdot k_1 - k_1\cdot k_4\varepsilon_4\cdot k_2)\Big] \\
& +\dot{G}_{B12}\dot{G}_{B21}Z_2(12)\left[(2G_{B31} - \tfrac{1}{3})(\varepsilon_3\cdot k_4\varepsilon_4\cdot k_1 - \tfrac{1}{2}\varepsilon_3\cdot\varepsilon_4 k_4\cdot k_1)\right. \\
& \left.+(2G_{B32} - \tfrac{1}{3})(\varepsilon_3\cdot k_4\varepsilon_4\cdot k_2 - \tfrac{1}{2}\varepsilon_3\cdot\varepsilon_4 k_4\cdot k_2)\right],
\end{aligned}
\tag{D.45}
$$

$$
\begin{aligned}
I^2_{(4)}(13) = {} & \tfrac{1}{3}\dot{G}_{B13}\dot{G}_{B31}Z_2(13)\Big[\\
& -\tfrac{1}{2}\varepsilon_2\cdot\varepsilon_4 k_2\cdot k_1\dot{G}_{B21}(\dot{G}_{B21}G_{B12}k_1\cdot k_4+\dot{G}_{B23}G_{B32}k_3\cdot k_4) \\
& -\tfrac{1}{2}\varepsilon_2\cdot\varepsilon_4 k_2\cdot k_3\dot{G}_{B23}(\dot{G}_{B21}G_{B12}k_1\cdot k_4+\dot{G}_{B23}G_{B32}k_3\cdot k_4) \\
& +\dot{G}_{B21}\dot{G}_{B21}G_{B12}\varepsilon_2\cdot k_1(k_1\cdot k_4\varepsilon_4\cdot k_2-k_2\cdot k_4\varepsilon_4\cdot k_1) \\
& +\dot{G}_{B21}\dot{G}_{B23}G_{B32}\varepsilon_2\cdot k_1(k_3\cdot k_4\varepsilon_4\cdot k_2-k_2\cdot k_4\varepsilon_4\cdot k_3) \\
& +\dot{G}_{B23}\dot{G}_{B21}G_{B12}\varepsilon_2\cdot k_3(k_1\cdot k_4\varepsilon_4\cdot k_2-k_2\cdot k_4\varepsilon_4\cdot k_1) \\
& +\dot{G}_{B23}\dot{G}_{B23}G_{B32}\varepsilon_2\cdot k_3(k_3\cdot k_4\varepsilon_4\cdot k_2-k_2\cdot k_4\varepsilon_4\cdot k_3) \\
& +\dot{G}_{B13}\dot{G}_{B21}G_{B31}\varepsilon_2\cdot k_1(k_3\cdot k_4\varepsilon_4\cdot k_1-k_1\cdot k_4\varepsilon_4\cdot k_3) \\
& +\dot{G}_{B13}\dot{G}_{B23}G_{B31}\varepsilon_2\cdot k_3(k_3\cdot k_4\varepsilon_4\cdot k_1-k_1\cdot k_4\varepsilon_4\cdot k_3)\Big] \\
& +\dot{G}_{B13}\dot{G}_{B31}Z_2(13)\left[(2G_{B21}-\tfrac{1}{3})(\varepsilon_2\cdot k_4\varepsilon_4\cdot k_1-\tfrac{1}{2}\varepsilon_2\cdot\varepsilon_4 k_4\cdot k_1)\right. \\
& \left.+(2G_{B23}-\tfrac{1}{3})(\varepsilon_2\cdot k_4\varepsilon_4\cdot k_3-\tfrac{1}{2}\varepsilon_2\cdot\varepsilon_4 k_4\cdot k_3)\right],
\end{aligned}
\tag{D.46}
$$

$$
\begin{aligned}
I^2_{(4)}(23) = {} & \tfrac{1}{3}\dot{G}_{B23}\dot{G}_{B32}Z_2(23)\Big[\\
& -\tfrac{1}{2}\varepsilon_1\cdot\varepsilon_4 k_1\cdot k_2\dot{G}_{B12}(\dot{G}_{B12}G_{B21}k_2\cdot k_4+\dot{G}_{B13}G_{B31}k_3\cdot k_4) \\
& -\tfrac{1}{2}\varepsilon_1\cdot\varepsilon_4 k_1\cdot k_3\dot{G}_{B13}(\dot{G}_{B12}G_{B21}k_2\cdot k_4+\dot{G}_{B13}G_{B31}k_3\cdot k_4) \\
& +\dot{G}_{B12}\dot{G}_{B12}G_{B21}\varepsilon_1\cdot k_2(k_2\cdot k_4\varepsilon_4\cdot k_1-k_1\cdot k_4\varepsilon_4\cdot k_2) \\
& +\dot{G}_{B12}\dot{G}_{B13}G_{B31}\varepsilon_1\cdot k_2(k_3\cdot k_4\varepsilon_4\cdot k_1-k_1\cdot k_4\varepsilon_4\cdot k_3) \\
& +\dot{G}_{B13}\dot{G}_{B12}G_{B21}\varepsilon_1\cdot k_3(k_2\cdot k_4\varepsilon_4\cdot k_1-k_1\cdot k_4\varepsilon_4\cdot k_2) \\
& +\dot{G}_{B13}\dot{G}_{B13}G_{B31}\varepsilon_1\cdot k_3(k_3\cdot k_4\varepsilon_4\cdot k_1-k_1\cdot k_4\varepsilon_4\cdot k_3) \\
& +\dot{G}_{B23}\dot{G}_{B12}G_{B32}\varepsilon_1\cdot k_2(k_3\cdot k_4\varepsilon_4\cdot k_2-k_2\cdot k_4\varepsilon_4\cdot k_3) \\
& +\dot{G}_{B23}\dot{G}_{B13}G_{B32}\varepsilon_1\cdot k_3(k_3\cdot k_4\varepsilon_4\cdot k_2-k_2\cdot k_4\varepsilon_4\cdot k_3)\Big] \\
& +\dot{G}_{B23}\dot{G}_{B32}Z_2(23)\left[(2G_{B12}-\tfrac{1}{3})(\varepsilon_1\cdot k_4\varepsilon_4\cdot k_2-\tfrac{1}{2}\varepsilon_1\cdot\varepsilon_4 k_4\cdot k_2)\right. \\
& \left.+\,(2G_{B13}-\tfrac{1}{3})(\varepsilon_1\cdot k_4\varepsilon_4\cdot k_3-\tfrac{1}{2}\varepsilon_1\cdot\varepsilon_4 k_4\cdot k_3)\right].
\end{aligned}
\tag{D.47}
$$

We use (B.5) to write the last equations in terms of f_4

$$
\begin{aligned}
I^2_{(4)}(12) = \tfrac{1}{3}\dot{G}_{B12}\dot{G}_{B21}Z_2(12)\Big[& \\
& -\tfrac{1}{2}\varepsilon_3\cdot\varepsilon_4 k_3\cdot k_1\dot{G}_{B31}(\dot{G}_{B31}G_{B13}k_1\cdot k_4+\dot{G}_{B32}G_{B23}k_2\cdot k_4)\\
& -\tfrac{1}{2}\varepsilon_3\cdot\varepsilon_4 k_3\cdot k_2\dot{G}_{B32}(\dot{G}_{B31}G_{B13}k_1\cdot k_4+\dot{G}_{B32}G_{B23}k_2\cdot k_4)\\
& +\dot{G}_{B31}\dot{G}_{B31}G_{B13}\varepsilon_3\cdot k_1 k_1\cdot f_4\cdot k_3+\dot{G}_{B32}\dot{G}_{B31}G_{B13}\varepsilon_3\cdot k_2 k_1\cdot f_4\cdot k_3\\
& +\dot{G}_{B31}\dot{G}_{B32}G_{B23}\varepsilon_3\cdot k_1 k_2\cdot f_4\cdot k_3+\dot{G}_{B32}\dot{G}_{B32}G_{B23}\varepsilon_3\cdot k_2 k_2\cdot f_4\cdot k_3\\
& +\dot{G}_{B12}\dot{G}_{B31}G_{B21}\varepsilon_3\cdot k_1 k_2\cdot f_4\cdot k_1+\dot{G}_{B12}\dot{G}_{B32}G_{B21}\varepsilon_3\cdot k_2 k_2\cdot f_4\cdot k_1\Big]\\
& +\dot{G}_{B12}\dot{G}_{B21}Z_2(12)\left[(2G_{B31}-\tfrac{1}{3})(\varepsilon_3\cdot k_4\varepsilon_4\cdot k_1-\tfrac{1}{2}\varepsilon_3\cdot\varepsilon_4 k_4\cdot k_1)\right.\\
& \left.+(2G_{B32}-\tfrac{1}{3})(\varepsilon_3\cdot k_4\varepsilon_4\cdot k_2-\tfrac{1}{2}\varepsilon_3\cdot\varepsilon_4 k_4\cdot k_2)\right],
\end{aligned}
\tag{D.48}
$$

$$
\begin{aligned}
I^2_{(4)}(13) = \tfrac{1}{3}\dot{G}_{B13}\dot{G}_{B31}Z_2(13)\Big[& \\
& -\tfrac{1}{2}\varepsilon_2\cdot\varepsilon_4 k_2\cdot k_1\dot{G}_{B21}(\dot{G}_{B21}G_{B12}k_1\cdot k_4+\dot{G}_{B23}G_{B32}k_3\cdot k_4)\\
& -\tfrac{1}{2}\varepsilon_2\cdot\varepsilon_4 k_2\cdot k_3\dot{G}_{B23}(\dot{G}_{B21}G_{B12}k_1\cdot k_4+\dot{G}_{B23}G_{B32}k_3\cdot k_4)\\
& +\dot{G}_{B21}\dot{G}_{B21}G_{B12}\varepsilon_2\cdot k_1 k_1\cdot f_4\cdot k_2+\dot{G}_{B21}\dot{G}_{B23}G_{B32}\varepsilon_2\cdot k_1 k_3\cdot f_4\cdot k_2\\
& +\dot{G}_{B23}\dot{G}_{B21}G_{B12}\varepsilon_2\cdot k_3 k_1\cdot f_4\cdot k_2+\dot{G}_{B23}\dot{G}_{B23}G_{B32}\varepsilon_2\cdot k_3 k_3\cdot f_4\cdot k_2\\
& +\dot{G}_{B13}\dot{G}_{B21}G_{B31}\varepsilon_2\cdot k_1 k_3\cdot f_4\cdot k_1+\dot{G}_{B13}\dot{G}_{B23}G_{B31}\varepsilon_2\cdot k_3 k_3\cdot f_4\cdot k_1\Big]\\
& +\dot{G}_{B13}\dot{G}_{B31}Z_2(13)\left[(2G_{B21}-\tfrac{1}{3})(\varepsilon_2\cdot k_4\varepsilon_4\cdot k_1-\tfrac{1}{2}\varepsilon_2\cdot\varepsilon_4 k_4\cdot k_1)\right.\\
& \left.+(2G_{B23}-\tfrac{1}{3})(\varepsilon_2\cdot k_4\varepsilon_4\cdot k_3-\tfrac{1}{2}\varepsilon_2\cdot\varepsilon_4 k_4\cdot k_3)\right],
\end{aligned}
\tag{D.49}
$$

$$
\begin{aligned}
I^2_{(4)}(23) = \tfrac{1}{3}\dot{G}_{B23}\dot{G}_{B32}Z_2(23)\Big[& \\
& -\tfrac{1}{2}\varepsilon_1\cdot\varepsilon_4 k_1\cdot k_2\dot{G}_{B12}(\dot{G}_{B12}G_{B21}k_2\cdot k_4+\dot{G}_{B13}G_{B31}k_3\cdot k_4)\\
& -\tfrac{1}{2}\varepsilon_1\cdot\varepsilon_4 k_1\cdot k_3\dot{G}_{B13}(\dot{G}_{B12}G_{B21}k_2\cdot k_4+\dot{G}_{B13}G_{B31}k_3\cdot k_4)\\
& +\dot{G}_{B12}\dot{G}_{B12}G_{B21}\varepsilon_1\cdot k_2 k_2\cdot f_4\cdot k_1+\dot{G}_{B12}\dot{G}_{B13}G_{B31}\varepsilon_1\cdot k_2 k_3\cdot f_4\cdot k_1\\
& +\dot{G}_{B13}\dot{G}_{B12}G_{B21}\varepsilon_1\cdot k_3 k_2\cdot f_4\cdot k_1+\dot{G}_{B13}\dot{G}_{B13}G_{B31}\varepsilon_1\cdot k_3 k_3\cdot f_4\cdot k_1\\
& +\dot{G}_{B23}\dot{G}_{B12}G_{B32}\varepsilon_1\cdot k_2 k_3\cdot f_4\cdot k_2+\dot{G}_{B23}\dot{G}_{B13}G_{B32}\varepsilon_1\cdot k_3 k_3\cdot f_4\cdot k_2\Big]\\
& +\dot{G}_{B23}\dot{G}_{B32}Z_2(23)\left[(2G_{B12}-\tfrac{1}{3})(\varepsilon_1\cdot k_4\varepsilon_4\cdot k_2-\tfrac{1}{2}\varepsilon_1\cdot\varepsilon_4 k_4\cdot k_2)\right.\\
& \left.+\,(2G_{B13}-\tfrac{1}{3})(\varepsilon_1\cdot k_4\varepsilon_4\cdot k_3-\tfrac{1}{2}\varepsilon_1\cdot\varepsilon_4 k_4\cdot k_3)\right].
\end{aligned}
\tag{D.50}
$$

Rearranging

$$
\begin{aligned}
I^2_{(4)}(12) = {} & \tfrac{1}{3}\dot{G}_{B12}\dot{G}_{B21}Z_2(12)\Big[\\
& -\tfrac{1}{2}\varepsilon_3\cdot\varepsilon_4 k_1\cdot k_4 \dot{G}_{B31}(\dot{G}_{B31}G_{B13}k_3\cdot k_1 + \dot{G}_{B32}G_{B13}k_3\cdot k_2) \\
& -\tfrac{1}{2}\varepsilon_3\cdot\varepsilon_4 k_2\cdot k_4 \dot{G}_{B32}(\dot{G}_{B31}G_{B23}k_3\cdot k_1 + \dot{G}_{B32}G_{B23}k_3\cdot k_2) \\
& +\dot{G}_{B31}\dot{G}_{B31}G_{B13}\varepsilon_3\cdot k_1 k_1\cdot f_4\cdot k_3 + \dot{G}_{B32}\dot{G}_{B31}G_{B13}\varepsilon_3\cdot k_2 k_1\cdot f_4\cdot k_3 \\
& +\dot{G}_{B31}\dot{G}_{B32}G_{B23}\varepsilon_3\cdot k_1 k_2\cdot f_4\cdot k_3 + \dot{G}_{B32}\dot{G}_{B32}G_{B23}\varepsilon_3\cdot k_2 k_2\cdot f_4\cdot k_3 \\
& +\dot{G}_{B12}\dot{G}_{B31}G_{B21}\varepsilon_3\cdot k_1 k_2\cdot f_4\cdot k_1 + \dot{G}_{B12}\dot{G}_{B32}G_{B21}\varepsilon_3\cdot k_2 k_2\cdot f_4\cdot k_1\Big] \\
& +\tfrac{1}{3}\dot{G}_{B12}\dot{G}_{B21}Z_2(12)\left[(6G_{B31}-1)(\varepsilon_3\cdot k_4\varepsilon_4\cdot k_1 - \tfrac{1}{2}\varepsilon_3\cdot\varepsilon_4 k_4\cdot k_1)\right. \\
& \left.+(6G_{B32}-1)(\varepsilon_3\cdot k_4\varepsilon_4\cdot k_2 - \tfrac{1}{2}\varepsilon_3\cdot\varepsilon_4 k_4\cdot k_2)\right],
\end{aligned}
\tag{D.51}
$$

$$
\begin{aligned}
I^2_{(4)}(13) = {} & \tfrac{1}{3}\dot{G}_{B13}\dot{G}_{B31}Z_2(13)\Big[\\
& -\tfrac{1}{2}\varepsilon_2\cdot\varepsilon_4 k_1\cdot k_4 \dot{G}_{B21}(\dot{G}_{B21}G_{B12}k_2\cdot k_1 + \dot{G}_{B23}G_{B12}k_2\cdot k_3) \\
& -\tfrac{1}{2}\varepsilon_2\cdot\varepsilon_4 k_3\cdot k_4 \dot{G}_{B23}(\dot{G}_{B21}G_{B32}k_2\cdot k_1 + \dot{G}_{B23}G_{B32}k_2\cdot k_3) \\
& +\dot{G}_{B21}\dot{G}_{B21}G_{B12}\varepsilon_2\cdot k_1 k_1\cdot f_4\cdot k_2 + \dot{G}_{B21}\dot{G}_{B23}G_{B32}\varepsilon_2\cdot k_1 k_3\cdot f_4\cdot k_2 \\
& +\dot{G}_{B23}\dot{G}_{B21}G_{B12}\varepsilon_2\cdot k_3 k_1\cdot f_4\cdot k_2 + \dot{G}_{B23}\dot{G}_{B23}G_{B32}\varepsilon_2\cdot k_3 k_3\cdot f_4\cdot k_2 \\
& +\dot{G}_{B13}\dot{G}_{B21}G_{B31}\varepsilon_2\cdot k_1 k_3\cdot f_4\cdot k_1 + \dot{G}_{B13}\dot{G}_{B23}G_{B31}\varepsilon_2\cdot k_3 k_3\cdot f_4\cdot k_1\Big] \\
& +\tfrac{1}{3}\dot{G}_{B13}\dot{G}_{B31}Z_2(13)\left[(6G_{B21}-1)(\varepsilon_2\cdot k_4\varepsilon_4\cdot k_1 - \tfrac{1}{2}\varepsilon_2\cdot\varepsilon_4 k_4\cdot k_1)\right. \\
& \left.+(6G_{B23}-1)(\varepsilon_2\cdot k_4\varepsilon_4\cdot k_3 - \tfrac{1}{2}\varepsilon_2\cdot\varepsilon_4 k_4\cdot k_3)\right],
\end{aligned}
\tag{D.52}
$$

$$
\begin{aligned}
I^2_{(4)}(23) = {} & \tfrac{1}{3}\dot{G}_{B23}\dot{G}_{B32}Z_2(23)\Big[\\
& -\tfrac{1}{2}\varepsilon_1\cdot\varepsilon_4 k_2\cdot k_4 \dot{G}_{B12}(\dot{G}_{B12}G_{B21}k_1\cdot k_2 + \dot{G}_{B13}G_{B21}k_1\cdot k_3) \\
& -\tfrac{1}{2}\varepsilon_1\cdot\varepsilon_4 k_3\cdot k_4 \dot{G}_{B13}(\dot{G}_{B12}G_{B31}k_1\cdot k_2 + \dot{G}_{B13}G_{B31}k_1\cdot k_3) \\
& +\dot{G}_{B12}\dot{G}_{B12}G_{B21}\varepsilon_1\cdot k_2 k_2\cdot f_4\cdot k_1 + \dot{G}_{B12}\dot{G}_{B13}G_{B31}\varepsilon_1\cdot k_2 k_3\cdot f_4\cdot k_1 \\
& +\dot{G}_{B13}\dot{G}_{B12}G_{B21}\varepsilon_1\cdot k_3 k_2\cdot f_4\cdot k_1 + \dot{G}_{B13}\dot{G}_{B13}G_{B31}\varepsilon_1\cdot k_3 k_3\cdot f_4\cdot k_1 \\
& +\dot{G}_{B23}\dot{G}_{B12}G_{B32}\varepsilon_1\cdot k_2 k_3\cdot f_4\cdot k_2 + \dot{G}_{B23}\dot{G}_{B13}G_{B32}\varepsilon_1\cdot k_3 k_3\cdot f_4\cdot k_2\Big] \\
& +\tfrac{1}{3}\dot{G}_{B23}\dot{G}_{B32}Z_2(23)\left[(6G_{B12}-1)(\varepsilon_1\cdot k_4\varepsilon_4\cdot k_2 - \tfrac{1}{2}\varepsilon_1\cdot\varepsilon_4 k_4\cdot k_2)\right. \\
& \left.+\,(6G_{B13}-1)(\varepsilon_1\cdot k_4\varepsilon_4\cdot k_3 - \tfrac{1}{2}\varepsilon_1\cdot\varepsilon_4 k_4\cdot k_3)\right].
\end{aligned}
\tag{D.53}
$$

Notice that in the last equations there are terms like the one that appears in the left side of the following expression

$$
\dot{G}_{B31}G_{B13}(\dot{G}_{B31}k_3\cdot k_1 + \dot{G}_{B32}k_3\cdot k_2) = (6G_{B31}-1)\,. \tag{D.54}
$$

It is worthwhile to point out that the last expression is valid only in our case because the ever truth equation is

$$\partial_3\left(\dot{G}_{B31}G_{B31}e^{(\cdot 3)}\right)=\left[\ddot{G}_{B31}G_{B13}+\dot{G}^2_{B31}+\dot{G}_{B31}G_{B13}(\dot{G}_{B31}k_3\cdot k_1+\dot{G}_{B32}k_3\cdot k_2)\right]e^{(\cdot 3)}, \tag{D.55}$$

in which we use the following relations

$$\ddot{G}_{B31}=2\delta(u_3-u_1)-2 \quad \text{y} \quad \dot{G}^2_{B31}=1-4G_{B31}, \tag{D.56}$$

and we see that neither the Dirac's delta nor the total derivative contribute to the integral, and with this we get (D.54). Then we use (D.54) to write these integrals as

$$\begin{aligned}
I^2_{(4)}(12)=&\tfrac{1}{3}\dot{G}_{B12}\dot{G}_{B21}Z_2(12)\Big\{\\
&+\dot{G}_{B31}\dot{G}_{B31}G_{B13}\varepsilon_3\cdot k_1k_1\cdot f_4\cdot k_3+\dot{G}_{B32}\dot{G}_{B31}G_{B13}\varepsilon_3\cdot k_2k_1\cdot f_4\cdot k_3\\
&+\dot{G}_{B31}\dot{G}_{B32}G_{B23}\varepsilon_3\cdot k_1k_2\cdot f_4\cdot k_3+\dot{G}_{B32}\dot{G}_{B32}G_{B23}\varepsilon_3\cdot k_2k_2\cdot f_4\cdot k_3\\
&+\dot{G}_{B12}\dot{G}_{B31}G_{B21}\varepsilon_3\cdot k_1k_2\cdot f_4\cdot k_1+\dot{G}_{B12}\dot{G}_{B32}G_{B21}\varepsilon_3\cdot k_2k_2\cdot f_4\cdot k_1\Big\}\\
&+\tfrac{1}{3}\dot{G}_{B12}\dot{G}_{B21}Z_2(12)\left[(6G_{B31}-1)(\varepsilon_3\cdot k_4\varepsilon_4\cdot k_1-\varepsilon_3\cdot\varepsilon_4k_4\cdot k_1)\right.\\
&\left.+(6G_{B32}-1)(\varepsilon_3\cdot k_4\varepsilon_4\cdot k_2-\varepsilon_3\cdot\varepsilon_4k_4\cdot k_2)\right],
\end{aligned} \tag{D.57}$$

$$\begin{aligned}
I^2_{(4)}(13)=&\tfrac{1}{3}\dot{G}_{B13}\dot{G}_{B31}Z_2(13)\Big\{\\
&+\dot{G}_{B21}\dot{G}_{B21}G_{B12}\varepsilon_2\cdot k_1k_1\cdot f_4\cdot k_2+\dot{G}_{B21}\dot{G}_{B23}G_{B32}\varepsilon_2\cdot k_1k_3\cdot f_4\cdot k_2\\
&+\dot{G}_{B23}\dot{G}_{B21}G_{B12}\varepsilon_2\cdot k_3k_1\cdot f_4\cdot k_2+\dot{G}_{B23}\dot{G}_{B23}G_{B32}\varepsilon_2\cdot k_3k_3\cdot f_4\cdot k_2\\
&+\dot{G}_{B13}\dot{G}_{B21}G_{B31}\varepsilon_2\cdot k_1k_3\cdot f_4\cdot k_1+\dot{G}_{B13}\dot{G}_{B23}G_{B31}\varepsilon_2\cdot k_3k_3\cdot f_4\cdot k_1\Big\}\\
&+\tfrac{1}{3}\dot{G}_{B13}\dot{G}_{B31}Z_2(13)\left[(6G_{B21}-1)(\varepsilon_2\cdot k_4\varepsilon_4\cdot k_1-\varepsilon_2\cdot\varepsilon_4k_4\cdot k_1)\right.\\
&\left.+(6G_{B23}-1)(\varepsilon_2\cdot k_4\varepsilon_4\cdot k_3-\varepsilon_2\cdot\varepsilon_4k_4\cdot k_3)\right],
\end{aligned} \tag{D.58}$$

$$\begin{aligned}
I^2_{(4)}(23)=&\tfrac{1}{3}\dot{G}_{B23}\dot{G}_{B32}Z_2(23)\Big\{\\
&+\dot{G}_{B12}\dot{G}_{B12}G_{B21}\varepsilon_1\cdot k_2k_2\cdot f_4\cdot k_1+\dot{G}_{B12}\dot{G}_{B13}G_{B31}\varepsilon_1\cdot k_2k_3\cdot f_4\cdot k_1\\
&+\dot{G}_{B13}\dot{G}_{B12}G_{B21}\varepsilon_1\cdot k_3k_2\cdot f_4\cdot k_1+\dot{G}_{B13}\dot{G}_{B13}G_{B31}\varepsilon_1\cdot k_3k_3\cdot f_4\cdot k_1\\
&+\dot{G}_{B23}\dot{G}_{B12}G_{B32}\varepsilon_1\cdot k_2k_3\cdot f_4\cdot k_2+\dot{G}_{B23}\dot{G}_{B13}G_{B32}\varepsilon_1\cdot k_3k_3\cdot f_4\cdot k_2\Big\}\\
&+\tfrac{1}{3}\dot{G}_{B23}\dot{G}_{B32}Z_2(23)\left[(6G_{B12}-1)(\varepsilon_1\cdot k_4\varepsilon_4\cdot k_2-\varepsilon_1\cdot\varepsilon_4k_4\cdot k_2)\right.\\
&\left.+\,(6G_{B13}-1)(\varepsilon_1\cdot k_4\varepsilon_4\cdot k_3-\varepsilon_1\cdot\varepsilon_4k_4\cdot k_3)\right].
\end{aligned} \tag{D.59}$$

Again we use (B.5) to write this in terms of f_4

$$
\begin{aligned}
I^2_{(4)}(12) = \tfrac{1}{3}\dot{G}_{B12}\dot{G}_{B21}Z_2(12)\Big\{ & \\
&+\dot{G}_{B31}\dot{G}_{B31}G_{B13}\varepsilon_3\cdot k_1k_1\cdot f_4\cdot k_3+\dot{G}_{B32}\dot{G}_{B31}G_{B13}\varepsilon_3\cdot k_2k_1\cdot f_4\cdot k_3\\
&+\dot{G}_{B31}\dot{G}_{B32}G_{B23}\varepsilon_3\cdot k_1k_2\cdot f_4\cdot k_3+\dot{G}_{B32}\dot{G}_{B32}G_{B23}\varepsilon_3\cdot k_2k_2\cdot f_4\cdot k_3\\
&+\dot{G}_{B12}\dot{G}_{B31}G_{B21}\varepsilon_3\cdot k_1k_2\cdot f_4\cdot k_1+\dot{G}_{B12}\dot{G}_{B32}G_{B21}\varepsilon_3\cdot k_2k_2\cdot f_4\cdot k_1\\
&+(6G_{B31}-1)(\varepsilon_3\cdot f_4\cdot k_1)+(6G_{B32}-1)(\varepsilon_3\cdot f_4\cdot k_2)\Big\},
\end{aligned}
\tag{D.60}
$$

$$
\begin{aligned}
I^2_{(4)}(13) = \tfrac{1}{3}\dot{G}_{B13}\dot{G}_{B31}Z_2(13)\Big\{ & \\
&+\dot{G}_{B21}\dot{G}_{B21}G_{B12}\varepsilon_2\cdot k_1k_1\cdot f_4\cdot k_2+\dot{G}_{B21}\dot{G}_{B23}G_{B32}\varepsilon_2\cdot k_1k_3\cdot f_4\cdot k_2\\
&+\dot{G}_{B23}\dot{G}_{B21}G_{B12}\varepsilon_2\cdot k_3k_1\cdot f_4\cdot k_2+\dot{G}_{B23}\dot{G}_{B23}G_{B32}\varepsilon_2\cdot k_3k_3\cdot f_4\cdot k_2\\
&+\dot{G}_{B13}\dot{G}_{B21}G_{B31}\varepsilon_2\cdot k_1k_3\cdot f_4\cdot k_1+\dot{G}_{B13}\dot{G}_{B23}G_{B31}\varepsilon_2\cdot k_3k_3\cdot f_4\cdot k_1\\
&+(6G_{B21}-1)(\varepsilon_2\cdot f_4\cdot k_1)+(6G_{B23}-1)(\varepsilon_2\cdot f_4\cdot k_3)\Big\},
\end{aligned}
\tag{D.61}
$$

$$
\begin{aligned}
I^2_{(4)}(23) = \tfrac{1}{3}\dot{G}_{B23}\dot{G}_{B32}Z_2(23)\Big\{ & \\
&+\dot{G}_{B12}\dot{G}_{B12}G_{B21}\varepsilon_1\cdot k_2k_2\cdot f_4\cdot k_1+\dot{G}_{B12}\dot{G}_{B13}G_{B31}\varepsilon_1\cdot k_2k_3\cdot f_4\cdot k_1\\
&+\dot{G}_{B13}\dot{G}_{B12}G_{B21}\varepsilon_1\cdot k_3k_2\cdot f_4\cdot k_1+\dot{G}_{B13}\dot{G}_{B13}G_{B31}\varepsilon_1\cdot k_3k_3\cdot f_4\cdot k_1\\
&+\dot{G}_{B23}\dot{G}_{B12}G_{B32}\varepsilon_1\cdot k_2k_3\cdot f_4\cdot k_2+\dot{G}_{B23}\dot{G}_{B13}G_{B32}\varepsilon_1\cdot k_3k_3\cdot f_4\cdot k_2\\
&+(6G_{B12}-1)(\varepsilon_1\cdot f_4\cdot k_2)+(6G_{B13}-1)(\varepsilon_1\cdot f_4\cdot k_3)\Big\}.
\end{aligned}
\tag{D.62}
$$

Further, notice that

$$
k_1\cdot f_4\cdot k_3\varepsilon_3\cdot k_1 = k_1\cdot f_4\cdot f_3\cdot k_1+k_1\cdot f_4\cdot\varepsilon_3k_3\cdot k_1, \tag{D.63}
$$

which will help us in the second and third line of the las three equations. For the fourth line it will be useful to notice that

$$
k_2\cdot f_4\cdot k_1r\cdot k_3\varepsilon_3\cdot k_1 = k_2\cdot f_4\cdot k_1(r\cdot f_3\cdot k_1+r\cdot\varepsilon_3k_3\cdot k_1), \tag{D.64}
$$

where "r" is an arbitrary vector that satisfy $r \cdot k_i \neq 0$. Then, we have

$$
\begin{aligned}
I^2_{(4)}(12) = \tfrac{1}{3}\dot{G}_{B12}\dot{G}_{B21}Z_2(12)\Big\{ & \\
&+\dot{G}_{B31}\dot{G}_{B31}G_{B13}(k_1\cdot f_4\cdot f_3\cdot k_1 + k_1\cdot f_4\cdot \varepsilon_3 k_3\cdot k_1)\\
&+\dot{G}_{B32}\dot{G}_{B31}G_{B13}(k_1\cdot f_4\cdot f_3\cdot k_2 + k_1\cdot f_4\cdot \varepsilon_3 k_3\cdot k_2)\\
&+\dot{G}_{B31}\dot{G}_{B32}G_{B23}(k_2\cdot f_4\cdot f_3\cdot k_1 + k_2\cdot f_4\cdot \varepsilon_3 k_3\cdot k_1)\\
&+\dot{G}_{B32}\dot{G}_{B32}G_{B23}(k_2\cdot f_4\cdot f_3\cdot k_2 + k_2\cdot f_4\cdot \varepsilon_3 k_3\cdot k_2)\\
&+\dot{G}_{B12}\dot{G}_{B31}G_{B21}\tfrac{k_2\cdot f_4\cdot k_1}{r_4\cdot k_3}(r_4\cdot f_3\cdot k_1 + r_4\cdot \varepsilon_3 k_3\cdot k_1)\\
&+\dot{G}_{B12}\dot{G}_{B32}G_{B21}\tfrac{k_2\cdot f_4\cdot k_1}{r_4\cdot k_3}(r_4\cdot f_3\cdot k_2 + r_4\cdot \varepsilon_3 k_3\cdot k_2)\\
&+(6G_{B31}-1)(\varepsilon_3\cdot f_4\cdot k_1) + (6G_{B32}-1)(\varepsilon_3\cdot f_4\cdot k_2)\Big\},
\end{aligned}
\tag{D.65}
$$

$$
\begin{aligned}
I^2_{(4)}(13) = \tfrac{1}{3}\dot{G}_{B13}\dot{G}_{B31}Z_2(13)\Big\{ & \\
&+\dot{G}_{B21}\dot{G}_{B21}G_{B12}(k_1\cdot f_4\cdot f_2\cdot k_1 + k_1\cdot f_4\cdot \varepsilon_2 k_2\cdot k_1)\\
&+\dot{G}_{B21}\dot{G}_{B23}G_{B32}(k_3\cdot f_4\cdot f_2\cdot k_1 + k_3\cdot f_4\cdot \varepsilon_2 k_2\cdot k_1)\\
&+\dot{G}_{B23}\dot{G}_{B21}G_{B12}(k_1\cdot f_4\cdot f_2\cdot k_3 + k_1\cdot f_4\cdot \varepsilon_2 k_2\cdot k_3)\\
&+\dot{G}_{B23}\dot{G}_{B23}G_{B32}(k_3\cdot f_4\cdot f_2\cdot k_3 + k_3\cdot f_4\cdot \varepsilon_2 k_2\cdot k_3)\\
&+\dot{G}_{B13}\dot{G}_{B21}G_{B31}\tfrac{k_3\cdot f_4\cdot k_1}{r_5\cdot k_2}(r_5\cdot f_2\cdot k_1 + r_5\cdot \varepsilon_2 k_2\cdot k_1)\\
&+\dot{G}_{B13}\dot{G}_{B23}G_{B31}\tfrac{k_3\cdot f_4\cdot k_1}{r_5\cdot k_2}(r_5\cdot f_2\cdot k_3 + r_5\cdot \varepsilon_2 k_2\cdot k_3)\\
&+(6G_{B21}-1)(\varepsilon_2\cdot f_4\cdot k_1) + (6G_{B23}-1)(\varepsilon_2\cdot f_4\cdot k_3)\Big\},
\end{aligned}
\tag{D.66}
$$

$$
\begin{aligned}
I^2_{(4)}(23) = \tfrac{1}{3}\dot{G}_{B23}\dot{G}_{B32}Z_2(23)\Big\{ & \\
&+\dot{G}_{B12}\dot{G}_{B12}G_{B21}(k_2\cdot f_4\cdot f_1\cdot k_2 + k_2\cdot f_4\cdot \varepsilon_1 k_1\cdot k_2)\\
&+\dot{G}_{B12}\dot{G}_{B13}G_{B31}(k_3\cdot f_4\cdot f_1\cdot k_2 + k_3\cdot f_4\cdot \varepsilon_1 k_1\cdot k_2)\\
&+\dot{G}_{B13}\dot{G}_{B12}G_{B21}(k_2\cdot f_4\cdot f_1\cdot k_3 + k_2\cdot f_4\cdot \varepsilon_1 k_1\cdot k_3)\\
&+\dot{G}_{B13}\dot{G}_{B13}G_{B31}(k_3\cdot f_4\cdot f_1\cdot k_3 + k_3\cdot f_4\cdot \varepsilon_1 k_1\cdot k_3)\\
&+\dot{G}_{B23}\dot{G}_{B12}G_{B32}\tfrac{k_3\cdot f_4\cdot k_2}{r_6\cdot k_1}(r_6\cdot f_1\cdot k_2 + r_6\cdot \varepsilon_1 k_1\cdot k_2)\\
&+\dot{G}_{B23}\dot{G}_{B13}G_{B32}\tfrac{k_3\cdot f_4\cdot k_2}{r_6\cdot k_1}(r_6\cdot f_1\cdot k_3 + r_6\cdot \varepsilon_1 k_1\cdot k_3)\\
&+(6G_{B12}-1)(\varepsilon_1\cdot f_4\cdot k_2) + (6G_{B13}-1)(\varepsilon_1\cdot f_4\cdot k_3)\Big\}.
\end{aligned}
\tag{D.67}
$$

Again, notice that in the previous equations there are terms that can be rewritten as it is showed in the next equations

$$
\begin{aligned}
& k_1\cdot f_4\cdot \varepsilon_3\left(\dot{G}_{B31}\dot{G}_{B31}G_{B13}k_3\cdot k_1 + \dot{G}_{B32}\dot{G}_{B31}G_{B13}k_3\cdot k_2\right)\\
&= k_1\cdot f_4\cdot \varepsilon_3\left[e^{-(\cdot 3)}\partial_3\left(\dot{G}_{B31}G_{B13}e^{(\cdot 3)}\right) + (6G_{B31}-1)\right],
\end{aligned}
\tag{D.68}
$$

and

$$\begin{aligned}&\frac{k_2\cdot f_4\cdot k_1}{r\cdot k_3}r\cdot\varepsilon_3\left(\dot{G}_{B12}\dot{G}_{B31}G_{B21}k_3\cdot k_1+\dot{G}_{B12}\dot{G}_{B32}G_{B21}k_3\cdot k_2\right)\\&=\frac{k_2\cdot f_4\cdot k_1}{r\cdot k_3}r\cdot\varepsilon_3 e^{-(\cdot 3)}\partial_3\left(\dot{G}_{B12}G_{B21}e^{(\cdot 3)}\right).\end{aligned}\tag{D.69}$$

Since this total derivatives do not contribute to the integral we cancel them and now we have

$$\begin{aligned}I^2_{(4)}(12)=&\tfrac{1}{3}\dot{G}_{B12}\dot{G}_{B21}Z_2(12)\Big\{\\&+\dot{G}_{B31}\dot{G}_{B31}G_{B13}k_1\cdot f_4\cdot f_3\cdot k_1+\dot{G}_{B32}\dot{G}_{B31}G_{B13}k_1\cdot f_4\cdot f_3\cdot k_2\\&+\dot{G}_{B31}\dot{G}_{B32}G_{B23}k_2\cdot f_4\cdot f_3\cdot k_1+\dot{G}_{B32}\dot{G}_{B32}G_{B23}k_2\cdot f_4\cdot f_3\cdot k_2\\&+\dot{G}_{B12}\dot{G}_{B31}G_{B21}\tfrac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_1}{r_4\cdot k_3}+\dot{G}_{B12}\dot{G}_{B32}G_{B21}\tfrac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_2}{r_4\cdot k_3}\\&+k_1\cdot f_4\cdot\varepsilon_3(6G_{B31}-1)+k_2\cdot f_4\cdot\varepsilon_3(6G_{B32}-1)\\&+(6G_{B31}-1)(\varepsilon_3\cdot f_4\cdot k_1)+(6G_{B32}-1)(\varepsilon_3\cdot f_4\cdot k_2)\Big\},\end{aligned}\tag{D.70}$$

$$\begin{aligned}I^2_{(4)}(13)=&\tfrac{1}{3}\dot{G}_{B13}\dot{G}_{B31}Z_2(13)\Big\{\\&+\dot{G}_{B21}\dot{G}_{B21}G_{B12}k_1\cdot f_4\cdot f_2\cdot k_1+\dot{G}_{B23}\dot{G}_{B21}G_{B12}k_1\cdot f_4\cdot f_2\cdot k_3\\&+\dot{G}_{B21}\dot{G}_{B23}G_{B32}k_3\cdot f_4\cdot f_2\cdot k_1+\dot{G}_{B23}\dot{G}_{B23}G_{B32}k_3\cdot f_4\cdot f_2\cdot k_3\\&+\dot{G}_{B13}\dot{G}_{B21}G_{B31}\tfrac{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_1}{r_5\cdot k_2}+\dot{G}_{B13}\dot{G}_{B23}G_{B31}\tfrac{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_3}{r_5\cdot k_2}\\&+k_1\cdot f_4\cdot\varepsilon_2(6G_{B21}-1)+k_3\cdot f_4\cdot\varepsilon_2(6G_{B23}-1)\\&+(6G_{B21}-1)(\varepsilon_2\cdot f_4\cdot k_1)+(6G_{B23}-1)(\varepsilon_2\cdot f_4\cdot k_3)\Big\},\end{aligned}\tag{D.71}$$

$$\begin{aligned}I^2_{(4)}(23)=&\tfrac{1}{3}\dot{G}_{B23}\dot{G}_{B32}Z_2(23)\Big\{\\&+\dot{G}_{B12}\dot{G}_{B12}G_{B21}k_2\cdot f_4\cdot f_1\cdot k_2+\dot{G}_{B13}\dot{G}_{B12}G_{B21}k_2\cdot f_4\cdot f_1\cdot k_3\\&+\dot{G}_{B12}\dot{G}_{B13}G_{B31}k_3\cdot f_4\cdot f_1\cdot k_2+\dot{G}_{B13}\dot{G}_{B13}G_{B31}k_3\cdot f_4\cdot f_1\cdot k_3\\&+\dot{G}_{B23}\dot{G}_{B12}G_{B32}\tfrac{k_3\cdot f_4\cdot k_2 r_6\cdot f_1\cdot k_2}{r_6\cdot k_1}+\dot{G}_{B23}\dot{G}_{B13}G_{B32}\tfrac{k_3\cdot f_4\cdot k_2 r_6\cdot f_1\cdot k_3}{r_6\cdot k_1}\\&+k_2\cdot f_4\cdot\varepsilon_1(6G_{B12}-1)+k_3\cdot f_4\cdot\varepsilon_1(6G_{B13}-1)\\&+(6G_{B12}-1)(\varepsilon_1\cdot f_4\cdot k_2)+(6G_{B13}-1)(\varepsilon_1\cdot f_4\cdot k_3)\Big\}.\end{aligned}\tag{D.72}$$

Now, notice that

$$k_1\cdot f_4\cdot\varepsilon_3=-\varepsilon_3\cdot f_4\cdot k_1,\tag{D.73}$$

then

$$
\begin{aligned}
I^2_{(4)}(12) = \tfrac{1}{3}\dot{G}_{B12}\dot{G}_{B21}Z_2(12)\Big\{ & \\
&+\dot{G}_{B31}\dot{G}_{B31}G_{B13}k_1\cdot f_4\cdot f_3\cdot k_1 + \dot{G}_{B32}\dot{G}_{B31}G_{B13}k_1\cdot f_4\cdot f_3\cdot k_2 \\
&+\dot{G}_{B31}\dot{G}_{B32}G_{B23}k_2\cdot f_4\cdot f_3\cdot k_1 + \dot{G}_{B32}\dot{G}_{B32}G_{B23}k_2\cdot f_4\cdot f_3\cdot k_2 \\
&+\dot{G}_{B12}\dot{G}_{B31}G_{B21}\tfrac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_1}{r_4\cdot k_3} + \dot{G}_{B12}\dot{G}_{B32}G_{B21}\tfrac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_2}{r_4\cdot k_3}\Big\},
\end{aligned}
\tag{D.74}
$$

$$
\begin{aligned}
I^2_{(4)}(13) = \tfrac{1}{3}\dot{G}_{B13}\dot{G}_{B31}Z_2(13)\Big\{ & \\
&+\dot{G}_{B21}\dot{G}_{B21}G_{B12}k_1\cdot f_4\cdot f_2\cdot k_1 + \dot{G}_{B23}\dot{G}_{B21}G_{B12}k_1\cdot f_4\cdot f_2\cdot k_3 \\
&+\dot{G}_{B21}\dot{G}_{B23}G_{B32}k_3\cdot f_4\cdot f_2\cdot k_1 + \dot{G}_{B23}\dot{G}_{B23}G_{B32}k_3\cdot f_4\cdot f_2\cdot k_3 \\
&+\dot{G}_{B13}\dot{G}_{B21}G_{B31}\tfrac{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_1}{r_5\cdot k_2} + \dot{G}_{B13}\dot{G}_{B23}G_{B31}\tfrac{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_3}{r_5\cdot k_2}\Big\},
\end{aligned}
\tag{D.75}
$$

$$
\begin{aligned}
I^2_{(4)}(23) = \tfrac{1}{3}\dot{G}_{B23}\dot{G}_{B32}Z_2(23)\Big\{ & \\
&+\dot{G}_{B12}\dot{G}_{B12}G_{B21}k_2\cdot f_4\cdot f_1\cdot k_2 + \dot{G}_{B13}\dot{G}_{B12}G_{B21}k_2\cdot f_4\cdot f_1\cdot k_3 \\
&+\dot{G}_{B12}\dot{G}_{B13}G_{B31}k_3\cdot f_4\cdot f_1\cdot k_2 + \dot{G}_{B13}\dot{G}_{B13}G_{B31}k_3\cdot f_4\cdot f_1\cdot k_3 \\
&+\dot{G}_{B23}\dot{G}_{B12}G_{B32}\tfrac{k_3\cdot f_4\cdot k_2 r_6\cdot f_1\cdot k_2}{r_6\cdot k_1} + \dot{G}_{B23}\dot{G}_{B13}G_{B32}\tfrac{k_3\cdot f_4\cdot k_2 r_6\cdot f_1\cdot k_3}{r_6\cdot k_1}\Big\}.
\end{aligned}
\tag{D.76}
$$

Now, we use (B.1) to write all in terms of $\dot{G}$'s (also we arrange the subscripts)

Result of $I^2_{(4)}(12)$, $I^2_{(4)}(13)$ and $I^2_{(4)}(23)$

$$
\begin{aligned}
& I^2_{(4)}(12) = \tfrac{-1}{12}\dot{G}^2_{B12}Z_2(12) \\
& \times\Big\{\dot{G}^2_{B31}\left(1-\dot{G}^2_{B31}\right)k_1\cdot f_4\cdot f_3\cdot k_1 - \dot{G}_{B23}\dot{G}_{B31}\left(1-\dot{G}^2_{B31}\right)k_1\cdot f_4\cdot f_3\cdot k_2 \\
& -\dot{G}_{B23}\dot{G}_{B31}\left(1-\dot{G}^2_{B23}\right)k_2\cdot f_4\cdot f_3\cdot k_1 + \dot{G}^2_{B23}\left(1-\dot{G}^2_{B23}\right)k_2\cdot f_4\cdot f_3\cdot k_2 \\
& +\dot{G}_{B12}\dot{G}_{B31}\left(1-\dot{G}^2_{B12}\right)\tfrac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_1}{r_4\cdot k_3} - \dot{G}_{B12}\dot{G}_{B23}\left(1-\dot{G}^2_{B12}\right)\tfrac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_2}{r_4\cdot k_3}\Big\},
\end{aligned}
\tag{D.77}
$$

$$
\begin{aligned}
&I^2_{(4)}(13) = \tfrac{-1}{12}\dot{G}^2_{B31} Z_2(13) \\
&\times\Big\{\dot{G}^2_{B12}\left(1-\dot{G}^2_{B12}\right) k_1\cdot f_4\cdot f_2\cdot k_1 - \dot{G}_{B12}\dot{G}_{B23}\left(1-\dot{G}^2_{B12}\right) k_1\cdot f_4\cdot f_2\cdot k_3 \\
&-\dot{G}_{B12}\dot{G}_{B23}\left(1-\dot{G}^2_{B23}\right) k_3\cdot f_4\cdot f_2\cdot k_1 + \dot{G}^2_{B23}\left(1-\dot{G}^2_{B23}\right) k_3\cdot f_4\cdot f_2\cdot k_3 \\
&+\dot{G}_{B12}\dot{G}_{B31}\left(1-\dot{G}^2_{B31}\right)\tfrac{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_1}{r_5\cdot k_2} - \dot{G}_{B23}\dot{G}_{B31}\left(1-\dot{G}^2_{B31}\right)\tfrac{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_3}{r_5\cdot k_2}\Big\},
\end{aligned}
\tag{D.78}
$$

$$
\begin{aligned}
&I^2_{(4)}(23) = \tfrac{-1}{12}\dot{G}^2_{B23} Z_2(23) \\
&\times\Big\{\dot{G}^2_{B12}\left(1-\dot{G}^2_{B12}\right) k_2\cdot f_4\cdot f_1\cdot k_2 - \dot{G}_{B12}\dot{G}_{B31}\left(1-\dot{G}^2_{B12}\right) k_2\cdot f_4\cdot f_1\cdot k_3 \\
&-\dot{G}_{B12}\dot{G}_{B31}\left(1-\dot{G}^2_{B31}\right) k_3\cdot f_4\cdot f_1\cdot k_2 + \dot{G}^2_{B31}\left(1-\dot{G}^2_{B31}\right) k_3\cdot f_4\cdot f_1\cdot k_3 \\
&+\dot{G}_{B12}\dot{G}_{B23}\left(1-\dot{G}^2_{B23}\right)\tfrac{k_3\cdot f_4\cdot k_2 r_6\cdot f_1\cdot k_2}{r_6\cdot k_1} - \dot{G}_{B23}\dot{G}_{B31}\left(1-\dot{G}^2_{B23}\right)\tfrac{k_3\cdot f_4\cdot k_2 r_6\cdot f_1\cdot k_3}{r_6\cdot k_1}\Big\},
\end{aligned}
\tag{D.79}
$$

Working with $I^2_{(4)}(14)$, $I^2_{(4)}(24)$ and $I^2_{(4)}(34)$

Similarly, we work with $I^2_{(4)}(14)$, $I^2_{(4)}(24)$ and $I^2_{(4)}(34)$ in order to write them in terms of the field strength tensor (2.12).

$$
\begin{aligned}
I^2_{(4)}(14) = -\tfrac{1}{3} Z_2(14) \Big[&\dot{G}_{B21}\dot{G}_{B31}\varepsilon_2\cdot k_1\varepsilon_3\cdot k_1 \\
&+\dot{G}_{B23}\dot{G}_{B31}\left(\varepsilon_2\cdot k_3\varepsilon_3\cdot k_1 - \tfrac{1}{2}\varepsilon_2\cdot\varepsilon_3 k_3\cdot k_1\right) \\
&+ \dot{G}_{B21}\dot{G}_{B32}\left(\varepsilon_2\cdot k_1\varepsilon_3\cdot k_2 - \tfrac{1}{2}\varepsilon_2\cdot\varepsilon_3 k_2\cdot k_1\right)\Big],
\end{aligned}
\tag{D.80}
$$

$$
\begin{aligned}
I^2_{(4)}(24) = -\tfrac{1}{3} Z_2(24) \Big[&\dot{G}_{B12}\dot{G}_{B32}\varepsilon_1\cdot k_2\varepsilon_3\cdot k_2 \\
&+\dot{G}_{B13}\dot{G}_{B32}\left(\varepsilon_1\cdot k_3\varepsilon_3\cdot k_2 - \tfrac{1}{2}\varepsilon_1\cdot\varepsilon_3 k_3\cdot k_2\right) \\
&+ \dot{G}_{B12}\dot{G}_{B31}\left(\varepsilon_1\cdot k_2\varepsilon_3\cdot k_1 - \tfrac{1}{2}\varepsilon_1\cdot\varepsilon_3 k_1\cdot k_2\right)\Big],
\end{aligned}
\tag{D.81}
$$

$$
\begin{aligned}
I^2_{(4)}(34) = -\tfrac{1}{3} Z_2(34) \Big[&\dot{G}_{B13}\dot{G}_{B23}\varepsilon_1\cdot k_3\varepsilon_2\cdot k_3 \\
&+\dot{G}_{B13}\dot{G}_{B21}\left(\varepsilon_1\cdot k_3\varepsilon_2\cdot k_1 - \tfrac{1}{2}\varepsilon_1\cdot\varepsilon_2 k_1\cdot k_3\right) \\
&+ \dot{G}_{B12}\dot{G}_{B23}\left(\varepsilon_1\cdot k_2\varepsilon_2\cdot k_3 - \tfrac{1}{2}\varepsilon_1\cdot\varepsilon_2 k_2\cdot k_3\right)\Big].
\end{aligned}
\tag{D.82}
$$

The las equations can be written as

$$
\begin{aligned}
I^2_{(4)}(14) = -\tfrac{1}{3} Z_2(14)\tfrac{r_7\cdot k_2 r_8\cdot k_3}{r_7\cdot k_2 r_8\cdot k_3} \Big[&\dot{G}_{B21}\dot{G}_{B31}\varepsilon_2\cdot k_1\varepsilon_3\cdot k_1 \\
&+\dot{G}_{B23}\dot{G}_{B31}\varepsilon_2\cdot k_3\varepsilon_3\cdot k_1 + \dot{G}_{B21}\dot{G}_{B32}\varepsilon_2\cdot k_1\varepsilon_3\cdot k_2 \\
&- \tfrac{1}{2}\dot{G}_{B23}\dot{G}_{B31}\varepsilon_2\cdot\varepsilon_3 k_3\cdot k_1 - \tfrac{1}{2}\dot{G}_{B21}\dot{G}_{B32}\varepsilon_2\cdot\varepsilon_3 k_2\cdot k_1\Big],
\end{aligned}
\tag{D.83}
$$

$$
\begin{aligned}
I^2_{(4)}(24) = -\tfrac{1}{3}Z_2(24)\tfrac{r_9\cdot k_1 r_0\cdot k_3}{r_9\cdot k_1 r_0\cdot k_3}\Big[&\dot{G}_{B12}\dot{G}_{B32}\varepsilon_1\cdot k_2\varepsilon_3\cdot k_2\\
&+\dot{G}_{B13}\dot{G}_{B32}\varepsilon_1\cdot k_3\varepsilon_3\cdot k_2+\dot{G}_{B12}\dot{G}_{B31}\varepsilon_1\cdot k_2\varepsilon_3\cdot k_1\\
&-\tfrac{1}{2}\dot{G}_{B13}\dot{G}_{B32}\varepsilon_1\cdot\varepsilon_3 k_3\cdot k_2-\tfrac{1}{2}\dot{G}_{B12}\dot{G}_{B31}\varepsilon_1\cdot\varepsilon_3 k_1\cdot k_2\Big],
\end{aligned}
\tag{D.84}
$$

$$
\begin{aligned}
I^2_{(4)}(34) = -\tfrac{1}{3}Z_2(34)\tfrac{r_{01}\cdot k_1 r_{02}\cdot k_2}{r_{01}\cdot k_1 r_{02}\cdot k_2}\Big[&\dot{G}_{B13}\dot{G}_{B23}\varepsilon_1\cdot k_3\varepsilon_2\cdot k_3\\
&+\dot{G}_{B13}\dot{G}_{B21}\varepsilon_1\cdot k_3\varepsilon_2\cdot k_1+\dot{G}_{B12}\dot{G}_{B23}\varepsilon_1\cdot k_2\varepsilon_2\cdot k_3\\
&-\tfrac{1}{2}\dot{G}_{B13}\dot{G}_{B21}\varepsilon_1\cdot\varepsilon_2 k_1\cdot k_3-\tfrac{1}{2}\dot{G}_{B12}\dot{G}_{B23}\varepsilon_1\cdot\varepsilon_2 k_2\cdot k_3\Big].
\end{aligned}
\tag{D.85}
$$

Using (B.5) and (B.6) in the las equations we could replace some terms as follows

$$
\begin{aligned}
2r\cdot k_2\varepsilon_2\cdot k_1 r'\cdot k_3\varepsilon_3\cdot k_1 = r'\cdot k_3\varepsilon_3\cdot k_1(r\cdot f_2\cdot k_1+r\cdot\varepsilon_2 k_2\cdot k_1)\\
+\,r\cdot k_2\varepsilon_2\cdot k_1(r'\cdot f_3\cdot k_1+r'\cdot\varepsilon_3 k_3\cdot k_1),
\end{aligned}
\tag{D.86}
$$

and

$$
\begin{aligned}
-r\cdot k_2\varepsilon_2\cdot\varepsilon_3 r'\cdot k_3 = r\cdot f_2\cdot f_3\cdot r'-r\cdot k_2\varepsilon_2\cdot k_3\varepsilon_3\cdot r'\\
+\,r\cdot\varepsilon_2 k_2\cdot k_3\varepsilon_3\cdot r'-r\cdot\varepsilon_2 k_2\cdot\varepsilon_3 k_3\cdot r',
\end{aligned}
\tag{D.87}
$$

then we have

$$
\begin{aligned}
I^2_{(4)}(14) = -\tfrac{Z_2(14)}{6r_7\cdot k_2 r_8\cdot k_3}\Big\{&\dot{G}_{B21}\dot{G}_{B31}[r_8\cdot k_3\varepsilon_3\cdot k_1(r_7\cdot f_2\cdot k_1+r_7\cdot\varepsilon_2 k_2\cdot k_1)\\
&+r_7\cdot k_2\varepsilon_2\cdot k_1(r_8\cdot f_3\cdot k_1+r_8\cdot\varepsilon_3 k_3\cdot k_1)]\\
&+\dot{G}_{B23}\dot{G}_{B31}[r_8\cdot k_3\varepsilon_3\cdot k_1(r_7\cdot f_2\cdot k_3+r_7\cdot\varepsilon_2 k_2\cdot k_3)\\
&+r_7\cdot k_2\varepsilon_2\cdot k_3(r_8\cdot f_3\cdot k_1+\cancel{r_8\cdot\varepsilon_3}k_3\cdot k_1)]\\
&+\dot{G}_{B21}\dot{G}_{B32}[r_8\cdot k_3\varepsilon_3\cdot k_2(r_7\cdot f_2\cdot k_1+\cancel{r_7\cdot\varepsilon_2}k_2\cdot k_1)\\
&+r_7\cdot k_2\varepsilon_2\cdot k_1(r_8\cdot f_3\cdot k_2+r_8\cdot\varepsilon_3 k_3\cdot k_2)]\\
&+\dot{G}_{B23}\dot{G}_{B31}(r_7\cdot f_2\cdot f_3\cdot r_8-r_7\cdot k_2\varepsilon_2\cdot k_3\cancel{\varepsilon_3\cdot r_8}\\
&+r_7\cdot\varepsilon_2 k_2\cdot k_3\varepsilon_3\cdot r_8-r_7\cdot\varepsilon_2 k_2\cdot\varepsilon_3 k_3\cdot r_8)k_3\cdot k_1\\
&+\dot{G}_{B21}\dot{G}_{B32}(r_7\cdot f_2\cdot f_3\cdot r_8-r_7\cdot k_2\varepsilon_2\cdot k_3\varepsilon_3\cdot r_8\\
&+r_7\cdot\varepsilon_2 k_2\cdot k_3\varepsilon_3\cdot r_8-\cancel{r_7\cdot\varepsilon_2}k_2\cdot\varepsilon_3 k_3\cdot r_8)k_2\cdot k_1\Big\},
\end{aligned}
\tag{D.88}
$$

$$
\begin{aligned}
I^2_{(4)}(24) = -\tfrac{Z_2(24)}{6r_9\cdot k_1 r_0\cdot k_3}\Big\{&\dot{G}_{B12}\dot{G}_{B32}[r_0\cdot k_3\varepsilon_3\cdot k_2(r_9\cdot f_1\cdot k_2 + r_9\cdot\varepsilon_1\, k_1\cdot k_2\,)\\
&+r_9\cdot k_1\varepsilon_1\cdot k_2(r_0\cdot f_3\cdot k_2 + r_0\cdot\varepsilon_3\, k_3\cdot k_2\,)]\\
&+\dot{G}_{B13}\dot{G}_{B32}[r_0\cdot k_3\varepsilon_3\cdot k_2(r_9\cdot f_1\cdot k_3 + r_9\cdot\varepsilon_1\, k_1\cdot k_3\,)\\
&+r_9\cdot k_1\varepsilon_1\cdot k_3(r_0\cdot f_3\cdot k_2 + \cancel{r_0\cdot\varepsilon_3 k_3}\cdot k_2)]\\
&+\dot{G}_{B12}\dot{G}_{B31}[r_0\cdot k_3\varepsilon_3\cdot k_1(r_9\cdot f_1\cdot k_2 + \cancel{r_9\cdot\varepsilon_1 k_1}\cdot k_2)\\
&+r_9\cdot k_1\varepsilon_1\cdot k_2(r_0\cdot f_3\cdot k_1 + r_0\cdot\varepsilon_3\, k_3\cdot k_1\,)]\\
&+\dot{G}_{B13}\dot{G}_{B32}(r_9\cdot f_1\cdot f_3\cdot r_0 - r_9\cdot k_1\varepsilon_1\cdot k_3\varepsilon_3\cdot\cancel{r_0}\\
&+r_9\cdot\varepsilon_1 k_1\cdot k_3\varepsilon_3\cdot r_0 - r_9\cdot\varepsilon_1 k_1\cdot\varepsilon_3 k_3\cdot r_0)k_3\cdot k_2\\
&+\dot{G}_{B12}\dot{G}_{B31}(r_9\cdot f_1\cdot f_3\cdot r_0 - r_9\cdot k_1\varepsilon_1\cdot k_3\varepsilon_3\cdot r_0\\
&+r_9\cdot\varepsilon_1 k_1\cdot k_3\varepsilon_3\cdot r_0 - \cancel{r_9\cdot\varepsilon_1 k_1}\cdot\varepsilon_3 k_3\cdot r_0)k_1\cdot k_2\Big\},
\end{aligned}
\tag{D.89}
$$

$$
\begin{aligned}
I^2_{(4)}(34) = -\tfrac{Z_2(34)}{6r_{01}\cdot k_1 r_{02}\cdot k_2}\Big\{&\dot{G}_{B13}\dot{G}_{B23}[r_{02}\cdot k_2\varepsilon_2\cdot k_3(r_{01}\cdot f_1\cdot k_3 + r_{01}\cdot\varepsilon_1\, k_1\cdot k_3\,)\\
&+r_{01}\cdot k_1\varepsilon_1\cdot k_3(r_{02}\cdot f_2\cdot k_3 + r_{02}\cdot\varepsilon_2\, k_2\cdot k_3\,)]\\
&+\dot{G}_{B12}\dot{G}_{B23}[r_{02}\cdot k_2\varepsilon_2\cdot k_3(r_{01}\cdot f_1\cdot k_2 + r_{01}\cdot\varepsilon_1\, k_1\cdot k_2\,)\\
&+r_{01}\cdot k_1\varepsilon_1\cdot k_2(r_{02}\cdot f_2\cdot k_3 + \cancel{r_{02}\cdot\varepsilon_2 k_2}\cdot k_3)]\\
&+\dot{G}_{B13}\dot{G}_{B21}[r_{02}\cdot k_2\varepsilon_2\cdot k_1(r_{01}\cdot f_1\cdot k_3 + \cancel{r_{01}\cdot\varepsilon_1 k_1}\cdot k_3)\\
&+r_{01}\cdot k_1\varepsilon_1\cdot k_3(r_{02}\cdot f_2\cdot k_1 + r_{02}\cdot\varepsilon_2\, k_2\cdot k_1\,)]\\
&+\dot{G}_{B12}\dot{G}_{B23}(r_{01}\cdot f_1\cdot f_2\cdot r_{02} - r_{01}\cdot k_1\varepsilon_1\cdot k_2\varepsilon_2\cdot\cancel{r_{02}}\\
&+r_{01}\cdot\varepsilon_1 k_1\cdot k_2\varepsilon_2\cdot r_{02} - r_{01}\cdot\varepsilon_1 k_1\cdot\varepsilon_2 k_2\cdot r_{02})k_2\cdot k_3\\
&+\dot{G}_{B13}\dot{G}_{B21}(r_{01}\cdot f_1\cdot f_2\cdot r_{02} - r_{01}\cdot k_1\varepsilon_1\cdot k_2\varepsilon_2\cdot r_{02}\\
&+r_{01}\cdot\varepsilon_1 k_1\cdot k_2\varepsilon_2\cdot r_{02} - \cancel{r_{01}\cdot\varepsilon_1 k_1}\cdot\varepsilon_2 k_2\cdot r_{02})k_1\cdot k_3\Big\}.
\end{aligned}
\tag{D.90}
$$

Notice that some terms cancel each other and the terms marked with green or yellow putting together form total derivatives that do not contribute to

the integral. Then, the terms that remains are

$$
\begin{aligned}
I^2_{(4)}(14) = -\tfrac{Z_2(14)}{6r_7\cdot k_2 r_8\cdot k_3}\Big[&\dot{G}_{B21}\dot{G}_{B31}[r_8\cdot k_3\varepsilon_3\cdot k_1 r_7\cdot f_2\cdot k_1\\
&+r_7\cdot k_2\varepsilon_2\cdot k_1 r_8\cdot f_3\cdot k_1]\\
&+\dot{G}_{B23}\dot{G}_{B31}[r_8\cdot k_3\varepsilon_3\cdot k_1 r_7\cdot f_2\cdot k_3+r_7\cdot k_2\varepsilon_2\cdot k_3 r_8\cdot f_3\cdot k_1]\\
&+\dot{G}_{B21}\dot{G}_{B32}[r_8\cdot k_3\varepsilon_3\cdot k_2 r_7\cdot f_2\cdot k_1+r_7\cdot k_2\varepsilon_2\cdot k_1 r_8\cdot f_3\cdot k_2]\\
&+\dot{G}_{B23}\dot{G}_{B31}(r_7\cdot f_2\cdot f_3\cdot r_8\\
&+r_7\cdot\varepsilon_2 k_2\cdot k_3\varepsilon_3\cdot r_8-r_7\cdot\varepsilon_2 k_2\cdot\varepsilon_3 k_3\cdot r_8)k_3\cdot k_1\\
&+\dot{G}_{B21}\dot{G}_{B32}(r_7\cdot f_2\cdot f_3\cdot r_8\\
&+r_7\cdot\varepsilon_2 k_2\cdot k_3\varepsilon_3\cdot r_8-r_7\cdot k_2\varepsilon_2\cdot k_3\varepsilon_3\cdot r_8)k_2\cdot k_1\Big],
\end{aligned}
\tag{D.91}
$$

$$
\begin{aligned}
I^2_{(4)}(24) = -\tfrac{Z_2(24)}{6r_9\cdot k_1 r_0\cdot k_3}\Big[&\dot{G}_{B12}\dot{G}_{B32}[r_0\cdot k_3\varepsilon_3\cdot k_2 r_9\cdot f_1\cdot k_2\\
&+r_9\cdot k_1\varepsilon_1\cdot k_2 r_0\cdot f_3\cdot k_2]\\
&+\dot{G}_{B13}\dot{G}_{B32}[r_0\cdot k_3\varepsilon_3\cdot k_2 r_9\cdot f_1\cdot k_3+r_9\cdot k_1\varepsilon_1\cdot k_3 r_0\cdot f_3\cdot k_2]\\
&+\dot{G}_{B12}\dot{G}_{B31}[r_0\cdot k_3\varepsilon_3\cdot k_1 r_9\cdot f_1\cdot k_2+r_9\cdot k_1\varepsilon_1\cdot k_2 r_0\cdot f_3\cdot k_1]\\
&+\dot{G}_{B13}\dot{G}_{B32}(r_9\cdot f_1\cdot f_3\cdot r_0\\
&+r_9\cdot\varepsilon_1 k_1\cdot k_3\varepsilon_3\cdot r_0-r_9\cdot\varepsilon_1 k_1\cdot\varepsilon_3 k_3\cdot r_0)k_3\cdot k_2\\
&+\dot{G}_{B12}\dot{G}_{B31}(r_9\cdot f_1\cdot f_3\cdot r_0\\
&+r_9\cdot\varepsilon_1 k_1\cdot k_3\varepsilon_3\cdot r_0-r_9\cdot k_1\varepsilon_1\cdot k_3\varepsilon_3\cdot r_0)k_1\cdot k_2\Big],
\end{aligned}
\tag{D.92}
$$

$$
\begin{aligned}
I^2_{(4)}(34) = -\tfrac{Z_2(34)}{6r_{01}\cdot k_1 r_{02}\cdot k_2}\Big[&\dot{G}_{B13}\dot{G}_{B23}[r_{02}\cdot k_2\varepsilon_2\cdot k_3 r_{01}\cdot f_1\cdot k_3\\
&+r_{01}\cdot k_1\varepsilon_1\cdot k_3 r_{02}\cdot f_2\cdot k_3]\\
&+\dot{G}_{B12}\dot{G}_{B23}[r_{02}\cdot k_2\varepsilon_2\cdot k_3 r_{01}\cdot f_1\cdot k_2+r_{01}\cdot k_1\varepsilon_1\cdot k_2 r_{02}\cdot f_2\cdot k_3]\\
&+\dot{G}_{B13}\dot{G}_{B21}[r_{02}\cdot k_2\varepsilon_2\cdot k_1 r_{01}\cdot f_1\cdot k_3+r_{01}\cdot k_1\varepsilon_1\cdot k_3 r_{02}\cdot f_2\cdot k_1]\\
&+\dot{G}_{B12}\dot{G}_{B23}(r_{01}\cdot f_1\cdot f_2\cdot r_{02}\\
&+r_{01}\cdot\varepsilon_1 k_1\cdot k_2\varepsilon_2\cdot r_{02}-r_{01}\cdot\varepsilon_1 k_1\cdot\varepsilon_2 k_2\cdot r_{02})k_2\cdot k_3\\
&+\dot{G}_{B13}\dot{G}_{B21}(r_{01}\cdot f_1\cdot f_2\cdot r_{02}\\
&+r_{01}\cdot\varepsilon_1 k_1\cdot k_2\varepsilon_2\cdot r_{02}-r_{01}\cdot k_1\varepsilon_1\cdot k_2\varepsilon_2\cdot r_{02})k_1\cdot k_3\Big].
\end{aligned}
\tag{D.93}
$$

Similarly, in the last equations there are terms that with the help of (B.5) can be replaced in the following way

$$
r\cdot k_3\varepsilon_3\cdot k_1 = r\cdot f_3\cdot k_1 + r\cdot\varepsilon_3 k_3\cdot k_1, \tag{D.94}
$$

then

$$
\begin{aligned}
I^2_{(4)}(14) = -\tfrac{Z_2(14)}{6r_7\cdot k_2 r_8\cdot k_3}\Big[&\dot{G}_{B21}\dot{G}_{B31}[(r_8\cdot f_3\cdot k_1 + r_8\cdot\varepsilon_3\, k_3\cdot k_1)r_7\cdot f_2\cdot k_1\\
&+(r_7\cdot f_2\cdot k_1 + r_7\cdot\varepsilon_2\, k_2\cdot k_1)r_8\cdot f_3\cdot k_1]\\
&+\dot{G}_{B23}\dot{G}_{B31}[(r_8\cdot f_3\cdot k_1 + r_8\cdot\varepsilon_3\, k_3\cdot k_1)r_7\cdot f_2\cdot k_3\\
&+(r_7\cdot f_2\cdot k_3 + r_7\cdot\varepsilon_2\, k_2\cdot k_3)r_8\cdot f_3\cdot k_1]\\
&+\dot{G}_{B21}\dot{G}_{B32}[(r_8\cdot f_3\cdot k_2 + r_8\cdot\varepsilon_3\, k_3\cdot k_2)r_7\cdot f_2\cdot k_1\\
&+(r_7\cdot f_2\cdot k_1 + r_7\cdot\varepsilon_2\, k_2\cdot k_1)r_8\cdot f_3\cdot k_2]\\
&+\dot{G}_{B23}\dot{G}_{B31}(r_7\cdot f_2\cdot f_3\cdot r_8 + r_7\cdot\varepsilon_2 k_2\cdot f_3\cdot\, r_8)k_3\cdot k_1\\
&+\dot{G}_{B21}\dot{G}_{B32}(r_7\cdot f_2\cdot f_3\cdot r_8 - r_7\cdot f_2\cdot k_3\varepsilon_3\cdot\, r_8)k_2\cdot k_1\Big],
\end{aligned}
\tag{D.95}
$$

$$
\begin{aligned}
I^2_{(4)}(24) = -\tfrac{Z_2(24)}{6r_9\cdot k_1 r_0\cdot k_3}\Big[&\dot{G}_{B12}\dot{G}_{B32}[(r_0\cdot f_3\cdot k_2 + r_0\cdot\varepsilon_3\, k_3\cdot k_2)r_9\cdot f_1\cdot k_2\\
&+(r_9\cdot f_1\cdot k_2 + r_9\cdot\varepsilon_1\, k_1\cdot k_2)r_0\cdot f_3\cdot k_2]\\
&+\dot{G}_{B13}\dot{G}_{B32}[(r_0\cdot f_3\cdot k_2 + r_0\cdot\varepsilon_3\, k_3\cdot k_2)r_9\cdot f_1\cdot k_3\\
&+(r_9\cdot f_1\cdot k_3 + r_9\cdot\varepsilon_1\, k_1\cdot k_3)r_0\cdot f_3\cdot k_2]\\
&+\dot{G}_{B12}\dot{G}_{B31}[(r_0\cdot f_3\cdot k_1 + r_0\cdot\varepsilon_3\, k_3\cdot k_1)r_9\cdot f_1\cdot k_2\\
&+(r_9\cdot f_1\cdot k_2 + r_9\cdot\varepsilon_1\, k_1\cdot k_2)r_0\cdot f_3\cdot k_1]\\
&+\dot{G}_{B13}\dot{G}_{B32}(r_9\cdot f_1\cdot f_3\cdot r_0 + r_9\cdot\varepsilon_1 k_1\cdot f_3\cdot\, r_0)k_3\cdot k_2\\
&+\dot{G}_{B12}\dot{G}_{B31}(r_9\cdot f_1\cdot f_3\cdot r_0 - r_9\cdot f_1\cdot k_3\varepsilon_3\cdot\, r_0)k_1\cdot k_2\Big],
\end{aligned}
\tag{D.96}
$$

$$
\begin{aligned}
I^2_{(4)}(34) = -\tfrac{Z_2(34)}{6r_{01}\cdot k_1 r_{02}\cdot k_2}\Big[&\dot{G}_{B13}\dot{G}_{B23}[(r_{02}\cdot f_2\cdot k_3 + r_{02}\cdot\varepsilon_2\, k_2\cdot k_3)r_{01}\cdot f_1\cdot k_3\\
&+(r_{01}\cdot f_1\cdot k_3 + r_{01}\cdot\varepsilon_1\, k_1\cdot k_3)r_{02}\cdot f_2\cdot k_3]\\
&+\dot{G}_{B12}\dot{G}_{B23}[(r_{02}\cdot f_2\cdot k_3 + r_{02}\cdot\varepsilon_2\, k_2\cdot k_3)r_{01}\cdot f_1\cdot k_2\\
&+(r_{01}\cdot f_1\cdot k_2 + r_{01}\cdot\varepsilon_1\, k_1\cdot k_2)r_{02}\cdot f_2\cdot k_3]\\
&+\dot{G}_{B13}\dot{G}_{B21}[(r_{02}\cdot f_2\cdot k_1 + r_{02}\cdot\varepsilon_2\, k_2\cdot k_1)r_{01}\cdot f_1\cdot k_3\\
&+(r_{01}\cdot f_1\cdot k_3 + r_{01}\cdot\varepsilon_1\, k_1\cdot k_3)r_{02}\cdot f_2\cdot k_1]\\
&+\dot{G}_{B12}\dot{G}_{B23}(r_{01}\cdot f_1\cdot f_2\cdot r_{02} + r_{01}\cdot\varepsilon_1 k_1\cdot f_2\cdot\, r_{02})k_2\cdot k_3\\
&+\dot{G}_{B13}\dot{G}_{B21}(r_{01}\cdot f_1\cdot f_2\cdot r_{02} - r_{01}\cdot f_1\cdot k_2\varepsilon_2\cdot\, r_{02})k_1\cdot k_3\Big].
\end{aligned}
\tag{D.97}
$$

Putting together the colored terms this cancel because they form total derivatives. And now we have

$$\begin{aligned} I^2_{(4)}(14) = -\tfrac{Z_2(14)}{3r_7\cdot k_2 r_8\cdot k_3}\Big[&\dot{G}_{B21}\dot{G}_{B31} r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_1 \\ &+\dot{G}_{B23}\dot{G}_{B31}\left(r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_3+\tfrac{1}{2}r_7\cdot f_2\cdot f_3\cdot r_8 k_3\cdot k_1\right) \\ &+\dot{G}_{B21}\dot{G}_{B32}\left(r_8\cdot f_3\cdot k_2 r_7\cdot f_2\cdot k_1+\tfrac{1}{2}r_7\cdot f_2\cdot f_3\cdot r_8 k_2\cdot k_1\right)\Big], \end{aligned} \tag{D.98}$$

$$\begin{aligned} I^2_{(4)}(24) = -\tfrac{Z_2(24)}{3r_9\cdot k_1 r_0\cdot k_3}\Big[&\dot{G}_{B12}\dot{G}_{B32} r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_2 \\ &+\dot{G}_{B13}\dot{G}_{B32}\left(r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_3+\tfrac{1}{2}r_9\cdot f_1\cdot f_3\cdot r_0 k_3\cdot k_2\right) \\ &+\dot{G}_{B12}\dot{G}_{B31}\left(r_0\cdot f_3\cdot k_1 r_9\cdot f_1\cdot k_2+\tfrac{1}{2}r_9\cdot f_1\cdot f_3\cdot r_0 k_1\cdot k_2\right)\Big], \end{aligned} \tag{D.99}$$

$$\begin{aligned} I^2_{(4)}(34) = -\tfrac{Z_2(34)}{3r_{01}\cdot k_1 r_{02}\cdot k_2}\Big[&\dot{G}_{B13}\dot{G}_{B23} r_{02}\cdot f_2\cdot k_3 r_{01}\cdot f_1\cdot k_3 \\ &+\dot{G}_{B12}\dot{G}_{B23}\left(r_{02}\cdot f_2\cdot k_3 r_{01}\cdot f_1\cdot k_2+\tfrac{1}{2}r_{01}\cdot f_1\cdot f_2\cdot r_{02}k_2\cdot k_3\right) \\ &+\dot{G}_{B13}\dot{G}_{B21}\left(r_{02}\cdot f_2\cdot k_1 r_{01}\cdot f_1\cdot k_3+\tfrac{1}{2}r_{01}\cdot f_1\cdot f_2\cdot r_{02}k_1\cdot k_3\right)\Big]. \end{aligned} \tag{D.100}$$

Finally, notice that we can use

$$\frac{1}{2}\dot{G}_{B12}k_2\cdot k_1-\frac{1}{2}\dot{G}_{B13}k_3\cdot k_1=-\dot{G}_{B12}k_2\cdot k_1-2\dot{G}_{B13}k_3\cdot k_1, \tag{D.101}$$

to forget about the "$\frac{1}{2}$" in the last equations and the verifications of our results is easier in this form.

Result of $I^2_{(4)}(14)$, $I^2_{(4)}(24)$ and $I^2_{(4)}(34)$

$$\begin{aligned} I^2_{(4)}(14) = -\tfrac{Z_2(14)}{3r_7\cdot k_2 r_8\cdot k_3}\Big[&-\dot{G}_{B12}\dot{G}_{B31} r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_1 \\ &+\dot{G}_{B23}\dot{G}_{B31}\left(r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_3+2r_7\cdot f_2\cdot f_3\cdot r_8 k_3\cdot k_1\right) \\ &+\dot{G}_{B12}\dot{G}_{B23}\left(r_8\cdot f_3\cdot k_2 r_7\cdot f_2\cdot k_1-r_7\cdot f_2\cdot f_3\cdot r_8 k_2\cdot k_1\right)\Big], \end{aligned} \tag{D.102}$$

$$\begin{aligned} I^2_{(4)}(24) = -\tfrac{Z_2(24)}{3r_9\cdot k_1 r_0\cdot k_3}\Big[&-\dot{G}_{B12}\dot{G}_{B23} r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_2 \\ &+\dot{G}_{B23}\dot{G}_{B31}\left(r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_3+2r_9\cdot f_1\cdot f_3\cdot r_0 k_3\cdot k_2\right) \\ &+\dot{G}_{B12}\dot{G}_{B31}\left(r_0\cdot f_3\cdot k_1 r_9\cdot f_1\cdot k_2-r_9\cdot f_1\cdot f_3\cdot r_0 k_1\cdot k_2\right)\Big], \end{aligned} \tag{D.103}$$

$$I^2_{(4)}(34) = -\frac{Z_2(34)}{3r_{01}\cdot k_1 r_{02}\cdot k_2}\Big[-\dot{G}_{B23}\dot{G}_{B31} r_{02}\cdot f_2\cdot k_3 r_{01}\cdot f_1\cdot k_3 \\ +\dot{G}_{B12}\dot{G}_{B23}\left(r_{02}\cdot f_2\cdot k_3 r_{01}\cdot f_1\cdot k_2 + 2r_{01}\cdot f_1\cdot f_2\cdot r_{02}k_2\cdot k_3\right) \\ +\dot{G}_{B12}\dot{G}_{B31}\left(r_{02}\cdot f_2\cdot k_1 r_{01}\cdot f_1\cdot k_3 - r_{01}\cdot f_1\cdot f_2\cdot r_{02}k_1\cdot k_3\right)\Big], \tag{D.104}$$

D.1.4 Integral of Q_4^{22}

Using (B.11), we have that

$$I^{22}_{(4)}(12,34) = -\frac{1}{3}\dot{G}_{B12}\dot{G}_{B21}Z_2(12)Z_2(34), \tag{D.105}$$

$$I^{22}_{(4)}(13,24) = -\frac{1}{3}\dot{G}_{B13}\dot{G}_{B31}Z_2(13)Z_2(24), \tag{D.106}$$

$$I^{22}_{(4)}(14,23) = -\frac{1}{3}\dot{G}_{B23}\dot{G}_{B32}Z_2(14)Z_2(23), \tag{D.107}$$

or

$$I^{22}_{(4)}(12,34) = \frac{1}{3}\dot{G}^2_{B12}Z_2(12)Z_2(34), \tag{D.108}$$

$$I^{22}_{(4)}(13,24) = \frac{1}{3}\dot{G}^2_{B31}Z_2(13)Z_2(24), \tag{D.109}$$

$$I^{22}_{(4)}(14,23) = \frac{1}{3}\dot{G}^2_{B23}Z_2(14)Z_2(23). \tag{D.110}$$

D.1.5 Result of $I_{(4)}$

We write the full expression for $I_{(4)}$

$$\begin{aligned} I^4_{(4)}(1234) &= \dot{G}_{B12}\dot{G}_{B23}\left(\tfrac{1}{6}-\tfrac{1}{2}\dot{G}^2_{B31}\right)Z_4(1234)\\ I^4_{(4)}(1243) &= \dot{G}_{B12}\dot{G}_{B31}\left(\tfrac{1}{6}-\tfrac{1}{2}\dot{G}^2_{B23}\right)Z_4(1243)\\ I^4_{(4)}(1324) &= \dot{G}_{B23}\dot{G}_{B31}\left(\tfrac{1}{6}-\tfrac{1}{2}\dot{G}^2_{B12}\right)Z_4(1324), \end{aligned} \tag{D.111}$$

$$\begin{aligned} I^3_{(4)}(123) = \tfrac{1}{12}\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31}Z_3(123)\Big[&\dot{G}_{B12}\left(1-\dot{G}^2_{B12}\right)k_2\cdot f_4\cdot k_1\\ &+\dot{G}_{B23}\left(1-\dot{G}^2_{B23}\right)k_3\cdot f_4\cdot k_2+\dot{G}_{B31}\left(1-\dot{G}^2_{B31}\right)k_1\cdot f_4\cdot k_3\Big], \end{aligned} \tag{D.112}$$

$$I^3_{(4)}(234) = \frac{1}{2}\dot{G}_{B23}\left(\frac{1}{3}-\dot{G}^2_{B23}\right)Z_3(234)\left(\dot{G}_{B12}\frac{r_1\cdot f_1\cdot k_2}{r_1\cdot k_1}-\dot{G}_{B31}\frac{r_1\cdot f_1\cdot k_3}{r_1\cdot k_1}\right), \tag{D.113}$$

$$I^3_{(4)}(341) = \frac{1}{2}\left(\frac{1}{3}-\dot{G}^2_{B31}\right)\dot{G}_{B31}Z_3(341)\left(\dot{G}_{B12}\frac{r_2\cdot f_2\cdot k_1}{r_2\cdot k_2}-\dot{G}_{B23}\frac{r_2\cdot f_2\cdot k_3}{r_2\cdot k_2}\right), \tag{D.114}$$

$$I^3_{(4)}(412) = \frac{1}{2}\left(\frac{1}{3}-\dot{G}^2_{B12}\right)\dot{G}_{B12}Z_3(412)\left(\dot{G}_{B31}\frac{r_3\cdot f_3\cdot k_1}{r_3\cdot k_3}-\dot{G}_{B23}\frac{r_3\cdot f_3\cdot k_2}{r_3\cdot k_3}\right). \tag{D.115}$$

$$\begin{aligned} I^2_{(4)}(12) = \tfrac{-1}{12}\dot{G}^2_{B12}Z_2(12)\Big[&\\ &+\dot{G}^2_{B31}\left(1-\dot{G}^2_{B31}\right)k_1\cdot f_4\cdot f_3\cdot k_1-\dot{G}_{B23}\dot{G}_{B31}\left(1-\dot{G}^2_{B31}\right)k_1\cdot f_4\cdot f_3\cdot k_2\\ &-\dot{G}_{B23}\dot{G}_{B31}\left(1-\dot{G}^2_{B23}\right)k_2\cdot f_4\cdot f_3\cdot k_1+\dot{G}^2_{B23}\left(1-\dot{G}^2_{B23}\right)k_2\cdot f_4\cdot f_3\cdot k_2\\ &+\dot{G}_{B12}\dot{G}_{B31}\left(1-\dot{G}^2_{B12}\right)\tfrac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_1}{r_4\cdot k_3}-\dot{G}_{B12}\dot{G}_{B23}\left(1-\dot{G}^2_{B12}\right)\tfrac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_2}{r_4\cdot k_3}\Big], \end{aligned} \tag{D.116}$$

$$\begin{aligned} I^2_{(4)}(13) = \tfrac{-1}{12}\dot{G}^2_{B31}Z_2(13)\Big[&\\ &+\dot{G}^2_{B12}\left(1-\dot{G}^2_{B12}\right)k_1\cdot f_4\cdot f_2\cdot k_1-\dot{G}_{B12}\dot{G}_{B23}\left(1-\dot{G}^2_{B12}\right)k_1\cdot f_4\cdot f_2\cdot k_3\\ &-\dot{G}_{B12}\dot{G}_{B23}\left(1-\dot{G}^2_{B23}\right)k_3\cdot f_4\cdot f_2\cdot k_1+\dot{G}^2_{B23}\left(1-\dot{G}^2_{B23}\right)k_3\cdot f_4\cdot f_2\cdot k_3\\ &+\dot{G}_{B12}\dot{G}_{B31}\left(1-\dot{G}^2_{B31}\right)\tfrac{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_1}{r_5\cdot k_2}-\dot{G}_{B23}\dot{G}_{B31}\left(1-\dot{G}^2_{B31}\right)\tfrac{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_3}{r_5\cdot k_2}\Big], \end{aligned} \tag{D.117}$$

$$
\begin{aligned}
I^2_{(4)}(23) = {} & \tfrac{-1}{12}\dot{G}^2_{B23}Z_2(23)\Big[\\
& +\dot{G}^2_{B12}\left(1-\dot{G}^2_{B12}\right)k_2\cdot f_4\cdot f_1\cdot k_2 - \dot{G}_{B12}\dot{G}_{B31}\left(1-\dot{G}^2_{B12}\right)k_2\cdot f_4\cdot f_1\cdot k_3 \\
& -\dot{G}_{B12}\dot{G}_{B31}\left(1-\dot{G}^2_{B31}\right)k_3\cdot f_4\cdot f_1\cdot k_2 + \dot{G}^2_{B31}\left(1-\dot{G}^2_{B31}\right)k_3\cdot f_4\cdot f_1\cdot k_3 \\
& +\dot{G}_{B12}\dot{G}_{B23}\left(1-\dot{G}^2_{B23}\right)\tfrac{k_3\cdot f_4\cdot k_2 r_6\cdot f_1\cdot k_2}{r_6\cdot k_1} - \dot{G}_{B23}\dot{G}_{B31}\left(1-\dot{G}^2_{B23}\right)\tfrac{k_3\cdot f_4\cdot k_2 r_6\cdot f_1\cdot k_3}{r_6\cdot k_1}\Big],
\end{aligned}
\tag{D.118}
$$

$$
\begin{aligned}
I^2_{(4)}(14) = {} & -\tfrac{Z_2(14)}{3r_7\cdot k_2 r_8\cdot k_3}\Big[-\dot{G}_{B12}\dot{G}_{B31} r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_1 \\
& +\dot{G}_{B23}\dot{G}_{B31}\left(r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_3 + 2r_7\cdot f_2\cdot f_3\cdot r_8 k_3\cdot k_1\right) \\
& +\dot{G}_{B12}\dot{G}_{B23}\left(r_8\cdot f_3\cdot k_2 r_7\cdot f_2\cdot k_1 - r_7\cdot f_2\cdot f_3\cdot r_8 k_2\cdot k_1\right)\Big],
\end{aligned}
\tag{D.119}
$$

$$
\begin{aligned}
I^2_{(4)}(24) = {} & -\tfrac{Z_2(24)}{3r_9\cdot k_1 r_0\cdot k_3}\Big[-\dot{G}_{B12}\dot{G}_{B23} r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_2 \\
& +\dot{G}_{B23}\dot{G}_{B31}\left(r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_3 + 2r_9\cdot f_1\cdot f_3\cdot r_0 k_3\cdot k_2\right) \\
& +\dot{G}_{B12}\dot{G}_{B31}\left(r_0\cdot f_3\cdot k_1 r_9\cdot f_1\cdot k_2 - r_9\cdot f_1\cdot f_3\cdot r_0 k_1\cdot k_2\right)\Big],
\end{aligned}
\tag{D.120}
$$

$$
\begin{aligned}
I^2_{(4)}(34) = {} & -\tfrac{Z_2(34)}{3r_{01}\cdot k_1 r_{02}\cdot k_2}\Big[-\dot{G}_{B23}\dot{G}_{B31} r_{02}\cdot f_2\cdot k_3 r_{01}\cdot f_1\cdot k_3 \\
& +\dot{G}_{B12}\dot{G}_{B23}\left(r_{02}\cdot f_2\cdot k_3 r_{01}\cdot f_1\cdot k_2 + 2r_{01}\cdot f_1\cdot f_2\cdot r_{02} k_2\cdot k_3\right) \\
& +\dot{G}_{B12}\dot{G}_{B31}\left(r_{02}\cdot f_2\cdot k_1 r_{01}\cdot f_1\cdot k_3 - r_{01}\cdot f_1\cdot f_2\cdot r_{02} k_1\cdot k_3\right)\Big],
\end{aligned}
\tag{D.121}
$$

$$
I^{22}_{(4)}(12,34) = \frac{1}{3}\dot{G}^2_{B12}Z_2(12)Z_2(34), \tag{D.122}
$$

$$
I^{22}_{(4)}(13,24) = \frac{1}{3}\dot{G}^2_{B31}Z_2(13)Z_2(24), \tag{D.123}
$$

$$
I^{22}_{(4)}(14,23) = \frac{1}{3}\dot{G}^2_{B23}Z_2(14)Z_2(23). \tag{D.124}
$$

D.1.6 $I_{(4)}$ in terms of G_{Bij} only

In the previous section we write $I_{(4)}$ in terms of $\dot{G}_{Bij}$ because in this form it is easier to perform the integrals when we take the low energy limit in some leg but we realize that if we want to perform the integrals without any assumption on the energy of the photons it is better to have this expressions in terms of G_{Bij} only. Therefore, lets look at the following total derivative of second order

$$
\partial_1^2 e^{(\bullet)} = \Big[\Big(\dot{G}_{B12}k_1\cdot k_2 + \dot{G}_{B13}k_1\cdot k_3\Big)^2 + \Big(\ddot{G}_{B12}k_1\cdot k_2 + \ddot{G}_{B13}k_1\cdot k_3\Big) \Big] e^{(\bullet)}, \tag{D.125}
$$

$(e^{(\bullet)} = e^{G_{B12}k_1\cdot k_2+G_{B13}k_1\cdot k_3+G_{B23}k_2\cdot k_3})$ from which we can obtain

$$2\dot{G}_{B12}\dot{G}_{B13}k_1\cdot k_2 k_1\cdot k_3 e^{(\bullet)} = \partial_1^2 e^{(\bullet)} - \Big[\Big(\dot{G}_{B12}k_1\cdot k_2\Big)^2 + \Big(\dot{G}_{B13}k_1\cdot k_3\Big)^2 + \Big(\ddot{G}_{B12}k_1\cdot k_2 + \ddot{G}_{B13}k_1\cdot k_3\Big)\Big]e^{(\bullet)}. \tag{D.126}$$

Since $\ddot{G}_{B12} = 2\delta(u_1-u_2) - 2$ and neither the Dirac's delta nor the total derivative contributes to the integral, we have

$$2\dot{G}_{B12}\dot{G}_{B13}k_1\cdot k_2 k_1\cdot k_3 = -\Big[\Big(\dot{G}_{B12}k_1\cdot k_2\Big)^2 + \Big(\dot{G}_{B13}k_1\cdot k_3\Big)^2 + (-2k_1\cdot k_2 - 2k_1\cdot k_3)\Big], \tag{D.127}$$

then we use (B.1) to obtain that

$$\begin{aligned}\dot{G}_{B12}\dot{G}_{B13} =& -\frac{1}{2}\Big[\dot{G}_{B12}^2\frac{k_1\cdot k_2}{k_1\cdot k_3} + \dot{G}_{B13}^2\frac{k_1\cdot k_3}{k_1\cdot k_2} + \left(-\frac{2}{k_1\cdot k_3} - \frac{2}{k_1\cdot k_2}\right)\Big] \\ =& 2G_{B12}\frac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13}\frac{k_1\cdot k_3}{k_1\cdot k_2} - \frac{1}{2}\Big[\frac{k_1\cdot k_2-2}{k_1\cdot k_3} + \frac{k_1\cdot k_3-2}{k_1\cdot k_2}\Big],\end{aligned} \tag{D.128}$$

similarly

$$\dot{G}_{B21}\dot{G}_{B23} = 2G_{B21}\frac{k_2\cdot k_1}{k_2\cdot k_3} + 2G_{B23}\frac{k_2\cdot k_3}{k_2\cdot k_1} - \frac{1}{2}\Big[\frac{k_2\cdot k_1-2}{k_2\cdot k_3} + \frac{k_2\cdot k_3-2}{k_2\cdot k_1}\Big], \tag{D.129}$$

$$\dot{G}_{B13}\dot{G}_{B23} = 2G_{B13}\frac{k_1\cdot k_3}{k_2\cdot k_3} + 2G_{B23}\frac{k_2\cdot k_3}{k_1\cdot k_3} - \frac{1}{2}\Big[\frac{k_1\cdot k_3-2}{k_2\cdot k_3} + \frac{k_2\cdot k_3-2}{k_1\cdot k_3}\Big], \tag{D.130}$$

Now, we define

$$a = -\frac{1}{2}\Big[\frac{k_1\cdot k_2-2}{k_1\cdot k_3} + \frac{k_1\cdot k_3-2}{k_1\cdot k_2}\Big], \tag{D.131}$$

$$b = -\frac{1}{2}\Big[\frac{k_2\cdot k_1-2}{k_2\cdot k_3} + \frac{k_2\cdot k_3-2}{k_2\cdot k_1}\Big], \tag{D.132}$$

$$c = -\frac{1}{2}\Big[\frac{k_1\cdot k_3-2}{k_2\cdot k_3} + \frac{k_2\cdot k_3-2}{k_1\cdot k_3}\Big], \tag{D.133}$$

then

$$\dot{G}_{B12}\dot{G}_{B13} = \left(2G_{B12}\frac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13}\frac{k_1\cdot k_3}{k_1\cdot k_2} + a\right), \tag{D.134}$$

$$\dot{G}_{B21}\dot{G}_{B23} = \left(2G_{B21}\frac{k_2\cdot k_1}{k_2\cdot k_3} + 2G_{B23}\frac{k_2\cdot k_3}{k_2\cdot k_1} + b\right), \tag{D.135}$$

$$\dot{G}_{B13}\dot{G}_{B23} = \left(2G_{B13}\frac{k_1\cdot k_3}{k_2\cdot k_3} + 2G_{B23}\frac{k_2\cdot k_3}{k_1\cdot k_3} + c\right). \tag{D.136}$$

Now, we make explicit the dependence of $\dot{G}_{B12}\dot{G}_{B13}$, $\dot{G}_{B21}\dot{G}_{B23}$ and $\dot{G}_{B13}\dot{G}_{B23}$ in (D.111)-(D.124) by rearranging this equations as

$$\begin{aligned} I^4_{(4)}(1234) &= -\dot{G}_{B21}\dot{G}_{B23}\left(\tfrac{1}{6} - \tfrac{1}{2}\dot{G}^2_{B31}\right) Z_4(1234) \\ I^4_{(4)}(1243) &= -\dot{G}_{B12}\dot{G}_{B13}\left(\tfrac{1}{6} - \tfrac{1}{2}\dot{G}^2_{B23}\right) Z_4(1243) \\ I^4_{(4)}(1324) &= -\dot{G}_{B13}\dot{G}_{B23}\left(\tfrac{1}{6} - \tfrac{1}{2}\dot{G}^2_{B12}\right) Z_4(1324), \end{aligned} \tag{D.137}$$

$$\begin{aligned} I^3_{(4)}(123) = \tfrac{-1}{12} Z_3(123) \Big[& \dot{G}_{B13}\dot{G}_{B23}\dot{G}^2_{B12}\left(1-\dot{G}^2_{B12}\right) k_2\cdot f_4\cdot k_1 \\ & + \dot{G}_{B12}\dot{G}_{B13}\dot{G}^2_{B23}\left(1-\dot{G}^2_{B23}\right) k_3\cdot f_4\cdot k_2 \\ & + \dot{G}_{B21}\dot{G}_{B23}\dot{G}^2_{B31}\left(1-\dot{G}^2_{B31}\right) k_1\cdot f_4\cdot k_3 \Big], \end{aligned} \tag{D.138}$$

$$I^3_{(4)}(234) = \tfrac{-1}{2}\left(\tfrac{1}{3} - \dot{G}^2_{B23}\right) Z_3(234)\left(\dot{G}_{B21}\dot{G}_{B23}\tfrac{r_1\cdot f_1\cdot k_2}{r_1\cdot k_1} - \dot{G}_{B13}\dot{G}_{B23}\tfrac{r_1\cdot f_1\cdot k_3}{r_1\cdot k_1}\right), \tag{D.139}$$

$$I^3_{(4)}(341) = \tfrac{-1}{2}\left(\tfrac{1}{3} - \dot{G}^2_{B31}\right) Z_3(341)\left(\dot{G}_{B12}\dot{G}_{B13}\tfrac{r_2\cdot f_2\cdot k_1}{r_2\cdot k_2} - \dot{G}_{B13}\dot{G}_{B23}\tfrac{r_2\cdot f_2\cdot k_3}{r_2\cdot k_2}\right), \tag{D.140}$$

$$I^3_{(4)}(412) = \tfrac{-1}{2}\left(\tfrac{1}{3} - \dot{G}^2_{B12}\right) Z_3(412)\left(\dot{G}_{B12}\dot{G}_{B13}\tfrac{r_3\cdot f_3\cdot k_1}{r_3\cdot k_3} - \dot{G}_{B21}\dot{G}_{B23}\tfrac{r_3\cdot f_3\cdot k_2}{r_3\cdot k_3}\right). \tag{D.141}$$

$$\begin{aligned} I^2_{(4)}(12) = \tfrac{-1}{12}\dot{G}^2_{B12} Z_2(12)\Big\{ & \\ & + \dot{G}^2_{B31}\left(1-\dot{G}^2_{B31}\right) k_1\cdot f_4\cdot f_3\cdot k_1 + \dot{G}^2_{B23}\left(1-\dot{G}^2_{B23}\right) k_2\cdot f_4\cdot f_3\cdot k_2 \\ & + \dot{G}_{B13}\dot{G}_{B23}\left[\left(1-\dot{G}^2_{B31}\right) k_1\cdot f_4\cdot f_3\cdot k_2 + \left(1-\dot{G}^2_{B23}\right) k_2\cdot f_4\cdot f_3\cdot k_1\right] \\ & - \dot{G}_{B12}\dot{G}_{B13}\left(1-\dot{G}^2_{B12}\right)\tfrac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_1}{r_4\cdot k_3} + \dot{G}_{B21}\dot{G}_{B23}\left(1-\dot{G}^2_{B12}\right)\tfrac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_2}{r_4\cdot k_3}\Big\}, \end{aligned} \tag{D.142}$$

$$
\begin{aligned}
I^2_{(4)}(13) = \tfrac{-1}{12}\dot{G}^2_{B31} Z_2(13)\Big\{ & \\
&+\dot{G}^2_{B12}\left(1-\dot{G}^2_{B12}\right) k_1\cdot f_4\cdot f_2\cdot k_1 + \dot{G}^2_{B23}\left(1-\dot{G}^2_{B23}\right) k_3\cdot f_4\cdot f_2\cdot k_3 \\
&+\dot{G}_{B21}\dot{G}_{B23}\left[\left(1-\dot{G}^2_{B12}\right) k_1\cdot f_4\cdot f_2\cdot k_3 + \left(1-\dot{G}^2_{B23}\right) k_3\cdot f_4\cdot f_2\cdot k_1\right] \\
&-\dot{G}_{B12}\dot{G}_{B13}\left(1-\dot{G}^2_{B31}\right)\tfrac{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_1}{r_5\cdot k_2} + \dot{G}_{B13}\dot{G}_{B23}\left(1-\dot{G}^2_{B31}\right)\tfrac{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_3}{r_5\cdot k_2}\Big\},
\end{aligned}
\tag{D.143}
$$

$$
\begin{aligned}
I^2_{(4)}(23) = \tfrac{-1}{12}\dot{G}^2_{B23} Z_2(23)\Big\{ & \\
&+\dot{G}^2_{B12}\left(1-\dot{G}^2_{B12}\right) k_2\cdot f_4\cdot f_1\cdot k_2 + \dot{G}^2_{B31}\left(1-\dot{G}^2_{B31}\right) k_3\cdot f_4\cdot f_1\cdot k_3 \\
&+\dot{G}_{B12}\dot{G}_{B13}\left[\left(1-\dot{G}^2_{B12}\right) k_2\cdot f_4\cdot f_1\cdot k_3 + \left(1-\dot{G}^2_{B31}\right) k_3\cdot f_4\cdot f_1\cdot k_2\right] \\
&-\dot{G}_{B21}\dot{G}_{B23}\left(1-\dot{G}^2_{B23}\right)\tfrac{k_3\cdot f_4\cdot k_2 r_6\cdot f_1\cdot k_2}{r_6\cdot k_1} + \dot{G}_{B13}\dot{G}_{B23}\left(1-\dot{G}^2_{B23}\right)\tfrac{k_3\cdot f_4\cdot k_2 r_6\cdot f_1\cdot k_3}{r_6\cdot k_1}\Big\},
\end{aligned}
\tag{D.144}
$$

$$
\begin{aligned}
I^2_{(4)}(14) = -\tfrac{Z_2(14)}{3r_7\cdot k_2 r_8\cdot k_3}\Big[& \dot{G}_{B12}\dot{G}_{B13} r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_1 \\
& -\dot{G}_{B13}\dot{G}_{B23}\left(r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_3 + 2r_7\cdot f_2\cdot f_3\cdot r_8 k_3\cdot k_1\right) \\
& -\dot{G}_{B21}\dot{G}_{B23}\left(r_8\cdot f_3\cdot k_2 r_7\cdot f_2\cdot k_1 - r_7\cdot f_2\cdot f_3\cdot r_8 k_2\cdot k_1\right)\Big],
\end{aligned}
\tag{D.145}
$$

$$
\begin{aligned}
I^2_{(4)}(24) = -\tfrac{Z_2(24)}{3r_9\cdot k_1 r_0\cdot k_3}\Big[& \dot{G}_{B21}\dot{G}_{B23} r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_2 \\
& -\dot{G}_{B13}\dot{G}_{B23}\left(r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_3 + 2r_9\cdot f_1\cdot f_3\cdot r_0 k_3\cdot k_2\right) \\
& -\dot{G}_{B12}\dot{G}_{B13}\left(r_0\cdot f_3\cdot k_1 r_9\cdot f_1\cdot k_2 - r_9\cdot f_1\cdot f_3\cdot r_0 k_1\cdot k_2\right)\Big],
\end{aligned}
\tag{D.146}
$$

$$
\begin{aligned}
I^2_{(4)}(34) = -\tfrac{Z_2(34)}{3r_{01}\cdot k_1 r_{02}\cdot k_2}\Big[& \dot{G}_{B13}\dot{G}_{B23} r_{02}\cdot f_2\cdot k_3 r_{01}\cdot f_1\cdot k_3 \\
& -\dot{G}_{B21}\dot{G}_{B23}\left(r_{02}\cdot f_2\cdot k_3 r_{01}\cdot f_1\cdot k_2 + 2r_{01}\cdot f_1\cdot f_2\cdot r_{02} k_2\cdot k_3\right) \\
& -\dot{G}_{B12}\dot{G}_{B13}\left(r_{02}\cdot f_2\cdot k_1 r_{01}\cdot f_1\cdot k_3 - r_{01}\cdot f_1\cdot f_2\cdot r_{02} k_1\cdot k_3\right)\Big],
\end{aligned}
\tag{D.147}
$$

$$
I^{22}_{(4)}(12,34) = \frac{1}{3}\dot{G}^2_{B12} Z_2(12) Z_2(34), \tag{D.148}
$$

$$
I^{22}_{(4)}(13,24) = \frac{1}{3}\dot{G}^2_{B31} Z_2(13) Z_2(24), \tag{D.149}
$$

$$
I^{22}_{(4)}(14,23) = \frac{1}{3}\dot{G}^2_{B23} Z_2(14) Z_2(23). \tag{D.150}
$$

And now, we simply use (B.1) and substitute (D.134)-(D.136) in (D.137)-(D.150), where it is needed, and we restore the "T" factors

$$
\begin{aligned}
I^4_{(4)}(1234) &= \left(2G_{B21}\tfrac{k_2\cdot k_1}{k_2\cdot k_3}+2G_{B23}\tfrac{k_2\cdot k_3}{k_2\cdot k_1}+b\right)\left(\tfrac{1}{3}-2G_{B31}\right)Z_4(1234)\\
I^4_{(4)}(1243) &= \left(2G_{B12}\tfrac{k_1\cdot k_2}{k_1\cdot k_3}+2G_{B13}\tfrac{k_1\cdot k_3}{k_1\cdot k_2}+a\right)\left(\tfrac{1}{3}-2G_{B23}\right)Z_4(1243)\\
I^4_{(4)}(1324) &= \left(2G_{B13}\tfrac{k_1\cdot k_3}{k_2\cdot k_3}+2G_{B23}\tfrac{k_2\cdot k_3}{k_1\cdot k_3}+c\right)\left(\tfrac{1}{3}-2G_{B12}\right)Z_4(1324),
\end{aligned}
\tag{D.151}
$$

$$
\begin{aligned}
I^3_{(4)}(123) = \tfrac{-T}{3}Z_3(123)\Big[&\left(2G_{B13}\tfrac{k_1\cdot k_3}{k_2\cdot k_3}+2G_{B23}\tfrac{k_2\cdot k_3}{k_1\cdot k_3}+c\right)(1-4G_{B12})\,G_{B12}k_2\cdot f_4\cdot k_1\\
&+\left(2G_{B12}\tfrac{k_1\cdot k_2}{k_1\cdot k_3}+2G_{B13}\tfrac{k_1\cdot k_3}{k_1\cdot k_2}+a\right)(1-4G_{B23})\,G_{B23}k_3\cdot f_4\cdot k_2\\
&+\left(2G_{B21}\tfrac{k_2\cdot k_1}{k_2\cdot k_3}+2G_{B23}\tfrac{k_2\cdot k_3}{k_2\cdot k_1}+b\right)(1-4G_{B31})\,G_{B31}k_1\cdot f_4\cdot k_3\Big],
\end{aligned}
\tag{D.152}
$$

$$
\begin{aligned}
I^3_{(4)}(234) = \left(\tfrac{1}{3}-2G_{B23}\right)Z_3(234)\Big(&\left(2G_{B21}\tfrac{k_2\cdot k_1}{k_2\cdot k_3}+2G_{B23}\tfrac{k_2\cdot k_3}{k_2\cdot k_1}+b\right)\tfrac{r_1\cdot f_1\cdot k_2}{r_1\cdot k_1}\\
&-\left(2G_{B13}\tfrac{k_1\cdot k_3}{k_2\cdot k_3}+2G_{B23}\tfrac{k_2\cdot k_3}{k_1\cdot k_3}+c\right)\tfrac{r_1\cdot f_1\cdot k_3}{r_1\cdot k_1}\Big),
\end{aligned}
\tag{D.153}
$$

$$
\begin{aligned}
I^3_{(4)}(341) = \left(\tfrac{1}{3}-2G_{B31}\right)Z_3(341)\Big[&\left(2G_{B12}\tfrac{k_1\cdot k_2}{k_1\cdot k_3}+2G_{B13}\tfrac{k_1\cdot k_3}{k_1\cdot k_2}+a\right)\tfrac{r_2\cdot f_2\cdot k_1}{r_2\cdot k_2}\\
&-\left(2G_{B13}\tfrac{k_1\cdot k_3}{k_2\cdot k_3}+2G_{B23}\tfrac{k_2\cdot k_3}{k_1\cdot k_3}+c\right)\tfrac{r_2\cdot f_2\cdot k_3}{r_2\cdot k_2}\Big],
\end{aligned}
\tag{D.154}
$$

$$
\begin{aligned}
I^3_{(4)}(412) = \left(\tfrac{1}{3}-2G_{B12}\right)Z_3(412)\Big[&\left(2G_{B12}\tfrac{k_1\cdot k_2}{k_1\cdot k_3}+2G_{B13}\tfrac{k_1\cdot k_3}{k_1\cdot k_2}+a\right)\tfrac{r_3\cdot f_3\cdot k_1}{r_3\cdot k_3}\\
&-\left(2G_{B21}\tfrac{k_2\cdot k_1}{k_2\cdot k_3}+2G_{B23}\tfrac{k_2\cdot k_3}{k_2\cdot k_1}+b\right)\tfrac{r_3\cdot f_3\cdot k_2}{r_3\cdot k_3}\Big].
\end{aligned}
\tag{D.155}
$$

$$
\begin{aligned}
I^2_{(4)}(12) = \tfrac{-T}{3}(1-4G_{B12})\,Z_2(12)\Big\{&\\
&+(1-4G_{B31})\,G_{B31}k_1\cdot f_4\cdot f_3\cdot k_1+(1-4G_{B23})\,G_{B23}k_2\cdot f_4\cdot f_3\cdot k_2\\
&+\left(2G_{B13}\tfrac{k_1\cdot k_3}{k_2\cdot k_3}+2G_{B23}\tfrac{k_2\cdot k_3}{k_1\cdot k_3}+c\right)[G_{B31}k_1\cdot f_4\cdot f_3\cdot k_2+G_{B23}k_2\cdot f_4\cdot f_3\cdot k_1]\\
&-\left(2G_{B12}\tfrac{k_1\cdot k_2}{k_1\cdot k_3}+2G_{B13}\tfrac{k_1\cdot k_3}{k_1\cdot k_2}+a\right)G_{B12}\tfrac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_1}{r_4\cdot k_3}\\
&+\left(2G_{B21}\tfrac{k_2\cdot k_1}{k_2\cdot k_3}+2G_{B23}\tfrac{k_2\cdot k_3}{k_2\cdot k_1}+b\right)G_{B12}\tfrac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_2}{r_4\cdot k_3}\Big\},
\end{aligned}
\tag{D.156}
$$

$$
\begin{aligned}
I^2_{(4)}(13) = {} & \tfrac{-T}{3}\,(1-4G_{B31})\,Z_2(13)\Big\{ \\
& + (1-4G_{B12})\,G_{B12} k_1\cdot f_4\cdot f_2\cdot k_1 + (1-4G_{B23})\,G_{B23} k_3\cdot f_4\cdot f_2\cdot k_3 \\
& + \left(2G_{B21}\tfrac{k_2\cdot k_1}{k_2\cdot k_3} + 2G_{B23}\tfrac{k_2\cdot k_3}{k_2\cdot k_1} + b\right)\left[G_{B12} k_1\cdot f_4\cdot f_2\cdot k_3 + G_{B23} k_3\cdot f_4\cdot f_2\cdot k_1\right] \\
& - \left(2G_{B12}\tfrac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13}\tfrac{k_1\cdot k_3}{k_1\cdot k_2} + a\right) G_{B31}\tfrac{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_1}{r_5\cdot k_2} \\
& + \left(2G_{B13}\tfrac{k_1\cdot k_3}{k_2\cdot k_3} + 2G_{B23}\tfrac{k_2\cdot k_3}{k_1\cdot k_3} + c\right) G_{B31}\tfrac{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_3}{r_5\cdot k_2}\Big\},
\end{aligned}
\tag{D.157}
$$

$$
\begin{aligned}
I^2_{(4)}(23) = {} & \tfrac{-T}{3}\,(1-4G_{B23})\,Z_2(23)\Big\{ \\
& + (1-4G_{B12})\,G_{B12} k_2\cdot f_4\cdot f_1\cdot k_2 + (1-4G_{B31})\,G_{B31} k_3\cdot f_4\cdot f_1\cdot k_3 \\
& + \left(2G_{B12}\tfrac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13}\tfrac{k_1\cdot k_3}{k_1\cdot k_2} + a\right)\left[G_{B12} k_2\cdot f_4\cdot f_1\cdot k_3 + G_{B31} k_3\cdot f_4\cdot f_1\cdot k_2\right] \\
& - \left(2G_{B21}\tfrac{k_2\cdot k_1}{k_2\cdot k_3} + 2G_{B23}\tfrac{k_2\cdot k_3}{k_2\cdot k_1} + b\right) G_{B23}\tfrac{k_3\cdot f_4\cdot k_2 r_6\cdot f_1\cdot k_2}{r_6\cdot k_1} \\
& + \left(2G_{B13}\tfrac{k_1\cdot k_3}{k_2\cdot k_3} + 2G_{B23}\tfrac{k_2\cdot k_3}{k_1\cdot k_3} + c\right) G_{B23}\tfrac{k_3\cdot f_4\cdot k_2 r_6\cdot f_1\cdot k_3}{r_6\cdot k_1}\Big\},
\end{aligned}
\tag{D.158}
$$

$$
\begin{aligned}
I^2_{(4)}(14) = {} & -\tfrac{Z_2(14)}{3r_7\cdot k_2 r_8\cdot k_3}\Big[\left(2G_{B12}\tfrac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13}\tfrac{k_1\cdot k_3}{k_1\cdot k_2} + a\right) r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_1 \\
& - \left(2G_{B13}\tfrac{k_1\cdot k_3}{k_2\cdot k_3} + 2G_{B23}\tfrac{k_2\cdot k_3}{k_1\cdot k_3} + c\right)\left(r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_3 + 2r_7\cdot f_2\cdot f_3\cdot r_8 k_3\cdot k_1\right) \\
& - \left(2G_{B21}\tfrac{k_2\cdot k_1}{k_2\cdot k_3} + 2G_{B23}\tfrac{k_2\cdot k_3}{k_2\cdot k_1} + b\right)\left(r_8\cdot f_3\cdot k_2 r_7\cdot f_2\cdot k_1 - r_7\cdot f_2\cdot f_3\cdot r_8 k_2\cdot k_1\right)\Big],
\end{aligned}
\tag{D.159}
$$

$$
\begin{aligned}
I^2_{(4)}(24) = {} & -\tfrac{Z_2(24)}{3r_9\cdot k_1 r_0\cdot k_3}\Big[\left(2G_{B21}\tfrac{k_2\cdot k_1}{k_2\cdot k_3} + 2G_{B23}\tfrac{k_2\cdot k_3}{k_2\cdot k_1} + b\right) r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_2 \\
& - \left(2G_{B13}\tfrac{k_1\cdot k_3}{k_2\cdot k_3} + 2G_{B23}\tfrac{k_2\cdot k_3}{k_1\cdot k_3} + c\right)\left(r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_3 + 2r_9\cdot f_1\cdot f_3\cdot r_0 k_3\cdot k_2\right) \\
& - \left(2G_{B12}\tfrac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13}\tfrac{k_1\cdot k_3}{k_1\cdot k_2} + a\right)\left(r_0\cdot f_3\cdot k_1 r_9\cdot f_1\cdot k_2 - r_9\cdot f_1\cdot f_3\cdot r_0 k_1\cdot k_2\right)\Big],
\end{aligned}
\tag{D.160}
$$

$$
\begin{aligned}
I^2_{(4)}(34) = {} & -\tfrac{Z_2(34)}{3r_{01}\cdot k_1 r_{02}\cdot k_2}\Big[\left(2G_{B13}\tfrac{k_1\cdot k_3}{k_2\cdot k_3} + 2G_{B23}\tfrac{k_2\cdot k_3}{k_1\cdot k_3} + c\right) r_{02}\cdot f_2\cdot k_3 r_{01}\cdot f_1\cdot k_3 \\
& - \left(2G_{B21}\tfrac{k_2\cdot k_1}{k_2\cdot k_3} + 2G_{B23}\tfrac{k_2\cdot k_3}{k_2\cdot k_1} + b\right)\left(r_{02}\cdot f_2\cdot k_3 r_{01}\cdot f_1\cdot k_2 + 2r_{01}\cdot f_1\cdot f_2\cdot r_{02} k_2\cdot k_3\right) \\
& - \left(2G_{B12}\tfrac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13}\tfrac{k_1\cdot k_3}{k_1\cdot k_2} + a\right)\left(r_{02}\cdot f_2\cdot k_1 r_{01}\cdot f_1\cdot k_3 - r_{01}\cdot f_1\cdot f_2\cdot r_{02} k_1\cdot k_3\right)\Big],
\end{aligned}
\tag{D.161}
$$

$$
I^{22}_{(4)}(12,34) = \frac{1}{3}\,(1-4G_{B12})\,Z_2(12)Z_2(34), \tag{D.162}
$$

$$
I^{22}_{(4)}(13,24) = \frac{1}{3}\,(1-4G_{B31})\,Z_2(13)Z_2(24), \tag{D.163}
$$

$$I^{22}_{(4)}(14,23) = \frac{1}{3}\left(1 - 4G_{B23}\right) Z_2(14) Z_2(23). \tag{D.164}$$

D.2 Two low energy photons

Since we have already integrated u_4 with the assumption that k_4 is low energy, now we do the same for k_3. Then, we cancel all the terms $O(k_3^2)$ in $I_{(4)}$ (in terms of $\dot{G}_{Bij}$ and with $T = 1$)

$$\begin{aligned}
I^4_{(4)}(1234) &= \dot{G}_{B12}\dot{G}_{B23}\left(\tfrac{1}{6} - \tfrac{1}{2}\dot{G}^2_{B31}\right) Z_4(1234),\\
I^4_{(4)}(1243) &= \dot{G}_{B12}\dot{G}_{B31}\left(\tfrac{1}{6} - \tfrac{1}{2}\dot{G}^2_{B23}\right) Z_4(1243),\\
I^4_{(4)}(1324) &= \dot{G}_{B23}\dot{G}_{B31}\left(\tfrac{1}{6} - \tfrac{1}{2}\dot{G}^2_{B12}\right) Z_4(1324),
\end{aligned} \tag{D.165}$$

$$\begin{aligned}
I^3_{(4)}(123) = \tfrac{1}{12}\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31} Z_3(123) \Big[&\dot{G}_{B12}\left(1 - \dot{G}^2_{B12}\right) k_2 \cdot f_4 \cdot k_1\\
&+ \dot{G}_{B23}\left(1 - \dot{G}^2_{B23}\right) \cancel{k_3 \cdot f_4 \cdot k_2} + \dot{G}_{B31}\left(1 - \dot{G}^2_{B31}\right) \cancel{k_1 \cdot f_4 \cdot k_3}\Big],
\end{aligned} \tag{D.166}$$

$$I^3_{(4)}(234) = \frac{1}{2}\dot{G}_{B23}\left(\frac{1}{3} - \dot{G}^2_{B23}\right) Z_3(234)\left(\dot{G}_{B12}\frac{r_1 \cdot f_1 \cdot k_2}{r_1 \cdot k_1} - \dot{G}_{B31}\frac{\cancel{r_1 \cdot f_1 \cdot k_3}}{r_1 \cdot k_1}\right), \tag{D.167}$$

$$I^3_{(4)}(341) = \frac{1}{2}\left(\frac{1}{3} - \dot{G}^2_{B31}\right)\dot{G}_{B31} Z_3(341)\left(\dot{G}_{B12}\frac{r_2 \cdot f_2 \cdot k_1}{r_2 \cdot k_2} - \dot{G}_{B23}\frac{\cancel{r_2 \cdot f_2 \cdot k_3}}{r_2 \cdot k_2}\right), \tag{D.168}$$

$$I^3_{(4)}(412) = \frac{1}{2}\left(\frac{1}{3} - \dot{G}^2_{B12}\right)\dot{G}_{B12} Z_3(412)\left(\dot{G}_{B31}\frac{r_3 \cdot f_3 \cdot k_1}{r_3 \cdot k_3} - \dot{G}_{B23}\frac{r_3 \cdot f_3 \cdot k_2}{r_3 \cdot k_3}\right), \tag{D.169}$$

$$\begin{aligned}
I^2_{(4)}(12) = \tfrac{-1}{12}\dot{G}^2_{B12} Z_2(12)\Big\{&\\
&+\dot{G}^2_{B31}\left(1 - \dot{G}^2_{B31}\right) k_1 \cdot f_4 \cdot f_3 \cdot k_1 - \dot{G}_{B23}\dot{G}_{B31}\left(1 - \dot{G}^2_{B31}\right) k_1 \cdot f_4 \cdot f_3 \cdot k_2\\
&-\dot{G}_{B23}\dot{G}_{B31}\left(1 - \dot{G}^2_{B23}\right) k_2 \cdot f_4 \cdot f_3 \cdot k_1 + \dot{G}^2_{B23}\left(1 - \dot{G}^2_{B23}\right) k_2 \cdot f_4 \cdot f_3 \cdot k_2\\
&+\dot{G}_{B12}\dot{G}_{B31}\left(1 - \dot{G}^2_{B12}\right)\tfrac{k_2 \cdot f_4 \cdot k_1 r_4 \cdot f_3 \cdot k_1}{r_4 \cdot k_3} - \dot{G}_{B12}\dot{G}_{B23}\left(1 - \dot{G}^2_{B12}\right)\tfrac{k_2 \cdot f_4 \cdot k_1 r_4 \cdot f_3 \cdot k_2}{r_4 \cdot k_3}\Big\},
\end{aligned} \tag{D.170}$$

$$I^2_{(4)}(13) = \tfrac{-1}{12}\dot{G}^2_{B31}Z_2(13)\Big\{ \\ +\dot{G}^2_{B12}\left(1-\dot{G}^2_{B12}\right)k_1\cdot f_4\cdot f_2\cdot k_1 - \dot{G}_{B12}\dot{G}_{B23}\left(1-\dot{G}^2_{B12}\right)\cancel{k_1\cdot f_4\cdot f_2\cdot k_3} \\ -\dot{G}_{B12}\dot{G}_{B23}\left(1-\dot{G}^2_{B23}\right)\cancel{k_3\cdot f_4\cdot f_2\cdot k_1} + \dot{G}^2_{B23}\left(1-\dot{G}^2_{B23}\right)k_3\cdot f_4\cdot f_2\cdot k_3 \\ +\dot{G}_{B12}\dot{G}_{B31}\left(1-\dot{G}^2_{B31}\right)\cancel{\tfrac{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_1}{r_5\cdot k_2}} - \dot{G}_{B23}\dot{G}_{B31}\left(1-\dot{G}^2_{B31}\right)\cancel{\tfrac{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_3}{r_5\cdot k_2}}\Big\}, \tag{D.171}$$

$$I^2_{(4)}(23) = \tfrac{-1}{12}\dot{G}^2_{B23}Z_2(23)\Big\{ \\ +\dot{G}^2_{B12}\left(1-\dot{G}^2_{B12}\right)k_2\cdot f_4\cdot f_1\cdot k_2 - \dot{G}_{B12}\dot{G}_{B31}\left(1-\dot{G}^2_{B12}\right)\cancel{k_2\cdot f_4\cdot f_1\cdot k_3} \\ -\dot{G}_{B12}\dot{G}_{B31}\left(1-\dot{G}^2_{B31}\right)\cancel{k_3\cdot f_4\cdot f_1\cdot k_2} + \dot{G}^2_{B31}\left(1-\dot{G}^2_{B31}\right)k_3\cdot f_4\cdot f_1\cdot k_3 \\ +\dot{G}_{B12}\dot{G}_{B23}\left(1-\dot{G}^2_{B23}\right)\cancel{\tfrac{k_3\cdot f_4\cdot k_2 r_6\cdot f_1\cdot k_2}{r_6\cdot k_1}} - \dot{G}_{B23}\dot{G}_{B31}\left(1-\dot{G}^2_{B23}\right)\cancel{\tfrac{k_3\cdot f_4\cdot k_2 r_6\cdot f_1\cdot k_3}{r_6\cdot k_1}}\Big\}, \tag{D.172}$$

$$I^2_{(4)}(14) = -\tfrac{Z_2(14)}{3r_7\cdot k_2 r_8\cdot k_3}\Big[-\dot{G}_{B12}\dot{G}_{B31}r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_1 \\ +\dot{G}_{B23}\dot{G}_{B31}\left(r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_3 + 2r_7\cdot f_2\cdot f_3\cdot r_8 k_3\cdot k_1\right) \\ +\dot{G}_{B12}\dot{G}_{B23}\left(r_8\cdot f_3\cdot k_2 r_7\cdot f_2\cdot k_1 - r_7\cdot f_2\cdot f_3\cdot r_8 k_2\cdot k_1\right)\Big], \tag{D.173}$$

$$I^2_{(4)}(24) = -\tfrac{Z_2(24)}{3r_9\cdot k_1 r_0\cdot k_3}\Big[-\dot{G}_{B12}\dot{G}_{B23}r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_2 \\ +\dot{G}_{B23}\dot{G}_{B31}\left(r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_3 + 2r_9\cdot f_1\cdot f_3\cdot r_0 k_3\cdot k_2\right) \\ +\dot{G}_{B12}\dot{G}_{B31}\left(r_0\cdot f_3\cdot k_1 r_9\cdot f_1\cdot k_2 - r_9\cdot f_1\cdot f_3\cdot r_0 k_1\cdot k_2\right)\Big], \tag{D.174}$$

$$I^2_{(4)}(34) = -\tfrac{Z_2(34)}{3r_{01}\cdot k_1 r_{02}\cdot k_2}\Big[-\dot{G}_{B23}\dot{G}_{B31}\cancel{r_{02}\cdot f_2\cdot k_3 r_{01}\cdot f_1\cdot k_3} \\ +\dot{G}_{B12}\dot{G}_{B23}\left(\cancel{r_{02}\cdot f_2\cdot k_3 r_{01}\cdot f_1\cdot k_2} + \cancel{2r_{01}\cdot f_1\cdot f_2\cdot r_{02}k_2\cdot k_3}\right) \\ +\dot{G}_{B12}\dot{G}_{B31}\left(r_{02}\cdot f_2\cdot k_1\cancel{r_{01}\cdot f_1\cdot k_3} - r_{01}\cdot f_1\cdot f_2\cdot r_{02}\cancel{k_1\cdot k_3}\right)\Big], \tag{D.175}$$

$$I^{22}_{(4)}(12,34) = \frac{1}{3}\dot{G}^2_{B12}Z_2(12)Z_2(34), \tag{D.176}$$

$$I^{22}_{(4)}(13,24) = \frac{1}{3}\dot{G}^2_{B31}Z_2(13)Z_2(24), \tag{D.177}$$

$$I^{22}_{(4)}(14,23) = \frac{1}{3}\dot{G}^2_{B23}Z_2(14)Z_2(23). \tag{D.178}$$

This leaves us with the following terms to be integrated

$$\begin{aligned} I^4_{(34)}(1234) &= \textstyle\int_0^1 \dot{G}_{B12}\dot{G}_{B23}\left(\frac{1}{6}-\frac{1}{2}\dot{G}^2_{B31}\right) Z_4(1234)du_3, \\ I^4_{(34)}(1243) &= \textstyle\int_0^1 \dot{G}_{B12}\dot{G}_{B31}\left(\frac{1}{6}-\frac{1}{2}\dot{G}^2_{B23}\right) Z_4(1243)du_3, \\ I^4_{(34)}(1324) &= \textstyle\int_0^1 \dot{G}_{B23}\dot{G}_{B31}\left(\frac{1}{6}-\frac{1}{2}\dot{G}^2_{B12}\right) Z_4(1324)du_3, \end{aligned} \tag{D.179}$$

$$I^3_{(34)}(123) = \frac{1}{12}\int_0^1 \dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31}Z_3(123)\dot{G}_{B12}\left(1-\dot{G}^2_{B12}\right) k_2\cdot f_4\cdot k_1 du_3, \tag{D.180}$$

$$I^3_{(34)}(234) = \frac{1}{2}\int_0^1 \dot{G}_{B23}\left(\cancel{\frac{1}{3}}-\cancel{\dot{G}^2_{B23}}\right) Z_3(234)\dot{G}_{B12}\frac{r_1\cdot f_1\cdot k_2}{r_1\cdot k_1}du_3, \tag{D.181}$$

$$I^3_{(34)}(341) = \frac{1}{2}\int_0^1 \left(\cancel{\frac{1}{3}}-\cancel{\dot{G}^2_{B31}}\right) \dot{G}_{B31}Z_3(341)\dot{G}_{B12}\frac{r_2\cdot f_2\cdot k_1}{r_2\cdot k_2}du_3, \tag{D.182}$$

$$\begin{aligned} I^3_{(34)}(412) =& \frac{1}{2}\int_0^1 \left(\frac{1}{3}-\dot{G}^2_{B12}\right)\dot{G}_{B12}Z_3(412) \\ &\times\left(\dot{G}_{B31}\frac{r_3\cdot f_3\cdot k_1}{r_3\cdot k_3}-\dot{G}_{B23}\frac{r_3\cdot f_3\cdot k_2}{r_3\cdot k_3}\right)e^{(\cdot 3)}du_3, \end{aligned} \tag{D.183}$$

$$\begin{aligned} &I^2_{(34)}(12) = \tfrac{-1}{12}\textstyle\int_0^1 \dot{G}^2_{B12}Z_2(12)\Big\{ \\ &+\dot{G}^2_{B31}\left(1-\dot{G}^2_{B31}\right)k_1\cdot f_4\cdot f_3\cdot k_1 - \dot{G}_{B23}\dot{G}_{B31}\left(1-\dot{G}^2_{B31}\right)k_1\cdot f_4\cdot f_3\cdot k_2 \\ &-\dot{G}_{B23}\dot{G}_{B31}\left(1-\dot{G}^2_{B23}\right)k_2\cdot f_4\cdot f_3\cdot k_1 + \dot{G}^2_{B23}\left(1-\dot{G}^2_{B23}\right)k_2\cdot f_4\cdot f_3\cdot k_2 \\ &+\Big[\dot{G}_{B12}\dot{G}_{B31}\left(1-\dot{G}^2_{B12}\right)\tfrac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_1}{r_4\cdot k_3} \\ &\quad -\dot{G}_{B12}\dot{G}_{B23}\left(1-\dot{G}^2_{B12}\right)\tfrac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_2}{r_4\cdot k_3}\Big]e^{(\cdot 3)}\Big\}du_3, \end{aligned} \tag{D.184}$$

$$I^2_{(34)}(13) = \frac{-1}{12}\int_0^1 \dot{G}^2_{B31}Z_2(13)\dot{G}^2_{B12}\left(1-\dot{G}^2_{B12}\right)k_1\cdot f_4\cdot f_2\cdot k_1 du_3, \tag{D.185}$$

$$I^2_{(34)}(23) = \frac{-1}{12}\int_0^1 \dot{G}^2_{B23}Z_2(23)\dot{G}^2_{B12}\left(1-\dot{G}^2_{B12}\right)k_2\cdot f_4\cdot f_1\cdot k_2 du_3, \tag{D.186}$$

$$\begin{aligned} I^2_{(34)}(14) =& -\tfrac{Z_2(14)}{3r_7\cdot k_2 r_8\cdot k_3}\textstyle\int_0^1\Big\{\Big[-\dot{G}_{B12}\dot{G}_{B31}r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_1 \\ &+\dot{G}_{B12}\dot{G}_{B23}\left(r_8\cdot f_3\cdot k_2 r_7\cdot f_2\cdot k_1 - r_7\cdot f_2\cdot f_3\cdot r_8 k_2\cdot k_1\right)\Big]e^{(\cdot 3)} \\ &+\dot{G}_{B23}\dot{G}_{B31}\left(r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_3 + 2r_7\cdot f_2\cdot f_3\cdot r_8 k_3\cdot k_1\right)\Big\}du_3, \end{aligned} \tag{D.187}$$

$$I^2_{(34)}(24) = -\tfrac{Z_2(24)}{3r_9\cdot k_1 r_0\cdot k_3}\int_0^1 \Big\{\Big[-\dot G_{B12}\dot G_{B23} r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_2 \\ +\dot G_{B12}\dot G_{B31}\left(r_0\cdot f_3\cdot k_1 r_9\cdot f_1\cdot k_2 - r_9\cdot f_1\cdot f_3\cdot r_0 k_1\cdot k_2\right)\Big] e^{(\cdot 3)} \\ +\dot G_{B23}\dot G_{B31}\left(r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_3 + 2 r_9\cdot f_1\cdot f_3\cdot r_0 k_3\cdot k_2\right)\Big\} du_3, \tag{D.188}$$

$$I^{22}_{(34)}(12,34) = \frac{1}{3}\int_0^1 \dot G^2_{B12} Z_2(12) Z_2(34) du_3, \tag{D.189}$$

$$I^{22}_{(34)}(13,24) = \frac{1}{3}\int_0^1 \dot G^2_{B31} Z_2(13) Z_2(24) du_3, \tag{D.190}$$

$$I^{22}_{(34)}(14,23) = \frac{1}{3}\int_0^1 \dot G^2_{B23} Z_2(14) Z_2(23) du_3. \tag{D.191}$$

Some terms cancel because of the integration. We use some particular cases of (B.11) enlisted in appendix B in order to perform the integrals over u_3, and we obtain

$$\begin{aligned} I^4_{(34)}(1234) &= \tfrac{1}{2}\dot G_{B12}\tfrac{1}{3}\left(\dot G_{B12} - \dot G^3_{B12}\right) Z_4(1234), \\ I^4_{(34)}(1243) &= \tfrac{1}{2}\dot G_{B12}\tfrac{1}{3}\left(\dot G_{B12} - \dot G^3_{B12}\right) Z_4(1243), \\ I^4_{(34)}(1324) &= \left(\tfrac{1}{6} - \tfrac{1}{2}\dot G^2_{B12}\right)\left(\tfrac{1}{6} - \tfrac{1}{2}\dot G^2_{B21}\right) Z_4(1324), \end{aligned} \tag{D.192}$$

$$I^3_{(34)}(123) = \frac{1}{12}\left(\dot G^2_{B12} - \dot G^4_{B12}\right)\left(\frac{1}{6} - \frac{1}{2}\dot G^2_{B12}\right) Z_3(123) k_2\cdot f_4\cdot k_1, \tag{D.193}$$

$$I^3_{(34)}(412) = \tfrac{1}{2}\left(\tfrac{1}{3} - \dot G^2_{B12}\right)\dot G_{B12} Z_3(412)\Big[\tfrac{k_2\cdot k_3}{12}\left(\dot G_{B12} - \dot G^3_{B12}\right)\tfrac{r_3\cdot f_3\cdot k_1}{r_3\cdot k_3} \\ -\tfrac{k_3\cdot k_1}{12}\left(\dot G_{B12} - \dot G^3_{B12}\right)\tfrac{r_3\cdot f_3\cdot k_2}{r_3\cdot k_3}\Big], \tag{D.194}$$

$$I^2_{(34)}(12) = \tfrac{-1}{12}\dot G^2_{B12} Z_2(12)\Big\{ \\ +\left(\tfrac{1}{3} - \tfrac{1}{5}\right) k_1\cdot f_4\cdot f_3\cdot k_1 - \left[\left(\tfrac{1}{6} - \tfrac{1}{2}\dot G^2_{B12}\right) - \left(\tfrac{3!}{5!} - \tfrac{1}{4}\dot G^4_{B12}\right)\right] k_1\cdot f_4\cdot f_3\cdot k_2 \\ -\left[\left(\tfrac{1}{6} - \tfrac{1}{2}\dot G^2_{B12}\right) - \left(\tfrac{3!}{5!} - \tfrac{1}{4}\dot G^4_{B12}\right)\right] k_2\cdot f_4\cdot f_3\cdot k_1 + \left(\tfrac{1}{3} - \tfrac{1}{5}\right) k_2\cdot f_4\cdot f_3\cdot k_2 \\ +\Big[\dot G_{B12}\tfrac{k_2\cdot k_3}{12}\left(\dot G_{B12} - \dot G^3_{B12}\right)\left(1 - \dot G^2_{B12}\right)\tfrac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_1}{r_4\cdot k_3} \\ -\dot G_{B12}\tfrac{k_3\cdot k_1}{12}\left(\dot G_{B12} - \dot G^3_{B12}\right)\left(1 - \dot G^2_{B12}\right)\tfrac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_2}{r_4\cdot k_3}\Big]\Big\}, \tag{D.195}$$

$$I^2_{(34)}(13) = \frac{-1}{36} Z_2(13) \dot{G}^2_{B12} \left(1 - \dot{G}^2_{B12}\right) k_1 \cdot f_4 \cdot f_2 \cdot k_1, \tag{D.196}$$

$$I^2_{(34)}(23) = \frac{-1}{36} Z_2(23) \dot{G}^2_{B12} \left(1 - \dot{G}^2_{B12}\right) k_2 \cdot f_4 \cdot f_1 \cdot k_2, \tag{D.197}$$

$$\begin{aligned} I^2_{(34)}(14) = -\tfrac{Z_2(14)}{3r_7 \cdot k_2 r_8 \cdot k_3} \Big[& - \dot{G}_{B12} \tfrac{k_2 \cdot k_3}{12} \left(\dot{G}_{B12} - \dot{G}^3_{B12}\right) r_8 \cdot f_3 \cdot k_1 r_7 \cdot f_2 \cdot k_1 \\ & + \dot{G}_{B12} \tfrac{k_3 \cdot k_1}{12} \left(\dot{G}_{B12} - \dot{G}^3_{B12}\right) (r_8 \cdot f_3 \cdot k_2 r_7 \cdot f_2 \cdot k_1 - r_7 \cdot f_2 \cdot f_3 \cdot r_8 k_2 \cdot k_1) \\ & + \left(\tfrac{1}{6} - \tfrac{1}{2} \dot{G}^2_{B12}\right) (r_8 \cdot f_3 \cdot k_1 r_7 \cdot f_2 \cdot k_3 + 2 r_7 \cdot f_2 \cdot f_3 \cdot r_8 k_3 \cdot k_1) \Big], \end{aligned} \tag{D.198}$$

$$\begin{aligned} I^2_{(34)}(24) = -\tfrac{Z_2(24)}{3r_9 \cdot k_1 r_0 \cdot k_3} \Big[& - \dot{G}_{B12} \tfrac{k_3 \cdot k_1}{12} \left(\dot{G}_{B12} - \dot{G}^3_{B12}\right) r_0 \cdot f_3 \cdot k_2 r_9 \cdot f_1 \cdot k_2 \\ & + \dot{G}_{B12} \tfrac{k_2 \cdot k_3}{12} \left(\dot{G}_{B12} - \dot{G}^3_{B12}\right) (r_0 \cdot f_3 \cdot k_1 r_9 \cdot f_1 \cdot k_2 - r_9 \cdot f_1 \cdot f_3 \cdot r_0 k_1 \cdot k_2) \\ & + \left(\tfrac{1}{6} - \tfrac{1}{2} \dot{G}^2_{B12}\right) (r_0 \cdot f_3 \cdot k_2 r_9 \cdot f_1 \cdot k_3 + 2 r_9 \cdot f_1 \cdot f_3 \cdot r_0 k_3 \cdot k_2) \Big], \end{aligned} \tag{D.199}$$

$$I^{22}_{(34)}(12, 34) = \frac{1}{3} \dot{G}^2_{B12} Z_2(12) Z_2(34), \tag{D.200}$$

$$I^{22}_{(34)}(13, 24) = \frac{1}{9} Z_2(13) Z_2(24), \tag{D.201}$$

$$I^{22}_{(34)}(14, 23) = \frac{1}{9} Z_2(14) Z_2(23). \tag{D.202}$$

Making simpler

$$\begin{aligned} I^4_{(34)}(1234) &= \tfrac{1}{6} \left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) Z_4(1234), \\ I^4_{(34)}(1243) &= \tfrac{1}{6} \left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) Z_4(1243), \\ I^4_{(34)}(1324) &= \tfrac{1}{6^2} \left(1 - 6\dot{G}^2_{B12} + 9\dot{G}^4_{B21}\right) Z_4(1324), \end{aligned} \tag{D.203}$$

$$I^3_{(34)}(123) = \frac{1}{72} \left(\dot{G}^2_{B12} - 4\dot{G}^4_{B12} + 3\dot{G}^6_{B12}\right) Z_3(123) k_2 \cdot f_4 \cdot k_1, \tag{D.204}$$

$$\begin{aligned} I^3_{(34)}(412) = & \frac{1}{72} \left(\dot{G}^2_{B12} - 4\dot{G}^4_{B12} + 3\dot{G}^6_{B12}\right) Z_3(412) \\ & \times \left(\frac{k_2 \cdot k_3 r_3 \cdot f_3 \cdot k_1}{r_3 \cdot k_3} - \frac{k_3 \cdot k_1 r_3 \cdot f_3 \cdot k_2}{r_3 \cdot k_3}\right), \end{aligned} \tag{D.205}$$

$$I^2_{(34)}(12) = \tfrac{-1}{12}\dot{G}^2_{B12}Z_2(12)\Big[\tfrac{2}{15}\left(k_1\cdot f_4\cdot f_3\cdot k_1 + k_2\cdot f_4\cdot f_3\cdot k_2\right) - \tfrac{1}{12}\left(\tfrac{7}{5} - 6\dot{G}^2_{B12} + 3\dot{G}^4_{B12}\right)\left(k_1\cdot f_4\cdot f_3\cdot k_2 + k_2\cdot f_4\cdot f_3\cdot k_1\right) + \tfrac{1}{12}\left(\dot{G}^2_{B12} - 2\dot{G}^4_{B12} + \dot{G}^6_{B12}\right) k_2\cdot f_4\cdot k_1\left(\tfrac{k_2\cdot k_3 r_4\cdot f_3\cdot k_1}{r_4\cdot k_3} - \tfrac{k_3\cdot k_1 r_4\cdot f_3\cdot k_2}{r_4\cdot k_3}\right)\Big] \tag{D.206}$$

$$I^2_{(34)}(13) = \frac{-1}{36}Z_2(13)\dot{G}^2_{B12}\left(1-\dot{G}^2_{B12}\right)k_1\cdot f_4\cdot f_2\cdot k_1, \tag{D.207}$$

$$I^2_{(34)}(23) = \frac{-1}{36}Z_2(23)\dot{G}^2_{B12}\left(1-\dot{G}^2_{B12}\right)k_2\cdot f_4\cdot f_1\cdot k_2, \tag{D.208}$$

$$I^2_{(34)}(14) = -\tfrac{Z_2(14)}{36r_7\cdot k_2 r_8\cdot k_3}\Big[-\left(\dot{G}^2_{B12}-\dot{G}^4_{B12}\right)k_2\cdot k_3 r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_1 + \left(\dot{G}^2_{B12}-\dot{G}^4_{B12}\right)k_3\cdot k_1\left(r_8\cdot f_3\cdot k_2 r_7\cdot f_2\cdot k_1 - r_7\cdot f_2\cdot f_3\cdot r_8 k_2\cdot k_1\right) + \left(2-6\dot{G}^2_{B12}\right)\left(r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_3 + 2r_7\cdot f_2\cdot f_3\cdot r_8 k_3\cdot k_1\right)\Big], \tag{D.209}$$

$$I^2_{(34)}(24) = -\tfrac{Z_2(24)}{36r_9\cdot k_1 r_0\cdot k_3}\Big[-\left(\dot{G}^2_{B12}-\dot{G}^4_{B12}\right)k_3\cdot k_1 r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_2 + \left(\dot{G}^2_{B12}-\dot{G}^4_{B12}\right)k_2\cdot k_3\left(r_0\cdot f_3\cdot k_1 r_9\cdot f_1\cdot k_2 - r_9\cdot f_1\cdot f_3\cdot r_0 k_1\cdot k_2\right) + \left(2-6\dot{G}^2_{B12}\right)\left(r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_3 + 2r_9\cdot f_1\cdot f_3\cdot r_0 k_3\cdot k_2\right)\Big], \tag{D.210}$$

$$I^{22}_{(34)}(12,34) = \frac{1}{3}\dot{G}^2_{B12}Z_2(12)Z_2(34), \tag{D.211}$$

$$I^{22}_{(34)}(13,24) = \frac{1}{9}Z_2(13)Z_2(24), \tag{D.212}$$

$$I^{22}_{(34)}(14,23) = \frac{1}{9}Z_2(14)Z_2(23). \tag{D.213}$$

Notice that, in (D.205) and (D.206) we can make the following replacement

$$\frac{k_2\cdot k_3 r\cdot f_3\cdot k_1}{r\cdot k_3} - \frac{k_3\cdot k_1 r\cdot f_3\cdot k_2}{r\cdot k_3} = k_2\cdot f_3\cdot k_1 \tag{D.214}$$

performing this replacement we have

D.2.1 Result of $I_{(34)}$

$$\begin{aligned} I^4_{(34)}(1234) &= \tfrac{1}{6}\left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) Z_4(1234), \\ I^4_{(34)}(1243) &= \tfrac{1}{6}\left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) Z_4(1243), \\ I^4_{(34)}(1324) &= \tfrac{1}{6^2}\left(1 - 6\dot{G}^2_{B12} + 9\dot{G}^4_{B21}\right) Z_4(1324), \end{aligned} \tag{D.215}$$

$$I^3_{(34)}(123) = \frac{1}{72}\left(\dot{G}^2_{B12} - 4\dot{G}^4_{B12} + 3\dot{G}^6_{B12}\right) Z_3(123) k_2 \cdot f_4 \cdot k_1, \tag{D.216}$$

$$I^3_{(34)}(412) = \frac{1}{72}\left(\dot{G}^2_{B12} - 4\dot{G}^4_{B12} + 3\dot{G}^6_{B12}\right) Z_3(412) k_2 \cdot f_3 \cdot k_1, \tag{D.217}$$

$$\begin{aligned} I^2_{(34)}(12) = \tfrac{-1}{12}\dot{G}^2_{B12} Z_2(12)\Big[& \tfrac{2}{15}\left(k_1 \cdot f_4 \cdot f_3 \cdot k_1 + k_2 \cdot f_4 \cdot f_3 \cdot k_2\right) \\ & - \tfrac{1}{12}\left(\tfrac{7}{5} - 6\dot{G}^2_{B12} + 3\dot{G}^4_{B12}\right)\left(k_1 \cdot f_4 \cdot f_3 \cdot k_2 + k_2 \cdot f_4 \cdot f_3 \cdot k_1\right) \\ & + \tfrac{1}{12}\left(\dot{G}^2_{B12} - 2\dot{G}^4_{B12} + \dot{G}^6_{B12}\right) k_2 \cdot f_4 \cdot k_1 k_2 \cdot f_3 \cdot k_1 \Big], \end{aligned} \tag{D.218}$$

$$I^2_{(34)}(13) = \frac{-1}{36} Z_2(13)\dot{G}^2_{B12}\left(1 - \dot{G}^2_{B12}\right) k_1 \cdot f_4 \cdot f_2 \cdot k_1, \tag{D.219}$$

$$I^2_{(34)}(23) = \frac{-1}{36} Z_2(23)\dot{G}^2_{B12}\left(1 - \dot{G}^2_{B12}\right) k_2 \cdot f_4 \cdot f_1 \cdot k_2, \tag{D.220}$$

$$\begin{aligned} I^2_{(34)}(14) = -\tfrac{Z_2(14)}{36 r_7 \cdot k_2 r_8 \cdot k_3}\Big[& -\left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) k_2 \cdot k_3 r_8 \cdot f_3 \cdot k_1 r_7 \cdot f_2 \cdot k_1 \\ & + \left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) k_3 \cdot k_1 \left(r_8 \cdot f_3 \cdot k_2 r_7 \cdot f_2 \cdot k_1 - r_7 \cdot f_2 \cdot f_3 \cdot r_8 k_2 \cdot k_1\right) \\ & + \left(2 - 6\dot{G}^2_{B12}\right)\left(r_8 \cdot f_3 \cdot k_1 r_7 \cdot f_2 \cdot k_3 + 2 r_7 \cdot f_2 \cdot f_3 \cdot r_8 k_3 \cdot k_1\right) \Big], \end{aligned} \tag{D.221}$$

$$\begin{aligned} I^2_{(34)}(24) = -\tfrac{Z_2(24)}{36 r_9 \cdot k_1 r_0 \cdot k_3}\Big[& -\left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) k_3 \cdot k_1 r_0 \cdot f_3 \cdot k_2 r_9 \cdot f_1 \cdot k_2 \\ & + \left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) k_2 \cdot k_3 \left(r_0 \cdot f_3 \cdot k_1 r_9 \cdot f_1 \cdot k_2 - r_9 \cdot f_1 \cdot f_3 \cdot r_0 k_1 \cdot k_2\right) \\ & + \left(2 - 6\dot{G}^2_{B12}\right)\left(r_0 \cdot f_3 \cdot k_2 r_9 \cdot f_1 \cdot k_3 + 2 r_9 \cdot f_1 \cdot f_3 \cdot r_0 k_3 \cdot k_2\right) \Big], \end{aligned} \tag{D.222}$$

$$I^{22}_{(34)}(12,34) = \frac{1}{3}\dot{G}^2_{B12} Z_2(12) Z_2(34), \tag{D.223}$$

$$I^{22}_{(34)}(13,24) = \frac{1}{9} Z_2(13) Z_2(24), \tag{D.224}$$

$$I^{22}_{(34)}(14,23) = \frac{1}{9} Z_2(14) Z_2(23). \tag{D.225}$$

D.2.2 $I_{(34)}$ in terms of G_{Bij} only

As we can see that in chapter 5 we need $I_{(34)}$ in terms of G_{Bij} to perform the remaining integrals. Hence, we use (B.1) to write the equations above in terms of G_{Bij} (we restore the T factors also)

$$\begin{aligned} I^4_{(34)}(1234) &= \tfrac{2}{3}\left(G_{B12} - 4G^2_{B12}\right) Z_4(1234), \\ I^4_{(34)}(1243) &= \tfrac{2}{3}\left(G_{B12} - 4G^2_{B12}\right) Z_4(1243), \\ I^4_{(34)}(1324) &= \tfrac{1}{9}\left(1 - 12G_{B12} + 36G^2_{B21}\right) Z_4(1324), \end{aligned} \tag{D.226}$$

$$I^3_{(34)}(123) = \frac{-T}{9}\left(G_{B12} - 10G^2_{B12} + 24G^3_{B12}\right) Z_3(123) k_2 \cdot f_4 \cdot k_1, \tag{D.227}$$

$$I^3_{(34)}(412) = \frac{-T}{9}\left(G_{B12} - 10G^2_{B12} + 24G^3_{B12}\right) Z_3(412) k_2 \cdot f_3 \cdot k_1, \tag{D.228}$$

$$\begin{aligned} I^2_{(34)}(12) = \tfrac{-T}{90} Z_2(12) \Big[& \left(1 - 4G_{B12}\right)\left(k_1 \cdot f_4 \cdot f_3 \cdot k_1 + k_2 \cdot f_4 \cdot f_3 \cdot k_2\right) \\ & + \left(1 - 4G_{B12} - 30G^2_{B12} + 120G^3_{B12}\right)\left(k_1 \cdot f_4 \cdot f_3 \cdot k_2 + k_2 \cdot f_4 \cdot f_3 \cdot k_1\right) \\ & + T\left(10G^2_{B12} - 80G^3_{B12} + 160G^4_{B12}\right) k_2 \cdot f_4 \cdot k_1 k_2 \cdot f_3 \cdot k_1 \Big], \end{aligned} \tag{D.229}$$

$$I^2_{(34)}(13) = \frac{-T}{9} Z_2(13)\left(G_{B12} - 4G^2_{B12}\right) k_1 \cdot f_4 \cdot f_2 \cdot k_1, \tag{D.230}$$

$$I^2_{(34)}(23) = \frac{-T}{9} Z_2(23)\left(G_{B12} - 4G^2_{B12}\right) k_2 \cdot f_4 \cdot f_1 \cdot k_2, \tag{D.231}$$

$$\begin{aligned} I^2_{(34)}(14) = \tfrac{Z_2(14)}{9 r_7 \cdot k_2 r_8 \cdot k_3} \Big[& T\left(G_{B12} - 4G^2_{B12}\right) k_2 \cdot k_3 r_8 \cdot f_3 \cdot k_1 r_7 \cdot f_2 \cdot k_1 \\ & + T\left(G_{B12} - 4G^2_{B12}\right) k_3 \cdot k_1 \left(r_7 \cdot f_2 \cdot f_3 \cdot r_8 k_2 \cdot k_1 - r_8 \cdot f_3 \cdot k_2 r_7 \cdot f_2 \cdot k_1\right) \\ & + \left(1 - 6G_{B12}\right)\left(r_8 \cdot f_3 \cdot k_1 r_7 \cdot f_2 \cdot k_3 + 2 r_7 \cdot f_2 \cdot f_3 \cdot r_8 k_3 \cdot k_1\right) \Big], \end{aligned} \tag{D.232}$$

$$\begin{aligned} I^2_{(34)}(24) = \tfrac{Z_2(24)}{9 r_9 \cdot k_1 r_0 \cdot k_3} \Big[& T\left(G_{B12} - 4G^2_{B12}\right) k_3 \cdot k_1 r_0 \cdot f_3 \cdot k_2 r_9 \cdot f_1 \cdot k_2 \\ & + T\left(G_{B12} - 4G^2_{B12}\right) k_2 \cdot k_3 \left(r_9 \cdot f_1 \cdot f_3 \cdot r_0 k_1 \cdot k_2 - r_0 \cdot f_3 \cdot k_1 r_9 \cdot f_1 \cdot k_2\right) \\ & + \left(1 - 6G_{B12}\right)\left(r_0 \cdot f_3 \cdot k_2 r_9 \cdot f_1 \cdot k_3 + 2 r_9 \cdot f_1 \cdot f_3 \cdot r_0 k_3 \cdot k_2\right) \Big], \end{aligned} \tag{D.233}$$

$$I^{22}_{(34)}(12,34) = \frac{1}{3}\left(1 - 4G_{B12}\right) Z_2(12) Z_2(34), \tag{D.234}$$

$$I^{22}_{(34)}(13,24) = \frac{1}{9} Z_2(13) Z_2(24), \tag{D.235}$$

$$I^{22}_{(34)}(14,23) = \frac{1}{9} Z_2(14) Z_2(23). \tag{D.236}$$

D.3 Three low energy photons

Here, we take the low energy limit in the leg corresponding to the integral over u_2. First, we cancel all the terms $O(k_2^2)$ in $I_{(34)}$ (in terms of $\dot{G}_{Bij}$)

$$\begin{aligned} I^4_{(34)}(1234) &= \tfrac{1}{6}\left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) Z_4(1234), \\ I^4_{(34)}(1243) &= \tfrac{1}{6}\left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) Z_4(1243), \\ I^4_{(34)}(1324) &= \tfrac{1}{6^2}\left(1 - 6\dot{G}^2_{B12} + 9\dot{G}^4_{B21}\right) Z_4(1324), \end{aligned} \tag{D.237}$$

$$I^3_{(34)}(123) = \frac{1}{72}\left(\dot{G}^2_{B12} - 4\dot{G}^4_{B12} + 3\dot{G}^6_{B12}\right) Z_3(123) k_2 \cdot f_4 \cdot k_1, \tag{D.238}$$

$$I^3_{(34)}(412) = \frac{1}{72}\left(\dot{G}^2_{B12} - 4\dot{G}^4_{B12} + 3\dot{G}^6_{B12}\right) Z_3(412) k_2 \cdot f_3 \cdot k_1, \tag{D.239}$$

$$\begin{aligned} I^2_{(34)}(12) = \tfrac{-1}{12}\dot{G}^2_{B12} Z_2(12) \Big[& \tfrac{2}{15}\left(k_1 \cdot f_4 \cdot f_3 \cdot k_1 + k_2 \cdot f_4 \cdot f_3 \cdot k_2\right) \\ & - \tfrac{1}{12}\left(\tfrac{7}{5} - 6\dot{G}^2_{B12} + 3\dot{G}^4_{B12}\right)\left(k_1 \cdot f_4 \cdot f_3 \cdot k_2 + k_2 \cdot f_4 \cdot f_3 \cdot k_1\right) \\ & + \tfrac{1}{12}\left(\dot{G}^2_{B12} - 2\dot{G}^4_{B12} + \dot{G}^6_{B12}\right) k_2 \cdot f_4 \cdot k_1 k_2 \cdot f_3 \cdot k_1 \Big], \end{aligned} \tag{D.240}$$

$$I^2_{(34)}(13) = \frac{-1}{36} Z_2(13) \dot{G}^2_{B12}\left(1 - \dot{G}^2_{B12}\right) k_1 \cdot f_4 \cdot f_2 \cdot k_1, \tag{D.241}$$

$$I^2_{(34)}(23) = \tfrac{-1}{36} Z_2(23) \dot{G}^2_{B12}\left(1 - \dot{G}^2_{B12}\right) k_2 \cdot f_4 \cdot f_1 \cdot k_2, \tag{D.242}$$

$$\begin{aligned} I^2_{(34)}(14) = -\tfrac{Z_2(14)}{36 r_7 \cdot k_2 r_8 \cdot k_3} \Big[& -\left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) k_2 \cdot k_3 r_8 \cdot f_3 \cdot k_1 r_7 \cdot f_2 \cdot k_1 \\ & + \left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) k_3 \cdot k_1 \left(r_8 \cdot f_3 \cdot k_2 r_7 \cdot f_2 \cdot k_1 - r_7 \cdot f_2 \cdot f_3 \cdot r_8 k_2 \cdot k_1\right) \\ & + \left(2 - 6\dot{G}^2_{B12}\right)\left(r_8 \cdot f_3 \cdot k_1 r_7 \cdot f_2 \cdot k_3 + 2 r_7 \cdot f_2 \cdot f_3 \cdot r_8 k_3 \cdot k_1\right) \Big], \end{aligned} \tag{D.243}$$

$$\begin{aligned} I^2_{(34)}(24) = -\tfrac{Z_2(24)}{36 r_9 \cdot k_1 r_0 \cdot k_3} \Big[& -\left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) k_3 \cdot k_1 r_0 \cdot f_3 \cdot k_2 r_9 \cdot f_1 \cdot k_2 \\ & + \left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) k_2 \cdot k_3 \left(r_0 \cdot f_3 \cdot k_1 r_9 \cdot f_1 \cdot k_2 - r_9 \cdot f_1 \cdot f_3 \cdot r_0 k_1 \cdot k_2\right) \\ & + \left(2 - 6\dot{G}^2_{B12}\right)\left(r_0 \cdot f_3 \cdot k_2 r_9 \cdot f_1 \cdot k_3 + 2 r_9 \cdot f_1 \cdot f_3 \cdot r_0 k_3 \cdot k_2\right) \Big], \end{aligned} \tag{D.244}$$

$$I^{22}_{(34)}(12,34) = \frac{1}{3}\dot{G}^2_{B12} Z_2(12) Z_2(34), \tag{D.245}$$

$$I^{22}_{(34)}(13,24) = \frac{1}{9} Z_2(13) Z_2(24), \tag{D.246}$$

$$I^{22}_{(34)}(14,23) = \frac{1}{9} Z_2(14) Z_2(23). \tag{D.247}$$

Then, what remains to be integrated is

$$\begin{aligned} I^4_{(234)}(1234) &= \tfrac{1}{6}\int_0^1 \left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) Z_4(1234) du_2, \\ I^4_{(234)}(1243) &= \tfrac{1}{6}\int_0^1 \left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) Z_4(1243) du_2, \\ I^4_{(234)}(1324) &= \tfrac{1}{6^2}\int_0^1 \left(1 - 6\dot{G}^2_{B12} + 9\dot{G}^4_{B21}\right) Z_4(1324) du_2, \end{aligned} \tag{D.248}$$

$$I^2_{(234)}(12) = \frac{-1}{12}\frac{2}{15}\int_0^1 \dot{G}^2_{B12} Z_2(12) k_1 \cdot f_4 \cdot f_3 \cdot k_1 du_2, \tag{D.249}$$

$$I^2_{(234)}(13) = \frac{-1}{36}\int_0^1 Z_2(13)\dot{G}^2_{B12}\left(1 - \dot{G}^2_{B12}\right) k_1 \cdot f_4 \cdot f_2 \cdot k_1 du_2, \tag{D.250}$$

$$\begin{aligned} I^2_{(234)}(14) = -\tfrac{Z_2(14)}{36 r_7 \cdot k_2 r_8 \cdot k_3}\int_0^1 \Big[&- \left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) k_2 \cdot k_3 r_8 \cdot f_3 \cdot k_1 r_7 \cdot f_2 \cdot k_1 \\ &+ \left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) k_3 \cdot k_1 \left(r_8 \cdot f_3 \cdot k_2 r_7 \cdot f_2 \cdot k_1 - r_7 \cdot f_2 \cdot f_3 \cdot r_8 k_2 \cdot k_1\right) \\ &+ \left(2 - 6\dot{G}^2_{B12}\right) e^{()} \left(r_8 \cdot f_3 \cdot k_1 r_7 \cdot f_2 \cdot k_3 + 2 r_7 \cdot f_2 \cdot f_3 \cdot r_8 k_3 \cdot k_1\right) \Big] du_2, \end{aligned} \tag{D.251}$$

$$I^{22}_{(234)}(12,34) = \frac{1}{3}\int_0^1 \dot{G}^2_{B12} Z_2(12) Z_2(34) du_2, \tag{D.252}$$

$$I^{22}_{(234)}(13,24) = \frac{1}{9}\int_0^1 Z_2(13) Z_2(24) du_2, \tag{D.253}$$

$$I^{22}_{(234)}(14,23) = \frac{1}{9}\int_0^1 Z_2(14) Z_2(23) du_2. \tag{D.254}$$

We perform this integrals using some particular cases of (B.11) and we obtain

$$\begin{aligned} I^4_{(234)}(1234) &= \tfrac{1}{6}\left(\tfrac{1}{3} - \tfrac{1}{5}\right) Z_4(1234), \\ I^4_{(234)}(1243) &= \tfrac{1}{6}\left(\tfrac{1}{3} - \tfrac{1}{5}\right) Z_4(1243), \\ I^4_{(234)}(1324) &= \tfrac{1}{6^2}\left(1 - \tfrac{6}{3} + \tfrac{9}{5}\right) Z_4(1324), \end{aligned} \tag{D.255}$$

$$I^2_{(234)}(12) = \frac{-1}{12}\frac{2}{15}\frac{1}{3}Z_2(12)k_1 \cdot f_4 \cdot f_3 \cdot k_1, \tag{D.256}$$

$$I^2_{(234)}(13) = \frac{-1}{36}Z_2(13)\left(\frac{1}{3}-\frac{1}{5}\right)k_1 \cdot f_4 \cdot f_2 \cdot k_1, \tag{D.257}$$

$$\begin{aligned} I^2_{(234)}(14) = -\tfrac{Z_2(14)}{36r_7\cdot k_2 r_8\cdot k_3}\Big\{ &-\left(\tfrac{1}{3}-\tfrac{1}{5}\right)k_2\cdot k_3 r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_1 \\ &+\left(\tfrac{1}{3}-\tfrac{1}{5}\right)k_3\cdot k_1\left(r_8\cdot f_3\cdot k_2 r_7\cdot f_2\cdot k_1 - r_7\cdot f_2\cdot f_3\cdot r_8 k_2\cdot k_1\right) \\ &+\left[\tfrac{2}{6}-\tfrac{6}{4}\left(\tfrac{1}{3}-\tfrac{1}{5}\right)\right]k_1\cdot k_2\left(r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_3 + 2r_7\cdot f_2\cdot f_3\cdot r_8 k_3\cdot k_1\right)\Big\}, \end{aligned} \tag{D.258}$$

$$I^{22}_{(234)}(12,34) = \frac{1}{9}Z_2(12)Z_2(34), \tag{D.259}$$

$$I^{22}_{(234)}(13,24) = \frac{1}{9}Z_2(13)Z_2(24), \tag{D.260}$$

$$I^{22}_{(234)}(14,23) = \frac{1}{9}Z_2(14)Z_2(23). \tag{D.261}$$

Making simpler

D.3.1 Result of $I_{(234)}$

$$\begin{aligned} I^4_{(234)}(1234) &= \tfrac{1}{45}Z_4(1234), \\ I^4_{(234)}(1243) &= \tfrac{1}{45}Z_4(1243), \\ I^4_{(234)}(1324) &= \tfrac{1}{45}Z_4(1324), \end{aligned} \tag{D.262}$$

$$I^2_{(234)}(12) = \frac{-1}{45\cdot 6}Z_2(12)k_1 \cdot f_4 \cdot f_3 \cdot k_1, \tag{D.263}$$

$$I^2_{(234)}(13) = \frac{-1}{45\cdot 6}Z_2(13)k_1 \cdot f_4 \cdot f_2 \cdot k_1, \tag{D.264}$$

$$\begin{aligned} I^2_{(234)}(14) = -\tfrac{Z_2(14)}{36r_7\cdot k_2 r_8\cdot k_3}\Big\{ &-\tfrac{2}{15}k_2\cdot k_3 r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_1 \\ &+\tfrac{2}{15}k_3\cdot k_1\left(r_8\cdot f_3\cdot k_2 r_7\cdot f_2\cdot k_1 - r_7\cdot f_2\cdot f_3\cdot r_8 k_2\cdot k_1\right) \\ &+\tfrac{2}{15}k_1\cdot k_2\left(r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_3 + 2r_7\cdot f_2\cdot f_3\cdot r_8 k_3\cdot k_1\right)\Big\}, \end{aligned} \tag{D.265}$$

$$I^{22}_{(234)}(12,34) = \frac{1}{9}Z_2(12)Z_2(34), \tag{D.266}$$

$$I^{22}_{(234)}(13,24) = \frac{1}{9}Z_2(13)Z_2(24), \tag{D.267}$$

$$I^{22}_{(234)}(14,23) = \frac{1}{9} Z_2(14) Z_2(23). \tag{D.268}$$

However, as we already mention when we have three photons of low energy the remaining one must be also of low energy because of momentum conservation.

D.4 Four low energy photons

First, we cancel all the terms $O(k_1^2)$ in $I_{(234)}$

$$\begin{aligned} I^4_{(234)}(1234) &= \tfrac{1}{45} Z_4(1234), \\ I^4_{(234)}(1243) &= \tfrac{1}{45} Z_4(1243), \\ I^4_{(234)}(1324) &= \tfrac{1}{45} Z_4(1324), \end{aligned} \tag{D.269}$$

$$I^2_{(234)}(12) = \frac{-1}{45 \cdot 6} Z_2(12) \cancel{k_1 \cdot f_4 \cdot f_3 \cdot k_1}, \tag{D.270}$$

$$I^2_{(234)}(13) = \frac{-1}{45 \cdot 6} Z_2(13) \cancel{k_1 \cdot f_4 \cdot f_2 \cdot k_1}, \tag{D.271}$$

$$\begin{aligned} I^2_{(234)}(14) = -\tfrac{Z_2(14)}{36 r_7 \cdot k_2 r_8 \cdot k_3} \Big\{ & -\tfrac{2}{15} k_2 \cdot k_3 \cancel{r_8 \cdot f_3 \cdot k_1} r_7 \cdot f_2 \cdot k_1 \\ & + \tfrac{2}{15} \cancel{k_3 \cdot k_1} (r_8 \cdot f_3 \cdot k_2 r_7 \cdot f_2 \cdot k_1 - r_7 \cdot f_2 \cdot f_3 \cdot r_8 k_2 \cdot k_1) \\ & + \tfrac{2}{15} \cancel{k_1 \cdot k_2} (r_8 \cdot f_3 \cdot k_1 r_7 \cdot f_2 \cdot k_3 + 2 r_7 \cdot f_2 \cdot f_3 \cdot r_8 k_3 \cdot k_1) \Big\}, \end{aligned} \tag{D.272}$$

$$I^{22}_{(234)}(12,34) = \frac{1}{9} Z_2(12) Z_2(34), \tag{D.273}$$

$$I^{22}_{(234)}(13,24) = \frac{1}{9} Z_2(13) Z_2(24), \tag{D.274}$$

$$I^{22}_{(234)}(14,23) = \frac{1}{9} Z_2(14) Z_2(23). \tag{D.275}$$

Since there is no dependence in u_1 anymore, because of the translation invariance, we have the following result for this case

D.4.1 Result of $I_{(1234)}$

$$\begin{aligned} I^4_{(1234)}(1234) &= \tfrac{1}{45} Z_4(1234), \\ I^4_{(1234)}(1243) &= \tfrac{1}{45} Z_4(1243), \\ I^4_{(1234)}(1324) &= \tfrac{1}{45} Z_4(1324), \end{aligned} \tag{D.276}$$

$$I^{22}_{(1234)}(12,34) = \frac{1}{9} Z_2(12) Z_2(34), \tag{D.277}$$

$$I^{22}_{(1234)}(13,24) = \frac{1}{9} Z_2(13) Z_2(24), \tag{D.278}$$

$$I^{22}_{(1234)}(14,23) = \frac{1}{9} Z_2(14) Z_2(23). \tag{D.279}$$

Appendix E

Explicit calculation of low energy integrals (spinor case)

In this appendix, we present explicitly all the calculations that we require, to obtain all the formulas presented in Chapter 3, in the spinor cases. First, using the replacement rule (described in section 2.2) in Q_4 (section 3.1.1) and we obtain $\tilde{Q}_4$

$$\begin{aligned}
\tilde{Q}_4^4(1234) &= \left(\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B34}\dot{G}_{B41} - G_{F12}G_{F23}G_{F34}G_{F41}\right) Z_4(1234),\\
\tilde{Q}_4^4(1243) &= \left(\dot{G}_{B12}\dot{G}_{B24}\dot{G}_{B43}\dot{G}_{B31} - G_{F12}G_{F24}G_{F43}G_{F31}\right) Z_4(1243),\\
\tilde{Q}_4^4(1324) &= \left(\dot{G}_{B13}\dot{G}_{B32}\dot{G}_{B24}\dot{G}_{B41} - G_{F13}G_{F32}G_{F24}G_{F41}\right) Z_4(1324),
\end{aligned} \tag{E.1}$$

$$\begin{aligned}
\tilde{Q}_4^3(123) = &\left(\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31} - G_{F12}G_{F23}G_{F31}\right) Z_3(123)\\
&\times \left(\dot{G}_{B41}\varepsilon_4\cdot k_1 + \dot{G}_{B42}\varepsilon_4\cdot k_2 + \dot{G}_{B43}\varepsilon_4\cdot k_3\right),
\end{aligned} \tag{E.2}$$

$$\begin{aligned}
\tilde{Q}_4^3(234) = &\left(\dot{G}_{B23}\dot{G}_{B34}\dot{G}_{B42} - G_{F23}G_{F34}G_{F42}\right) Z_3(234)\\
&\times \left(\dot{G}_{B12}\varepsilon_1\cdot k_2 + \dot{G}_{B13}\varepsilon_1\cdot k_3 + \dot{G}_{B14}\varepsilon_1\cdot k_4\right),
\end{aligned} \tag{E.3}$$

$$\begin{aligned}\tilde{Q}_4^3(341) = &\left(\dot{G}_{B34}\dot{G}_{B41}\dot{G}_{B13} - G_{F34}G_{F41}G_{F13}\right) Z_3(341) \\ &\times \left(\dot{G}_{B21}\varepsilon_2 \cdot k_1 + \dot{G}_{B23}\varepsilon_2 \cdot k_3 + \dot{G}_{B24}\varepsilon_2 \cdot k_4\right),\end{aligned} \tag{E.4}$$

$$\begin{aligned}\tilde{Q}_4^3(412) = &\left(\dot{G}_{B41}\dot{G}_{B12}\dot{G}_{B24} - G_{F41}G_{F12}G_{F24}\right) Z_3(412) \\ &\times \left(\dot{G}_{B31}\varepsilon_3 \cdot k_1 + \dot{G}_{B32}\varepsilon_3 \cdot k_2 + \dot{G}_{B34}\varepsilon_3 \cdot k_4\right),\end{aligned} \tag{E.5}$$

$$\begin{aligned}\tilde{Q}_4^2(12) = &\left(\dot{G}_{B12}\dot{G}_{B21} - G_{F12}G_{F21}\right) Z_2(12) \Big[\dot{G}_{B31}\varepsilon_3 \cdot k_1 \dot{G}_{B41}\varepsilon_4 \cdot k_1 \\ &+\dot{G}_{B31}\varepsilon_3 \cdot k_1 \dot{G}_{B42}\varepsilon_4 \cdot k_2 \\ &+\dot{G}_{B31}\varepsilon_3 \cdot k_1 \dot{G}_{B43}\varepsilon_4 \cdot k_3 + \dot{G}_{B32}\varepsilon_3 \cdot k_2 \dot{G}_{B41}\varepsilon_4 \cdot k_1 \\ &+\dot{G}_{B32}\varepsilon_3 \cdot k_2 \dot{G}_{B42}\varepsilon_4 \cdot k_2 + \dot{G}_{B32}\varepsilon_3 \cdot k_2 \dot{G}_{B43}\varepsilon_4 \cdot k_3 \\ &+\dot{G}_{B34}\varepsilon_3 \cdot k_4 \dot{G}_{B41}\varepsilon_4 \cdot k_1 + \dot{G}_{B34}\varepsilon_3 \cdot k_4 \dot{G}_{B42}\varepsilon_4 \cdot k_2 \\ &+\tfrac{1}{2}\dot{G}_{B34}\varepsilon_3 \cdot \varepsilon_4 \left(\dot{G}_{B31}k_3 \cdot k_1 - \dot{G}_{B41}k_4 \cdot k_1\right) \\ &+ \tfrac{1}{2}\dot{G}_{B34}\varepsilon_3 \cdot \varepsilon_4 \left(\dot{G}_{B32}k_3 \cdot k_2 - \dot{G}_{B42}k_4 \cdot k_2\right)\Big],\end{aligned} \tag{E.6}$$

$$\begin{aligned}\tilde{Q}_4^2(13) = &\left(\dot{G}_{B13}\dot{G}_{B31} - G_{F13}G_{F31}\right) Z_2(13) \Big[\dot{G}_{B21}\varepsilon_2 \cdot k_1 \dot{G}_{B41}\varepsilon_4 \cdot k_1 \\ &+\dot{G}_{B21}\varepsilon_2 \cdot k_1 \dot{G}_{B42}\varepsilon_4 \cdot k_2 \\ &+\dot{G}_{B21}\varepsilon_2 \cdot k_1 \dot{G}_{B43}\varepsilon_4 \cdot k_3 + \dot{G}_{B23}\varepsilon_2 \cdot k_3 \dot{G}_{B41}\varepsilon_4 \cdot k_1 \\ &+\dot{G}_{B23}\varepsilon_2 \cdot k_3 \dot{G}_{B42}\varepsilon_4 \cdot k_2 + \dot{G}_{B23}\varepsilon_2 \cdot k_3 \dot{G}_{B43}\varepsilon_4 \cdot k_3 \\ &+\dot{G}_{B24}\varepsilon_2 \cdot k_4 \dot{G}_{B41}\varepsilon_4 \cdot k_1 + \dot{G}_{B24}\varepsilon_2 \cdot k_4 \dot{G}_{B43}\varepsilon_4 \cdot k_3 \\ &+\tfrac{1}{2}\dot{G}_{B24}\varepsilon_2 \cdot \varepsilon_4 \left(\dot{G}_{B21}k_2 \cdot k_1 - \dot{G}_{B41}k_4 \cdot k_1\right) \\ &+ \tfrac{1}{2}\dot{G}_{B24}\varepsilon_2 \cdot \varepsilon_4 \left(\dot{G}_{B23}k_2 \cdot k_3 - \dot{G}_{B43}k_4 \cdot k_3\right)\Big],\end{aligned} \tag{E.7}$$

$$\begin{aligned}\tilde{Q}_4^2(14) = &\left(\dot{G}_{B14}\dot{G}_{B41} - G_{F14}G_{F41}\right) Z_2(14) \Big[\dot{G}_{B21}\varepsilon_2 \cdot k_1 \dot{G}_{B31}\varepsilon_3 \cdot k_1 \\ &+\dot{G}_{B21}\varepsilon_2 \cdot k_1 \dot{G}_{B32}\varepsilon_3 \cdot k_2 \\ &+\dot{G}_{B21}\varepsilon_2 \cdot k_1 \dot{G}_{B34}\varepsilon_3 \cdot k_4 + \dot{G}_{B23}\varepsilon_2 \cdot k_3 \dot{G}_{B31}\varepsilon_3 \cdot k_1 \\ &+\dot{G}_{B23}\varepsilon_2 \cdot k_3 \dot{G}_{B34}\varepsilon_3 \cdot k_4 + \dot{G}_{B24}\varepsilon_2 \cdot k_4 \dot{G}_{B31}\varepsilon_3 \cdot k_1 \\ &+\dot{G}_{B24}\varepsilon_2 \cdot k_4 \dot{G}_{B32}\varepsilon_3 \cdot k_2 + \dot{G}_{B24}\varepsilon_2 \cdot k_4 \dot{G}_{B34}\varepsilon_3 \cdot k_4 \\ &+\tfrac{1}{2}\dot{G}_{B23}\varepsilon_2 \cdot \varepsilon_3 \left(\dot{G}_{B21}k_2 \cdot k_1 - \dot{G}_{B31}k_3 \cdot k_1\right) \\ &+ \tfrac{1}{2}\dot{G}_{B23}\varepsilon_2 \cdot \varepsilon_3 \left(\dot{G}_{B24}k_2 \cdot k_4 - \dot{G}_{B34}k_3 \cdot k_4\right)\Big],\end{aligned} \tag{E.8}$$

$$\begin{aligned}\tilde{Q}_4^2(23) = &\left(\dot{G}_{B23}\dot{G}_{B32} - G_{F23}G_{F32}\right) Z_2(23) \Big[\dot{G}_{B12}\varepsilon_1 \cdot k_2 \dot{G}_{B41}\varepsilon_4 \cdot k_1 \\ &+\dot{G}_{B12}\varepsilon_1 \cdot k_2 \dot{G}_{B42}\varepsilon_4 \cdot k_2 \\ &+\dot{G}_{B12}\varepsilon_1 \cdot k_2 \dot{G}_{B43}\varepsilon_4 \cdot k_3 + \dot{G}_{B13}\varepsilon_1 \cdot k_3 \dot{G}_{B41}\varepsilon_4 \cdot k_1 \\ &+\dot{G}_{B13}\varepsilon_1 \cdot k_3 \dot{G}_{B42}\varepsilon_4 \cdot k_2 + \dot{G}_{B13}\varepsilon_1 \cdot k_3 \dot{G}_{B43}\varepsilon_4 \cdot k_3 \\ &+\dot{G}_{B14}\varepsilon_1 \cdot k_4 \dot{G}_{B42}\varepsilon_4 \cdot k_2 + \dot{G}_{B14}\varepsilon_1 \cdot k_4 \dot{G}_{B43}\varepsilon_4 \cdot k_3 \\ &+\tfrac{1}{2}\dot{G}_{B14}\varepsilon_1 \cdot \varepsilon_4 \left(\dot{G}_{B12}k_1 \cdot k_2 - \dot{G}_{B42}k_4 \cdot k_2\right) \\ &+ \tfrac{1}{2}\dot{G}_{B14}\varepsilon_1 \cdot \varepsilon_4 \left(\dot{G}_{B13}k_1 \cdot k_3 - \dot{G}_{B43}k_4 \cdot k_3\right)\Big],\end{aligned} \tag{E.9}$$

$$\begin{aligned}\tilde{Q}_4^2(24) = &\left(\dot{G}_{B24}\dot{G}_{B42} - G_{F24}G_{F42}\right) Z_2(24) \Big[\dot{G}_{B12}\varepsilon_1 \cdot k_2 \dot{G}_{B31}\varepsilon_3 \cdot k_1 \\ &+\dot{G}_{B12}\varepsilon_1 \cdot k_2 \dot{G}_{B32}\varepsilon_3 \cdot k_2 \\ &+\dot{G}_{B12}\varepsilon_1 \cdot k_2 \dot{G}_{B34}\varepsilon_3 \cdot k_4 + \dot{G}_{B13}\varepsilon_1 \cdot k_3 \dot{G}_{B32}\varepsilon_3 \cdot k_2 \\ &+\dot{G}_{B13}\varepsilon_1 \cdot k_3 \dot{G}_{B34}\varepsilon_3 \cdot k_4 + \dot{G}_{B14}\varepsilon_1 \cdot k_4 \dot{G}_{B31}\varepsilon_3 \cdot k_1 \\ &+\dot{G}_{B14}\varepsilon_1 \cdot k_4 \dot{G}_{B32}\varepsilon_3 \cdot k_2 + \dot{G}_{B14}\varepsilon_1 \cdot k_4 \dot{G}_{B34}\varepsilon_3 \cdot k_4 \\ &+\tfrac{1}{2}\dot{G}_{B13}\varepsilon_1 \cdot \varepsilon_3 \left(\dot{G}_{B12}k_1 \cdot k_2 - \dot{G}_{B32}k_3 \cdot k_2\right) \\ &+ \tfrac{1}{2}\dot{G}_{B13}\varepsilon_1 \cdot \varepsilon_3 \left(\dot{G}_{B14}k_1 \cdot k_4 - \dot{G}_{B34}k_3 \cdot k_4\right)\Big],\end{aligned} \tag{E.10}$$

$$\begin{aligned}\tilde{Q}_4^2(34) = &\left(\dot{G}_{B34}\dot{G}_{B43} - G_{F34}G_{F43}\right) Z_2(34) \Big[\dot{G}_{B12}\varepsilon_1 \cdot k_2 \dot{G}_{B23}\varepsilon_2 \cdot k_3 \\ &+\dot{G}_{B12}\varepsilon_1 \cdot k_2 \dot{G}_{B24}\varepsilon_2 \cdot k_4 \\ &+\dot{G}_{B13}\varepsilon_1 \cdot k_3 \dot{G}_{B21}\varepsilon_2 \cdot k_1 + \dot{G}_{B13}\varepsilon_1 \cdot k_3 \dot{G}_{B23}\varepsilon_2 \cdot k_3 \\ &+\dot{G}_{B13}\varepsilon_1 \cdot k_3 \dot{G}_{B24}\varepsilon_2 \cdot k_4 + \dot{G}_{B14}\varepsilon_1 \cdot k_4 \dot{G}_{B21}\varepsilon_2 \cdot k_1 \\ &+\dot{G}_{B14}\varepsilon_1 \cdot k_4 \dot{G}_{B23}\varepsilon_2 \cdot k_3 + \dot{G}_{B14}\varepsilon_1 \cdot k_4 \dot{G}_{B24}\varepsilon_2 \cdot k_4 \\ &+\tfrac{1}{2}\dot{G}_{B12}\varepsilon_1 \cdot \varepsilon_2 \left(\dot{G}_{B13}k_1 \cdot k_3 - \dot{G}_{B23}k_2 \cdot k_3\right) \\ &+ \tfrac{1}{2}\dot{G}_{B12}\varepsilon_1 \cdot \varepsilon_2 \left(\dot{G}_{B14}k_1 \cdot k_4 - \dot{G}_{B24}k_2 \cdot k_4\right)\Big],\end{aligned} \tag{E.11}$$

$$\begin{aligned}\tilde{Q}_4^{22}(12,34) &= \left(\dot{G}_{B12}\dot{G}_{B21} - G_{F12}G_{F21}\right) Z_2(12) \left(\dot{G}_{B34}\dot{G}_{B43} - G_{F34}G_{F43}\right) Z_2(34), \\ \tilde{Q}_4^{22}(13,24) &= \left(\dot{G}_{B13}\dot{G}_{B31} - G_{F13}G_{F31}\right) Z_2(13) \left(\dot{G}_{B24}\dot{G}_{B42} - G_{F24}G_{F42}\right) Z_2(24), \\ \tilde{Q}_4^{22}(14,23) &= \left(\dot{G}_{B14}\dot{G}_{B41} - G_{F14}G_{F41}\right) Z_2(14) \left(\dot{G}_{B23}\dot{G}_{B32} - G_{F23}G_{F32}\right) Z_2(23).\end{aligned} \tag{E.12}$$

Now, we use (B.2), (B.3) and (B.4) to write all in terms of the bosonic Green's function

$$\begin{aligned}
\tilde{Q}_4^4(1234) &= \Big[\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B34}\dot{G}_{B41} \\
&+ \Big(\dot{G}_{B12}+\dot{G}_{B23}+\dot{G}_{B31}\Big)\Big(\dot{G}_{B34}+\dot{G}_{B41}+\dot{G}_{B13}\Big)\Big] Z_4(1234), \\
\tilde{Q}_4^4(1243) &= \Big[\dot{G}_{B12}\dot{G}_{B24}\dot{G}_{B43}\dot{G}_{B31} \\
&+ \Big(\dot{G}_{B12}+\dot{G}_{B24}+\dot{G}_{B41}\Big)\Big(\dot{G}_{B43}+\dot{G}_{B31}+\dot{G}_{B14}\Big)\Big] Z_4(1243), \\
\tilde{Q}_4^4(1324) &= \Big[\dot{G}_{B13}\dot{G}_{B32}\dot{G}_{B24}\dot{G}_{B41} \\
&+ \Big(\dot{G}_{B13}+\dot{G}_{B32}+\dot{G}_{B21}\Big)\Big(\dot{G}_{B24}+\dot{G}_{B41}+\dot{G}_{B12}\Big)\Big] Z_4(1324),
\end{aligned} \tag{E.13}$$

$$\begin{aligned}
\tilde{Q}_4^3(123) &= \Big(\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31}+\dot{G}_{B12}+\dot{G}_{B23}+\dot{G}_{B31}\Big) Z_3(123) \\
&\quad\times\Big(\dot{G}_{B41}\varepsilon_4\cdot k_1+\dot{G}_{B42}\varepsilon_4\cdot k_2+\dot{G}_{B43}\varepsilon_4\cdot k_3\Big),
\end{aligned} \tag{E.14}$$

$$\begin{aligned}
\tilde{Q}_4^3(234) &= \Big(\dot{G}_{B23}\dot{G}_{B34}\dot{G}_{B42}+\dot{G}_{B23}+\dot{G}_{B34}+\dot{G}_{B42}\Big) Z_3(234) \\
&\quad\times\Big(\dot{G}_{B12}\varepsilon_1\cdot k_2+\dot{G}_{B13}\varepsilon_1\cdot k_3+\dot{G}_{B14}\varepsilon_1\cdot k_4\Big),
\end{aligned} \tag{E.15}$$

$$\begin{aligned}
\tilde{Q}_4^3(341) &= \Big(\dot{G}_{B34}\dot{G}_{B41}\dot{G}_{B13}+\dot{G}_{B34}+\dot{G}_{B41}+\dot{G}_{B13}\Big) Z_3(341) \\
&\quad\times\Big(\dot{G}_{B21}\varepsilon_2\cdot k_1+\dot{G}_{B23}\varepsilon_2\cdot k_3+\dot{G}_{B24}\varepsilon_2\cdot k_4\Big),
\end{aligned} \tag{E.16}$$

$$\begin{aligned}
\tilde{Q}_4^3(412) &= \Big(\dot{G}_{B41}\dot{G}_{B12}\dot{G}_{B24}+\dot{G}_{B41}+\dot{G}_{B12}+\dot{G}_{B24}\Big) Z_3(412) \\
&\quad\times\Big(\dot{G}_{B31}\varepsilon_3\cdot k_1+\dot{G}_{B32}\varepsilon_3\cdot k_2+\dot{G}_{B34}\varepsilon_3\cdot k_4\Big),
\end{aligned} \tag{E.17}$$

$$\begin{aligned}
\tilde{Q}_4^2(12) &= \Big(\dot{G}_{B12}\dot{G}_{B21}+1\Big) Z_2(12)\Big[\dot{G}_{B31}\varepsilon_3\cdot k_1\dot{G}_{B41}\varepsilon_4\cdot k_1 \\
&\quad+\dot{G}_{B31}\varepsilon_3\cdot k_1\dot{G}_{B42}\varepsilon_4\cdot k_2 \\
&\quad+\dot{G}_{B31}\varepsilon_3\cdot k_1\dot{G}_{B43}\varepsilon_4\cdot k_3+\dot{G}_{B32}\varepsilon_3\cdot k_2\dot{G}_{B41}\varepsilon_4\cdot k_1 \\
&\quad+\dot{G}_{B32}\varepsilon_3\cdot k_2\dot{G}_{B42}\varepsilon_4\cdot k_2+\dot{G}_{B32}\varepsilon_3\cdot k_2\dot{G}_{B43}\varepsilon_4\cdot k_3 \\
&\quad+\dot{G}_{B34}\varepsilon_3\cdot k_4\dot{G}_{B41}\varepsilon_4\cdot k_1+\dot{G}_{B34}\varepsilon_3\cdot k_4\dot{G}_{B42}\varepsilon_4\cdot k_2 \\
&\quad+\tfrac{1}{2}\dot{G}_{B34}\varepsilon_3\cdot\varepsilon_4\Big(\dot{G}_{B31}k_3\cdot k_1-\dot{G}_{B41}k_4\cdot k_1\Big) \\
&\quad+\tfrac{1}{2}\dot{G}_{B34}\varepsilon_3\cdot\varepsilon_4\Big(\dot{G}_{B32}k_3\cdot k_2-\dot{G}_{B42}k_4\cdot k_2\Big)\Big],
\end{aligned} \tag{E.18}$$

$$\begin{aligned}\tilde{Q}_4^2(13) = &\left(\dot{G}_{B13}\dot{G}_{B31}+1\right) Z_2(13)\left[\dot{G}_{B21}\varepsilon_2\cdot k_1\dot{G}_{B41}\varepsilon_4\cdot k_1\right.\\
&+\dot{G}_{B21}\varepsilon_2\cdot k_1\dot{G}_{B42}\varepsilon_4\cdot k_2\\
&+\dot{G}_{B21}\varepsilon_2\cdot k_1\dot{G}_{B43}\varepsilon_4\cdot k_3+\dot{G}_{B23}\varepsilon_2\cdot k_3\dot{G}_{B41}\varepsilon_4\cdot k_1\\
&+\dot{G}_{B23}\varepsilon_2\cdot k_3\dot{G}_{B42}\varepsilon_4\cdot k_2+\dot{G}_{B23}\varepsilon_2\cdot k_3\dot{G}_{B43}\varepsilon_4\cdot k_3\\
&+\dot{G}_{B24}\varepsilon_2\cdot k_4\dot{G}_{B41}\varepsilon_4\cdot k_1+\dot{G}_{B24}\varepsilon_2\cdot k_4\dot{G}_{B43}\varepsilon_4\cdot k_3\\
&+\tfrac{1}{2}\dot{G}_{B24}\varepsilon_2\cdot\varepsilon_4\left(\dot{G}_{B21}k_2\cdot k_1-\dot{G}_{B41}k_4\cdot k_1\right)\\
&\left.+\tfrac{1}{2}\dot{G}_{B24}\varepsilon_2\cdot\varepsilon_4\left(\dot{G}_{B23}k_2\cdot k_3-\dot{G}_{B43}k_4\cdot k_3\right)\right],\end{aligned}\tag{E.19}$$

$$\begin{aligned}\tilde{Q}_4^2(14) = &\left(\dot{G}_{B14}\dot{G}_{B41}+1\right) Z_2(14)\left[\dot{G}_{B21}\varepsilon_2\cdot k_1\dot{G}_{B31}\varepsilon_3\cdot k_1\right.\\
&+\dot{G}_{B21}\varepsilon_2\cdot k_1\dot{G}_{B32}\varepsilon_3\cdot k_2\\
&+\dot{G}_{B21}\varepsilon_2\cdot k_1\dot{G}_{B34}\varepsilon_3\cdot k_4+\dot{G}_{B23}\varepsilon_2\cdot k_3\dot{G}_{B31}\varepsilon_3\cdot k_1\\
&+\dot{G}_{B23}\varepsilon_2\cdot k_3\dot{G}_{B34}\varepsilon_3\cdot k_4+\dot{G}_{B24}\varepsilon_2\cdot k_4\dot{G}_{B31}\varepsilon_3\cdot k_1\\
&+\dot{G}_{B24}\varepsilon_2\cdot k_4\dot{G}_{B32}\varepsilon_3\cdot k_2+\dot{G}_{B24}\varepsilon_2\cdot k_4\dot{G}_{B34}\varepsilon_3\cdot k_4\\
&+\tfrac{1}{2}\dot{G}_{B23}\varepsilon_2\cdot\varepsilon_3\left(\dot{G}_{B21}k_2\cdot k_1-\dot{G}_{B31}k_3\cdot k_1\right)\\
&\left.+\tfrac{1}{2}\dot{G}_{B23}\varepsilon_2\cdot\varepsilon_3\left(\dot{G}_{B24}k_2\cdot k_4-\dot{G}_{B34}k_3\cdot k_4\right)\right],\end{aligned}\tag{E.20}$$

$$\begin{aligned}\tilde{Q}_4^2(23) = &\left(\dot{G}_{B23}\dot{G}_{B32}+1\right) Z_2(23)\left[\dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B41}\varepsilon_4\cdot k_1\right.\\
&+\dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B42}\varepsilon_4\cdot k_2\\
&+\dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B43}\varepsilon_4\cdot k_3+\dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B41}\varepsilon_4\cdot k_1\\
&+\dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B42}\varepsilon_4\cdot k_2+\dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B43}\varepsilon_4\cdot k_3\\
&+\dot{G}_{B14}\varepsilon_1\cdot k_4\dot{G}_{B42}\varepsilon_4\cdot k_2+\dot{G}_{B14}\varepsilon_1\cdot k_4\dot{G}_{B43}\varepsilon_4\cdot k_3\\
&+\tfrac{1}{2}\dot{G}_{B14}\varepsilon_1\cdot\varepsilon_4\left(\dot{G}_{B12}k_1\cdot k_2-\dot{G}_{B42}k_4\cdot k_2\right)\\
&\left.+\tfrac{1}{2}\dot{G}_{B14}\varepsilon_1\cdot\varepsilon_4\left(\dot{G}_{B13}k_1\cdot k_3-\dot{G}_{B43}k_4\cdot k_3\right)\right],\end{aligned}\tag{E.21}$$

$$\begin{aligned}\tilde{Q}_4^2(24) = &\left(\dot{G}_{B24}\dot{G}_{B42}+1\right) Z_2(24)\left[\dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B31}\varepsilon_3\cdot k_1\right.\\
&+\dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B32}\varepsilon_3\cdot k_2\\
&+\dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B34}\varepsilon_3\cdot k_4+\dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B32}\varepsilon_3\cdot k_2\\
&+\dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B34}\varepsilon_3\cdot k_4+\dot{G}_{B14}\varepsilon_1\cdot k_4\dot{G}_{B31}\varepsilon_3\cdot k_1\\
&+\dot{G}_{B14}\varepsilon_1\cdot k_4\dot{G}_{B32}\varepsilon_3\cdot k_2+\dot{G}_{B14}\varepsilon_1\cdot k_4\dot{G}_{B34}\varepsilon_3\cdot k_4\\
&+\tfrac{1}{2}\dot{G}_{B13}\varepsilon_1\cdot\varepsilon_3\left(\dot{G}_{B12}k_1\cdot k_2-\dot{G}_{B32}k_3\cdot k_2\right)\\
&\left.+\tfrac{1}{2}\dot{G}_{B13}\varepsilon_1\cdot\varepsilon_3\left(\dot{G}_{B14}k_1\cdot k_4-\dot{G}_{B34}k_3\cdot k_4\right)\right],\end{aligned}\tag{E.22}$$

$$\begin{aligned}\tilde{Q}_4^2(34) = &\left(\dot{G}_{B34}\dot{G}_{B43}+1\right) Z_2(34)\left[\dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B23}\varepsilon_2\cdot k_3\right.\\ &+\dot{G}_{B12}\varepsilon_1\cdot k_2\dot{G}_{B24}\varepsilon_2\cdot k_4\\ &+\dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B21}\varepsilon_2\cdot k_1+\dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B23}\varepsilon_2\cdot k_3\\ &+\dot{G}_{B13}\varepsilon_1\cdot k_3\dot{G}_{B24}\varepsilon_2\cdot k_4+\dot{G}_{B14}\varepsilon_1\cdot k_4\dot{G}_{B21}\varepsilon_2\cdot k_1\\ &+\dot{G}_{B14}\varepsilon_1\cdot k_4\dot{G}_{B23}\varepsilon_2\cdot k_3+\dot{G}_{B14}\varepsilon_1\cdot k_4\dot{G}_{B24}\varepsilon_2\cdot k_4\\ &+\tfrac{1}{2}\dot{G}_{B12}\varepsilon_1\cdot\varepsilon_2\left(\dot{G}_{B13}k_1\cdot k_3-\dot{G}_{B23}k_2\cdot k_3\right)\\ &\left.+\tfrac{1}{2}\dot{G}_{B12}\varepsilon_1\cdot\varepsilon_2\left(\dot{G}_{B14}k_1\cdot k_4-\dot{G}_{B24}k_2\cdot k_4\right)\right],\end{aligned} \tag{E.23}$$

$$\begin{aligned}\tilde{Q}_4^{22}(12,34) &= \left(\dot{G}_{B12}\dot{G}_{B21}+1\right) Z_2(12)\left(\dot{G}_{B34}\dot{G}_{B43}+1\right) Z_2(34),\\ \tilde{Q}_4^{22}(13,24) &= \left(\dot{G}_{B13}\dot{G}_{B31}+1\right) Z_2(13)\left(\dot{G}_{B24}\dot{G}_{B42}+1\right) Z_2(24),\\ \tilde{Q}_4^{22}(14,23) &= \left(\dot{G}_{B14}\dot{G}_{B41}+1\right) Z_2(14)\left(\dot{G}_{B23}\dot{G}_{B32}+1\right) Z_2(23).\end{aligned} \tag{E.24}$$

E.1 One low energy photon

In this section, we perform the integral over u_4 of (3.4) with the assumption that the corresponding photon is in the low energy limit, its means that we only consider the linear terms in k_4. Then, we define

$$\tilde{I}_{(4)} = \int_0^1 du_4 \tilde{Q}_4 e^{(\cdot 4)} = \tilde{I}^4_{(4)} + \tilde{I}^3_{(4)} + \tilde{I}^2_{(4)} + \tilde{I}^{22}_{(4)}, \tag{E.25}$$

with

$$e^{(\cdot 4)} = e^{T(G_{B14}k_1\cdot k_4 + G_{B24}k_2\cdot k_4 + G_{B34}k_3\cdot k_4)}, \tag{E.26}$$

and

$$\tilde{I}^i_{(4)} = \int_0^1 du_4 \tilde{Q}^i_4 e^{(\cdot 4)}, \quad i = 4,3,2,22. \tag{E.27}$$

(All this calculations are very similar to the scalar case.)

E.1.1 Integral of $\tilde{Q}_4^4$

Since $\tilde{Q}_4^4$ is linear in k_4, we can disregard the exponential. Then we expand this expressions

$$\begin{aligned}
\tilde{I}^4_{(4)}(1234) &= \int_0^1 \Big[\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B34}\dot{G}_{B41} + \Big(\dot{G}_{B12}+\dot{G}_{B23}+\dot{G}_{B31}\Big)\dot{G}_{B13} \\
&+ \Big(\dot{G}_{B12}+\dot{G}_{B23}+\dot{G}_{B31}\Big)\Big(\dot{G}_{B34}+\dot{G}_{B41}\Big)\Big] Z_4(1234)du_4, \\
\tilde{I}^4_{(4)}(1243) &= \int_0^1 \Big[\dot{G}_{B12}\dot{G}_{B24}\dot{G}_{B43}\dot{G}_{B31} + \Big(\dot{G}_{B24}+\dot{G}_{B41}\Big)\Big(\dot{G}_{B43}+\dot{G}_{B14}\Big) \\
&+\dot{G}_{B12}\dot{G}_{B31} + \Big(\dot{G}_{B24}+\dot{G}_{B41}\Big)\dot{G}_{B31} + \dot{G}_{B12}\Big(\dot{G}_{B43}+\dot{G}_{B14}\Big)\Big] Z_4(1243)du_4, \\
\tilde{I}^4_{(4)}(1324) &= \int_0^1 \Big[\dot{G}_{B13}\dot{G}_{B32}\dot{G}_{B24}\dot{G}_{B41} + \Big(\dot{G}_{B13}+\dot{G}_{B32}+\dot{G}_{B21}\Big)\dot{G}_{B12} \\
&+ \Big(\dot{G}_{B13}+\dot{G}_{B32}+\dot{G}_{B21}\Big)\Big(\dot{G}_{B24}+\dot{G}_{B41}\Big)\Big] Z_4(1324)du_4,
\end{aligned} \tag{E.28}$$

and we see that some terms cancel due to (B.8) and for the other terms we use (B.15) to integrate them

Result of $\tilde{I}^4_{(4)}$

$$\begin{aligned}
\tilde{I}^4_{(4)}(1234) &= Z_4(1234)\Big[\dot{G}_{B12}\dot{G}_{B23}\Big(\tfrac{1}{6}-\tfrac{1}{2}\dot{G}^2_{B31}\Big) \\
&+ \Big(\dot{G}_{B12}+\dot{G}_{B23}+\dot{G}_{B31}\Big)\dot{G}_{B13}\Big], \\
\tilde{I}^4_{(4)}(1243) &= Z_4(1243)\Big[\dot{G}_{B12}\dot{G}_{B31}\Big(\tfrac{1}{6}-\tfrac{1}{2}\dot{G}^2_{B23}\Big) \\
&+\tfrac{1}{2}\Big(\dot{G}^2_{B12}+\dot{G}^2_{B31}-\dot{G}^2_{B23}-1\Big)+\dot{G}_{B12}\dot{G}_{B31}\Big], \\
\tilde{I}^4_{(4)}(1324) &= Z_4(1324)\Big[\dot{G}_{B23}\dot{G}_{B31}\Big(\tfrac{1}{6}-\tfrac{1}{2}\dot{G}^2_{B12}\Big) \\
&+ \Big(\dot{G}_{B13}+\dot{G}_{B32}+\dot{G}_{B21}\Big)\dot{G}_{B12}\Big],
\end{aligned} \tag{E.29}$$

E.1.2 Integral of $\tilde{Q}_4^3$

First, we identify and cancel the nonlinear terms in k_4 (marked with a diagonal) and the terms that become zero after the integration over u_4 (marked with "X")

$$\begin{aligned}
\tilde{Q}^3_4(123) = &\Big(\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31}+\dot{G}_{B12}+\dot{G}_{B23}+\dot{G}_{B31}\Big)Z_3(123) \\
&\times\Big(\dot{G}_{B41}\varepsilon_4\cdot k_1+\dot{G}_{B42}\varepsilon_4\cdot k_2+\dot{G}_{B43}\varepsilon_4\cdot k_3\Big),
\end{aligned} \tag{E.30}$$

$$\tilde{Q}_4^3(234) = \left(\dot{G}_{B23}\dot{G}_{B34}\dot{G}_{B42} + \dot{G}_{B23} + \cancel{\dot{G}_{B34}} + \cancel{\dot{G}_{B42}}\right) Z_3(234) \times \left(\dot{G}_{B12}\varepsilon_1 \cdot k_2 + \dot{G}_{B13}\varepsilon_1 \cdot k_3 + \cancel{\dot{G}_{B14}\varepsilon_1 \cdot k_4}\right), \tag{E.31}$$

$$\tilde{Q}_4^3(341) = \left(\dot{G}_{B34}\dot{G}_{B41}\dot{G}_{B13} + \cancel{\dot{G}_{B34}} + \cancel{\dot{G}_{B41}} + \dot{G}_{B13}\right) Z_3(341) \times \left(\dot{G}_{B21}\varepsilon_2 \cdot k_1 + \dot{G}_{B23}\varepsilon_2 \cdot k_3 + \cancel{\dot{G}_{B24}\varepsilon_2 \cdot k_4}\right), \tag{E.32}$$

$$\tilde{Q}_4^3(412) = \left(\dot{G}_{B41}\dot{G}_{B12}\dot{G}_{B24} + \cancel{\dot{G}_{B41}} + \dot{G}_{B12} + \cancel{\dot{G}_{B24}}\right) Z_3(412) \times \left(\dot{G}_{B31}\varepsilon_3 \cdot k_1 + \dot{G}_{B32}\varepsilon_3 \cdot k_2 + \cancel{\dot{G}_{B34}\varepsilon_3 \cdot k_4}\right), \tag{E.33}$$

Now, we use (B.5) to write the terms that do not cancel in the following way

$$\tilde{Q}_4^3(123) = \left(\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31} + \dot{G}_{B12} + \dot{G}_{B23} + \dot{G}_{B31}\right) Z_3(123) \times \left(\dot{G}_{B41}\varepsilon_4 \cdot k_1 + \dot{G}_{B42}\varepsilon_4 \cdot k_2 + \dot{G}_{B43}\varepsilon_4 \cdot k_3\right), \tag{E.34}$$

$$\tilde{Q}_4^3(234) = \left(\dot{G}_{B23}\dot{G}_{B34}\dot{G}_{B42} + \dot{G}_{B23}\right) Z_3(234) \left(\dot{G}_{B12}\frac{r_1 \cdot f_1 \cdot k_2}{r_1 \cdot k_1} + \dot{G}_{B12}\frac{r_1 \cdot \varepsilon_1 k_1 \cdot k_2}{r_1 \cdot k_1} + \dot{G}_{B13}\frac{r_1 \cdot f_1 \cdot k_3}{r_1 \cdot k_1} + \dot{G}_{B13}\frac{r_1 \cdot \varepsilon_1 k_1 \cdot k_3}{r_1 \cdot k_1}\right), \tag{E.35}$$

$$\tilde{Q}_4^3(341) = \left(\dot{G}_{B34}\dot{G}_{B41}\dot{G}_{B13} + \dot{G}_{B13}\right) Z_3(341) \left(\dot{G}_{B21}\frac{r_2 \cdot f_2 \cdot k_1}{r_2 \cdot k_2} + \dot{G}_{B21}\frac{r_2 \cdot \varepsilon_2 k_2 \cdot k_1}{r_2 \cdot k_2} + \dot{G}_{B23}\frac{r_2 \cdot f_2 \cdot k_3}{r_2 \cdot k_2} + \dot{G}_{B23}\frac{r_2 \cdot \varepsilon_2 k_2 \cdot k_3}{r_2 \cdot k_2}\right), \tag{E.36}$$

$$\tilde{Q}_4^3(412) = \left(\dot{G}_{B41}\dot{G}_{B12}\dot{G}_{B24} + \dot{G}_{B12}\right) Z_3(412) \left(\dot{G}_{B31}\frac{r_3 \cdot f_3 \cdot k_1}{r_3 \cdot k_3} + \dot{G}_{B31}\frac{r_3 \cdot \varepsilon_3 k_3 \cdot k_1}{r_3 \cdot k_3} + \dot{G}_{B32}\frac{r_3 \cdot f_3 \cdot k_2}{r_3 \cdot k_3} + \dot{G}_{B32}\frac{r_3 \cdot \varepsilon_3 k_3 \cdot k_2}{r_3 \cdot k_3}\right), \tag{E.37}$$

Then, we use (B.21), with $T = 1$, to compute the integral of $\tilde{Q}_4^3(123)$, since this is a zero order term in k_4

$$\begin{aligned}\tilde{I}^3_{(4)}(123) &= \left(\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31} + \dot{G}_{B12} + \dot{G}_{B23} + \dot{G}_{B31}\right) Z_3(123) \\ &\times \left[\tfrac{1}{3}\textstyle\sum_{i=1}^3 \dot{G}_{B1i}G_{Bi1}k_i \cdot k_4\varepsilon_4 \cdot k_1 + \tfrac{1}{3}\sum_{i=1}^3 \dot{G}_{B2i}G_{Bi2}k_i \cdot k_4\varepsilon_4 \cdot k_2\right. \\ &\quad \left. + \tfrac{1}{3}\textstyle\sum_{i=1}^3 \dot{G}_{B3i}G_{Bi3}k_i \cdot k_4\varepsilon_4 \cdot k_3\right],\end{aligned} \tag{E.38}$$

we expand the sums

$$\begin{aligned}&\tilde{I}^3_{(4)}(123) = \tfrac{1}{3}\left(\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31} + \dot{G}_{B12} + \dot{G}_{B23} + \dot{G}_{B31}\right) Z_3(123) \\ &\times \left[\dot{G}_{B12}G_{B21}k_2 \cdot k_4\varepsilon_4 \cdot k_1 + \dot{G}_{B13}G_{B31}k_3 \cdot k_4\varepsilon_4 \cdot k_1 + \dot{G}_{B21}G_{B12}k_1 \cdot k_4\varepsilon_4 \cdot k_2\right. \\ &\left. + \dot{G}_{B23}G_{B32}k_3 \cdot k_4\varepsilon_4 \cdot k_2\dot{G}_{B31}G_{B13}k_1 \cdot k_4\varepsilon_4 \cdot k_3 + + \dot{G}_{B32}G_{B23}k_2 \cdot k_4\varepsilon_4 \cdot k_3\right],\end{aligned} \tag{E.39}$$

factorizing

$$\begin{aligned}\tilde{I}^3_{(4)}(123) &= \tfrac{1}{3}\left(\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31} + \dot{G}_{B12} + \dot{G}_{B23} + \dot{G}_{B31}\right) Z_3(123) \\ &\times\Big[\dot{G}_{B12}G_{B21}(k_2 \cdot k_4\varepsilon_4 \cdot k_1 - k_1 \cdot k_4\varepsilon_4 \cdot k_2) \\ &+\dot{G}_{B23}G_{B32}(k_3 \cdot k_4\varepsilon_4 \cdot k_2 - k_2 \cdot k_4\varepsilon_4 \cdot k_3) \\ &+\dot{G}_{B31}G_{B13}(k_1 \cdot k_4\varepsilon_4 \cdot k_3 - k_3 \cdot k_4\varepsilon_4 \cdot k_1)\Big],\end{aligned} \tag{E.40}$$

and finally, we use (B.5) to write the previous expression as

$$\begin{aligned}\tilde{I}^3_{(4)}(123) &= \tfrac{1}{3}\left(\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31} + \dot{G}_{B12} + \dot{G}_{B23} + \dot{G}_{B31}\right) Z_3(123) \\ &\times \left[\dot{G}_{B12}G_{B21}k_2 \cdot f_4 \cdot k_1 + \dot{G}_{B23}G_{B32}k_3 \cdot f_4 \cdot k_2 + \dot{G}_{B31}G_{B13}k_1 \cdot f_4 \cdot k_3\right].\end{aligned} \tag{E.41}$$

For the others equations, we use (B.11) and (B.12) to integrate them, here we cancel the terms without f_i's because these are total derivatives

$$\tilde{I}^3_{(4)}(234) = \dot{G}_{B23}\left(2G_{B32} - \frac{1}{3} + 1\right) Z_3(234)\left(\dot{G}_{B12}\frac{r_1 \cdot f_1 \cdot k_2}{r_1 \cdot k_1} + \dot{G}_{B13}\frac{r_1 \cdot f_1 \cdot k_3}{r_1 \cdot k_1}\right), \tag{E.42}$$

$$\tilde{I}^3_{(4)}(341) = \left(2G_{B31} - \frac{1}{3} + 1\right)\dot{G}_{B13}Z_3(341)\left(\dot{G}_{B21}\frac{r_2 \cdot f_2 \cdot k_1}{r_2 \cdot k_2} + \dot{G}_{B23}\frac{r_2 \cdot f_2 \cdot k_3}{r_2 \cdot k_2}\right), \tag{E.43}$$

$$\tilde{I}^3_{(4)}(412) = \left(2G_{B12} - \frac{1}{3} + 1\right) \dot{G}_{B12} Z_3(412) \left(\dot{G}_{B31} \frac{r_3 \cdot f_3 \cdot k_1}{r_3 \cdot k_3} + \dot{G}_{B32} \frac{r_3 \cdot f_3 \cdot k_2}{r_3 \cdot k_3}\right). \tag{E.44}$$

Now, we use (B.1) to write all in terms of $\dot{G}$'s

Result of $\tilde{I}^3_{(4)}$

$$\begin{aligned}\tilde{I}^3_{(4)}(123) = {} & \tfrac{1}{12}\left(\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31} + \dot{G}_{B12} + \dot{G}_{B23} + \dot{G}_{B31}\right) Z_3(123) \\ & \times \Big[\dot{G}_{B12}\left(1 - \dot{G}^2_{B12}\right) k_2 \cdot f_4 \cdot k_1 + \dot{G}_{B23}\left(1 - \dot{G}^2_{B23}\right) k_3 \cdot f_4 \cdot k_2 \\ & + \dot{G}_{B31}\left(1 - \dot{G}^2_{B31}\right) k_1 \cdot f_4 \cdot k_3\Big],\end{aligned} \tag{E.45}$$

$$\tilde{I}^3_{(4)}(234) = \frac{1}{2}\dot{G}_{B23}\left(\frac{7}{3} - \dot{G}^2_{B23}\right) Z_3(234) \left(\dot{G}_{B12}\frac{r_1 \cdot f_1 \cdot k_2}{r_1 \cdot k_1} - \dot{G}_{B31}\frac{r_1 \cdot f_1 \cdot k_3}{r_1 \cdot k_1}\right), \tag{E.46}$$

$$\tilde{I}^3_{(4)}(341) = \frac{1}{2}\left(\frac{7}{3} - \dot{G}^2_{B31}\right) \dot{G}_{B31} Z_3(341) \left(\dot{G}_{B12}\frac{r_2 \cdot f_2 \cdot k_1}{r_2 \cdot k_2} - \dot{G}_{B23}\frac{r_2 \cdot f_2 \cdot k_3}{r_2 \cdot k_2}\right), \tag{E.47}$$

$$\tilde{I}^3_{(4)}(412) = \frac{1}{2}\left(\frac{7}{3} - \dot{G}^2_{B12}\right) \dot{G}_{B12} Z_3(412) \left(\dot{G}_{B31}\frac{r_3 \cdot f_3 \cdot k_1}{r_3 \cdot k_3} - \dot{G}_{B23}\frac{r_3 \cdot f_3 \cdot k_2}{r_3 \cdot k_3}\right). \tag{E.48}$$

E.1.3 Integral of $\tilde{Q}^2_4$

Similarly, to the scalar case (section D.1.3) we omit all the nonlinear terms and we see that the remaining terms to be integrated are

$$\begin{aligned}\tilde{I}^2_{(4)}(12) = {} & \left(\dot{G}_{B12}\dot{G}_{B21} + 1\right) Z_2(12) \\ & \times \Big\{\int_0^1 \Big[\dot{G}_{B31}\varepsilon_3 \cdot k_1 \dot{G}_{B41}\varepsilon_4 \cdot k_1 + \dot{G}_{B31}\varepsilon_3 \cdot k_1 \dot{G}_{B42}\varepsilon_4 \cdot k_2 \\ & + \dot{G}_{B31}\varepsilon_3 \cdot k_1 \dot{G}_{B43}\varepsilon_4 \cdot k_3 + \dot{G}_{B32}\varepsilon_3 \cdot k_2 \dot{G}_{B41}\varepsilon_4 \cdot k_1 \\ & + \dot{G}_{B32}\varepsilon_3 \cdot k_2 \dot{G}_{B42}\varepsilon_4 \cdot k_2 + \dot{G}_{B32}\varepsilon_3 \cdot k_2 \dot{G}_{B43}\varepsilon_4 \cdot k_3 \\ & + \tfrac{1}{2}\dot{G}_{B34}\varepsilon_3 \cdot \varepsilon_4 \dot{G}_{B31} k_3 \cdot k_1 + \tfrac{1}{2}\dot{G}_{B34}\varepsilon_3 \cdot \varepsilon_4 \dot{G}_{B32} k_3 \cdot k_2\Big] e^{(i4)} du_4 \\ & + \int_0^1 \Big[\dot{G}_{B34}\varepsilon_3 \cdot k_4 \dot{G}_{B41}\varepsilon_4 \cdot k_1 + \dot{G}_{B34}\varepsilon_3 \cdot k_4 \dot{G}_{B42}\varepsilon_4 \cdot k_2 \\ & - \tfrac{1}{2}\left(\dot{G}_{B34}\varepsilon_3 \cdot \varepsilon_4 \dot{G}_{B41} k_4 \cdot k_1 + \dot{G}_{B34}\varepsilon_3 \cdot \varepsilon_4 \dot{G}_{B42} k_4 \cdot k_2\right)\Big] du_4\Big\},\end{aligned} \tag{E.49}$$

$$\tilde{I}^2_{(4)}(13) = \left(\dot{G}_{B13}\dot{G}_{B31}+1\right) Z_2(13)$$
$$\times \Big\{ \int_0^1 \Big[\dot{G}_{B21}\varepsilon_2 \cdot k_1 \dot{G}_{B41}\varepsilon_4 \cdot k_1 + \dot{G}_{B21}\varepsilon_2 \cdot k_1 \dot{G}_{B42}\varepsilon_4 \cdot k_2$$
$$+\dot{G}_{B21}\varepsilon_2 \cdot k_1 \dot{G}_{B43}\varepsilon_4 \cdot k_3 + \dot{G}_{B23}\varepsilon_2 \cdot k_3 \dot{G}_{B41}\varepsilon_4 \cdot k_1$$
$$+\dot{G}_{B23}\varepsilon_2 \cdot k_3 \dot{G}_{B42}\varepsilon_4 \cdot k_2 + \dot{G}_{B23}\varepsilon_2 \cdot k_3 \dot{G}_{B43}\varepsilon_4 \cdot k_3$$
$$+ \tfrac{1}{2}\dot{G}_{B24}\varepsilon_2 \cdot \varepsilon_4 \dot{G}_{B21} k_2 \cdot k_1 + \tfrac{1}{2}\dot{G}_{B24}\varepsilon_2 \cdot \varepsilon_4 \dot{G}_{B23} k_2 \cdot k_3 \Big] e^{(i4)} du_4$$
$$+ \int_0^1 \Big[\dot{G}_{B24}\varepsilon_2 \cdot k_4 \dot{G}_{B41}\varepsilon_4 \cdot k_1 + \dot{G}_{B24}\varepsilon_2 \cdot k_4 \dot{G}_{B43}\varepsilon_4 \cdot k_3$$
$$- \tfrac{1}{2}\left(\dot{G}_{B24}\varepsilon_2 \cdot \varepsilon_4 \dot{G}_{B41} k_4 \cdot k_1 + \dot{G}_{B24}\varepsilon_2 \cdot \varepsilon_4 \dot{G}_{B43} k_4 \cdot k_3\right)\Big] du_4 \Big\}, \tag{E.50}$$

$$\tilde{I}^2_{(4)}(23) = \left(\dot{G}_{B23}\dot{G}_{B32}+1\right) Z_2(23)$$
$$\times \Big\{ \int_0^1 \Big[\dot{G}_{B12}\varepsilon_1 \cdot k_2 \dot{G}_{B41}\varepsilon_4 \cdot k_1 + \dot{G}_{B12}\varepsilon_1 \cdot k_2 \dot{G}_{B42}\varepsilon_4 \cdot k_2$$
$$+\dot{G}_{B12}\varepsilon_1 \cdot k_2 \dot{G}_{B43}\varepsilon_4 \cdot k_3 + \dot{G}_{B13}\varepsilon_1 \cdot k_3 \dot{G}_{B41}\varepsilon_4 \cdot k_1$$
$$+\dot{G}_{B13}\varepsilon_1 \cdot k_3 \dot{G}_{B42}\varepsilon_4 \cdot k_2 + \dot{G}_{B13}\varepsilon_1 \cdot k_3 \dot{G}_{B43}\varepsilon_4 \cdot k_3$$
$$+ \tfrac{1}{2}\dot{G}_{B14}\varepsilon_1 \cdot \varepsilon_4 \dot{G}_{B12} k_1 \cdot k_2 + \tfrac{1}{2}\dot{G}_{B14}\varepsilon_1 \cdot \varepsilon_4 \dot{G}_{B13} k_1 \cdot k_3 \Big] e^{(i4)} du_4$$
$$+ \int_0^1 \Big[\dot{G}_{B14}\varepsilon_1 \cdot k_4 \dot{G}_{B42}\varepsilon_4 \cdot k_2 + \dot{G}_{B14}\varepsilon_1 \cdot k_4 \dot{G}_{B43}\varepsilon_4 \cdot k_3$$
$$- \tfrac{1}{2}\left(\dot{G}_{B14}\varepsilon_1 \cdot \varepsilon_4 \dot{G}_{B42} k_4 \cdot k_2 + \dot{G}_{B14}\varepsilon_1 \cdot \varepsilon_4 \dot{G}_{B43} k_4 \cdot k_3\right)\Big] du_4 \Big\}, \tag{E.51}$$

$$\tilde{I}^2_{(4)}(14) = \int_0^1 \left(\dot{G}_{B14}\dot{G}_{B41}+1\right) Z_2(14) du_4 \Big[\dot{G}_{B21}\varepsilon_2 \cdot k_1 \dot{G}_{B31}\varepsilon_3 \cdot k_1$$
$$+\dot{G}_{B23}\varepsilon_2 \cdot k_3 \dot{G}_{B31}\varepsilon_3 \cdot k_1 + \dot{G}_{B21}\varepsilon_2 \cdot k_1 \dot{G}_{B32}\varepsilon_3 \cdot k_2$$
$$+ \tfrac{1}{2}\dot{G}_{B23}\varepsilon_2 \cdot \varepsilon_3 \left(\dot{G}_{B21} k_2 \cdot k_1 - \dot{G}_{B31} k_3 \cdot k_1\right)\Big], \tag{E.52}$$

$$\tilde{I}^2_{(4)}(24) = \int_0^1 \left(\dot{G}_{B24}\dot{G}_{B42}+1\right) Z_2(24) du_4 \Big[\dot{G}_{B12}\varepsilon_1 \cdot k_2 \dot{G}_{B31}\varepsilon_3 \cdot k_1$$
$$+\dot{G}_{B13}\varepsilon_1 \cdot k_3 \dot{G}_{B32}\varepsilon_3 \cdot k_2 + \dot{G}_{B12}\varepsilon_1 \cdot k_2 \dot{G}_{B32}\varepsilon_3 \cdot k_2$$
$$+ \tfrac{1}{2}\dot{G}_{B13}\varepsilon_1 \cdot \varepsilon_3 \left(\dot{G}_{B12} k_1 \cdot k_2 - \dot{G}_{B32} k_3 \cdot k_2\right)\Big], \tag{E.53}$$

$$\tilde{I}^2_{(4)}(34) = \int_0^1 \left(\dot{G}_{B34}\dot{G}_{B43}+1\right) Z_2(34) du_4 \Big[\dot{G}_{B12}\varepsilon_1 \cdot k_2 \dot{G}_{B23}\varepsilon_2 \cdot k_3$$
$$+\dot{G}_{B13}\varepsilon_1 \cdot k_3 \dot{G}_{B21}\varepsilon_2 \cdot k_1 + \dot{G}_{B13}\varepsilon_1 \cdot k_3 \dot{G}_{B23}\varepsilon_2 \cdot k_3$$
$$+ \tfrac{1}{2}\dot{G}_{B12}\varepsilon_1 \cdot \varepsilon_2 \left(\dot{G}_{B13} k_1 \cdot k_3 - \dot{G}_{B23} k_2 \cdot k_3\right)\Big]. \tag{E.54}$$

Notice that in all this cases we must follow the same steps of the scalar case, and that the only difference is that we change $\dot{G}_{Bij}\dot{G}_{Bji}$ by $\dot{G}_{Bij}\dot{G}_{Bji}+1$. In the first three equations of the previous six, this change does not mix with the integral over u_4 while in the last three equation, after the integration over u_4 we have to change the factor of $-\frac{1}{3}$ that appears in the scalar case by $\frac{2}{3}$ to have the correct result in the spinor case, so we can directly write,

Result of $\tilde{I}^2_{(4)}$

$$\begin{aligned}
\tilde{I}^2_{(4)}(12) &= \tfrac{1}{12}\left(1-\dot{G}^2_{B12}\right) Z_2(12)\Big\{ \\
&+\dot{G}^2_{B31}\left(1-\dot{G}^2_{B31}\right) k_1\cdot f_4\cdot f_3\cdot k_1 - \dot{G}_{B23}\dot{G}_{B31}\left(1-\dot{G}^2_{B31}\right) k_1\cdot f_4\cdot f_3\cdot k_2 \\
&-\dot{G}_{B23}\dot{G}_{B31}\left(1-\dot{G}^2_{B23}\right) k_2\cdot f_4\cdot f_3\cdot k_1 + \dot{G}^2_{B23}\left(1-\dot{G}^2_{B23}\right) k_2\cdot f_4\cdot f_3\cdot k_2 \\
&+\dot{G}_{B12}\dot{G}_{B31}\left(1-\dot{G}^2_{B12}\right) \tfrac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_1}{r_4\cdot k_3} - \dot{G}_{B12}\dot{G}_{B23}\left(1-\dot{G}^2_{B12}\right) \tfrac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_2}{r_4\cdot k_3}\Big\},
\end{aligned} \tag{E.55}$$

$$\begin{aligned}
\tilde{I}^2_{(4)}(13) &= \tfrac{1}{12}\left(1-\dot{G}^2_{B31}\right) Z_2(13)\Big\{ \\
&+\dot{G}^2_{B12}\left(1-\dot{G}^2_{B12}\right) k_1\cdot f_4\cdot f_2\cdot k_1 - \dot{G}_{B12}\dot{G}_{B23}\left(1-\dot{G}^2_{B12}\right) k_1\cdot f_4\cdot f_2\cdot k_3 \\
&-\dot{G}_{B12}\dot{G}_{B23}\left(1-\dot{G}^2_{B23}\right) k_3\cdot f_4\cdot f_2\cdot k_1 + \dot{G}^2_{B23}\left(1-\dot{G}^2_{B23}\right) k_3\cdot f_4\cdot f_2\cdot k_3 \\
&+\dot{G}_{B12}\dot{G}_{B31}\left(1-\dot{G}^2_{B31}\right) \tfrac{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_1}{r_5\cdot k_2} - \dot{G}_{B23}\dot{G}_{B31}\left(1-\dot{G}^2_{B31}\right) \tfrac{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_3}{r_5\cdot k_2}\Big\},
\end{aligned} \tag{E.56}$$

$$\begin{aligned}
\tilde{I}^2_{(4)}(23) &= \tfrac{1}{12}\left(1-\dot{G}^2_{B23}\right) Z_2(23)\Big\{ \\
&+\dot{G}^2_{B12}\left(1-\dot{G}^2_{B12}\right) k_2\cdot f_4\cdot f_1\cdot k_2 - \dot{G}_{B12}\dot{G}_{B31}\left(1-\dot{G}^2_{B12}\right) k_2\cdot f_4\cdot f_1\cdot k_3 \\
&-\dot{G}_{B12}\dot{G}_{B31}\left(1-\dot{G}^2_{B31}\right) k_3\cdot f_4\cdot f_1\cdot k_2 + \dot{G}^2_{B31}\left(1-\dot{G}^2_{B31}\right) k_3\cdot f_4\cdot f_1\cdot k_3 \\
&+\dot{G}_{B12}\dot{G}_{B23}\left(1-\dot{G}^2_{B23}\right) \tfrac{k_3\cdot f_4\cdot k_2 r_6\cdot f_1\cdot k_2}{r_6\cdot k_1} - \dot{G}_{B23}\dot{G}_{B31}\left(1-\dot{G}^2_{B23}\right) \tfrac{k_3\cdot f_4\cdot k_2 r_6\cdot f_1\cdot k_3}{r_6\cdot k_1}\Big\},
\end{aligned} \tag{E.57}$$

$$\begin{aligned}
\tilde{I}^2_{(4)}(14) = \tfrac{2Z_2(14)}{3r_7\cdot k_2 r_8\cdot k_3}\Big[&-\dot{G}_{B12}\dot{G}_{B31} r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_1 \\
&+\dot{G}_{B23}\dot{G}_{B31}\left(r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_3 + 2r_7\cdot f_2\cdot f_3\cdot r_8 k_3\cdot k_1\right) \\
&+\dot{G}_{B12}\dot{G}_{B23}\left(r_8\cdot f_3\cdot k_2 r_7\cdot f_2\cdot k_1 - r_7\cdot f_2\cdot f_3\cdot r_8 k_2\cdot k_1\right)\Big],
\end{aligned} \tag{E.58}$$

$$\begin{aligned}\tilde{I}^{2}_{(4)}(24) = \tfrac{2Z_2(24)}{3r_9\cdot k_1 r_0\cdot k_3}\Big[&- \dot{G}_{B12}\dot{G}_{B23} r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_2 \\ &+\dot{G}_{B23}\dot{G}_{B31}\left(r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_3 + 2r_9\cdot f_1\cdot f_3\cdot r_0 k_3\cdot k_2\right) \\ &+\dot{G}_{B12}\dot{G}_{B31}\left(r_0\cdot f_3\cdot k_1 r_9\cdot f_1\cdot k_2 - r_9\cdot f_1\cdot f_3\cdot r_0 k_1\cdot k_2\right)\Big],\end{aligned} \tag{E.59}$$

$$\begin{aligned}\tilde{I}^{2}_{(4)}(34) = \tfrac{2Z_2(34)}{3r_{01}\cdot k_1 r_{02}\cdot k_2}\Big[&- \dot{G}_{B23}\dot{G}_{B31} r_{02}\cdot f_2\cdot k_3 r_{01}\cdot f_1\cdot k_3 \\ &+\dot{G}_{B12}\dot{G}_{B23}\left(r_{02}\cdot f_2\cdot k_3 r_{01}\cdot f_1\cdot k_2 + 2r_{01}\cdot f_1\cdot f_2\cdot r_{02} k_2\cdot k_3\right) \\ &+\dot{G}_{B12}\dot{G}_{B31}\left(r_{02}\cdot f_2\cdot k_1 r_{01}\cdot f_1\cdot k_3 - r_{01}\cdot f_1\cdot f_2\cdot r_{02} k_1\cdot k_3\right)\Big],\end{aligned} \tag{E.60}$$

E.1.4 Integral of $\tilde{Q}^{22}_4$

Using (B.11), we obtain that

$$\tilde{I}^{22}_{(4)}(12,34) = \left(\dot{G}_{B12}\dot{G}_{B21}+1\right) Z_2(12)\left(\frac{-1}{3}+1\right) Z_2(34), \tag{E.61}$$

$$\tilde{I}^{22}_{(4)}(13,24) = \left(\dot{G}_{B13}\dot{G}_{B31}+1\right) Z_2(13)\left(\frac{-1}{3}+1\right) Z_2(24), \tag{E.62}$$

$$\tilde{I}^{22}_{(4)}(14,23) = \left(\frac{-1}{3}+1\right) Z_2(14)\left(\dot{G}_{B23}\dot{G}_{B32}+1\right) Z_2(23), \tag{E.63}$$

simplifying

Result of $\tilde{I}^{22}_{(4)}$

$$\tilde{I}^{22}_{(4)}(12,34) = \frac{2}{3}\left(1-\dot{G}^2_{B12}\right) Z_2(12) Z_2(34), \tag{E.64}$$

$$\tilde{I}^{22}_{(4)}(13,24) = \frac{2}{3}\left(1-\dot{G}^2_{B31}\right) Z_2(13) Z_2(24), \tag{E.65}$$

$$\tilde{I}^{22}_{(4)}(14,23) = \frac{2}{3}\left(1-\dot{G}^2_{B23}\right) Z_2(14) Z_2(23), \tag{E.66}$$

E.1.5 Results of $\tilde{I}_{(4)}$

$$\begin{aligned}
\tilde{I}^4_{(4)}(1234) &= Z_4(1234)\left[\dot{G}_{B12}\dot{G}_{B23}\left(\tfrac{1}{6}-\tfrac{1}{2}\dot{G}^2_{B31}\right)\right.\\
&\left.+\left(\dot{G}_{B12}+\dot{G}_{B23}+\dot{G}_{B31}\right)\dot{G}_{B13}\right],\\
\tilde{I}^4_{(4)}(1243) &= Z_4(1243)\left[\dot{G}_{B12}\dot{G}_{B31}\left(\tfrac{1}{6}-\tfrac{1}{2}\dot{G}^2_{B23}\right)\right.\\
&\left.+\tfrac{1}{2}\left(\dot{G}^2_{B12}+\dot{G}^2_{B31}-\dot{G}^2_{B23}-1\right)+\dot{G}_{B12}\dot{G}_{B31}\right],\\
\tilde{I}^4_{(4)}(1324) &= Z_4(1324)\left[\dot{G}_{B23}\dot{G}_{B31}\left(\tfrac{1}{6}-\tfrac{1}{2}\dot{G}^2_{B12}\right)\right.\\
&\left.+\left(\dot{G}_{B13}+\dot{G}_{B32}+\dot{G}_{B21}\right)\dot{G}_{B12}\right],
\end{aligned} \tag{E.67}$$

$$\begin{aligned}
\tilde{I}^3_{(4)}(123) &= \tfrac{1}{12}\left(\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31}+\dot{G}_{B12}+\dot{G}_{B23}+\dot{G}_{B31}\right)Z_3(123)\\
&\times\left[\dot{G}_{B12}\left(1-\dot{G}^2_{B12}\right)k_2\cdot f_4\cdot k_1+\dot{G}_{B23}\left(1-\dot{G}^2_{B23}\right)k_3\cdot f_4\cdot k_2\right.\\
&\left.+\dot{G}_{B31}\left(1-\dot{G}^2_{B31}\right)k_1\cdot f_4\cdot k_3\right],
\end{aligned} \tag{E.68}$$

$$\tilde{I}^3_{(4)}(234)=\frac{1}{2}\dot{G}_{B23}\left(\frac{7}{3}-\dot{G}^2_{B23}\right)Z_3(234)\left(\dot{G}_{B12}\frac{r_1\cdot f_1\cdot k_2}{r_1\cdot k_1}-\dot{G}_{B31}\frac{r_1\cdot f_1\cdot k_3}{r_1\cdot k_1}\right), \tag{E.69}$$

$$\tilde{I}^3_{(4)}(341)=\frac{1}{2}\left(\frac{7}{3}-\dot{G}^2_{B31}\right)\dot{G}_{B31}Z_3(341)\left(\dot{G}_{B12}\frac{r_2\cdot f_2\cdot k_1}{r_2\cdot k_2}-\dot{G}_{B23}\frac{r_2\cdot f_2\cdot k_3}{r_2\cdot k_2}\right), \tag{E.70}$$

$$\tilde{I}^3_{(4)}(412)=\frac{1}{2}\left(\frac{7}{3}-\dot{G}^2_{B12}\right)\dot{G}_{B12}Z_3(412)\left(\dot{G}_{B31}\frac{r_3\cdot f_3\cdot k_1}{r_3\cdot k_3}-\dot{G}_{B23}\frac{r_3\cdot f_3\cdot k_2}{r_3\cdot k_3}\right). \tag{E.71}$$

$$\begin{aligned}
\tilde{I}^2_{(4)}(12) &= \tfrac{1}{12}\left(1-\dot{G}^2_{B12}\right)Z_2(12)\Big\{\\
&+\dot{G}^2_{B31}\left(1-\dot{G}^2_{B31}\right)k_1\cdot f_4\cdot f_3\cdot k_1-\dot{G}_{B23}\dot{G}_{B31}\left(1-\dot{G}^2_{B31}\right)k_1\cdot f_4\cdot f_3\cdot k_2\\
&-\dot{G}_{B23}\dot{G}_{B31}\left(1-\dot{G}^2_{B23}\right)k_2\cdot f_4\cdot f_3\cdot k_1+\dot{G}^2_{B23}\left(1-\dot{G}^2_{B23}\right)k_2\cdot f_4\cdot f_3\cdot k_2\\
&+\dot{G}_{B12}\dot{G}_{B31}\left(1-\dot{G}^2_{B12}\right)\tfrac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_1}{r_4\cdot k_3}-\dot{G}_{B12}\dot{G}_{B23}\left(1-\dot{G}^2_{B12}\right)\tfrac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_2}{r_4\cdot k_3}\Big\},
\end{aligned} \tag{E.72}$$

$$\tilde{I}^2_{(4)}(13) = \tfrac{1}{12}\left(1-\dot{G}^2_{B31}\right) Z_2(13)\Big\{ \\ +\dot{G}^2_{B12}\left(1-\dot{G}^2_{B12}\right) k_1\cdot f_4\cdot f_2\cdot k_1 - \dot{G}_{B12}\dot{G}_{B23}\left(1-\dot{G}^2_{B12}\right) k_1\cdot f_4\cdot f_2\cdot k_3 \\ -\dot{G}_{B12}\dot{G}_{B23}\left(1-\dot{G}^2_{B23}\right) k_3\cdot f_4\cdot f_2\cdot k_1 + \dot{G}^2_{B23}\left(1-\dot{G}^2_{B23}\right) k_3\cdot f_4\cdot f_2\cdot k_3 \\ +\dot{G}_{B12}\dot{G}_{B31}\left(1-\dot{G}^2_{B31}\right) \tfrac{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_1}{r_5\cdot k_2} - \dot{G}_{B23}\dot{G}_{B31}\left(1-\dot{G}^2_{B31}\right) \tfrac{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_3}{r_5\cdot k_2}\Big\}, \tag{E.73}$$

$$\tilde{I}^2_{(4)}(23) = \tfrac{1}{12}\left(1-\dot{G}^2_{B23}\right) Z_2(23)\Big\{ \\ +\dot{G}^2_{B12}\left(1-\dot{G}^2_{B12}\right) k_2\cdot f_4\cdot f_1\cdot k_2 - \dot{G}_{B12}\dot{G}_{B31}\left(1-\dot{G}^2_{B12}\right) k_2\cdot f_4\cdot f_1\cdot k_3 \\ -\dot{G}_{B12}\dot{G}_{B31}\left(1-\dot{G}^2_{B31}\right) k_3\cdot f_4\cdot f_1\cdot k_2 + \dot{G}^2_{B31}\left(1-\dot{G}^2_{B31}\right) k_3\cdot f_4\cdot f_1\cdot k_3 \\ +\dot{G}_{B12}\dot{G}_{B23}\left(1-\dot{G}^2_{B23}\right) \tfrac{k_3\cdot f_4\cdot k_2 r_6\cdot f_1\cdot k_2}{r_6\cdot k_1} - \dot{G}_{B23}\dot{G}_{B31}\left(1-\dot{G}^2_{B23}\right) \tfrac{k_3\cdot f_4\cdot k_2 r_6\cdot f_1\cdot k_3}{r_6\cdot k_1}\Big\}, \tag{E.74}$$

$$\begin{aligned}\tilde{I}^2_{(4)}(14) = \tfrac{2Z_2(14)}{3r_7\cdot k_2 r_8\cdot k_3}\Big[&-\dot{G}_{B12}\dot{G}_{B31} r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_1 \\ &+\dot{G}_{B23}\dot{G}_{B31}\left(r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_3 + 2r_7\cdot f_2\cdot f_3\cdot r_8 k_3\cdot k_1\right) \\ &+\dot{G}_{B12}\dot{G}_{B23}\left(r_8\cdot f_3\cdot k_2 r_7\cdot f_2\cdot k_1 - r_7\cdot f_2\cdot f_3\cdot r_8 k_2\cdot k_1\right)\Big],\end{aligned} \tag{E.75}$$

$$\begin{aligned}\tilde{I}^2_{(4)}(24) = \tfrac{2Z_2(24)}{3r_9\cdot k_1 r_0\cdot k_3}\Big[&-\dot{G}_{B12}\dot{G}_{B23} r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_2 \\ &+\dot{G}_{B23}\dot{G}_{B31}\left(r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_3 + 2r_9\cdot f_1\cdot f_3\cdot r_0 k_3\cdot k_2\right) \\ &+\dot{G}_{B12}\dot{G}_{B31}\left(r_0\cdot f_3\cdot k_1 r_9\cdot f_1\cdot k_2 - r_9\cdot f_1\cdot f_3\cdot r_0 k_1\cdot k_2\right)\Big],\end{aligned} \tag{E.76}$$

$$\begin{aligned}\tilde{I}^2_{(4)}(34) = \tfrac{2Z_2(34)}{3r_{01}\cdot k_1 r_{02}\cdot k_2}\Big[&-\dot{G}_{B23}\dot{G}_{B31} r_{02}\cdot f_2\cdot k_3 r_{01}\cdot f_1\cdot k_3 \\ &+\dot{G}_{B12}\dot{G}_{B23}\left(r_{02}\cdot f_2\cdot k_3 r_{01}\cdot f_1\cdot k_2 + 2r_{01}\cdot f_1\cdot f_2\cdot r_{02} k_2\cdot k_3\right) \\ &+\dot{G}_{B12}\dot{G}_{B31}\left(r_{02}\cdot f_2\cdot k_1 r_{01}\cdot f_1\cdot k_3 - r_{01}\cdot f_1\cdot f_2\cdot r_{02} k_1\cdot k_3\right)\Big],\end{aligned} \tag{E.77}$$

$$\tilde{I}^{22}_{(4)}(12,34) = \frac{2}{3}\left(1-\dot{G}^2_{B12}\right) Z_2(12) Z_2(34), \tag{E.78}$$

$$\tilde{I}^{22}_{(4)}(13,24) = \frac{2}{3}\left(1-\dot{G}^2_{B31}\right) Z_2(13) Z_2(24), \tag{E.79}$$

$$\tilde{I}^{22}_{(4)}(14,23) = \frac{2}{3}\left(1-\dot{G}^2_{B23}\right) Z_2(14) Z_2(23). \tag{E.80}$$

E.1.6 $\tilde{I}_{(4)}$ in terms of G_{Bij} only

In the previous section, we write $\tilde{I}_{(4)}$ in terms of $\dot{G}_{Bij}$ because in this form it is easier to perform the integrals when we take the low energy limit in some leg but we realize that if we want to perform the integrals without any assumption on the energy of the photons it is better to have this expressions in terms of G_{Bij} only. Since, the spinor case have the same terms that the scalar case plus some extra terms we only need to use the equations (D.134), (D.135) and (D.136) in these extra terms then we sum this to our result from the scalar case ($I_{(4)}$ in terms of G_{Bij}) and we obtain

$$\begin{aligned}
\tilde{I}^4_{(4)}(1234) &= Z_4(1234)\Big[\left(2G_{B21}\tfrac{k_2\cdot k_1}{k_2\cdot k_3}+2G_{B23}\tfrac{k_2\cdot k_3}{k_2\cdot k_1}+b\right)\left(\tfrac{1}{3}-2G_{B13}\right)\\
&\quad+\left(2G_{B12}\tfrac{k_1\cdot k_2}{k_1\cdot k_3}+2G_{B13}\tfrac{k_1\cdot k_3}{k_1\cdot k_2}+a\right)+\left(2G_{B13}\tfrac{k_1\cdot k_3}{k_2\cdot k_3}+2G_{B23}\tfrac{k_2\cdot k_3}{k_1\cdot k_3}+c\right)\\
&\quad-(1-4G_{B13})\Big]\\
\tilde{I}^4_{(4)}(1243) &= Z_4(1243)\Big[\left(2G_{B12}\tfrac{k_1\cdot k_2}{k_1\cdot k_3}+2G_{B13}\tfrac{k_1\cdot k_3}{k_1\cdot k_2}+a\right)\left(\tfrac{1}{3}-2G_{B23}\right)\\
&+2\left(G_{B23}-G_{B12}-G_{B13}\right)-\left(2G_{B12}\tfrac{k_1\cdot k_2}{k_1\cdot k_3}+2G_{B13}\tfrac{k_1\cdot k_3}{k_1\cdot k_2}+a\right)\Big]\\
\tilde{I}^4_{(4)}(1324) &= Z_4(1324)\Big[\left(2G_{B13}\tfrac{k_1\cdot k_3}{k_2\cdot k_3}+2G_{B23}\tfrac{k_2\cdot k_3}{k_1\cdot k_3}+c\right)\left(\tfrac{1}{3}-2G_{B12}\right)\\
&\quad+\left(2G_{B12}\tfrac{k_1\cdot k_2}{k_1\cdot k_3}+2G_{B13}\tfrac{k_1\cdot k_3}{k_1\cdot k_2}+a\right)+\left(2G_{B21}\tfrac{k_2\cdot k_1}{k_2\cdot k_3}+2G_{B23}\tfrac{k_2\cdot k_3}{k_2\cdot k_1}+b\right)\\
&\quad-(1-4G_{B12})\Big],
\end{aligned}\tag{E.81}$$

$$
\begin{aligned}
\tilde{I}^3_{(4)}(123) = \tfrac{-T}{3} Z_3(123) \big[& -(1-4G_{B12})\, G_{B12} k_2 \cdot f_4 \cdot k_1 \\
& -(1-4G_{B23})\, G_{B23} k_3 \cdot f_4 \cdot k_2 - (1-4G_{B31})\, G_{B31} k_1 \cdot f_4 \cdot k_3 \\
& + \left(2G_{B13} \tfrac{k_1\cdot k_3}{k_2\cdot k_3} + 2G_{B23} \tfrac{k_2\cdot k_3}{k_1\cdot k_3} + c\right) (1-4G_{B12})\, G_{B12} k_2 \cdot f_4 \cdot k_1 \\
& + \left(2G_{B12} \tfrac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13} \tfrac{k_1\cdot k_3}{k_1\cdot k_2} + a\right) (1-4G_{B23})\, G_{B23} k_3 \cdot f_4 \cdot k_2 \\
& + \left(2G_{B21} \tfrac{k_2\cdot k_1}{k_2\cdot k_3} + 2G_{B23} \tfrac{k_2\cdot k_3}{k_2\cdot k_1} + b\right) (1-4G_{B31})\, G_{B31} k_1 \cdot f_4 \cdot k_3 \\
& + \left(2G_{B21} \tfrac{k_2\cdot k_1}{k_2\cdot k_3} + 2G_{B23} \tfrac{k_2\cdot k_3}{k_2\cdot k_1} + b\right) G_{B23} k_3 \cdot f_4 \cdot k_2 \\
& + \left(2G_{B12} \tfrac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13} \tfrac{k_1\cdot k_3}{k_1\cdot k_2} + a\right) G_{B31} k_1 \cdot f_4 \cdot k_3 \\
& + \left(2G_{B21} \tfrac{k_2\cdot k_1}{k_2\cdot k_3} + 2G_{B23} \tfrac{k_2\cdot k_3}{k_2\cdot k_1} + b\right) G_{B12} k_2 \cdot f_4 \cdot k_1 \\
& + \left(2G_{B13} \tfrac{k_1\cdot k_3}{k_2\cdot k_3} + 2G_{B23} \tfrac{k_2\cdot k_3}{k_1\cdot k_3} + c\right) G_{B31} k_1 \cdot f_4 \cdot k_3 \\
& + \left(2G_{B12} \tfrac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13} \tfrac{k_1\cdot k_3}{k_1\cdot k_2} + a\right) G_{B12} k_2 \cdot f_4 \cdot k_1 \\
& + \left(2G_{B13} \tfrac{k_1\cdot k_3}{k_2\cdot k_3} + 2G_{B23} \tfrac{k_2\cdot k_3}{k_1\cdot k_3} + c\right) G_{B23} k_3 \cdot f_4 \cdot k_2 \Big],
\end{aligned}
\tag{E.82}
$$

$$
\begin{aligned}
\tilde{I}^3_{(4)}(234) = & -\left(\tfrac{2}{3} + 2G_{B23}\right) Z_3(234) \left[\left(2G_{B21} \tfrac{k_2\cdot k_1}{k_2\cdot k_3} + 2G_{B23} \tfrac{k_2\cdot k_3}{k_2\cdot k_1} + b\right) \tfrac{r_1\cdot f_1\cdot k_2}{r_1\cdot k_1} \right. \\
& \left. - \left(2G_{B13} \tfrac{k_1\cdot k_3}{k_2\cdot k_3} + 2G_{B23} \tfrac{k_2\cdot k_3}{k_1\cdot k_3} + c\right) \tfrac{r_1\cdot f_1\cdot k_3}{r_1\cdot k_1} \right],
\end{aligned}
\tag{E.83}
$$

$$
\begin{aligned}
\tilde{I}^3_{(4)}(341) = & -\left(\tfrac{2}{3} + 2G_{B31}\right) Z_3(341) \left[\left(2G_{B12} \tfrac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13} \tfrac{k_1\cdot k_3}{k_1\cdot k_2} + a\right) \tfrac{r_2\cdot f_2\cdot k_1}{r_2\cdot k_2} \right. \\
& \left. - \left(2G_{B13} \tfrac{k_1\cdot k_3}{k_2\cdot k_3} + 2G_{B23} \tfrac{k_2\cdot k_3}{k_1\cdot k_3} + c\right) \tfrac{r_2\cdot f_2\cdot k_3}{r_2\cdot k_2} \right],
\end{aligned}
\tag{E.84}
$$

$$
\begin{aligned}
\tilde{I}^3_{(4)}(412) = & -\left(\tfrac{2}{3} + 2G_{B12}\right) Z_3(412) \left[\left(2G_{B12} \tfrac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13} \tfrac{k_1\cdot k_3}{k_1\cdot k_2} + a\right) \tfrac{r_3\cdot f_3\cdot k_1}{r_3\cdot k_3} \right. \\
& \left. - \left(2G_{B21} \tfrac{k_2\cdot k_1}{k_2\cdot k_3} + 2G_{B23} \tfrac{k_2\cdot k_3}{k_2\cdot k_1} + b\right) \tfrac{r_3\cdot f_3\cdot k_2}{r_3\cdot k_3} \right].
\end{aligned}
\tag{E.85}
$$

$$
\begin{aligned}
& \tilde{I}^2_{(4)}(12) = \tfrac{4}{3} T G_{B12} Z_2(12) \Big\{ \\
& + (1-4G_{B31})\, G_{B31} k_1 \cdot f_4 \cdot f_3 \cdot k_1 + (1-4G_{B23})\, G_{B23} k_2 \cdot f_4 \cdot f_3 \cdot k_2 \\
& + \left(2G_{B13} \tfrac{k_1\cdot k_3}{k_2\cdot k_3} + 2G_{B23} \tfrac{k_2\cdot k_3}{k_1\cdot k_3} + c\right) \left[G_{B31} k_1 \cdot f_4 \cdot f_3 \cdot k_2 + G_{B23} k_2 \cdot f_4 \cdot f_3 \cdot k_1 \right] \\
& - \left(2G_{B12} \tfrac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13} \tfrac{k_1\cdot k_3}{k_1\cdot k_2} + a\right) G_{B12} \tfrac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_1}{r_4\cdot k_3} \\
& + \left(2G_{B21} \tfrac{k_2\cdot k_1}{k_2\cdot k_3} + 2G_{B23} \tfrac{k_2\cdot k_3}{k_2\cdot k_1} + b\right) G_{B12} \tfrac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_2}{r_4\cdot k_3} \Big\},
\end{aligned}
\tag{E.86}
$$

$$\tilde{I}^2_{(4)}(13) = \tfrac{4}{3}TG_{B31}Z_2(13)\Big\{$$
$$+ (1-4G_{B12})\,G_{B12}k_1\cdot f_4\cdot f_2\cdot k_1 + (1-4G_{B23})\,G_{B23}k_3\cdot f_4\cdot f_2\cdot k_3$$
$$+ \left(2G_{B21}\tfrac{k_2\cdot k_1}{k_2\cdot k_3} + 2G_{B23}\tfrac{k_2\cdot k_3}{k_2\cdot k_1} + b\right)[G_{B12}k_1\cdot f_4\cdot f_2\cdot k_3 + G_{B23}k_3\cdot f_4\cdot f_2\cdot k_1]$$
$$- \left(2G_{B12}\tfrac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13}\tfrac{k_1\cdot k_3}{k_1\cdot k_2} + a\right)G_{B31}\tfrac{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_1}{r_5\cdot k_2}$$
$$+ \left(2G_{B13}\tfrac{k_1\cdot k_3}{k_2\cdot k_3} + 2G_{B23}\tfrac{k_2\cdot k_3}{k_1\cdot k_3} + c\right)G_{B31}\tfrac{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_3}{r_5\cdot k_2}\Big\}, \tag{E.87}$$

$$\tilde{I}^2_{(4)}(23) = \tfrac{4}{3}TG_{B23}Z_2(23)\Big\{$$
$$+ (1-4G_{B12})\,G_{B12}k_2\cdot f_4\cdot f_1\cdot k_2 + (1-4G_{B31})\,G_{B31}k_3\cdot f_4\cdot f_1\cdot k_3$$
$$+ \left(2G_{B12}\tfrac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13}\tfrac{k_1\cdot k_3}{k_1\cdot k_2} + a\right)[G_{B12}k_2\cdot f_4\cdot f_1\cdot k_3 + G_{B31}k_3\cdot f_4\cdot f_1\cdot k_2]$$
$$- \left(2G_{B21}\tfrac{k_2\cdot k_1}{k_2\cdot k_3} + 2G_{B23}\tfrac{k_2\cdot k_3}{k_2\cdot k_1} + b\right)G_{B23}\tfrac{k_3\cdot f_4\cdot k_2 r_6\cdot f_1\cdot k_2}{r_6\cdot k_1}$$
$$+ \left(2G_{B13}\tfrac{k_1\cdot k_3}{k_2\cdot k_3} + 2G_{B23}\tfrac{k_2\cdot k_3}{k_1\cdot k_3} + c\right)G_{B23}\tfrac{k_3\cdot f_4\cdot k_2 r_6\cdot f_1\cdot k_3}{r_6\cdot k_1}\Big\}, \tag{E.88}$$

$$\tilde{I}^2_{(4)}(14) = \tfrac{2Z_2(14)}{3r_7\cdot k_2 r_8\cdot k_3}\Big[\left(2G_{B12}\tfrac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13}\tfrac{k_1\cdot k_3}{k_1\cdot k_2} + a\right)r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_1$$
$$- \left(2G_{B13}\tfrac{k_1\cdot k_3}{k_2\cdot k_3} + 2G_{B23}\tfrac{k_2\cdot k_3}{k_1\cdot k_3} + c\right)(r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_3 + 2r_7\cdot f_2\cdot f_3\cdot r_8 k_3\cdot k_1)$$
$$- \left(2G_{B21}\tfrac{k_2\cdot k_1}{k_2\cdot k_3} + 2G_{B23}\tfrac{k_2\cdot k_3}{k_2\cdot k_1} + b\right)(r_8\cdot f_3\cdot k_2 r_7\cdot f_2\cdot k_1 - r_7\cdot f_2\cdot f_3\cdot r_8 k_2\cdot k_1)\Big], \tag{E.89}$$

$$\tilde{I}^2_{(4)}(24) = \tfrac{2Z_2(24)}{3r_9\cdot k_1 r_0\cdot k_3}\Big[\left(2G_{B21}\tfrac{k_2\cdot k_1}{k_2\cdot k_3} + 2G_{B23}\tfrac{k_2\cdot k_3}{k_2\cdot k_1} + b\right)r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_2$$
$$- \left(2G_{B13}\tfrac{k_1\cdot k_3}{k_2\cdot k_3} + 2G_{B23}\tfrac{k_2\cdot k_3}{k_1\cdot k_3} + c\right)(r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_3 + 2r_9\cdot f_1\cdot f_3\cdot r_0 k_3\cdot k_2)$$
$$- \left(2G_{B12}\tfrac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13}\tfrac{k_1\cdot k_3}{k_1\cdot k_2} + a\right)(r_0\cdot f_3\cdot k_1 r_9\cdot f_1\cdot k_2 - r_9\cdot f_1\cdot f_3\cdot r_0 k_1\cdot k_2)\Big], \tag{E.90}$$

$$\tilde{I}^2_{(4)}(34) = \tfrac{2Z_2(34)}{3r_{01}\cdot k_1 r_{02}\cdot k_2}\Big[\left(2G_{B13}\tfrac{k_1\cdot k_3}{k_2\cdot k_3} + 2G_{B23}\tfrac{k_2\cdot k_3}{k_1\cdot k_3} + c\right)r_{02}\cdot f_2\cdot k_3 r_{01}\cdot f_1\cdot k_3$$
$$- \left(2G_{B21}\tfrac{k_2\cdot k_1}{k_2\cdot k_3} + 2G_{B23}\tfrac{k_2\cdot k_3}{k_2\cdot k_1} + b\right)(r_{02}\cdot f_2\cdot k_3 r_{01}\cdot f_1\cdot k_2 + 2r_{01}\cdot f_1\cdot f_2\cdot r_{02}k_2\cdot k_3)$$
$$- \left(2G_{B12}\tfrac{k_1\cdot k_2}{k_1\cdot k_3} + 2G_{B13}\tfrac{k_1\cdot k_3}{k_1\cdot k_2} + a\right)(r_{02}\cdot f_2\cdot k_1 r_{01}\cdot f_1\cdot k_3 - r_{01}\cdot f_1\cdot f_2\cdot r_{02}k_1\cdot k_3)\Big], \tag{E.91}$$

$$\tilde{I}^{22}_{(4)}(12,34) = \frac{8}{3}G_{B12}Z_2(12)Z_2(34), \tag{E.92}$$

$$\tilde{I}^{22}_{(4)}(13,24) = \frac{8}{3}G_{B31}Z_2(13)Z_2(24), \tag{E.93}$$

$$\tilde{I}^{22}_{(4)}(14,23) = \frac{8}{3} G_{B23} Z_2(14) Z_2(23). \tag{E.94}$$

E.2 Two low energy photons

Since, we have already integrated u_4 with the assumption that k_4 is low energy. Now, we do the same for k_3. Then, we cancel all the terms $O(k_3^2)$ in $\tilde{I}_{(4)}$ (in terms of $\dot{G}_{Bij}$ and with $T = 1$)

$$\begin{aligned}
\tilde{I}^4_{(4)}(1234) &= Z_4(1234) \left[\dot{G}_{B12}\dot{G}_{B23} \left(\tfrac{1}{6} - \tfrac{1}{2}\dot{G}^2_{B31} \right) \right. \\
&\left. + \left(\dot{G}_{B12} + \dot{G}_{B23} + \dot{G}_{B31} \right) \dot{G}_{B13} \right], \\
\tilde{I}^4_{(4)}(1243) &= Z_4(1243) \left[\dot{G}_{B12}\dot{G}_{B31} \left(\tfrac{1}{6} - \tfrac{1}{2}\dot{G}^2_{B23} \right) \right. \\
&\left. + \tfrac{1}{2} \left(\dot{G}^2_{B12} + \dot{G}^2_{B31} - \dot{G}^2_{B23} - 1 \right) + \dot{G}_{B12}\dot{G}_{B31} \right], \\
\tilde{I}^4_{(4)}(1324) &= Z_4(1324) \left[\dot{G}_{B23}\dot{G}_{B31} \left(\tfrac{1}{6} - \tfrac{1}{2}\dot{G}^2_{B12} \right) \right. \\
&\left. + \left(\dot{G}_{B13} + \dot{G}_{B32} + \dot{G}_{B21} \right) \dot{G}_{B12} \right],
\end{aligned} \tag{E.95}$$

$$\begin{aligned}
\tilde{I}^3_{(4)}(123) &= \tfrac{1}{12} \left(\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31} + \dot{G}_{B12} + \dot{G}_{B23} + \dot{G}_{B31} \right) Z_3(123) \\
&\times \left[\dot{G}_{B12} \left(1 - \dot{G}^2_{B12} \right) k_2 \cdot f_4 \cdot k_1 + \dot{G}_{B23} \left(1 - \dot{G}^2_{B23} \right) \cancel{k_3 \cdot f_4 \cdot k_2} \right. \\
&\left. + \dot{G}_{B31} \left(1 - \dot{G}^2_{B31} \right) \cancel{k_1 \cdot f_4 \cdot k_3} \right],
\end{aligned} \tag{E.96}$$

$$\tilde{I}^3_{(4)}(234) = \frac{1}{2}\dot{G}_{B23} \left(\frac{7}{3} - \dot{G}^2_{B23} \right) Z_3(234) \left(\dot{G}_{B12} \frac{r_1 \cdot f_1 \cdot k_2}{r_1 \cdot k_1} - \dot{G}_{B31} \frac{\cancel{r_1 \cdot f_1 \cdot k_3}}{r_1 \cdot k_1} \right), \tag{E.97}$$

$$\tilde{I}^3_{(4)}(341) = \frac{1}{2} \left(\frac{7}{3} - \dot{G}^2_{B31} \right) \dot{G}_{B31} Z_3(341) \left(\dot{G}_{B12} \frac{r_2 \cdot f_2 \cdot k_1}{r_2 \cdot k_2} - \dot{G}_{B23} \frac{\cancel{r_2 \cdot f_2 \cdot k_3}}{r_2 \cdot k_2} \right), \tag{E.98}$$

$$\tilde{I}^3_{(4)}(412) = \frac{1}{2} \left(\frac{7}{3} - \dot{G}^2_{B12} \right) \dot{G}_{B12} Z_3(412) \left(\dot{G}_{B31} \frac{r_3 \cdot f_3 \cdot k_1}{r_3 \cdot k_3} - \dot{G}_{B23} \frac{r_3 \cdot f_3 \cdot k_2}{r_3 \cdot k_3} \right). \tag{E.99}$$

$$\begin{aligned}\tilde{I}^2_{(4)}(12) = \tfrac{1}{12}\left(1-\dot{G}^2_{B12}\right) Z_2(12)\Big\{ \\ +\dot{G}^2_{B31}\left(1-\dot{G}^2_{B31}\right) k_1\cdot f_4\cdot f_3\cdot k_1 - \dot{G}_{B23}\dot{G}_{B31}\left(1-\dot{G}^2_{B31}\right) k_1\cdot f_4\cdot f_3\cdot k_2 \\ -\dot{G}_{B23}\dot{G}_{B31}\left(1-\dot{G}^2_{B23}\right) k_2\cdot f_4\cdot f_3\cdot k_1 + \dot{G}^2_{B23}\left(1-\dot{G}^2_{B23}\right) k_2\cdot f_4\cdot f_3\cdot k_2 \\ +\dot{G}_{B12}\dot{G}_{B31}\left(1-\dot{G}^2_{B12}\right) \tfrac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_1}{r_4\cdot k_3} - \dot{G}_{B12}\dot{G}_{B23}\left(1-\dot{G}^2_{B12}\right) \tfrac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_2}{r_4\cdot k_3}\Big\},\end{aligned} \tag{E.100}$$

$$\begin{aligned}\tilde{I}^2_{(4)}(13) = \tfrac{1}{12}\left(1-\dot{G}^2_{B31}\right) Z_2(13)\Big\{ \\ +\dot{G}^2_{B12}\left(1-\dot{G}^2_{B12}\right) k_1\cdot f_4\cdot f_2\cdot k_1 - \dot{G}_{B12}\dot{G}_{B23}\left(1-\dot{G}^2_{B12}\right) \cancel{k_1\cdot f_4\cdot f_2\cdot k_3} \\ -\dot{G}_{B12}\dot{G}_{B23}\left(1-\dot{G}^2_{B23}\right) \cancel{k_3\cdot f_4\cdot f_2\cdot k_1} + \dot{G}^2_{B23}\left(1-\dot{G}^2_{B23}\right) \cancel{k_3\cdot f_4\cdot f_2\cdot k_3} \\ +\dot{G}_{B12}\dot{G}_{B31}\left(1-\dot{G}^2_{B31}\right) \tfrac{\cancel{k_3\cdot f_4\cdot k_1} r_5\cdot f_2\cdot k_1}{r_5\cdot k_2} - \dot{G}_{B23}\dot{G}_{B31}\left(1-\dot{G}^2_{B31}\right) \tfrac{\cancel{k_3\cdot f_4\cdot k_1 r_5\cdot f_2\cdot k_3}}{r_5\cdot k_2}\Big\},\end{aligned} \tag{E.101}$$

$$\begin{aligned}\tilde{I}^2_{(4)}(23) = \tfrac{1}{12}\left(1-\dot{G}^2_{B23}\right) Z_2(23)\Big\{ \\ +\dot{G}^2_{B12}\left(1-\dot{G}^2_{B12}\right) k_2\cdot f_4\cdot f_1\cdot k_2 - \dot{G}_{B12}\dot{G}_{B31}\left(1-\dot{G}^2_{B12}\right) \cancel{k_2\cdot f_4\cdot f_1\cdot k_3} \\ -\dot{G}_{B12}\dot{G}_{B31}\left(1-\dot{G}^2_{B31}\right) \cancel{k_3\cdot f_4\cdot f_1\cdot k_2} + \dot{G}^2_{B31}\left(1-\dot{G}^2_{B31}\right) \cancel{k_3\cdot f_4\cdot f_1\cdot k_3} \\ +\dot{G}_{B12}\dot{G}_{B23}\left(1-\dot{G}^2_{B23}\right) \tfrac{\cancel{k_3\cdot f_4\cdot k_2} r_6\cdot f_1\cdot k_2}{r_6\cdot k_1} - \dot{G}_{B23}\dot{G}_{B31}\left(1-\dot{G}^2_{B23}\right) \tfrac{\cancel{k_3\cdot f_4\cdot k_2 r_6\cdot f_1\cdot k_3}}{r_6\cdot k_1}\Big\},\end{aligned} \tag{E.102}$$

$$\begin{aligned}\tilde{I}^2_{(4)}(14) = \tfrac{2Z_2(14)}{3r_7\cdot k_2 r_8\cdot k_3}\Big[-\dot{G}_{B12}\dot{G}_{B31} r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_1 \\ +\dot{G}_{B23}\dot{G}_{B31}\left(r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_3 + 2r_7\cdot f_2\cdot f_3\cdot r_8 k_3\cdot k_1\right) \\ +\dot{G}_{B12}\dot{G}_{B23}\left(r_8\cdot f_3\cdot k_2 r_7\cdot f_2\cdot k_1 - r_7\cdot f_2\cdot f_3\cdot r_8 k_2\cdot k_1\right)\Big],\end{aligned} \tag{E.103}$$

$$\begin{aligned}\tilde{I}^2_{(4)}(24) = \tfrac{2Z_2(24)}{3r_9\cdot k_1 r_0\cdot k_3}\Big[-\dot{G}_{B12}\dot{G}_{B23} r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_2 \\ +\dot{G}_{B23}\dot{G}_{B31}\left(r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_3 + 2r_9\cdot f_1\cdot f_3\cdot r_0 k_3\cdot k_2\right) \\ +\dot{G}_{B12}\dot{G}_{B31}\left(r_0\cdot f_3\cdot k_1 r_9\cdot f_1\cdot k_2 - r_9\cdot f_1\cdot f_3\cdot r_0 k_1\cdot k_2\right)\Big],\end{aligned} \tag{E.104}$$

$$\begin{aligned}\tilde{I}^2_{(4)}(34) = \tfrac{2Z_2(34)}{3r_{01}\cdot k_1 r_{02}\cdot k_2}\Big[-\dot{G}_{B23}\dot{G}_{B31}\cancel{r_{02}\cdot f_2\cdot k_3 r_{01}\cdot f_1\cdot k_3} \\ +\dot{G}_{B12}\dot{G}_{B23}\left(\cancel{r_{02}\cdot f_2\cdot k_3} r_{01}\cdot f_1\cdot k_2 + 2r_{01}\cdot f_1\cdot f_2\cdot r_{02}\cancel{k_2\cdot k_3}\right) \\ +\dot{G}_{B12}\dot{G}_{B31}\left(r_{02}\cdot f_2\cdot k_1\cancel{r_{01}\cdot f_1\cdot k_3} - r_{01}\cdot f_1\cdot f_2\cdot r_{02}\cancel{k_1\cdot k_3}\right)\Big],\end{aligned} \tag{E.105}$$

$$\tilde{I}^{22}_{(4)}(12,34) = \frac{2}{3}\left(1-\dot{G}^2_{B12}\right) Z_2(12) Z_2(34), \tag{E.106}$$

$$\tilde{I}^{22}_{(4)}(13,24) = \frac{2}{3}\left(1-\dot{G}^2_{B31}\right) Z_2(13)Z_2(24), \tag{E.107}$$

$$\tilde{I}^{22}_{(4)}(14,23) = \frac{2}{3}\left(1-\dot{G}^2_{B23}\right) Z_2(14)Z_2(23). \tag{E.108}$$

This leaves us with the following terms to be integrated

$$\begin{aligned}
&\tilde{I}^4_{(34)}(1234) = Z_4(1234)\int_0^1 \left[\dot{G}_{B12}\dot{G}_{B23}\left(\tfrac{1}{6}-\tfrac{1}{2}\dot{G}^2_{B31}\right)\right.\\
&\left.+\left(\dot{G}_{B12}+\dot{G}_{B23}+\dot{G}_{B31}\right)\dot{G}_{B13}\right]du_3,\\
&\tilde{I}^4_{(34)}(1243) = Z_4(1243)\int_0^1 \left[\dot{G}_{B12}\dot{G}_{B31}\left(\tfrac{1}{6}-\tfrac{1}{2}\dot{G}^2_{B23}\right)\right.\\
&\left.+\tfrac{1}{2}\left(\dot{G}^2_{B12}+\dot{G}^2_{B31}-\dot{G}^2_{B23}-1\right)+\dot{G}_{B12}\dot{G}_{B31}\right]du_3,\\
&\tilde{I}^4_{(34)}(1324) = Z_4(1324)\int_0^1 \left[\dot{G}_{B23}\dot{G}_{B31}\left(\tfrac{1}{6}-\tfrac{1}{2}\dot{G}^2_{B12}\right)\right.\\
&\left.+\left(\dot{G}_{B13}+\dot{G}_{B32}+\dot{G}_{B21}\right)\dot{G}_{B12}\right]du_3,
\end{aligned} \tag{E.109}$$

$$\begin{aligned}
\tilde{I}^3_{(34)}(123) =& \frac{1}{12}\int_0^1 \left(\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31}+\dot{G}_{B12}+\dot{G}_{B23}+\dot{G}_{B31}\right)\\
&\times Z_3(123)\dot{G}_{B12}\left(1-\dot{G}^2_{B12}\right)k_2\cdot f_4\cdot k_1 du_3,
\end{aligned} \tag{E.110}$$

$$\tilde{I}^3_{(34)}(234) = \frac{1}{2}\int_0^1 \dot{G}_{B23}\left(\frac{7}{3}-\dot{G}^2_{B23}\right) Z_3(234)\dot{G}_{B12}\frac{r_1\cdot f_1\cdot k_2}{r_1\cdot k_1}du_3, \tag{E.111}$$

$$\tilde{I}^3_{(34)}(341) = \frac{1}{2}\int_0^1 \left(\frac{7}{3}-\dot{G}^2_{B31}\right)\dot{G}_{B31}Z_3(341)\dot{G}_{B12}\frac{r_2\cdot f_2\cdot k_1}{r_2\cdot k_2}du_3, \tag{E.112}$$

$$\begin{aligned}
\tilde{I}^3_{(34)}(412) =& \frac{1}{2}\int_0^1 \left(\frac{7}{3}-\dot{G}^2_{B12}\right)\dot{G}_{B12}Z_3(412)\\
&\times\left(\dot{G}_{B31}\frac{r_3\cdot f_3\cdot k_1}{r_3\cdot k_3}-\dot{G}_{B23}\frac{r_3\cdot f_3\cdot k_2}{r_3\cdot k_3}\right)e^{()}du_3,
\end{aligned} \tag{E.113}$$

$$\begin{aligned}
&\tilde{I}^2_{(34)}(12) = \tfrac{1}{12}\int_0^1 \left(1-\dot{G}^2_{B12}\right)Z_2(12)\Big\{\\
&+\dot{G}^2_{B31}\left(1-\dot{G}^2_{B31}\right)k_1\cdot f_4\cdot f_3\cdot k_1-\dot{G}_{B23}\dot{G}_{B31}\left(1-\dot{G}^2_{B31}\right)k_1\cdot f_4\cdot f_3\cdot k_2\\
&-\dot{G}_{B23}\dot{G}_{B31}\left(1-\dot{G}^2_{B23}\right)k_2\cdot f_4\cdot f_3\cdot k_1+\dot{G}^2_{B23}\left(1-\dot{G}^2_{B23}\right)k_2\cdot f_4\cdot f_3\cdot k_2\\
&+\left[\dot{G}_{B12}\dot{G}_{B31}\left(1-\dot{G}^2_{B12}\right)\tfrac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_1}{r_4\cdot k_3}\right.\\
&\left.\quad-\dot{G}_{B12}\dot{G}_{B23}\left(1-\dot{G}^2_{B12}\right)\tfrac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_2}{r_4\cdot k_3}\right]e^{()}\Big\}du_3,
\end{aligned} \tag{E.114}$$

$$\tilde{I}^2_{(34)}(13) = \frac{1}{12}\int_0^1 \left(1-\dot{G}^2_{B31}\right) Z_2(13)\dot{G}^2_{B12}\left(1-\dot{G}^2_{B12}\right) k_1\cdot f_4\cdot f_2\cdot k_1 du_3, \tag{E.115}$$

$$\tilde{I}^2_{(34)}(23) = \frac{1}{12}\int_0^1 \left(1-\dot{G}^2_{B23}\right) Z_2(23)\dot{G}^2_{B12}\left(1-\dot{G}^2_{B12}\right) k_2\cdot f_4\cdot f_1\cdot k_2 du_3, \tag{E.116}$$

$$\begin{aligned}\tilde{I}^2_{(34)}(14) = \tfrac{2Z_2(14)}{3r_7\cdot k_2 r_8\cdot k_3}\int_0^1 \Big\{\Big[&-\dot{G}_{B12}\dot{G}_{B31} r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_1 \\ &+\dot{G}_{B12}\dot{G}_{B23}\left(r_8\cdot f_3\cdot k_2 r_7\cdot f_2\cdot k_1 - r_7\cdot f_2\cdot f_3\cdot r_8 k_2\cdot k_1\right)\Big] e^{()} \\ &+\dot{G}_{B23}\dot{G}_{B31}\left(r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_3 + 2r_7\cdot f_2\cdot f_3\cdot r_8 k_3\cdot k_1\right)\Big\} du_3,\end{aligned} \tag{E.117}$$

$$\begin{aligned}\tilde{I}^2_{(34)}(24) = \tfrac{2Z_2(24)}{3r_9\cdot k_1 r_0\cdot k_3}\int_0^1 \Big\{\Big[&-\dot{G}_{B12}\dot{G}_{B23} r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_2 \\ &+\dot{G}_{B12}\dot{G}_{B31}\left(r_0\cdot f_3\cdot k_1 r_9\cdot f_1\cdot k_2 - r_9\cdot f_1\cdot f_3\cdot r_0 k_1\cdot k_2\right)\Big] e^{()} \\ &+\dot{G}_{B23}\dot{G}_{B31}\left(r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_3 + 2r_9\cdot f_1\cdot f_3\cdot r_0 k_3\cdot k_2\right)\Big\} du_3,\end{aligned} \tag{E.118}$$

$$\tilde{I}^{22}_{(34)}(12,34) = \frac{2}{3}\int_0^1 \left(1-\dot{G}^2_{B12}\right) Z_2(12)Z_2(34) du_3, \tag{E.119}$$

$$\tilde{I}^{22}_{(34)}(13,24) = \frac{2}{3}\int_0^1 \left(1-\dot{G}^2_{B31}\right) Z_2(13)Z_2(24) du_3, \tag{E.120}$$

$$\tilde{I}^{22}_{(34)}(14,23) = \frac{2}{3}\int_0^1 \left(1-\dot{G}^2_{B23}\right) Z_2(14)Z_2(23) du_3. \tag{E.121}$$

Some terms cancel because of the integration. We use some particular cases of (B.11) enlisted in appendix B in order to perform the integrals over u_3, and we obtain

$$\begin{aligned}\tilde{I}^4_{(34)}(1234) &= \left[\tfrac{1}{6}\dot{G}_{B12}\left(\dot{G}_{B12}-\dot{G}^3_{B12}\right) - \left(\tfrac{1}{6}-\tfrac{1}{2}\dot{G}^2_{B12}\right) - \tfrac{1}{3}\right] Z_4(1234), \\ \tilde{I}^4_{(34)}(1243) &= \left[\tfrac{1}{6}\dot{G}_{B12}\left(\dot{G}_{B12}-\dot{G}^3_{B12}\right) + \tfrac{1}{2}\left(\dot{G}^2_{B12}-1\right)\right] Z_4(1243), \\ \tilde{I}^4_{(34)}(1324) &= \left[\left(\tfrac{1}{6}-\tfrac{1}{2}\dot{G}^2_{B12}\right)\left(\tfrac{1}{6}-\tfrac{1}{2}\dot{G}^2_{B21}\right) - \dot{G}^2_{B12}\right] Z_4(1324),\end{aligned} \tag{E.122}$$

$$\tilde{I}^3_{(34)}(123) = \frac{1}{12}\left(\dot{G}^2_{B12}-\dot{G}^4_{B12}\right)\left(\frac{1}{6}-\frac{1}{2}\dot{G}^2_{B12}+1\right) Z_3(123) k_2\cdot f_4\cdot k_1, \tag{E.123}$$

$$\tilde{I}^{3}_{(34)}(412) = \tfrac{1}{2}\left(\tfrac{7}{3} - \dot{G}^{2}_{B12}\right)\dot{G}_{B12}Z_3(412)\Big[\tfrac{k_2\cdot k_3}{12}\left(\dot{G}_{B12} - \dot{G}^{3}_{B12}\right)\tfrac{r_3\cdot f_3\cdot k_1}{r_3\cdot k_3} - \tfrac{k_3\cdot k_1}{12}\left(\dot{G}_{B12} - \dot{G}^{3}_{B12}\right)\tfrac{r_3\cdot f_3\cdot k_2}{r_3\cdot k_3}\Big], \tag{E.124}$$

$$\begin{aligned}\tilde{I}^{2}_{(34)}(12) = {} & \tfrac{1}{12}\left(1 - \dot{G}^{2}_{B12}\right)Z_2(12)\Big\{ \\ & + \left(\tfrac{1}{3} - \tfrac{1}{5}\right)k_1\cdot f_4\cdot f_3\cdot k_1 - \left[\left(\tfrac{1}{6} - \tfrac{1}{2}\dot{G}^{2}_{B12}\right) - \left(\tfrac{3!}{5!} - \tfrac{1}{4}\dot{G}^{4}_{B12}\right)\right]k_1\cdot f_4\cdot f_3\cdot k_2 \\ & - \left[\left(\tfrac{1}{6} - \tfrac{1}{2}\dot{G}^{2}_{B12}\right) - \left(\tfrac{3!}{5!} - \tfrac{1}{4}\dot{G}^{4}_{B12}\right)\right]k_2\cdot f_4\cdot f_3\cdot k_1 + \left(\tfrac{1}{3} - \tfrac{1}{5}\right)k_2\cdot f_4\cdot f_3\cdot k_2 \\ & + \Big[\dot{G}_{B12}\tfrac{k_2\cdot k_3}{12}\left(\dot{G}_{B12} - \dot{G}^{3}_{B12}\right)\left(1 - \dot{G}^{2}_{B12}\right)\tfrac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_1}{r_4\cdot k_3} \\ & - \dot{G}_{B12}\tfrac{k_3\cdot k_1}{12}\left(\dot{G}_{B12} - \dot{G}^{3}_{B12}\right)\left(1 - \dot{G}^{2}_{B12}\right)\tfrac{k_2\cdot f_4\cdot k_1 r_4\cdot f_3\cdot k_2}{r_4\cdot k_3}\Big]\Big\},\end{aligned} \tag{E.125}$$

$$\tilde{I}^{2}_{(34)}(13) = \frac{1}{18}Z_2(13)\dot{G}^{2}_{B12}\left(1 - \dot{G}^{2}_{B12}\right)k_1\cdot f_4\cdot f_2\cdot k_1, \tag{E.126}$$

$$\tilde{I}^{2}_{(34)}(23) = \frac{1}{18}Z_2(23)\dot{G}^{2}_{B12}\left(1 - \dot{G}^{2}_{B12}\right)k_2\cdot f_4\cdot f_1\cdot k_2, \tag{E.127}$$

$$\begin{aligned}\tilde{I}^{2}_{(34)}(14) = {} & \tfrac{2Z_2(14)}{3r_7\cdot k_2 r_8\cdot k_3}\Big[- \dot{G}_{B12}\tfrac{k_2\cdot k_3}{12}\left(\dot{G}_{B12} - \dot{G}^{3}_{B12}\right)r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_1 \\ & + \dot{G}_{B12}\tfrac{k_3\cdot k_1}{12}\left(\dot{G}_{B12} - \dot{G}^{3}_{B12}\right)(r_8\cdot f_3\cdot k_2 r_7\cdot f_2\cdot k_1 - r_7\cdot f_2\cdot f_3\cdot r_8 k_2\cdot k_1) \\ & + \left(\tfrac{1}{6} - \tfrac{1}{2}\dot{G}^{2}_{B12}\right)(r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_3 + 2r_7\cdot f_2\cdot f_3\cdot r_8 k_3\cdot k_1)\Big],\end{aligned} \tag{E.128}$$

$$\begin{aligned}\tilde{I}^{2}_{(34)}(24) = {} & \tfrac{2Z_2(24)}{3r_9\cdot k_1 r_0\cdot k_3}\Big[- \dot{G}_{B12}\tfrac{k_3\cdot k_1}{12}\left(\dot{G}_{B12} - \dot{G}^{3}_{B12}\right)r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_2 \\ & + \dot{G}_{B12}\tfrac{k_2\cdot k_3}{12}\left(\dot{G}_{B12} - \dot{G}^{3}_{B12}\right)(r_0\cdot f_3\cdot k_1 r_9\cdot f_1\cdot k_2 - r_9\cdot f_1\cdot f_3\cdot r_0 k_1\cdot k_2) \\ & + \left(\tfrac{1}{6} - \tfrac{1}{2}\dot{G}^{2}_{B12}\right)(r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_3 + 2r_9\cdot f_1\cdot f_3\cdot r_0 k_3\cdot k_2)\Big],\end{aligned} \tag{E.129}$$

$$\tilde{I}^{22}_{(34)}(12,34) = \frac{2}{3}\left(1 - \dot{G}^{2}_{B12}\right)Z_2(12)Z_2(34), \tag{E.130}$$

$$\tilde{I}^{22}_{(34)}(13,24) = \frac{4}{9}Z_2(13)Z_2(24), \tag{E.131}$$

$$\tilde{I}^{22}_{(34)}(14,23) = \frac{4}{9}Z_2(14)Z_2(23). \tag{E.132}$$

Simplifying

$$\begin{aligned}
\tilde{I}^4_{(34)}(1234) &= \tfrac{1}{6}\left(4\dot{G}^2_{B12} - \dot{G}^4_{B12} - 3\right) Z_4(1234),\\
\tilde{I}^4_{(34)}(1243) &= \tfrac{1}{6}\left(4\dot{G}^2_{B12} - \dot{G}^4_{B12} - 3\right) Z_4(1243),\\
\tilde{I}^4_{(34)}(1324) &= \tfrac{1}{6^2}\left(1 - 42\dot{G}^2_{B12} + 9\dot{G}^4_{B21}\right) Z_4(1324),
\end{aligned} \tag{E.133}$$

$$\tilde{I}^3_{(34)}(123) = \frac{1}{72}\left(7\dot{G}^2_{B12} - 10\dot{G}^4_{B12} + 3\dot{G}^6_{B12}\right) Z_3(123) k_2 \cdot f_4 \cdot k_1, \tag{E.134}$$

$$\begin{aligned}
\tilde{I}^3_{(34)}(412) =& \frac{1}{72}\left(7\dot{G}^2_{B12} - 10\dot{G}^4_{B12} + 3\dot{G}^6_{B12}\right) Z_3(412)\\
&\times \left(\frac{k_2 \cdot k_3 r_3 \cdot f_3 \cdot k_1}{r_3 \cdot k_3} - \frac{k_3 \cdot k_1 r_3 \cdot f_3 \cdot k_2}{r_3 \cdot k_3}\right),
\end{aligned} \tag{E.135}$$

$$\begin{aligned}
\tilde{I}^2_{(34)}(12) =& \tfrac{1}{12}\left(1 - \dot{G}^2_{B12}\right) Z_2(12)\Big[\tfrac{2}{15}\left(k_1 \cdot f_4 \cdot f_3 \cdot k_1 + k_2 \cdot f_4 \cdot f_3 \cdot k_2\right)\\
&-\tfrac{1}{12}\left(\tfrac{7}{5} - 6\dot{G}^2_{B12} + 3\dot{G}^4_{B12}\right)\left(k_1 \cdot f_4 \cdot f_3 \cdot k_2 + k_2 \cdot f_4 \cdot f_3 \cdot k_1\right)\\
&+\tfrac{1}{12}\left(\dot{G}^2_{B12} - 2\dot{G}^4_{B12} + \dot{G}^6_{B12}\right) k_2 \cdot f_4 \cdot k_1 \left(\tfrac{k_2 \cdot k_3 r_4 \cdot f_3 \cdot k_1}{r_4 \cdot k_3} - \tfrac{k_3 \cdot k_1 r_4 \cdot f_3 \cdot k_2}{r_4 \cdot k_3}\right)\Big]
\end{aligned} \tag{E.136}$$

$$\tilde{I}^2_{(34)}(13) = \frac{1}{18} Z_2(13) \dot{G}^2_{B12}\left(1 - \dot{G}^2_{B12}\right) k_1 \cdot f_4 \cdot f_2 \cdot k_1, \tag{E.137}$$

$$\tilde{I}^2_{(34)}(23) = \frac{1}{18} Z_2(23) \dot{G}^2_{B12}\left(1 - \dot{G}^2_{B12}\right) k_2 \cdot f_4 \cdot f_1 \cdot k_2, \tag{E.138}$$

$$\begin{aligned}
\tilde{I}^2_{(34)}(14) =& \tfrac{2Z_2(14)}{36 r_7 \cdot k_2 r_8 \cdot k_3}\Big[-\left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) k_2 \cdot k_3 r_8 \cdot f_3 \cdot k_1 r_7 \cdot f_2 \cdot k_1\\
&+\left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) k_3 \cdot k_1 \left(r_8 \cdot f_3 \cdot k_2 r_7 \cdot f_2 \cdot k_1 - r_7 \cdot f_2 \cdot f_3 \cdot r_8 k_2 \cdot k_1\right)\\
&+\left(2 - 6\dot{G}^2_{B12}\right)\left(r_8 \cdot f_3 \cdot k_1 r_7 \cdot f_2 \cdot k_3 + 2 r_7 \cdot f_2 \cdot f_3 \cdot r_8 k_3 \cdot k_1\right)\Big],
\end{aligned} \tag{E.139}$$

$$\begin{aligned}
\tilde{I}^2_{(34)}(24) =& \tfrac{2Z_2(24)}{36 r_9 \cdot k_1 r_0 \cdot k_3}\Big[-\left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) k_3 \cdot k_1 r_0 \cdot f_3 \cdot k_2 r_9 \cdot f_1 \cdot k_2\\
&+\left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) k_2 \cdot k_3 \left(r_0 \cdot f_3 \cdot k_1 r_9 \cdot f_1 \cdot k_2 - r_9 \cdot f_1 \cdot f_3 \cdot r_0 k_1 \cdot k_2\right)\\
&+\left(2 - 6\dot{G}^2_{B12}\right)\left(r_0 \cdot f_3 \cdot k_2 r_9 \cdot f_1 \cdot k_3 + 2 r_9 \cdot f_1 \cdot f_3 \cdot r_0 k_3 \cdot k_2\right)\Big],
\end{aligned} \tag{E.140}$$

$$\tilde{I}^{22}_{(34)}(12,34) = \frac{2}{3}\left(1 - \dot{G}^2_{B12}\right) Z_2(12) Z_2(34), \tag{E.141}$$

$$\tilde{I}^{22}_{(34)}(13,24) = \frac{4}{9} Z_2(13) Z_2(24), \tag{E.142}$$

$$\tilde{I}^{22}_{(34)}(14,23) = \frac{4}{9} Z_2(14) Z_2(23). \tag{E.143}$$

Notice that, in (E.135) and (E.136) we can make the following replacement

$$\frac{k_2 \cdot k_3 r \cdot f_3 \cdot k_1}{r \cdot k_3} - \frac{k_3 \cdot k_1 r \cdot f_3 \cdot k_2}{r \cdot k_3} = k_2 \cdot f_3 \cdot k_1 \tag{E.144}$$

performing this replacement we have

E.2.1 Result of $\tilde{I}_{(34)}$

$$\begin{aligned}
\tilde{I}^4_{(34)}(1234) &= \tfrac{1}{6}\left(4\dot{G}^2_{B12} - \dot{G}^4_{B12} - 3\right) Z_4(1234), \\
\tilde{I}^4_{(34)}(1243) &= \tfrac{1}{6}\left(4\dot{G}^2_{B12} - \dot{G}^4_{B12} - 3\right) Z_4(1243), \\
\tilde{I}^4_{(34)}(1324) &= \tfrac{1}{6^2}\left(1 - 42\dot{G}^2_{B12} + 9\dot{G}^4_{B21}\right) Z_4(1324),
\end{aligned} \tag{E.145}$$

$$\tilde{I}^3_{(34)}(123) = \frac{1}{72}\left(7\dot{G}^2_{B12} - 10\dot{G}^4_{B12} + 3\dot{G}^6_{B12}\right) Z_3(123) k_2 \cdot f_4 \cdot k_1, \tag{E.146}$$

$$\tilde{I}^3_{(34)}(412) = \frac{1}{72}\left(7\dot{G}^2_{B12} - 10\dot{G}^4_{B12} + 3\dot{G}^6_{B12}\right) Z_3(412) k_2 \cdot f_3 \cdot k_1, \tag{E.147}$$

$$\begin{aligned}
\tilde{I}^2_{(34)}(12) = \tfrac{1}{12}\left(1 - \dot{G}^2_{B12}\right) Z_2(12) \Big[& \tfrac{2}{15}\left(k_1 \cdot f_4 \cdot f_3 \cdot k_1 + k_2 \cdot f_4 \cdot f_3 \cdot k_2\right) \\
& - \tfrac{1}{12}\left(\tfrac{7}{5} - 6\dot{G}^2_{B12} + 3\dot{G}^4_{B12}\right)\left(k_1 \cdot f_4 \cdot f_3 \cdot k_2 + k_2 \cdot f_4 \cdot f_3 \cdot k_1\right) \\
& + \tfrac{1}{12}\left(\dot{G}^2_{B12} - 2\dot{G}^4_{B12} + \dot{G}^6_{B12}\right) k_2 \cdot f_4 \cdot k_1 k_2 \cdot f_3 \cdot k_1 \Big],
\end{aligned} \tag{E.148}$$

$$\tilde{I}^2_{(34)}(13) = \frac{1}{18} Z_2(13) \dot{G}^2_{B12}\left(1 - \dot{G}^2_{B12}\right) k_1 \cdot f_4 \cdot f_2 \cdot k_1, \tag{E.149}$$

$$\tilde{I}^2_{(34)}(23) = \frac{1}{18} Z_2(23) \dot{G}^2_{B12}\left(1 - \dot{G}^2_{B12}\right) k_2 \cdot f_4 \cdot f_1 \cdot k_2, \tag{E.150}$$

$$\begin{aligned}
\tilde{I}^2_{(34)}(14) = \tfrac{2Z_2(14)}{36 r_7 \cdot k_2 r_8 \cdot k_3} \Big[& -\left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) k_2 \cdot k_3 r_8 \cdot f_3 \cdot k_1 r_7 \cdot f_2 \cdot k_1 \\
& + \left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) k_3 \cdot k_1 \left(r_8 \cdot f_3 \cdot k_2 r_7 \cdot f_2 \cdot k_1 - r_7 \cdot f_2 \cdot f_3 \cdot r_8 k_2 \cdot k_1\right) \\
& + \left(2 - 6\dot{G}^2_{B12}\right)\left(r_8 \cdot f_3 \cdot k_1 r_7 \cdot f_2 \cdot k_3 + 2 r_7 \cdot f_2 \cdot f_3 \cdot r_8 k_3 \cdot k_1\right) \Big],
\end{aligned} \tag{E.151}$$

$$\begin{aligned}\tilde{I}^2_{(34)}(24) = \tfrac{2Z_2(24)}{36r_9\cdot k_1 r_0\cdot k_3}\Big[&-\left(\dot{G}^2_{B12}-\dot{G}^4_{B12}\right) k_3\cdot k_1 r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_2\\ &+\left(\dot{G}^2_{B12}-\dot{G}^4_{B12}\right) k_2\cdot k_3\,(r_0\cdot f_3\cdot k_1 r_9\cdot f_1\cdot k_2 - r_9\cdot f_1\cdot f_3\cdot r_0 k_1\cdot k_2)\\ &+\left(2-6\dot{G}^2_{B12}\right)(r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_3 + 2r_9\cdot f_1\cdot f_3\cdot r_0 k_3\cdot k_2)\Big],\end{aligned} \tag{E.152}$$

$$\tilde{I}^{22}_{(34)}(12,34) = \frac{2}{3}\left(1-\dot{G}^2_{B12}\right) Z_2(12)Z_2(34), \tag{E.153}$$

$$\tilde{I}^{22}_{(34)}(13,24) = \frac{4}{9} Z_2(13)Z_2(24), \tag{E.154}$$

$$\tilde{I}^{22}_{(34)}(14,23) = \frac{4}{9} Z_2(14)Z_2(23). \tag{E.155}$$

E.2.2 $\tilde{I}_{(34)}$ in terms of G_{Bij} only

As we can see that in chapter 5 we need $\tilde{I}_{(34)}$ in terms of G_{Bij} to perform the remaining integrals. Hence, we use (B.1) to write the equations above in terms of G_{Bij} (we restore the T factors also)

$$\begin{aligned}\tilde{I}^4_{(34)}(1234) &= -\tfrac{4}{3}\left(G_{B12}-2G^2_{B12}\right) Z_4(1234),\\ \tilde{I}^4_{(34)}(1243) &= -\tfrac{4}{3}\left(G_{B12}-2G^2_{B12}\right) Z_4(1243),\\ \tilde{I}^4_{(34)}(1324) &= -\tfrac{4}{9}\left(2-6G_{B12}-9G^2_{B21}\right) Z_4(1324),\end{aligned} \tag{E.156}$$

$$\tilde{I}^3_{(34)}(123) = \frac{2T}{9}\left(G_{B12}-G^2_{B12}-12G^3_{B12}\right) Z_3(123) k_2\cdot f_4\cdot k_1, \tag{E.157}$$

$$\tilde{I}^3_{(34)}(412) = \frac{2T}{9}\left(G_{B12}-G^2_{B12}-12G^3_{B12}\right) Z_3(412) k_2\cdot f_3\cdot k_1, \tag{E.158}$$

$$\begin{aligned}\tilde{I}^2_{(34)}(12) = \tfrac{T}{3} G_{B12} Z_2(12)\Big[&\tfrac{2}{15}\left(k_1\cdot f_4\cdot f_3\cdot k_1 + k_2\cdot f_4\cdot f_3\cdot k_2\right)\\ &+\tfrac{2}{15}\left(1-30G^2_{B12}\right)\left(k_1\cdot f_4\cdot f_3\cdot k_2 + k_2\cdot f_4\cdot f_3\cdot k_1\right)\\ &+\tfrac{4T}{3}\left(G^2_{B12}-4G^3_{B12}\right) k_2\cdot f_4\cdot k_1 k_2\cdot f_3\cdot k_1\Big],\end{aligned} \tag{E.159}$$

$$\tilde{I}^2_{(34)}(13) = \frac{2T}{9} Z_2(13)\left(G_{B12}-4G^2_{B12}\right) k_1\cdot f_4\cdot f_2\cdot k_1, \tag{E.160}$$

$$\tilde{I}^2_{(34)}(23) = \frac{2T}{9} Z_2(23)\left(G_{B12}-4G^2_{B12}\right) k_2\cdot f_4\cdot f_1\cdot k_2, \tag{E.161}$$

$$\begin{aligned}\tilde{I}^2_{(34)}(14) = \tfrac{-2Z_2(14)}{9r_7\cdot k_2 r_8\cdot k_3}\Big[&T\left(G_{B12}-4G^2_{B12}\right)k_2\cdot k_3 r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_1\\ &+T\left(G_{B12}-4G^2_{B12}\right)k_3\cdot k_1\left(r_7\cdot f_2\cdot f_3\cdot r_8 k_2\cdot k_1 - r_8\cdot f_3\cdot k_2 r_7\cdot f_2\cdot k_1\right)\\ &+\left(1-6G_{B12}\right)\left(r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_3 + 2r_7\cdot f_2\cdot f_3\cdot r_8 k_3\cdot k_1\right)\Big],\end{aligned} \tag{E.162}$$

$$\begin{aligned}\tilde{I}^2_{(34)}(24) = \tfrac{-2Z_2(24)}{9r_9\cdot k_1 r_0\cdot k_3}\Big[&T\left(G_{B12}-4G^2_{B12}\right)k_3\cdot k_1 r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_2\\ &+T\left(G_{B12}-4G^2_{B12}\right)k_2\cdot k_3\left(r_9\cdot f_1\cdot f_3\cdot r_0 k_1\cdot k_2 - r_0\cdot f_3\cdot k_1 r_9\cdot f_1\cdot k_2\right)\\ &+\left(1-6G_{B12}\right)\left(r_0\cdot f_3\cdot k_2 r_9\cdot f_1\cdot k_3 + 2r_9\cdot f_1\cdot f_3\cdot r_0 k_3\cdot k_2\right)\Big],\end{aligned} \tag{E.163}$$

$$\tilde{I}^{22}_{(34)}(12,34) = \frac{8}{3}G_{B12}Z_2(12)Z_2(34), \tag{E.164}$$

$$\tilde{I}^{22}_{(34)}(13,24) = \frac{4}{9}Z_2(13)Z_2(24), \tag{E.165}$$

$$\tilde{I}^{22}_{(34)}(14,23) = \frac{4}{9}Z_2(14)Z_2(23). \tag{E.166}$$

E.3 Three low energy photons

Here, we take the low energy limit in the leg corresponding to the integral over u_2. First, we cancel all the terms $O(k_2^2)$ in $\tilde{I}_{(34)}$ (in terms of $\dot{G}_{Bij}$)

$$\begin{aligned}\tilde{I}^4_{(34)}(1234) &= \tfrac{1}{6}\left(4\dot{G}^2_{B12}-\dot{G}^4_{B12}-3\right)Z_4(1234),\\ \tilde{I}^4_{(34)}(1243) &= \tfrac{1}{6}\left(4\dot{G}^2_{B12}-\dot{G}^4_{B12}-3\right)Z_4(1243),\\ \tilde{I}^4_{(34)}(1324) &= \tfrac{1}{6^2}\left(1-42\dot{G}^2_{B12}+9\dot{G}^4_{B21}\right)Z_4(1324),\end{aligned} \tag{E.167}$$

$$\tilde{I}^3_{(34)}(123) = \frac{1}{72}\left(7\dot{G}^2_{B12}-10\dot{G}^4_{B12}+3\dot{G}^6_{B12}\right)Z_3(123)\cancel{k_2\cdot f_4\cdot k_1}, \tag{E.168}$$

$$\tilde{I}^3_{(34)}(412) = \frac{1}{72}\left(7\dot{G}^2_{B12}-10\dot{G}^4_{B12}+3\dot{G}^6_{B12}\right)Z_3(412)\cancel{k_2\cdot f_3\cdot k_1}, \tag{E.169}$$

$$\begin{aligned}\tilde{I}^2_{(34)}(12) = \tfrac{1}{12}\left(1-\dot{G}^2_{B12}\right)Z_2(12)\Big[&\tfrac{2}{15}\left(k_1\cdot f_4\cdot f_3\cdot k_1 + \cancel{k_2\cdot f_4\cdot f_3\cdot k_2}\right)\\ &-\tfrac{1}{12}\left(\tfrac{7}{5}-6\dot{G}^2_{B12}+3\dot{G}^4_{B12}\right)\left(\cancel{k_1\cdot f_4\cdot f_3\cdot k_2}+\cancel{k_2\cdot f_4\cdot f_3\cdot k_1}\right)\\ &+\tfrac{1}{12}\left(\dot{G}^2_{B12}-2\dot{G}^4_{B12}+\dot{G}^6_{B12}\right)\cancel{k_2\cdot f_4\cdot k_1 k_2\cdot f_3\cdot k_1}\Big],\end{aligned} \tag{E.170}$$

$$\tilde{I}^2_{(34)}(13) = \frac{1}{18} Z_2(13) \dot{G}^2_{B12} \left(1 - \dot{G}^2_{B12}\right) k_1 \cdot f_4 \cdot f_2 \cdot k_1, \tag{E.171}$$

$$\tilde{I}^2_{(34)}(23) = \frac{1}{18} Z_2(23) \dot{G}^2_{B12} \left(1 - \dot{G}^2_{B12}\right) k_2 \cdot f_4 \cdot f_1 \cdot k_2, \tag{E.172}$$

$$\begin{aligned}\tilde{I}^2_{(34)}(14) = \tfrac{2Z_2(14)}{36 r_7 \cdot k_2 r_8 \cdot k_3} \Big[& - \left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) k_2 \cdot k_3 r_8 \cdot f_3 \cdot k_1 r_7 \cdot f_2 \cdot k_1 \\ & + \left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) k_3 \cdot k_1 \left(r_8 \cdot f_3 \cdot k_2 r_7 \cdot f_2 \cdot k_1 - r_7 \cdot f_2 \cdot f_3 \cdot r_8 k_2 \cdot k_1\right) \\ & + \left(2 - 6\dot{G}^2_{B12}\right) \left(r_8 \cdot f_3 \cdot k_1 r_7 \cdot f_2 \cdot k_3 + 2 r_7 \cdot f_2 \cdot f_3 \cdot r_8 k_3 \cdot k_1\right) \Big],\end{aligned} \tag{E.173}$$

$$\begin{aligned}\tilde{I}^2_{(34)}(24) = \tfrac{2Z_2(24)}{36 r_9 \cdot k_1 r_0 \cdot k_3} \Big[& - \left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) k_3 \cdot k_1 r_0 \cdot f_3 \cdot k_2 r_9 \cdot f_1 \cdot k_2 \\ & + \left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) k_2 \cdot k_3 \left(r_0 \cdot f_3 \cdot k_1 r_9 \cdot f_1 \cdot k_2 - r_9 \cdot f_1 \cdot f_3 \cdot r_0 k_1 \cdot k_2\right) \\ & + \left(2 - 6\dot{G}^2_{B12}\right) \left(r_0 \cdot f_3 \cdot k_2 r_9 \cdot f_1 \cdot k_3 + 2 r_9 \cdot f_1 \cdot f_3 \cdot r_0 k_3 \cdot k_2\right) \Big],\end{aligned} \tag{E.174}$$

$$\tilde{I}^{22}_{(34)}(12,34) = \frac{2}{3} \left(1 - \dot{G}^2_{B12}\right) Z_2(12) Z_2(34), \tag{E.175}$$

$$\tilde{I}^{22}_{(34)}(13,24) = \frac{4}{9} Z_2(13) Z_2(24), \tag{E.176}$$

$$\tilde{I}^{22}_{(34)}(14,23) = \frac{4}{9} Z_2(14) Z_2(23). \tag{E.177}$$

Then, what we need to integrate is

$$\begin{aligned}\tilde{I}^4_{(234)}(1234) &= \tfrac{1}{6} \textstyle\int_0^1 \left(4\dot{G}^2_{B12} - \dot{G}^4_{B12} - 3\right) Z_4(1234) du_2, \\ \tilde{I}^4_{(234)}(1243) &= \tfrac{1}{6} \textstyle\int_0^1 \left(4\dot{G}^2_{B12} - \dot{G}^4_{B12} - 3\right) Z_4(1243) du_2, \\ \tilde{I}^4_{(234)}(1324) &= \tfrac{1}{6^2} \textstyle\int_0^1 \left(1 - 42\dot{G}^2_{B12} + 9\dot{G}^4_{B21}\right) Z_4(1324) du_2,\end{aligned} \tag{E.178}$$

$$\tilde{I}^2_{(234)}(12) = \frac{1}{12} \frac{2}{15} \int_0^1 \left(1 - \dot{G}^2_{B12}\right) Z_2(12) k_1 \cdot f_4 \cdot f_3 \cdot k_1 du_2, \tag{E.179}$$

$$\tilde{I}^2_{(234)}(13) = \frac{1}{18} \int_0^1 Z_2(13) \dot{G}^2_{B12} \left(1 - \dot{G}^2_{B12}\right) k_1 \cdot f_4 \cdot f_2 \cdot k_1 du_2, \tag{E.180}$$

$$\begin{aligned}\tilde{I}^2_{(234)}(14) = \tfrac{2Z_2(14)}{36 r_7 \cdot k_2 r_8 \cdot k_3} \textstyle\int_0^1 \Big[& - \left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) k_2 \cdot k_3 r_8 \cdot f_3 \cdot k_1 r_7 \cdot f_2 \cdot k_1 \\ & + \left(\dot{G}^2_{B12} - \dot{G}^4_{B12}\right) k_3 \cdot k_1 \left(r_8 \cdot f_3 \cdot k_2 r_7 \cdot f_2 \cdot k_1 - r_7 \cdot f_2 \cdot f_3 \cdot r_8 k_2 \cdot k_1\right) \\ & + \left(2 - 6\dot{G}^2_{B12}\right) e^{()} \left(r_8 \cdot f_3 \cdot k_1 r_7 \cdot f_2 \cdot k_3 + 2 r_7 \cdot f_2 \cdot f_3 \cdot r_8 k_3 \cdot k_1\right) \Big] du_2,\end{aligned} \tag{E.181}$$

$$\tilde{I}^{22}_{(234)}(12,34) = \frac{2}{3}\int_0^1 \left(1 - \dot{G}^2_{B12}\right) Z_2(12)Z_2(34)du_2, \tag{E.182}$$

$$\tilde{I}^{22}_{(234)}(13,24) = \frac{4}{9}\int_0^1 Z_2(13)Z_2(24)du_2, \tag{E.183}$$

$$\tilde{I}^{22}_{(234)}(14,23) = \frac{4}{9}\int_0^1 Z_2(14)Z_2(23)du_2. \tag{E.184}$$

We perform this integrals using some particular cases of (B.11) and we obtain

$$\begin{array}{l} \tilde{I}^{4}_{(234)}(1234) = \frac{1}{6}\left(\frac{4}{3} - \frac{1}{5} - 3\right) Z_4(1234), \\ \tilde{I}^{4}_{(234)}(1243) = \frac{1}{6}\left(\frac{4}{3} - \frac{1}{5} - 3\right) Z_4(1243), \\ \tilde{I}^{4}_{(234)}(1324) = \frac{1}{6^2}\left(1 - \frac{42}{3} + \frac{9}{5}\right) Z_4(1324), \end{array} \tag{E.185}$$

$$\tilde{I}^{2}_{(234)}(12) = \frac{1}{12}\frac{2}{15}\frac{2}{3} Z_2(12) k_1 \cdot f_4 \cdot f_3 \cdot k_1, \tag{E.186}$$

$$\tilde{I}^{2}_{(234)}(13) = \frac{1}{18} Z_2(13)\left(\frac{1}{3} - \frac{1}{5}\right) k_1 \cdot f_4 \cdot f_2 \cdot k_1, \tag{E.187}$$

$$\begin{aligned} \tilde{I}^{2}_{(234)}(14) = \tfrac{2Z_2(14)}{36 r_7 \cdot k_2 r_8 \cdot k_3}\Big\{ & -\left(\tfrac{1}{3} - \tfrac{1}{5}\right) k_2 \cdot k_3 r_8 \cdot f_3 \cdot k_1 r_7 \cdot f_2 \cdot k_1 \\ & + \left(\tfrac{1}{3} - \tfrac{1}{5}\right) k_3 \cdot k_1 \left(r_8 \cdot f_3 \cdot k_2 r_7 \cdot f_2 \cdot k_1 - r_7 \cdot f_2 \cdot f_3 \cdot r_8 k_2 \cdot k_1\right) \\ & + \left[\tfrac{2}{6} - \tfrac{6}{4}\left(\tfrac{1}{3} - \tfrac{1}{5}\right)\right] k_1 \cdot k_2 \left(r_8 \cdot f_3 \cdot k_1 r_7 \cdot f_2 \cdot k_3 + 2 r_7 \cdot f_2 \cdot f_3 \cdot r_8 k_3 \cdot k_1\right) \Big\}, \end{aligned} \tag{E.188}$$

$$\tilde{I}^{22}_{(234)}(12,34) = \frac{2}{3}\frac{2}{3} Z_2(12)Z_2(34), \tag{E.189}$$

$$\tilde{I}^{22}_{(234)}(13,24) = \frac{4}{9} Z_2(13)Z_2(24), \tag{E.190}$$

$$\tilde{I}^{22}_{(234)}(14,23) = \frac{4}{9} Z_2(14)Z_2(23). \tag{E.191}$$

Making simpler

E.3.1 Result of $\tilde{I}_{(234)}$

$$\begin{array}{l} \tilde{I}^{4}_{(234)}(1234) = \frac{-14}{45} Z_4(1234), \\ \tilde{I}^{4}_{(234)}(1243) = \frac{-14}{45} Z_4(1243), \\ \tilde{I}^{4}_{(234)}(1324) = \frac{-14}{45} Z_4(1324), \end{array} \tag{E.192}$$

$$\tilde{I}^2_{(234)}(12) = \frac{1}{15\cdot 9} Z_2(12) k_1\cdot f_4\cdot f_3\cdot k_1, \tag{E.193}$$

$$\tilde{I}^2_{(234)}(13) = \frac{1}{15\cdot 9} Z_2(13) k_1\cdot f_4\cdot f_2\cdot k_1, \tag{E.194}$$

$$\begin{aligned}\tilde{I}^2_{(234)}(14) = \tfrac{Z_2(14)}{18 r_7\cdot k_2 r_8\cdot k_3}\Big\{ &-\tfrac{2}{15} k_2\cdot k_3 r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_1 \\ &+\tfrac{2}{15} k_3\cdot k_1\left(r_8\cdot f_3\cdot k_2 r_7\cdot f_2\cdot k_1 - r_7\cdot f_2\cdot f_3\cdot r_8 k_2\cdot k_1\right) \\ &+\tfrac{2}{15} k_1\cdot k_2\left(r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_3 + 2 r_7\cdot f_2\cdot f_3\cdot r_8 k_3\cdot k_1\right)\Big\},\end{aligned} \tag{E.195}$$

$$\tilde{I}^{22}_{(234)}(12,34) = \frac{4}{9} Z_2(12) Z_2(34), \tag{E.196}$$

$$\tilde{I}^{22}_{(234)}(13,24) = \frac{4}{9} Z_2(13) Z_2(24), \tag{E.197}$$

$$\tilde{I}^{22}_{(234)}(14,23) = \frac{4}{9} Z_2(14) Z_2(23). \tag{E.198}$$

However, as we already mention when we have three photons of low energy the remaining one must be also of low energy because of momentum conservation.

E.4 Four low energy photons

First, we cancel all the terms $O(k_1^2)$ in

$$\begin{aligned}\tilde{I}^4_{(234)}(1234) &= \tfrac{-14}{45} Z_4(1234), \\ \tilde{I}^4_{(234)}(1243) &= \tfrac{-14}{45} Z_4(1243), \\ \tilde{I}^4_{(234)}(1324) &= \tfrac{-14}{45} Z_4(1324),\end{aligned} \tag{E.199}$$

$$\tilde{I}^2_{(234)}(12) = \frac{1}{15\cdot 9} Z_2(12) \cancel{k_1\cdot f_4\cdot f_3\cdot k_1}, \tag{E.200}$$

$$\tilde{I}^2_{(234)}(13) = \frac{1}{15\cdot 9} Z_2(13) \cancel{k_1\cdot f_4\cdot f_2\cdot k_1}, \tag{E.201}$$

$$\begin{aligned}\tilde{I}^2_{(234)}(14) = \cancel{\tfrac{Z_2(14)}{18 r_7\cdot k_2 r_8\cdot k_3}}\Big\{ &-\tfrac{2}{15} k_2\cdot k_3 r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_1 \\ &+\tfrac{2}{15} k_3\cdot k_1\left(r_8\cdot f_3\cdot k_2 r_7\cdot f_2\cdot k_1 - r_7\cdot f_2\cdot f_3\cdot r_8 k_2\cdot k_1\right) \\ &+\tfrac{2}{15} k_1\cdot k_2\left(r_8\cdot f_3\cdot k_1 r_7\cdot f_2\cdot k_3 + 2 r_7\cdot f_2\cdot f_3\cdot r_8 k_3\cdot k_1\right)\Big\},\end{aligned} \tag{E.202}$$

$$\tilde{I}^{22}_{(234)}(12,34) = \frac{4}{9} Z_2(12) Z_2(34), \tag{E.203}$$

$$\tilde{I}^{22}_{(234)}(13,24) = \frac{4}{9} Z_2(13) Z_2(24), \tag{E.204}$$

$$\tilde{I}^{22}_{(234)}(14,23) = \frac{4}{9} Z_2(14) Z_2(23). \tag{E.205}$$

Since, there is no dependence in u_1 anymore, because of the translation invariance, we have the following result for this case

E.4.1 Result of $\tilde{I}_{(1234)}$

$$\begin{array}{l} \tilde{I}^{4}_{(1234)}(1234) = \frac{-14}{45} Z_4(1234), \\ \tilde{I}^{4}_{(1234)}(1243) = \frac{-14}{45} Z_4(1243), \\ \tilde{I}^{4}_{(1234)}(1324) = \frac{-14}{45} Z_4(1324), \end{array} \tag{E.206}$$

$$\tilde{I}^{22}_{(1234)}(12,34) = \frac{4}{9} Z_2(12) Z_2(34), \tag{E.207}$$

$$\tilde{I}^{22}_{(1234)}(13,24) = \frac{4}{9} Z_2(13) Z_2(24), \tag{E.208}$$

$$\tilde{I}^{22}_{(1234)}(14,23) = \frac{4}{9} Z_2(14) Z_2(23). \tag{E.209}$$

Appendix F

Hypergeometric ${}_2F_1$

In this appendix, we proof (5.5), and we proceed as follows: First, we Taylor expand the function to be integrate and we use the Pochhammer symbol

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1), \quad (a)_0 = 1, \tag{F.1}$$

to write this expansion in terms of an infinity sum. Second, we use the following relation

$$B(p+1, q+1) = \int_0^1 dt\ t^p(1-t)^q, \tag{F.2}$$

to perform the integral (where B is the "beta function", do not confuse with β-function, they are different). Third, we use the definition of beta function

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \tag{F.3}$$

to write the result in terms of factorial function, since

$$\Gamma(n) = n! \tag{F.4}$$

for n integer. Fourth, we use that

$$(2n+1)! = 2^{2n} n! \left(\frac{3}{2}\right)_n, \tag{F.5}$$

to cancel some factorial terms. Finally, we compare with

$$ {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \tag{F.6} $$

and we obtain the final result.

Now, we present this explicit calculation below

$$ \begin{aligned} \int_0^1 du \frac{1}{(1-G_B x)^\lambda} &= \int_0^1 du \frac{1}{[1-u(1-u)x]^\lambda} \\ &= \int_0^1 du \sum_{n=0}^{\infty} \frac{(\lambda)_n [u(1-u)x]^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} x^n B(n+1,n+1) \\ &= \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \frac{\Gamma(n+1)\Gamma(n+1)}{\Gamma(2n+2)} x^n \\ &= \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \frac{(n!)^2}{(2n+1)!} x^n \\ &= \sum_{n=0}^{\infty} \frac{(\lambda)_n}{\cancel{n!}} \frac{\cancel{(n!)^2}}{2^{2n}\cancel{n!}\left(\frac{3}{2}\right)_n} x^n \\ &= \sum_{n=0}^{\infty} \frac{(\lambda)_n}{\left(\frac{3}{2}\right)_n} \left(\frac{x}{4}\right)^n \\ &= {}_2F_1\left(1,\lambda;\frac{3}{2};\frac{x}{4}\right). \end{aligned} \tag{F.7} $$

In [43], it is possible to find the equations (F.1), (F.2), (F.3), (F.6), and

the relation (F.5) can be derived in the following way

$$\begin{aligned}(2n+1)! &= 1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(2n-2)(2n-1)(2n)(2n+1)\\ &= 2^{2n}\left[1\cdot 1\cdot\frac{3}{2}\cdot 2\cdot\frac{5}{2}\cdots(n-1)\left(n-\frac{1}{2}\right)n\left(n+\frac{1}{2}\right)\right]\\ &= 2^{2n}n!\left[\frac{3}{2}\cdot\left(\frac{3}{2}+1\right)\cdots\left(\frac{3}{2}+n-2\right)n\left(\frac{3}{2}+n-1\right)\right]\\ &= 2^{2n}n!\left(\frac{3}{2}\right)_n .\end{aligned} \tag{F.8}$$

Appendix G

Some explicit integral results

In Chapter 5, we derived the formula that solve our integrals

$$Y_{nl} = \frac{\Gamma\left(n-\frac{D}{2}\right)}{m^{2n-D}(n-l-\frac{D}{2})_l}\frac{d^l}{dx^l}\ {}_2F_1\left(1, n-l-\frac{D}{2}; \frac{3}{2}; \frac{x}{4}\right). \tag{G.1}$$

Thus, we give the explicit expression for some particular cases of Y_{nl} are

$$Y_{40} = \frac{\Gamma\left(4-\frac{D}{2}\right)}{m^{8-D}}\ {}_2F_1\left(1, 4-\frac{D}{2}, \frac{3}{2}, \frac{x}{4}\right), \tag{G.2}$$

$$Y_{41} = \frac{\Gamma\left(4-\frac{D}{2}\right)}{6m^{8-D}}\ {}_2F_1\left(2, 4-\frac{D}{2}, \frac{5}{2}, \frac{x}{4}\right), \tag{G.3}$$

$$Y_{42} = \frac{\Gamma\left(4-\frac{D}{2}\right)}{30m^{8-D}}\ {}_2F_1\left(3, 4-\frac{D}{2}, \frac{7}{2}, \frac{x}{4}\right), \tag{G.4}$$

$$Y_{50} = \frac{\Gamma\left(5-\frac{D}{2}\right)}{m^{10-D}}\ {}_2F_1\left(1, 5-\frac{D}{2}, \frac{3}{2}, \frac{x}{4}\right), \tag{G.5}$$

$$Y_{51} = \frac{\Gamma\left(5-\frac{D}{2}\right)}{6m^{10-D}}\ {}_2F_1\left(2, 5-\frac{D}{2}, \frac{5}{2}, \frac{x}{4}\right), \tag{G.6}$$

$$Y_{52} = \frac{\Gamma\left(5-\frac{D}{2}\right)}{30m^{10-D}}\ {}_2F_1\left(3, 5-\frac{D}{2}, \frac{7}{2}, \frac{x}{4}\right), \tag{G.7}$$

$$Y_{53} = \frac{\Gamma\left(5-\frac{D}{2}\right)}{140m^{10-D}}\ {}_2F_1\left(4, 5-\frac{D}{2}, \frac{9}{2}, \frac{x}{4}\right), \tag{G.8}$$

$$Y_{62} = \frac{\Gamma\left(6-\frac{D}{2}\right)}{30m^{12-D}}\ {}_2F_1\left(3, 6-\frac{D}{2}, \frac{7}{2}, \frac{x}{4}\right), \tag{G.9}$$

$$Y_{63} = \frac{\Gamma\left(6-\frac{D}{2}\right)}{140m^{12-D}}\ {}_2F_1\left(4, 6-\frac{D}{2}, \frac{9}{2}, \frac{x}{4}\right), \tag{G.10}$$

$$Y_{64} = \frac{\Gamma\left(6-\frac{D}{2}\right)}{630m^{12-D}}\ {}_2F_1\left(5, 6-\frac{D}{2}, \frac{11}{2}, \frac{x}{4}\right). \tag{G.11}$$

The previous expression can be computed in Mathematica easily, we only need to write (G.1) in Mathematica.

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