Multi-photon amplitudes

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To my parents.

Contents

Abstract.

This work is focussed on the light-by-light scattering: in particular we study N-photon amplitudes in the low energy limit at the tree-level and one-loop level. We explore the advantages of helicity formulas involving field strength tensors to compute photon amplitudes, at the tree level we present a new master formula to compute N-photon amplitudes for arbitrary N and any helicity assignments. We compare our results to a low energy expansion of the known amplitude for small values of N and find complete agreement; nonetheless the new formula presented here offers significant computational advantages at higher values of N.
Finally, we rederive an explicit formula for the one-loop on-shell N-photon amplitude and we study a double summation that appears by the use of a Taylor expansion to make progress towards simplifying its form.

Resumen:

Este trabajo se centra en la dispersin de luz por luz: en particular, estudiamos las amplitudes de fotones N en el lmite de baja energa a nivel de rbol y nivel de un ciclo. Exploramos las ventajas de las frmulas de helicidad que incluyen tensores de intensidad de campo para computar las amplitudes de fotones, a nivel de rbol presentamos una nueva frmula maestra para computar las amplitudes de fotn-N para N arbitraria y cualquier asignacin de helicidad. Comparamos nuestros resultados con una expansin de baja energa de la amplitud conocida para valores pequeos de N y encontramos un acuerdo completo; sin embargo, la nueva frmula presentada aqu ofrece ventajas computacionales significativas a valores ms altos de N. Finalmente, redirigimos una frmula explcita para la amplitud de fotones N en la carcasa de un bucle y estudiamos una suma doble que aparece mediante el uso de una expansin de Taylor para avanzar hacia la simplificacin de su forma.
Palabras clave: Lagrangiano de Euler-Heisenberg, Electrodinámica cuántica, límite de bajas energías, nivel de un lazo, nivel árbol.

Common symbols	
$\eta^{\mu\nu} = \mathrm{diag}(-,+,+,+)$	space-time metric convention in Minkowski space
$\delta^\mu_\nu, \delta_{ij}$	fully anti-symmetric
$\varepsilon^{0123} = -\varepsilon_{0123} = 1$	unit tensor
$a_\mu b^\mu = -a_0 b_0 + \mathbf{ab}$	scalar product of two four-vectors
$x^\mu = (t, \mathbf{x})$	four-position vector
$d^D x$	D-dimensional volume element
$\partial_\mu = \frac{\partial}{\partial x^\mu}$	differentiation operator
$[A, B] = AB - BA$	commutator
$\{A, B\} = AB + BA$	anticommutator
γ^μ	Dirac gamma matrices
$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$	fifth gamma matrix
$\mathbb{1}$	unit matrix
$\not{p} = \gamma^\mu p_\mu$	slash notation
$\delta(x)$	Dirac delta function
tr , det	trace/ determinant of a matrix
$F^{\mu\nu}$	background field tensor
$f_i^{\mu\nu}$	constant background field tensors
A^μ	background four-potential
$\|p], \|p>$	four-momenta bracket notation for positive and negative-helicity
$[p\|, <p\|$	
χ_+, χ_-	spinor helicity parameters
Δ_{ij}	Worldline Green's function on the open line
$\sigma(\tau_i - \tau_j)$	Sign function

Tab. 0: Summary of commonly used mathematical symbols.

Introduction

Photonic amplitudes

It is said that Physics was born with Aristoteles and that it is the discipline whose principal object of study is nature. This is the reason why the methods used by the physicist are strongly related to the meaning of the word Nature [1]. However the meanings of words depend enormously on time and culture, for example our concept of the atom is very different from that of the Greek philosophers. Even so chemical experiments can be described and classified by the atomic hypothesis of ancient times that consider the atoms as the building blocks of matter. The modern attitude to nature is directly influenced by modern science and technology, and nowadays the phrase used by scientists "description of nature" became to mean the mathematical description of nature [2]. If we consider that Nature is constituted by matter, a mathematical description of matter can be tested with experiments and observations, however, these observations can leave us with new concepts, for instance, when we look at the universe our mathematical description of the universe has to introduce the concept of invisible dark matter in order to fits our observations, thus to answer the question: What are the constituents of visible and invisible matter? is a very difficult task that physicists have.

Today, theoretical physicists try to make a mathematical formulation to arrive at laws that describe the structure of matter; on the other hand, experimentalist are developing clever and bigger accelerators to probe these laws, and they have found some features: matter is formed by elementary particles*. Some natural questions emerge: Which is the particular force law that describes the interaction between elementary particles? and How many fundamental forces in nature exist? Until now we know that there are just four fundamental forces in nature: strong, electromagnetic, weak †,

*The elementary particle concept is a little problematic since the arrival of new technology can bring the experimental physicist to new ways of testing the structure of the particles, for example in the late sixties experimental physicist found evidence that the charge of the proton is concentrated in three lumps "quarks".

†The theory of weak interactions treats weak and electromagnetic interactions as different manifestations of a single electroweak force, in this sense the four fundamental forces can be reduced to

and gravitational. Each force is mediated by the exchange of a particle therefore they belong to different physical theory; for a complete introduction to particle physics see [3].
This work is related with the electromagnetic force, this force has as a mediator the photon, and is described by quantum electrodynamics (QED).
QED is one of the most successful of the dynamical theories since it is the best tested theory we have‡. In this introduction we attempt to present how the light-by-light scattering concept appears in QED, what has been done so far and how we are going to study this problem.
When Dirac attempted to construct the relativistic quantum theory of an electron moving in a given electromagnetic field, he found that the solutions of his wave equation admit an equal number of positive and negative energy solutions, thus some fundamental alteration in the interpretation of negative eigenvalues had to be done. The appearance of this new interpretation problem led Dirac to the prediction of the positron in 1929 [4], and it was found experimentally in 1932 [5]. The study of the Dirac's theory of the positron led Euler and Heisenberg to define the term"critical field" [6]. This term corresponds to the amplitude of a constant electric field which spontaneously creates an electron-positron pair from vacuum§. Note that the critical field is not an umbral, because the probability of pair creation still exists for small amplitudes but this probability is exponentially suppressed. The study of the fact that an electric field can create a real electron-positron pair was performed by Sauter in 1931[7], and J. Schwinger in 1951[8] ¶.This phenomenon is related to the problem of the scattering of light by light, and the differential cross section for this scattering was first calculated by R. Karplus and M. Neuman in 1951 [9].
Contrary to classical theory in QED the vacuum has a certain rich structure where particle/antiparticle pairs "live". These pairs act as dipoles and thus the vacuum becomes polarizable. Although the classical theory of electrodynamics is governed by linear equations, when we consider these new quantum effects we can talk about nonlinear effects in QED.
These nonlinear effects have several implications that have no classical counterparts. For example, in atomic spectroscopy where one has the opportunity to verify any new model or to test any new approach by the study of the "simplest" atoms, in 1947 Rabi and his colleagues discovered the anomalous magnetic moment of the electron [10]. By measuring the hyperfine structure separation of the atomic hydrogen and deuterium they found an important difference between theory and experiment .

three.

‡The consistency of measurements of the Electron anomalous magnetic moment and the value predicted by QED theory is one of the reasons why QED is the most successful quantum field theory.

§The QED critical field strength $\mathcal{E}_{cr} = m^2c^3/|e|\hbar \approx 1.3 \times 10^{16} V/cm$ which corresponds to a laser intensity of $I_{cr} = 4.6 \times 10^{29} W/cm^2$ unfortunately this intensity is not yet attainable nowadays.

¶The production of a real electron-positron pair is known as Sauter-Schwinger pair production.

Another non-classical effect is the so called Delbrück scattering; this effect refers to the scattering of light by a static field generated by a nucleus. When a photon interacts with a Coulomb field, this photon is converted into a pair of electron and positron that then interact with the nucleus via virtual photons, and at the end, by conservation of energy, this pair forms a final photon with the same energy as that of the incident photon, see [11] for a experimental description of this effect.
Other nonlinear processes of QED include photon splitting, where again an initial photon is converted into a virtual pair and then it transforms into two photons [12], and nonlinear Compton scattering, see [13] for a modern study of this effect.
All these effects show us the importance of the study of light in our understanding of the structure of matter. Among the different nonlinear QED effects, the emission and absorption of photons by matter is always present; this converts the scattering of light-by-light into one of the most interesting nonlinear QED effects.
The quantum corrections related to photon-photon scattering are encoded in the effective Lagrangian. It was in 1936 that Heisenberg and Euler obtained a closed-form integral expression (see Section 2.2) for the nonlinear correction to the Maxwell Lagrangian at one loop order [14]. The expansion terms of this Lagrangian can be used to describe the light-by-light scattering in the low energy limit. Even though light-by-light scattering is one of the oldest predictions of QED, the experimental physicists have found the first direct evidence of this process just a few years ago in the ATLAS detector at the LHC[15].
Nowadays the scattering of light by light is an important phenomenon in many areas of modern physics, and this converts the photon into the ideal spin-one particle to study. Albeit QED is valid for arbitrary physical systems which underline the electromagnetic interaction, in the present thesis we will focus on the N-photon amplitudes in the low energy limit at tree and one-loop level a study of two-loop N-photon amplitudes can be found in [24].
In chapter 1, since we are interested in exploring the advantages of the spinor helicity technology, we begin with an introduction to this technology and present some useful identities in order to manipulate the standard integral representation of the Euler-Heisenberg Lagrangian to use it to compute photon amplitudes in the low energy limit; we obtain a compact expression in terms of some coefficients that contain a double summation, and we will try to find a way to simplify it. In the large N limit we find substantial simplifications.
In chapter 2 we study the photon amplitudes in the low energy limit at tree-level. We begin with a derivation of the scalar propagator in its integral representation and then we expand this propagator and, analogous to the one-loop case presented in the chapter 1, we use spinor helicity in order to construct a master formula to compute N-photon amplitudes for all helicity assignments and arbitrary N, and finally we present the conclusions of this work. This thesis also contains some appendices, in appendix A we present our conventions, appendix B is dedicated to the spinor

helicity computations, and in appendix C we present the methods used to study the coefficients of the one-loop N-photon amplitudes.

Chapter 1

One-loop level photon amplitudes in the low energy limit.

The understanding of the dynamics of elementary particles is based on the calculation of decay rates and scattering cross sections. The particle decay refers to the process of one particle transforming into other particles, but this decay is restricted by some conservation laws whereas when two particles interact an area transverse to their relative motion within they are going to scatter from each other can be calculated and is called the scattering cross section. This work is dedicated to the study of light-by-light scattering particularly to the study of photon amplitudes. To determine a scattering amplitude an evaluation of the relevant Feynman diagrams has to be preformed; these diagrams are obtained from a perturbative expansion in terms of one coupling constant as we are going to see in chapter 2. This chapter is related to photon amplitudes at the one-loop level; in the computation of one-loop massive amplitudes little is known beyond the four-point case, however in the limit of low photon energy the N photon amplitude for arbitrary N can be obtained as we will see.

One way to simplify calculations of amplitudes that include massless particles is the use of spinor helicity so first we are going to introduce this technology provides very useful identities, then we are going to extract information of the N-photon amplitude from the Euler-Heisenberg Lagrangian since it is well-known that in the low energy limit the QED effective Lagrangian for a background field with a constant field strength tensor contained the information of the photon amplitudes and finally we will try to derive simple closed-form expressions for these amplitudes.

1.1 Spinor helicity technology

All out conventions and definitions used to present this technology are extracted from [16]. The starting point is the definition of helicity: the helicity of a particle is defined as the component of the particle's spin measured along the axis specified by its three-momentum. For any massless four-momentum p we can write:

$$-\not{p} = |p\rangle\,[p| + |p]\,\langle p|\,. \tag{1.1}$$

We introduce a bracket notation where the square-bracket $|p]$,$[p|$ and the angle-bracket $|p\rangle$, $\langle p|$ represent positive-helicity and negative helicity respectively. The angle and square spinors are the core of what is known as the spinor helicity technology. These bra-kets are nothing to be scared of: they are simply 2-component commuting spinors that solve the massless Weyl equation[17]. With this spinor helicity technology we can express the photon polarization vectors in terms of the bracket notation:

$$\varepsilon_{+}^{\mu}(k,q) = -\frac{\langle q|\,\gamma^{\mu}|k]}{\sqrt{2}\,\langle qk\rangle}, \tag{1.2}$$

$$\varepsilon_{-}^{\mu}(k,q) = -\frac{[q|\gamma^{\mu}|k\rangle}{\sqrt{2}\,[qk]}, \tag{1.3}$$

where q is an arbitrary reference momentum. When we want to compute photon amplitudes using spinor helicity we have to assign to each photon a field strength tensor given by:

$$f_{\mu\nu}^{\pm} \equiv k_{\mu}\varepsilon_{\nu}^{\pm} - \varepsilon_{\mu}^{\pm}k_{\nu}. \tag{1.4}$$

Using spinor helicity we find some useful identities for the anti-commutators (see Appendix B for the complete derivation):

$$\{f_i^{+}, f_j^{+}\} = -\frac{1}{2}[k_i k_j]^2\eta, \tag{1.5}$$

$$\{f_i^{-}, f_j^{-}\} = -\frac{1}{2}\,\langle k_i k_j\rangle^2\,\eta, \tag{1.6}$$

Where η is the metric[‡‡].
In the following, we are going to apply the spinor helicity notation to make the amplitude computations easier and more compact.

1.2 The Euler Heisenberg Effective Lagrangian

As was mentioned in the introduction the fact that an electromagnetic field can create virtual pairs of particles leads to a change of the Maxwell equations in the vacuum. If we want to study these quantum fluctuations we can use as a starting point the effective Euler-Heisenberg Lagrangian (EHL) because it takes into account the interaction of the electromagnetic field with these vacuum fluctuations. For instance the EHL encodes quantum corrections, these corrections are assumed to be local, at least for slowly varying field indeed we can used the EHL to compute photon amplitudes in the limit of vanishing photon energy involving constant external fields. Around the 30's Heisenberg began investigating the consequences of the positron theory and in 1936 Heisenberg and Euler published the famous paper *Consequences of Dirac's Theory of the Positron* where they give an analytic expression of the one-loop Effective Lagrangian written as a function of the invariants $E^2 - B^2$ and $(E \cdot B)^2$ [14]. In QED the constant electromagnetic field is one of our favorite objects by being one field configuration for which the Dirac equation can be solved exactly. We are looking for information about the low energy limit of the one-loop photon S-matrix so we will need to study this Lagrangian. It was by Heisenberg and Euler efforts that we have the following expression for spinor QED in the low energy [18]:

$$\mathcal{L}_{spin}^{EH} = -\frac{1}{8\pi^2} \int_0^\infty \frac{dT}{T} e^{-m^2 T} \left(\frac{e^2 ab}{\tanh(eaT)\tan(ebT)} - \frac{e^2}{3}(a^2 - b^2) - \frac{1}{T^2} \right). \tag{1.7}$$

Naturally, we have an analogous expression for scalar QED that was computed by Weisskopf and it is expressed as [18]:

$$\mathcal{L}_{scalar}^{EH} = \frac{1}{16\pi^2} \int_0^\infty \frac{dT}{T} e^{-m^2 T} \left(\frac{e^2 abT}{\sinh(eaT)\sin(ebT)} + \frac{e^2}{6}(a^2 - b^2) - \frac{1}{T^2} \right). \tag{1.8}$$

Where a and b are related to the two invariants of the Maxwell field by:

$$a^2 - b^2 = \vec{E}^2 - \vec{B}^2, \tag{1.9}$$

[‡‡]See appendix A where the conventions are given.

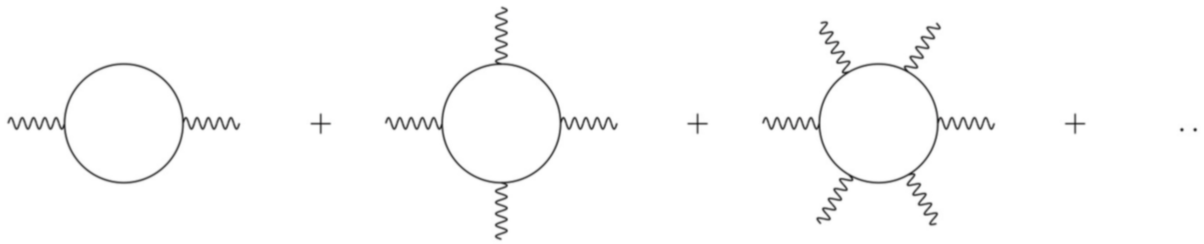

Figure 1.1: Perturbative expansion of the one-loop effective action. The number of external photons lines is even due to Furrys theorem

$$ab = \vec{E} \cdot \vec{B}. \tag{1.10}$$

In the previous expressions, $\vec{E}$ and $\vec{B}$ denote the electric and the magnetic field respectively, $F_{\mu\nu}$ the electromagnetic field strength and $\tilde{F}$ is the dual field strength tensor defined by:

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}. \tag{1.11}$$

The importance of the EHL lies in the fact that it contains the information on non-linear QED effects such as photon-photon scattering, photon dispersion, and photon splitting, thus it can be used to compute on-shell one-loop N-photon amplitudes. The last two Lagrangians involve a summation of all one-loop diagrams therefore the one-loop effective action has a natural perturbative expansion in powers of the external photon field A_μ as illustrated the Feynman diagrams of Fig 1.1.
The real part of the effective Lagrangian describes dispersive effects and its imaginary part the absorptive ones, we are going to study how photon amplitudes are directly related to the EHL. In order to obtain the amplitude with photon momenta $k_1, ..., k_N$ and polarisation vectors $\varepsilon_1, ..., \varepsilon_N$ we will use the field strength tensor definition (1.4) to introduce the total field strength tensor as:

$$F \equiv \sum_{i=1}^{N} f_i. \tag{1.12}$$

The corresponding photon-amplitudes can be obtained by inserting F into the effective Lagrangian and extracting the terms involving each $f_1, ..., f_N$ precisely once

[18]:

$$\Gamma^{(EH)}[k_1, \varepsilon_1; ...; k_N, \varepsilon_N] = \mathcal{L}(iF)|_{f1...f_N}. \tag{1.13}$$

In order to employ this effective action to compute photon amplitudes in a constant electromagnetic background field for both scalar and spinor QED by using spinor helicity, we are going to use the results of the last section.

1.3 From the Effective Lagrangian to Photon Amplitudes

In this part the EH Lagrangian is particularly studied to find an explicit formula for the one-loop N-photon amplitude in the low energy limit. It was shown in [18] that after some manipulations, the formal expression (Eq. 1.7) can be used to compute the N-photon amplitude for arbitrary N. There is found a compact expression for (Eq. 1.7) expressed in terms of some coefficients that contain the Bernoulli numbers, following this derivation we are going to try to simplify a double summation that appears in these coefficients. The standard integral representation of the EHL is our most important tool:

$$\mathcal{L}_{Spin}^{EH} = -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3}e^{-m^2T}\left(\frac{e^2abT^2}{\tanh(eaT)\tan(ebT)} - \frac{e^2}{3}(a^2-b^2) - \frac{1}{T^2}\right). \tag{1.14}$$

Here m and T denote the mass and the proper time of the fermion loop respectively. To use this Lagrangian density to compute scattering amplitudes the terms of zeroth and second order in a and b have to be subtracted because the former corresponds to vacuum polarisation and the later does not contribute. As was mentioned in section 1.2 spinor helicity technology can be used to compute the low energy limit of the on-shell N-photon amplitude. In this limit the background field has a constant field strength tensor $f_{\mu\nu}$. Looking for an amplitude for a fixed and arbitrary N number of photons we write the total constant field strength as:

$$F = \sum_{i=1}^{K} f_i^+ + \sum_{j=K+1}^{K+L} f_j^- = f^+ + f^-. \tag{1.15}$$

In the previous expression we have assigned helicity $'+'$ to legs $1, ..., K$ and the re-

maining ones helicity $'-'$*.
The two invariants of the Maxwell field can be expressed in a simple way by introducing some notation, first note that using (Eq. 1.16):

$$F_{\mu\nu}F^{\mu\nu} = -\mathrm{tr}(F^2) = -tr[(f^+)^2] - \mathrm{tr}[(f^-)^2] - tr[(f^+)(f^-)] - \mathrm{tr}[(f^-)(f^+)]. \quad (1.16)$$

With spinor helicity it is easy to check that the last two terms are zero, the terms that do not vanish are:

$$\mathrm{tr}[(f^+)^2] = \sum_{i,j} \mathrm{tr}[(f_i^+ f_j^+)] = \frac{1}{2}\sum_{i,j} \mathrm{tr}[\{f_i^+, f_j^+\}]. \quad (1.17)$$

Using (Eq. 1.5)

$$\mathrm{tr}[(f^+)^2] = -\frac{1}{2}\sum_{i,j}[k_i k_j]^2 \mathrm{tr}[\eta] = -2\sum_{i,j}[k_i k_j]^2. \quad (1.18)$$

Analogously

$$\mathrm{tr}[(f^-)^2] = -\frac{1}{2}\sum_{i,j}\langle k_i k_j\rangle^2 \mathrm{tr}[\eta] = -2\sum_{i,j}\langle k_i k_j\rangle^2. \quad (1.19)$$

So:

$$\frac{F_{\mu\nu}F^{\mu\nu}}{4} = \frac{1}{2}\sum_{i,j}[k_i k_j]^2 + \frac{1}{2}\sum_{i,j}\langle k_i k_j\rangle^2. \quad (1.20)$$

Now it is convenient to introduce a new notation:

$$\chi_+ \equiv \frac{1}{2}\sum_{1\le i<j\le N}[k_i k_j]^2, \quad \chi_- \equiv \frac{1}{2}\sum_{1\le i<j\le N}\langle k_i k_j\rangle^2. \quad (1.21)$$

Using this new notation:

*Note that photon labels are completely arbitrary

$$\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \chi_+ + \chi_-, \quad (1.22)$$
$$\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} = -i(\chi_+ - \chi_-). \quad (1.23)$$

The Maxwell field invariants are related to the field strength tensors and its dual by:

$$a^2 = \frac{1}{4}\sqrt{(F_{\mu\nu}F^{\mu\nu})^2 + (F_{\mu\nu}\tilde{F}^{\mu\nu})^2} + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad (1.24)$$
$$b^2 = \frac{1}{4}\sqrt{(F_{\mu\nu}F^{\mu\nu})^2 + (F_{\mu\nu}\tilde{F}^{\mu\nu})^2} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \quad (1.25)$$

Using (Eq. 1.23) and (Eq. 1.24):

$$a = \sqrt{\chi_+} + \sqrt{\chi_-}, \quad b = -i(\sqrt{\chi_+} - \sqrt{\chi_-}). \quad (1.26)$$

Note that these Maxwell invariants appear squared in the EHL, thus the choice of their sign does not matter.
In terms of χ_+ and χ_- the Lagrangian density at one-loop level for the spinor QED case is expressed by :

$$\mathcal{L}^{(1)}_{spin}(iF_{tot}) = -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T}e^{-m^2T}\frac{(\sqrt{\chi_+} + \sqrt{\chi_-})(\sqrt{\chi_+} - \sqrt{\chi_-})}{\tan((\sqrt{\chi_+} + \sqrt{\chi_-})T)\tan((\sqrt{\chi_+} - \sqrt{\chi_-})T)}. \quad (1.27)$$

To obtain the N-photon amplitude one must expand this expression in powers of χ_+ and χ_-. The expansion is carried out through the use of the following Taylor series:

$$\frac{x}{\tan x} = \sum_{n=0}^{\infty}(-1)^n \frac{2^{2n}B_{2n}}{(2n)!}x^{2n}, \quad (1.28)$$

where B_{2n} are the Bernoulli numbers. Then keeping only the part of order F^N and extracting those terms involving each individual f_i just once:

$$\mathcal{L}^{(1)}_{spin}(iF_{tot}) = -\frac{m^4}{8\pi^2}\sum_{N=4}^{\infty}\left(\frac{2e}{m^2}\right)^N \sum_{K=0}^{N} c^{(1)}_{spin}\left(\frac{K}{2},\frac{N-K}{2}\right)\chi_+^{\frac{K}{2}}\chi_-^{\frac{N-K}{2}}, \tag{1.29}$$

where

$$c^{(1)}_{\text{spin}}\left(\frac{K}{2},\frac{N-K}{2}\right) = (-1)^{\frac{N}{2}}(N-3)!\sum_{k=0}^{K}\sum_{l=0}^{N-K}(-1)^{N-K-l}\frac{\mathcal{B}_{k+l}\mathcal{B}_{N-k-l}}{k!l!(K-k)!(N-K-l)!}. \tag{1.30}$$

Picking out the terms multilinear in the $f_i's$ implies that all amplitudes with an odd number of '+' helicities vanish in the low energy limit because such terms exist only if K is an even number. The final result for the amplitude in the spinor case can be written as:

$$\Gamma^{(1)(EH)}_{spin}[\varepsilon_1^+;...;\varepsilon_K^+;\varepsilon_{K+1}^-;...;\varepsilon_N^-] = -\frac{m^4}{8\pi^2}\left(\frac{2e}{m^2}\right)^N c^{(1)}_{\text{spin}}\left(\frac{K}{2},\frac{N-K}{2}\right)\chi_K^+\chi_{N-K}^-, \tag{1.31}$$

where:

$$\begin{aligned}\chi_K^+ &\equiv (\chi_+)^{\frac{K}{2}}\big|_{\text{all different}}\\ &= \frac{\left(\frac{K}{2}\right)!}{2^{K/2}}\left\{[k_1k_2]^2[k_3k_4]^2\cdots[k_{K-1}k_K]^2 + \text{all permutations}\right\}, \qquad (1.32)\\ \chi_{N-K}^- &\equiv (\chi_-)^{\frac{N-K}{2}}\big|_{\text{all different}}. \qquad (1.33)\end{aligned}$$

Now we consider the scalar case which is completely analogous to the spinor case. The EHL integral representation for scalar QED was given in (Eq. 1.8):

$$\mathcal{L}^{(1)}_{scalar} = \frac{1}{16\pi^2}\int_0^{\infty}\frac{dT}{T}e^{-m^2T}\left(\frac{e^2ab}{\sinh(eaT)\sin(ebT)} + \frac{e^2}{6}(a^2-b^2) - \frac{1}{T^2}\right). \tag{1.34}$$

In this case we need the Taylor expansion:

$$\frac{x}{\sin x} = -\sum_{n=0}^{\infty}(-1)^n\frac{(2^{2n}-2)B_{2n}}{(2n)!}x^{2n}. \tag{1.35}$$

Following the same procedure as in the spinor QED case:

$$\Gamma^{(1)(EH)}_{scalar}[\varepsilon_1^+; ...; \varepsilon_K^+; \varepsilon_{K+1}^-; ...; \varepsilon_N^-] = \frac{m^4}{16\pi^2}\left(\frac{2e}{m^2}\right)^N c^{(1)}_{\text{scal}}\Big(\frac{K}{2}, \frac{N-K}{2}\Big)\chi_K^+ \chi_{N-K}^-. \quad (1.36)$$

Now, we define:

$$\begin{aligned} c^{(1)}_{\text{scal}}\Big(\frac{K}{2}, \frac{N-K}{2}\Big) &= (-1)^{\frac{N}{2}}(N-3)! \sum_{k=0}^{K}\sum_{l=0}^{N-K}(-1)^{N-K-l} \\ &\quad \times \frac{(1-2^{1-k-l})(1-2^{1-N+k+l})\mathcal{B}_{k+l}\mathcal{B}_{N-k-l}}{k!l!(K-k)!(N-K-l)!}. \end{aligned} \quad (1.37)$$

Once we have a simple expression for the one-loop N-photon amplitude in the low energy limit, we proceed to explore the double summation that appears in the coefficients of the (Eq. 1.31) with the purpose of simplifying this double summation:

$$S[K,L] = \sum_{k=0}^{K}\sum_{l=0}^{L}\frac{(-)^l \mathcal{B}_{k+l}\mathcal{B}_{N-k-l}}{k!l!(K-k)!(L-l)!}. \quad (1.38)$$

Preforming a change of variables $L = K - m$ and $k + l = x$ allow us to study the summation over k:

$$S[K,m] = \sum_{x=0}^{N}\mathcal{B}_x\mathcal{B}_{N-x} \sum_{max(0,x-L)<k<min(K,x)} \frac{(-)^{x-k}}{k!(x-k)!(K-k)!(K+k-m-x)!}. \quad (1.39)$$

Remember that K represents the photons with helicity '+' while L represents the ones that have helicity '-'. We start by exploring the case where $K = L$. Furthermore in [18] it is shown that all the amplitudes with an odd number of '+' helicities vanish in the low energy limit, this implies that in (Eq. 1.40) x and m must be even to have values different from zero, so after some manipulations using Mathematica (see

Appendix C) we find that the summation over k has a global factor:

$$\frac{(-1)^{x/2}}{(x/2)!\Gamma[2K+1]\Gamma[2K-\left(\frac{x-2}{2}+m\right)]}. \tag{1.40}$$

However this global factor comes with different polynomials every time you change the value of m so by denoting these polynomials $P_m(K,x)$ the summation results to be:

$$S[K,m]=\sum_{x=0}^{N}\mathcal{B}_x\mathcal{B}_{N-x}\frac{(-1)^{x/2}P_m(K,x)}{(x/2)!\Gamma[2K+1]\Gamma[2K-\left(\frac{x}{2}+m\right)]}. \tag{1.41}$$

Based in these polynomials it was found [††] that:

$$P_m(K,x):=\sum_{n=0}^{m}(-1)^{m-n}\binom{2m}{2n}(K-m-x/2-n+1)_n(x/2-m+n+1)_{m-n}, \tag{1.42}$$

where $(X)_n$ is the rising factorial function:

$$(X)_n:=X*(X+1)*....*(X+n-1). \tag{1.43}$$

After our manipulations we found another way to express Eq. (1.38). In the next section we presented a limit case where the double summation of Eq. (1.42) is solve.

1.4 Asymptotic limit

The high-order asymptotic behavior of the QED effective action is something that so far has not been widely explored, thus in this part we consider the limit were the number of photons is big to see if in the next summation can be simplify.

$$S[K,L]=\sum_{k=0}^{K}\sum_{l=0}^{L}\frac{(-1)^l\mathcal{B}_{k+l}\mathcal{B}_{N-k-l}}{k!l!(K-k)!(L-l)!}. \tag{1.44}$$

[††] The paper with the proof is in preparation we just corroborate that the formula works.

We are going to consider the Bernoulli numbers approximation :

$$\mathcal{B}_{2n} \propto \frac{(-1)^{n+1}2(2n)!}{(2\pi)^{2n}}. \tag{1.45}$$

In appendix C we present for which values of $2n$ this is a good approximation.
So:

$$\mathcal{B}_{k+l}\mathcal{B}_{N-k-l} = \mathcal{B}_{k+l}\mathcal{B}_{K+L-k-l} \propto \frac{(-1)^{(K+L)/2}4(k+l)!(K+L-k-l)!}{(2\pi)^{K+L}}. \tag{1.46}$$

Now we can replace the last expression in Eq. (1.45):

$$S^{\text{asymp}}_{\text{spin}}[K,L] = \sum_{k=0}^{K}\sum_{l=0}^{L} \frac{4(-1)^{l+(K+L)/2}(k+l)!(K+L-k-l)!}{(2\pi)^{K+L}k!l!(K-k)!(L-l)!}. \tag{1.47}$$

We have to exclude the case where $k+l$ takes odd values:

$$\begin{aligned} S^{\text{asymp}}_{\text{spin}} &= \sum_{k=0}^{K}\sum_{l=0}^{L}\left(\frac{(-1)^l+(-1)^k}{2}\right)\frac{4(-1)^{(K+L)/2}(k+l)!(K+L-k-l)!}{(2\pi)^{K+L}k!l!(K-k)!(L-l)!} \\ &= \frac{(-1)^{\frac{K+L}{2}}(K+L+2)!}{2^{K+1}(L+1)!(1)_{K/2}(3/2)_{K/2}} = (-1)^{N/2}\frac{2}{(2\pi)^N}\binom{N+2}{K+1}. \end{aligned} \tag{1.48}$$

Before starting on the asymptotic analysis, let us return to Eq. (1.31):

$$c^{(1)}_{\text{spin}}\left(\frac{K}{2},\frac{L}{2}\right) = (-1)^{N/2}(N-3)!S[K,L], \tag{1.49}$$

thus, we define:

$$\begin{aligned} c^{asymp}_{\text{spin}}\left(\frac{K}{2},\frac{L}{2}\right) &= (-1)^{N/2}(N-3)!S^{\text{asymp}}_{\text{spin}}[K,L] \\ &= \frac{2}{(2\pi)^N}(N-3)!\binom{N+2}{K+1}. \end{aligned} \tag{1.50}$$

Dividing this through the exact formula for the coefficients one expects the ratio to converge to unity if one fixes K or L and takes the other variable large. However, this is very difficult to see numerically, at least with MATHEMATICA. A detailed analysis shows that the main reason for this slow convergence are the terms involving the product $\mathcal{B}_0\mathcal{B}_N$, since our strategy of replacing $\zeta(2n)$ by unity in the formula:

$$\mathcal{B}_{2n} = \frac{(-1)^{n+1}2(2n)!}{(2\pi)^{2n}}\zeta(2n), \tag{1.51}$$

leads to the replacement of $\mathcal{B}_0 = 1$ by -2. We can correct this part of the error by adding a term $6\frac{\mathcal{B}_N}{K!L!}$ to $S[K, L]$, that is replacing c_{spin}^{asymp} by:

$$\tilde{c}_{\text{spin}}^{\text{asymp}} \equiv c_{\text{spin}}^{asymp} + 6(-1)^{N/2}(N-3)!\frac{\mathcal{B}_N}{K!L!}. \tag{1.52}$$

After this modification the convergence of the ratio to unity for K or L to infinity is already clearly visible in the numerics. However, this does not mean that the original formula did not already possess this property. Applying our asymptotic approximation to the $\mathcal{B}_N$ in Eq. (1.52) we can rewrite it for large N as:

$$\tilde{c}_{\text{spin}}^{\text{asymp}} \sim \frac{2}{(2\pi)^N}\frac{(N-3)!N!}{K!L!}\left[\frac{(N+2)(N-1)}{(K+1)(L+1)} - 6\right]. \tag{1.53}$$

This shows that the correction term is actually subleading in the large L limit. Thus its addition served only to accelerate the convergence of the ratio, it is not really necessary for convergence. Therefore for determining the limit itself we can return to Eq. (1.51). Using the asymptotic formula when $L \to \infty$:

$$(L+x)! \sim L^x L!, \tag{1.54}$$

we find:

$$c_{\text{spin}}^{asymp}\left(\frac{K}{2}, \frac{L}{2}\right) = \frac{2}{(2\pi)^N}\frac{(L+2K-2)!}{(K+1)!}. \tag{1.55}$$

This agrees with the known result for the $K = 0$ case. For $K > 0$ it is a new result.

Chapter 2

Tree-level photon amplitudes in the low energy limit.

In the standard formalism of quantum field theory (QFT) perturbative calculations are usually performed using second quantisation and Feynman diagrams but there also exist first quantised alternatives namely the Worldline formalism‖, this formalism have been shown to be extremely useful particularly in QED.
Compact master formulas for the one-loop N-photon amplitudes have been derived from the Worldline formalism and for scalar QED a master formula for the propagator dressed with N-photons in a constant field was obtained in [19].
Based on the results at the one-loop level, it is desirable to explore the N-photon amplitudes at tree-level in terms of the variables χ_+ and χ_- obtained in chapter 1 using spinor-helicity technology, so in this part of the thesis we manipulate the scalar propagator in a constant electromagnetic field to obtain photon amplitudes.
In QED the constant electromagnetic field is one of our favourite objects by being one field configuration for which the Dirac equation can be solved exactly. We are going to compute a proper time representation for the propagator of a scalar particle which moves from x to x' in such a constant background field.

2.1 Scalar propagator

Our starting point to compute the proper time representation of a scalar propagator is:

$$D^{xx'}(A) := \quad \langle x'| \frac{1}{m^2 + \Pi^2} |x\rangle \,, \tag{2.1}$$

‖In the literature this formalism is also known as string inspired formalism since the methods used have analogies to computations in string perturbation theory. Nervertheless the knowledge of string theory is not necessary for the application of the Worldline formalism.

where:

$$\Pi_\mu = p_\mu + eA_\mu = -i\partial_\mu + eA_\mu. \tag{2.2}$$

We write (2.1) in the proper time Fock-Schwinger representation:

$$D^{xx'}(A) = \langle x'| \int_0^\infty dT e^{-T(m^2+\Pi^2)} |x\rangle = \int_0^\infty dT e^{-Tm^2} \langle x'| e^{-T\Pi^2} |x\rangle . \tag{2.3}$$

The last expression contains the Kernel which is defined as:

$$K^{xx'}_{scal}(T;A) := \langle x'| e^{-T\Pi^2} |x\rangle . \tag{2.4}$$

This kernel has the path integral representation:

$$K^{xx'}_{scal} = \int_{x(0)=x}^{x(T)=x'} \mathcal{D}x(\tau) e^{-S[x(\tau)]}, \tag{2.5}$$

where

$$S = \int_0^T d\tau \left[\frac{\dot{x}_\mu \dot{x}^\mu}{4} + ie\dot{x}^\mu A_\mu\big(x(\tau)\big) \right]. \tag{2.6}$$

In the Fock-Schwinger gauge * , a unique relationship between the gauge potential $A_\mu(x)$ and the field strength $F_{\mu\nu}$ exists and this relation allows us to write the propagator in terms of the field strength instead of the gauge potential. Thus, in the Fock-Schwinger gauge the background is given by:

$$A^\mu\big(x(\tau)\big) = -\frac{1}{2} F^{\mu\nu}\big(x(\tau) - x\big)_\nu. \tag{2.7}$$

Putting this into (Eq. 2.6):

$$S = \int_0^T d\tau \left[\frac{\dot{x}^2}{4} - \frac{ie}{2}\dot{x}^\mu F_{\mu\nu}(x(\tau) - x)^\nu \right]. \tag{2.8}$$

Now, its convenient to introduce a change variables:

$$x(\tau) = x + \frac{\tau}{T} x_- + q(\tau), \quad x_- := x' - x, \quad q(T) = q(0) = 0. \tag{2.9}$$

*The Fock-Schwinger gauge is used to achieve manifest covariance in the calculation of the effective action.

So

$$\dot{x}(\tau) = \frac{x_-}{T} + \dot{q}(\tau), \tag{2.10}$$

and

$$S = \int_0^T d\tau \left[\frac{1}{4}\left(\frac{x_-}{T} + \dot{q}\right)^2 - \frac{ie}{2}\left(\frac{x_-}{T} + \dot{q}\right) F \left(\frac{\tau}{T}x_- + q\right)\right]. \tag{2.11}$$

After some algebraic manipulations and an integration by parts (IBP):

$$S = \int_0^T d\tau \left[\frac{x_-^2}{4T^2} + \frac{\dot{q}^2}{4} - \frac{ie}{T}x_-^\mu F_{\mu\nu} q^\nu + \frac{ie}{2} q^\mu F_{\mu\nu}\dot{q}^\nu\right]. \tag{2.12}$$

In terms of this change of variables the kernel can be expressed as:

$$K_{scal}^{xx'} = e^{-\frac{x_-^2}{4T}} \int \mathcal{D}q(\tau) e^{-\int_0^T d\tau \left[\frac{\dot{q}^2}{4} + \frac{ie}{2} q F \dot{q} - \frac{ie}{T} x_- F q\right]}, \tag{2.13}$$

or

$$K_{scal}^{xx'} = e^{-\frac{x_-^2}{4T}} \int \mathcal{D}q(\tau) e^{-\int_0^T d\tau \left[\frac{1}{4} q\left(-\frac{d^2}{d\tau^2} + 2ieF\frac{d}{d\tau}\right) q - \frac{ie}{T} x_- F q\right]}. \tag{2.14}$$

This integral is in the Gaussian form and can be preformed by the use of a Green function using string-inspired techniques and the result is:

$$K_{scal}^{xx'} = e^{-\frac{x_-^2}{4T}} (4\pi T)^{-D/2} \left[\det\left(\frac{eFT}{\sin(eFT)}\right)\right]^{1/2} e^{-\frac{e^2}{T^2}\int_0^T d\tau \int_0^T d\tau' x_- F \underline{\Delta}(\tau,\tau') F x_-}, \tag{2.15}$$

we have introduced the Green function:

$$\underline{\Delta}(\tau,\tau') = \langle \tau | \left(\frac{d^2}{d\tau^2} - 2ieF\frac{d}{d\tau}\right)^{-1} | \tau' \rangle . \tag{2.16}$$

This Green function satisfies the following identity:

$$\underline{\Delta}(\tau,\tau') = \frac{1}{2}\left[\mathcal{G}_B(\tau,\tau') - \mathcal{G}_B(\tau,0) - \mathcal{G}_B(0,\tau') + \mathcal{G}_B(0,0)\right], \tag{2.17}$$

where $\mathcal{G}_B$ is the bosonic Green function† in presence of an external field:

$$\mathcal{G}_B(\tau,\tau') = \frac{T}{2\mathcal{Z}^2}\left(\frac{\mathcal{Z}}{\sin\mathcal{Z}} e^{-i\mathcal{Z}\dot{G}_B(\tau,\tau')} + i\mathcal{Z}\dot{G}_B(\tau,\tau') - 1\right). \tag{2.18}$$

†The reader that is not familiar to this Green function is invited to read the section 5 of [20].

with $\mathcal{Z} = eFT$. Because of the "string inspired" boundary conditions:

$$\int_0^T d\tau\ \mathcal{G}_B(\tau,\tau') = \int_0^T d\tau'\ \mathcal{G}_B(\tau,\tau') = 0, \tag{2.19}$$

Eq. (2.15) turns out to be:

$$K_{scal}^{xx'} = e^{-\frac{x_-^2}{4T}} (4\pi T)^{-D/2} \left[\det\left(\frac{eFT}{\sin(eFT)}\right)\right]^{1/2} e^{-\frac{e^2}{2T^2}\int_0^T d\tau \int_0^T d\tau' x_- F\mathcal{G}_B(0,0)Fx_-}. \tag{2.20}$$

Performing the integrals over τ and τ':

$$K_{scal}^{xx'} = e^{-\frac{x_-^2}{4T}} (4\pi T)^{-D/2} \left[\det\left(\frac{eFT}{\sin(eFT)}\right)\right]^{1/2} e^{-\frac{e^2}{4}Tx_-F\left(\frac{\cot\mathcal{Z}}{\mathcal{Z}}-\frac{1}{\mathcal{Z}^2}\right)Fx_-}. \tag{2.21}$$

Writing the previous expression in terms of $\mathcal{Z}$

$$K_{scal}^{xx'} = (4\pi T)^{-D/2} \left[\det\left(\frac{\mathcal{Z}}{\sin\mathcal{Z}}\right)\right]^{1/2} e^{-\frac{1}{4T}x_-\mathcal{Z}\cot\mathcal{Z}x_-}. \tag{2.22}$$

Putting this in the propagator and Fourier transforming to momentum space :

$$D^{pp'}(A) = \int_0^\infty dT e^{-Tm^2}(2\pi)^4\delta(p+p') \left[\det\left(\frac{1}{\cos\mathcal{Z}}\right)\right]^{1/2} e^{-Tp\left(\frac{\tan(\mathcal{Z})}{\mathcal{Z}}\right)p}. \tag{2.23}$$

2.2 Master formula

We want to use Eq. (2.23) to obtain a new master formula to compute N-photon amplitudes for arbitrary N and any helicity assignment. The first step is to expand the exponential and the determinant term.
We are going to use spinor helicity technology as in the one-loop case where we define:

$$F = \sum_{i=1}^{K} f_i^+ + \sum_{j=K+1}^{K+L} f_j^- = f^+ + f^-. \tag{2.24}$$

Note that the expansion of $\tan(\mathcal{Z})/\mathcal{Z}$ contains just powers of F^{2n}. The case $n = 1$

give us:

$$\begin{aligned}
(F^2) &= (f^+ + f^-)(f^+ + f^-) = (f^+)^2 + (f^-)^2 + (f^+f^-) + (f^-f^+) && (2.25)\\
&= \sum_{i,k}(f_i^+ f_k^+) + \sum_{j,l}(f_j^- f_l^-) + (f^+f^-) + (f^-f^+) && (2.26)\\
&= -\chi_+\eta - \chi_-\eta + 2(f^+f^-). && (2.27)
\end{aligned}$$

To get the last line we use:

$$\begin{aligned}
\sum_{i,k}(f_i^+ f_k^+)^{\nu\mu} &= \sum_{i>k}\{f_i^+, f_k^+\}^{\nu\mu} = -\frac{1}{2}\sum_{i>k}[k_i k_k]^2\,\eta^{\mu\nu} = -\chi_+\,\eta, && (2.28)\\
\sum_{j,l}(f_j^- f_l^-)^{\nu\mu} &= \sum_{j>l}\{f_j^-, f_l^-\}^{\nu\mu} = -\frac{1}{2}\sum_{j,l}\langle k_j k_l\rangle^2\,\eta^{\mu\nu} = -\chi_-\,\eta. && (2.29)
\end{aligned}$$

Where η is the metric (see Appendix A for conventions).
For $n = 2$:

$$F^4 = (f^+f^+ + f^-f^- + 2f^+f^-)^2 = \chi_+^2\eta + \chi_-^2\eta + 6\chi_+\chi_-\eta - 4\chi_+f^+f^- - 4\chi_-f^+f^-. \qquad (2.30)$$

With the previous calculation we can infer that this powers depends only on the scalars χ_+ and χ_- and the matrices η and f^+f^-. We can derive a way to write F^{2n} for any value of n(see Appendix B):

$$\begin{aligned}
F^{2n} &= (f^+ + f^-)^{2n}\\
&= \frac{(-1)^n}{2}\Bigg(\left[(\sqrt{\chi_+} + \sqrt{\chi_-})^{2n} + (\sqrt{\chi_+} - \sqrt{\chi_-})^{2n}\right]\eta^{\mu\nu}\\
&\quad + \frac{(f_+f_-)^{\mu\nu}}{\sqrt{\chi_+\chi_-}}\left[(\sqrt{\chi_+} - \sqrt{\chi_-})^{2n} - (\sqrt{\chi_+} + \sqrt{\chi_-})^{2n}\right]\Bigg).
\end{aligned} \qquad (2.31)$$

The previous result can be used in:

$$\begin{aligned}
\frac{\tan(\mathcal{Z})}{\mathcal{Z}} &= \sum_{n=0}^{\infty} c_n \mathcal{Z}^{2n} = \sum_{n=0}^{\infty} c_n (eT)^{2n} F^{2n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n c_n (eT)^{2n}}{2} \Bigg(\left[(\sqrt{\chi_+} + \sqrt{\chi_-})^{2n} + (\sqrt{\chi_+} - \sqrt{\chi_-})^{2n} \right] \eta^{\mu\nu} + \\
&\quad \frac{(f_+ f_-)^{\mu\nu}}{\sqrt{\chi_+ \chi_-}} \left[(\sqrt{\chi_+} - \sqrt{\chi_-})^{2n} - (\sqrt{\chi_+} + \sqrt{\chi_-})^{2n} \right] \Bigg) \\
&= \frac{1}{2} \Bigg(\frac{\tanh(eT(\sqrt{\chi_+} + \sqrt{\chi_-}))}{eT(\sqrt{\chi_+} + \sqrt{\chi_-})} \left[\mathbb{1} - \frac{(f_+ f_-)^{\mu\nu}}{\sqrt{\chi_+ \chi_-}} \right] \\
&\quad + \frac{\tanh(eT(\sqrt{\chi_+} - \sqrt{\chi_-}))}{eT(\sqrt{\chi_+} - \sqrt{\chi_-})} \left[\mathbb{1} + \frac{(f_+ f_-)^{\mu\nu}}{\sqrt{\chi_+ \chi_-}} \right] \Bigg).
\end{aligned} \tag{2.32}$$

In Eq. (2.23) appears $e^{-Tp\frac{\tan(\mathcal{Z})}{\mathcal{Z}}p}$:

$$\begin{aligned}
e^{-Tp\frac{\tan(\mathcal{Z})}{\mathcal{Z}}p} &= e^{\frac{-T}{2} p^2 \left[\frac{\tanh(eT(\sqrt{\chi_+}+\sqrt{\chi_-}))}{eT(\sqrt{\chi_+}+\sqrt{\chi_-})} + \frac{\tanh(eT(\sqrt{\chi_+}-\sqrt{\chi_-}))}{eT(\sqrt{\chi_+}-\sqrt{\chi_-})} \right]} \\
&\quad \times e^{\frac{-Tp(f_+ f_-)p}{2\sqrt{\chi_+ \chi_-}} \left[\frac{\tanh(eT(\sqrt{\chi_+}-\sqrt{\chi_-}))}{eT(\sqrt{\chi_+}-\sqrt{\chi_-})} - \frac{\tanh(eT(\sqrt{\chi_+}+\sqrt{\chi_-}))}{eT(\sqrt{\chi_+}+\sqrt{\chi_-})} \right]}.
\end{aligned} \tag{2.33}$$

On the other hand the expansion of the determinant is:

$$\begin{aligned}
\mathrm{Det} \left(\frac{1}{\cos(\mathcal{Z})} \right)^{1/2} &= \mathrm{Det}\,(\sec(\mathcal{Z}))^{1/2} = e^{\log\left[\mathrm{Det}(\sec(\mathcal{Z}))^{1/2}\right]} = e^{\mathrm{tr}\left[\log(\sec(\mathcal{Z}))^{1/2}\right]} \\
&= e^{\frac{\mathrm{tr}}{2}[\log(\sec(\mathcal{Z}))]},
\end{aligned} \tag{2.34}$$

where:

$$\begin{aligned}
\frac{\mathrm{tr}}{2} \left[\log\left(\sec(\mathcal{Z})\right) \right] &= \frac{\mathrm{tr}}{2} \left[\sum b_n \mathcal{Z}^{2n} \right] = \frac{\mathrm{tr}}{2} \left[\sum b_n (eT)^{2n} F^{2n} \right] \\
&= \frac{\mathrm{tr}}{2} \left[\sum b_n (eT)^{2n} \left(\frac{(-1)^n}{2} \left[(\sqrt{\chi_+} + \sqrt{\chi_-})^{2n} + (\sqrt{\chi_+} - \sqrt{\chi_-})^{2n} \right] \eta \right) \right] \\
&= \frac{1}{4} \sum (-1)^n b_n (eT)^{2n} (\sqrt{\chi_+} + \sqrt{\chi_-})^{2n} \mathrm{tr}[\eta] \\
&\quad + \frac{1}{4} \sum (-1)^n b_n (eT)^{2n} (\sqrt{\chi_+} - \sqrt{\chi_-})^{2n} \mathrm{tr}[\eta] \\
&= \log(\mathrm{sech}(eT(\sqrt{\chi_+} + \sqrt{\chi_-}))) + \log(\mathrm{sech}(eT(\sqrt{\chi_+} - \sqrt{\chi_-}))) \\
&= \log(\mathrm{sech}(eT(\sqrt{\chi_+} + \sqrt{\chi_-}))\mathrm{sech}(eT(\sqrt{\chi_+} - \sqrt{\chi_-}))).
\end{aligned} \tag{2.35}$$

With the previous expansions, we can express the propagator (Eq. 2.23) as:

$$\begin{aligned} D^{pp'}(A) &= \int_0^\infty dT \Big\{ e^{-T(m^2+p^2)} \mathrm{sech}(eT(\sqrt{\chi_+}+\sqrt{\chi_-}))\mathrm{sech}(eT(\sqrt{\chi_+}-\sqrt{\chi_-})) \\ & e^{-T\frac{p^2}{2}\left[\frac{\tanh(eT(\sqrt{\chi_+}+\sqrt{\chi_-}))}{eT(\sqrt{\chi_+}+\sqrt{\chi_-})}+\frac{\tanh(eT(\sqrt{\chi_+}-\sqrt{\chi_-}))}{eT(\sqrt{\chi_+}-\sqrt{\chi_-})}-2\right]} \\ & e^{-T\frac{p(f_+f_-)p}{2\sqrt{\chi_+\chi_-}}\left[\frac{\tanh(eT(\sqrt{\chi_+}-\sqrt{\chi_-}))}{eT(\sqrt{\chi_+}-\sqrt{\chi_-})}-\frac{\tanh(eT(\sqrt{\chi_+}+\sqrt{\chi_-}))}{eT(\sqrt{\chi_+}+\sqrt{\chi_-})}\right]}\Big\} \\ &= \int_0^\infty dT e^{-T(m^2+p^2)} F[K,L], \end{aligned} \tag{2.36}$$

where $F[K,L]$ is a function of the number of photons with positive and negative helicity denoted by K and L respectively. To obtain photon amplitudes from (Eq. 2.36) we expand $F[K,L]$ using Mathematica (see the Appendix B to find the details of this computation), in the following we just present the computations for small values of N, namely the 2- and 4-photon amplitude.

The first example that we are going to compute is the two legs case where we have to consider just two configurations A(++) and A(+ -).

A(++):

$$\begin{aligned} \int_0^\infty dT e^{-T(m^2+p^2)} F[2,0] &= \frac{e^2p^2\chi_+}{3}\int_0^\infty dT e^{-T(m^2+p^2)}T^3 - e^2\chi_+\int_0^\infty dT e^{-T(m^2+p^2)}T^2 \\ &= -\frac{2e^2m^2\chi_+}{(m^2+p^2)^4} \end{aligned} \tag{2.37}$$

A(- -):

$$\begin{aligned} \int_0^\infty dT e^{-T(m^2+p^2)} F[0,2] &= \frac{e^2p^2\chi_-}{3}\int_0^\infty dT e^{-T(m^2+p^2)}T^3 - e^2\chi_-\int_0^\infty dT e^{-T(m^2+p^2)}T^2 \\ &= -\frac{2e^2m^2\chi_-}{(m^2+p^2)^4}. \end{aligned} \tag{2.38}$$

A(+-):

$$\begin{aligned} \int_0^\infty dT e^{-T(m^2+p^2)} F[1,1] &= -\frac{2e^2 p\cdot f_1^+ f_2^- \cdot p}{3}\int_0^\infty dT e^{-T(m^2+p^2)}T^3 \\ &= -\frac{4e^2 p\cdot f_1^+ f_2^- \cdot p}{(m^2+p^2)^4}. \end{aligned} \tag{2.39}$$

If we want to use the last results to obtain the two-photon scattering amplitude we will need to amputate the external scalar legs by LSZ and add the two possible configurations:

$$i\hat{D}^{pp'} = (p^2+m^2)^2\left(-\frac{2e^2m^2\chi_p}{(m^2+p^2)^4} - \frac{4e^2 p\cdot f_1^+ f_2^- \cdot p}{(m^2+p^2)^4}\right) \tag{2.40}$$

$$= -e^2\left(\frac{2m^2(\chi_+ + \chi_-)}{(m^2+p^2)^2} + \frac{4p\cdot f_1^+ f_2^- \cdot p}{(m^2+p^2)^2}\right). \tag{2.41}$$

Before proceed with the computation of more examples let us verify if the result obtained can be extracted from the worldline representation of the propagator of a scalar field coupled to an Abelian gauge field with gauge potential A^μ consisting of plane wave, in the low energy limit. In terms of the worldline Green function for Dirichlet boundary conditions as:

$$\begin{aligned}\mathcal{D}^{xx'}[k_1.\varepsilon_1;...;k_N,\varepsilon_N] &= (-ie)^N\int_0^\infty dT(4\pi T)^{-D/2}e^{-m^2T}e^{-\frac{(x-x')^2}{4T}} \\ &\quad \prod_{i=1}^N\int_0^T d\tau_i e^{\sum_{i=1}^N\left(ik_i\cdot(x'+(x-x')\frac{\tau_i}{T})+\frac{\varepsilon_i}{T}\cdot(x-x')\right)} \\ &\quad \times e^{\sum_{i,j=1}^N\left(\Delta_{ij}k_i\cdot k_j - 2i{}^\bullet\Delta_{ij}k_i\cdot k_j - {}^\bullet\Delta^\bullet_{ij}\varepsilon_i\cdot\varepsilon_j\right)}\Big|_{lin\varepsilon_1...\varepsilon_N},\end{aligned} \tag{2.42}$$

where, the Green function, its first and second derivative are:

$$\Delta_{ij} := \frac{1}{2}|\tau_i-\tau_j| - \frac{1}{2}(\tau_i+\tau_j)+\tau_i\tau_j, \tag{2.43}$$

$${}^\bullet\Delta_{ij} = \frac{\tau_j}{T}+\frac{1}{2}\sigma(\tau_i-\tau_j)-\frac{1}{2}, \tag{2.44}$$

$${}^\bullet\Delta^\bullet_{ij} = -\delta(\tau_i-\tau_j)+\frac{1}{T}. \tag{2.45}$$

Now if we fixed $N = 2$ and applied a change of variables $x_- = x - x'$ we obtained:

$$\begin{aligned}\mathcal{D}^{xx'}[k_1.\varepsilon_1; k_2, \varepsilon_2] &= -e^2 \int_0^\infty dT (4\pi T)^{-D/2} e^{-m^2 T} e^{-\frac{x_-^2}{4T}} e^{ik_1\cdot x'} e^{ik_2\cdot x'} \int_0^T d\tau_1 d\tau_2 \\ &\exp\Big\{\varepsilon_1 \cdot \frac{x_-}{T}\varepsilon_2 \cdot \frac{x_-}{T} - 2i\varepsilon_1 \cdot \frac{x_-}{T}\left(\varepsilon_2 \cdot k_1{}^\bullet\Delta_{12} + \varepsilon_2 \cdot k_2{}^\bullet\Delta_{22}\right) \\ &-2i\varepsilon_2 \cdot \frac{x_-}{T}\left(\varepsilon_1 \cdot k_1{}^\bullet\Delta_{11} + \varepsilon_1 \cdot k_2{}^\bullet\Delta_{12}\right) - 2\varepsilon_1 \cdot \varepsilon_2{}^\bullet\Delta_{12}^\bullet\Big\} \\ &\exp\Big\{ - 4\varepsilon_1 \cdot k_1\varepsilon_2 \cdot k_1{}^\bullet\Delta_{11}{}^\bullet\Delta_{21} - 4\varepsilon_1 \cdot k_1\varepsilon_2 \cdot k_2{}^\bullet\Delta_{11}{}^\bullet\Delta_{22} \\ &-4\varepsilon_1 \cdot k_2\varepsilon_2 \cdot k_1{}^\bullet\Delta_{12}{}^\bullet\Delta_{21} - 4\varepsilon_1 \cdot k_2\varepsilon_2 \cdot k_2{}^\bullet\Delta_{12}{}^\bullet\Delta_{22}\Big\}.\end{aligned} \tag{2.46}$$

It is more convenient to work in momentum space, so we transform the last expression:

$$\mathcal{D}^{pp'} = \int d^D x d^D x' \mathcal{D}^{xx'} e^{i(px+p'x')}. \tag{2.47}$$

After some algebraic manipulations and fixing $\tau_1 > \tau_2$:

$$\begin{aligned}\mathcal{D}^{pp'} &= -e^2(2\pi)^D\delta^D(p+p')\int_0^\infty dT e^{-m^2T} e^{-\frac{T(p-p'-k_1-k_2)^2}{4}} \\ &\int_0^T d\tau_1 \int_0^{\tau 1} d\tau_2 e^{-(p-p')(\tau_1 k_1+\tau_2 k_2)+|\tau_1-\tau_2|k_1\cdot k_2}\Big\{2\delta_{12}\varepsilon_1\cdot\varepsilon_2 \\ &-\varepsilon_1\cdot((p-p')-k_2)((p-p')+k_1)\cdot\varepsilon_2\Big\}.\end{aligned} \tag{2.48}$$

These integrals can be computed:

$$\mathcal{D}^{pp'} = -e^2\left\{\frac{32\varepsilon_1\cdot\varepsilon_2}{f_1 f_2} - \frac{64\varepsilon_1\cdot(\Delta p - k_2)(\Delta p + k_1)\cdot\varepsilon_2}{a_1 a_2 a_3}\right\}, \tag{2.49}$$

where:

$$\Delta p = p - p', \tag{2.50}$$

$$a_1 = (\Delta p - k_1 - k_2)^2 + 4m^2, \tag{2.51}$$

$$a_2 = (\Delta p - k_1 - k_2)^2 + 4(m^2 + \Delta p(k_1 + k_2)), \tag{2.52}$$

$$a_3 = (\Delta p - k_1 - k_2)^2 + 4(m^2 + (k_1(\Delta p - k_2)). \tag{2.53}$$

To extract the 2-photon amplitude in the low energy limit from Eq. 2.46 we must expand $(f_1)^{-1}, (f_2)^{-1}$ and $(f_3)^{-1}$, then we must consider the terms that contain linearly k_1 and k_2. In this limit $\Delta p = 2p$, with these considerations we finally found that:

$$\begin{aligned}\mathcal{D}^{pp'} &= -2e^2\frac{\varepsilon_1\cdot k_2\varepsilon_2 k_1 - 2\varepsilon_1\cdot\varepsilon_2 k_1\cdot k_2}{(p^2+m^2)^3} \\ &\quad -4e^2\frac{\varepsilon_1\cdot p\varepsilon_2\cdot k_1 p\cdot k_2 + \varepsilon_1\cdot k_2\varepsilon_2\cdot pp\cdot k_1 - \varepsilon_1\cdot\varepsilon_2 p\cdot k_1 p\cdot k_2 - \varepsilon_1\cdot p\varepsilon_2\cdot pk_1\cdot k_2}{(p^2+m^2)^4}.\end{aligned} \tag{2.54}$$

In terms of the field strength tensor:

$$\mathcal{D}^{pp'} = \frac{e^2\mathrm{tr}(f_1\cdot f_2)}{(p^2+m^2)^3} - \frac{4e^2 p\cdot f_1\cdot f_2\cdot p}{(p^2+m^2)^4}. \tag{2.55}$$

To compare the last equation with Eq. 2.40 we need to use spinor helicity, use the LSZ theorem and we found:

$$i\hat{\mathcal{D}}^{pp'} = -e^2\left(\frac{2m^2(\chi_+ + \chi_-)}{(m^2+p^2)^2} + \frac{4p\cdot f_1^+ f_2^-\cdot p}{(m^2+p^2)^2}\right). \tag{2.56}$$

With the previous calculations became clear that our master formula (Eq. 2.36) is more efficient than the master formula expressed in terms of the Green functions. In the following we are going to compute more cases.
In the four legs case we have to consider three configurations A(+ + ++),A(+ + + -),A(+ + - -) because the other configurations are related to these.
A(+ + + +):

$$\begin{aligned}\int_0^\infty dT e^{-T(m^2+p^2)}F[4,0] &= \frac{2e^4\chi_+^2}{3}\int_0^\infty dT e^{-T(m^2+p^2)}T^4 \\ &\quad -\frac{7e^4p^2\chi_+^2}{15}\int_0^\infty dT e^{-T(m^2+p^2)}T^5 \\ &\quad +\frac{e^4p^4\chi_+^2}{18}\int_0^\infty dT e^{-T(m^2+p^2)}T^6 \\ &= \frac{8e^4(2m^4-3m^2p^2)\chi_+^2}{(m^2+p^2)^7}.\end{aligned} \tag{2.57}$$

A(+ + + -):

$$\begin{aligned}\int_0^\infty dTe^{-T(m^2+p^2)}F[3,1] &= \frac{6e^4\chi_+ p\cdot(f^+f^-)\cdot p}{5}\int_0^\infty dTe^{-T(m^2+p^2)}T^5 \\ &\quad -\frac{2e^4p^2\chi_+ p\cdot(f^+f^-)\cdot p}{9}\int_0^\infty dTe^{-T(m^2+p^2)}T^6 \\ &= -\frac{16e^4(p^2-9m^2)\chi_+ p\cdot(f^+f^-)\cdot p}{(m^2+p^2)^7} \qquad (2.58)\end{aligned}$$

A(+ + - -):

$$\begin{aligned}\int_0^\infty dTe^{-T(m^2+p^2)}F[2,2] &= 2e^4\chi_+\chi_-\int_0^\infty dTe^{-T(m^2+p^2)}T^4 \\ &\quad -\frac{22e^4p^2\chi_+\chi_-}{15}\int_0^\infty dTe^{-T(m^2+p^2)}T^5 \\ &\quad +\frac{e^4p^4\chi_+\chi_-}{9}\int_0^\infty dTe^{-T(m^2+p^2)}T^6 \\ &\quad +\frac{2e^4p^2(p\cdot(f^+f^-)\cdot p)^2}{9}\int_0^\infty dTe^{-T(m^2+p^2)}T^6 \\ &= \frac{16e^4((3m^4-5m^2p^2-3p^4)\chi_+\chi_- + 10(p\cdot(f^+f^-)\cdot p)^2)}{(m^2+p^2)^7}. \qquad (2.59)\end{aligned}$$

The last example presented in this work is the 6 point case.
A(++++++), A(+++++-),A(++++--),A(+++---).
We start with A(++++++):

$$\int_0^\infty dTe^{-T(m^2+p^2)}F[6,0] = -\frac{16e^6}{(m^2+p^2)^{10}}(17m^6-78m^4p^2+45m^2p^4)\chi_+^3. \qquad (2.60)$$

A(+++++-):

$$\int_0^\infty dTe^{-T(m^2+p^2)}F[5,1] = -\frac{32e^6}{(m^2+p^2)^{10}}(205m^4-206m^2p^2+9p^4)\chi_+^2 p\cdot(f_+f_-)\cdot p. \qquad (2.61)$$

A(++++--):

$$\int_0^\infty dTe^{-T(m^2+p^2)}F[4,2] = \frac{16e^6\chi_+}{(m^2+p^2)^{10}} \Big((-105m^6+514m^4p^2+139m^2p^4-60p^6)\chi_+\chi_- +112(2p^2-13m^2)(p\cdot(f_+f_-)\cdot p)^2\Big). \tag{2.62}$$

Finally A(+++---):

$$\int_0^\infty dTe^{-T(m^2+p^2)}F[3,3] = -\frac{128e^6(p\cdot(f_+f_-)\cdot p)}{(m^2+p^2)^{10}} \Big((137m^4-146m^2p^2-73p^4)\chi_+\chi_- +140(p\cdot(f_+f_-)\cdot p)^2\Big). \tag{2.63}$$

With these examples it became clear that the master formula expressed in terms of the invariants χ_+, χ_- and f_+f_- is an efficient way to compute multi-photon amplitudes. Note that the spinor helicity technology help us with the treatment of photons (massless particles), but we did not use it for the massive scalar particle because the spinor helicity technology is often believed to be specific to the massless case. However it turns out that a massive momentum p^μ can be express in term of 2 massless momenta [21]. In the appendix B we explore this fact in order to find simplifications.

Conclusions and Outlook

In this work, by exploring the coefficients obtained in [18] for the spinor QED case, we rewrite them in terms of a relation of factorials, that seems to be unknown. Moreover, in the asymptotic limit of the Bernoulli numbers these coefficients can be simplify in a significant way, this is significant because it is quite rare that it is possible to study amplitudes in the limit of a large number of external legs, so we will explore the coefficients of the scalar QED case in the future.
We also studied the tree-level photon amplitudes and we derived a new master formula that can be used to compute N-photon amplitudes for arbitrary N and any choice of helicity. Our master formula shows significant computational advantage at higher values of N, and to small values of N we compare our results with the ones obtained with the Worldline master formula[23]. We would like to derive a similar master formula in terms of spinor helicity notation for the spinor case. With the help of spinor helicity, we present a compendium of formulas for the strength field tensor.

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Appendix A

Conventions

In this thesis the notation employed agrees with Mark Srednicki [16]. We use natural units $\hbar = c = 1$ and the four-dimensional space-time indices are denoted by lowercase Greek letters $(\alpha, \beta, ...)$. The metric for Minkowski space is $\eta^{\mu\nu} = diag(-, +, +, +)$.The Einstein summation convention is always implicitly assumed and the metric is used to raise and lower Lorentz indices $x_\mu = \eta_{\mu\nu} x^\nu$ and the totally anti-symmetric tensor is denoted $\varepsilon^{\mu\nu\alpha\beta}$ ($\varepsilon^{0123} = -\varepsilon_{0123} = 1$). Scalar product of two four-vectors is denoted by $x \cdot y = x^\mu y_\mu = x^\mu \eta_{\mu\nu} y^\nu$.

Appendix B

Spinor Helicity identities

The spinor helicity technology can be seen as a special notation for spinors and polarisation vectors of definite helicity for massless electrons and positrons and it can be used to simplify calculations in this way is important to have some identities at hand.
We define:

$$
\begin{aligned}
|p] &\equiv u_-(p) = v_+(p), && \text{(B.1)}\\
|p\rangle &\equiv u_+(p) = v_-(p), && \text{(B.2)}\\
[p| &\equiv \bar{u}_+(p) = \bar{v}_-(p), && \text{(B.3)}\\
\langle p| &\equiv \bar{u}_-(p) = \bar{v}_+(p). && \text{(B.4)}
\end{aligned}
$$

We then have a product:

$$
\begin{aligned}
[k||p] &= [kp], && \text{(B.5)}\\
\langle k||p\rangle &= \langle kp\rangle\,, && \text{(B.6)}\\
[k||p\rangle &= 0, && \text{(B.7)}\\
\langle k||p] &= 0. && \text{(B.8)}
\end{aligned}
$$

This product is antisymmetric:

$$
\begin{aligned}
[kp] &= -[pk], && \text{(B.9)}\\
\langle kp\rangle &= -\langle pk\rangle\,. && \text{(B.10)}
\end{aligned}
$$

For any massless four-momentum p we can write:

$$-\not{p} = |p\rangle\,[p| + |p]\,\langle p|\,. \tag{B.11}$$

In spinor helicity, the photon polarization vectors in terms of bracket notation:

$$\varepsilon_+^\mu(k,q) = -\frac{\langle q|\,\gamma^\mu|k]}{\sqrt{2}\,\langle qk\rangle}, \tag{B.12}$$

$$\varepsilon_-^\mu(k,q) = -\frac{[q|\gamma^\mu\,|k\rangle}{\sqrt{2}[qk]}, \tag{B.13}$$

where q is an arbitrary reference momentum. It is easy to see that:

$$\varepsilon_+(k;q)\cdot\varepsilon_+(k';q') = \frac{\langle qq'\rangle\,[kk']}{\langle qk\rangle\,\langle q'k'\rangle}, \tag{B.14}$$

$$\varepsilon_-(k;q)\cdot\varepsilon_-(k';q') = \frac{[qq']\,\langle kk'\rangle}{[qk][q'k']}, \tag{B.15}$$

$$\varepsilon_+(k;q)\cdot\varepsilon_-(k';q') = \frac{\langle qk'\rangle\,[kq']}{\langle qk\rangle\,[q'k']}. \tag{B.16}$$

Another useful relation in spinor helicity is:

$$-k\cdot p = \frac{1}{2}\,\langle kp\rangle\,[pk]. \tag{B.17}$$

We are going to derive the identities of the anti-commutator Eq. (1.5) and Eq. (1.6). For the derivation of these identities we will need to use:

$$(f_1^+ f_2^+)^{\mu\nu} = -\frac{[k_1k_2]}{4\,\langle k_1k_2\rangle}tr(P_-\not{k}_1\not{k}_2\gamma^\nu\gamma^\mu), \tag{B.18}$$

$$(f_1^- f_2^-)^{\mu\nu} = -\frac{\langle k_1k_2\rangle}{4[k_1k_2]}tr(P_+\not{k}_1\not{k}_2\gamma^\nu\gamma^\mu), \tag{B.19}$$

where:

$$P_\pm = \frac{1}{2}(\mathbb{1}\pm\gamma^5). \tag{B.20}$$

We are just going to show (B.18), and to simplify the notation we use $[k_ik_j] = [ij]$ and $\langle k_ik_j\rangle = \langle ij\rangle$. First lets compute the right side:

$$\begin{aligned}
-\frac{[12]}{4\langle 12\rangle}tr(P_-\not{k}_1\not{k}_2\gamma^\nu\gamma^\mu) &= -\frac{[12]}{4\langle 12\rangle}tr\left(\left[\frac{\mathbb{1}}{2}-\frac{\gamma^5}{2}\right]\not{k}_1\not{k}_2\gamma^\nu\gamma^\mu\right) \\
&= -\frac{[12]}{\langle 12\rangle}\frac{k_{1\alpha}k_{2\beta}}{8}\left(4(\eta^{\alpha\beta}\eta^{\nu\mu}-\eta^{\alpha\nu}\eta^{\beta\mu}+\eta^{\alpha\mu}\eta^{\beta\nu})+4i\varepsilon^{\alpha\beta\nu\mu}\right) \\
&= -\frac{[12]}{2\langle 12\rangle}\left(k_1\cdot k_2\eta^{\nu\mu}-k_1^\nu k_2^\mu+k_1^\mu k_2^\nu+ik_{1\alpha}k_{2\beta}\varepsilon^{\alpha\beta\nu\mu}\right).
\end{aligned} \tag{B.21}$$

Now we are going to compute the left side of (B.18):

$$(f_1^+f_2^+)^{\mu\nu} = (f_1^+)^{\mu\lambda}(f_2^+)_\lambda{}^\nu = (k_1^\mu\varepsilon_1^{+\lambda}-\varepsilon_1^{+\mu}k_1^\lambda)(k_{2\lambda}\varepsilon_2^{+\nu}-\varepsilon_{2\lambda}^+k_2^\nu). \tag{B.22}$$

Since the polarization vectors depend on arbitrary massless reference momentum q, we can simplify the last expression if we fix :

$$q_1 = k_2, \quad q_2 = k_1 \tag{B.23}$$

With this election (B.22) is reduced to:

$$(f_1^+f_2^+)^{\mu\nu} = -k_1^\mu k_2^\nu\varepsilon_1^+\cdot\varepsilon_2^+ - k_1\cdot k_2\varepsilon_1^{+\mu}\varepsilon_2^{+\nu}. \tag{B.24}$$

Using (B.12) and (B.14):

$$(f_1^+f_2^+)^{\mu\nu} = -\frac{[12]}{\langle 12\rangle}k_1^\mu k_2^\nu + \frac{k_1\cdot k_2}{2\langle 12\rangle^2}\langle k_2|\gamma^\mu|k_1]\langle k_1|\gamma^\nu|k_2]. \tag{B.25}$$

Now, we can use (B.11) to substitute $|k_1]\langle k_1|$:

$$(f_1^+f_2^+)^{\mu\nu} = -\frac{[12]}{\langle 12\rangle}k_1^\mu k_2^\nu - \frac{k_1\cdot k_2}{2\langle 12\rangle^2}\langle k_2|\gamma^\mu\not{k}_1\gamma^\nu|k_2] \tag{B.26}$$

$$= -\frac{[12]}{\langle 12\rangle}k_1^\mu k_2^\nu + \frac{k_1\cdot k_2}{2\langle 12\rangle^2}tr(P_-\not{k}_2\gamma^\mu\not{k}_1\gamma^\nu), \tag{B.27}$$

to get (B.28) we used the identity:

$$\langle k_2|\,\gamma^\mu \not{k}_1 \gamma^\nu |k_2] = -tr(P_- \not{k}_2 \gamma^\mu \not{k}_1 \gamma^\nu). \tag{B.28}$$

Let's compute separately the trace term:

$$\begin{aligned} \frac{k_1 \cdot k_2}{2\,\langle 12\rangle^2} tr(P_- \not{k}_2 \gamma^\mu \not{k}_1 \gamma^\nu) &= \frac{k_1 \cdot k_2}{2\,\langle 12\rangle^2} \frac{k_{2\alpha} k_{1\beta}}{2} \Big(tr(\gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\nu) - tr(\gamma^5 \gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\nu) \Big), \\ &= \frac{k_1 \cdot k_2}{\langle 12\rangle^2} \Big(k_2^\mu k_1^\nu - k_2 \cdot k_1 \eta^{\mu\nu} + k_2^\nu k_1 \mu + i k_{2\alpha} k_{1\beta} \varepsilon^{\alpha\mu\beta\nu} \Big). \end{aligned} \tag{B.29}$$

Using (B.17) we can simplify:

$$\frac{k_1 \cdot k_2}{2\,\langle 12\rangle^2} tr(P_- \not{k}_2 \gamma^\mu \not{k}_1 \gamma^\nu) = \frac{[12] k_1^\nu k_2^\mu}{2\,\langle 12\rangle} k_1^\nu k_2^\mu - \frac{[12]}{2\,\langle 12\rangle} k_1 \cdot k_2 \eta^{\mu\nu} + \frac{[12]}{2\,\langle 12\rangle} k_1^\mu k_2^\nu + i \frac{[12]}{2\,\langle 12\rangle} k_{1\beta} k_{2\alpha} \varepsilon^{\alpha\mu\beta\nu}. \tag{B.30}$$

Putting (B.31) in (B.28):

$$(f_1^+ f_2^+)^{\mu\nu} = -\frac{[12]}{2\,\langle 12\rangle} \Big(k_1 \cdot k_2 \eta^{\nu\mu} - k_1^\nu k_2^\mu + k_1^\mu k_2^\nu - i k_{1\beta} k_{2\alpha} \varepsilon^{\alpha\mu\beta\nu} \Big). \tag{B.31}$$

To show that (B.18) is valid, we need to compare (B.32) and (B.21) and is trivial to see that:

$$-i k_{1\beta} k_{2\alpha} \varepsilon^{\alpha\mu\beta\nu} = i k_{1\alpha} k_{2\beta} \varepsilon^{\alpha\beta\nu\mu}. \tag{B.32}$$

The procedure to show (B.19) is completely analogous and can be verified by the interested reader. So let's show the anti-commutator relation:

$$\{f_i^+, f_j^+\} = -\frac{1}{2}[ij]\eta. \tag{B.33}$$

We will use the following identity:

$$\{\not{k}_1, \not{k}_2\} = k_{1\alpha}k_{2\beta}\{\gamma^\alpha, \gamma^\beta\} = -2k_{1\alpha}k_{2\beta}\eta^{\alpha\beta}. \tag{B.34}$$

The anti-commutator is:

$$\begin{aligned}
\{f_1^+, f_2^+\} &= (f_1^+ f_2^+)^{\mu\nu} + (f_2^+ f_1^+)^{\mu\nu} = -\frac{[12]}{4\langle 12\rangle} tr(P_-\not{k}_1\not{k}_2\gamma^\nu\gamma^\mu) - \frac{[21]}{4\langle 21\rangle} tr(P_-\not{k}_2\not{k}_1\gamma^\nu\gamma^\mu) \\
&= -\frac{[12]}{4\langle 12\rangle}\left(tr(P_-\not{k}_1\not{k}_2\gamma^\nu\gamma^\mu) + tr(P_-\not{k}_2\not{k}_1\gamma^\nu\gamma^\mu)\right) = -\frac{[12]}{4\langle 12\rangle} tr(P_-\{\not{k}_1\not{k}_2\}\gamma^\nu\gamma^\mu) \\
&= -\frac{[12]}{4\langle 12\rangle}(-2k_1\cdot k_2) tr(P_-\gamma^\nu\gamma^\mu) \\
&= \frac{[12]}{2\langle 12\rangle} k_1\cdot k_2\left(\frac{1}{2}tr(\gamma^\nu\gamma^\mu) - \frac{1}{2}tr(\gamma^5\gamma^\nu\gamma^\mu)\right) = \frac{[12]}{4\langle 12\rangle} k_1\cdot k_2 tr(\gamma^\nu\gamma^\mu) && \text{(B.35)} \\
&= \frac{[12]}{4\langle 12\rangle} k_1\cdot k_2(-4\eta^{\nu\mu}) = -\frac{[12]}{\langle 12\rangle} k_1\cdot k_2\eta^{\nu\mu} = \frac{1}{2}\frac{[12]}{\langle 12\rangle}\langle 12\rangle [21]\eta^{\nu\mu} && \text{(B.36)} \\
&= -\frac{1}{2}[12]^2\eta^{\nu\mu} && \text{(B.37)}
\end{aligned}$$

In an analogous way, it is easy to show that

$$\{f_1^-, f_2^-\} = -\frac{1}{2}\langle 12\rangle^2 \eta^{\nu\mu}. \tag{B.38}$$

The freedom to choose the reference momentum of the polarization vectors led us with some interesting results. Note that in spinor helicity the product of momentum and polarization vectors looks like:

$$\begin{aligned}
p\cdot\varepsilon_+(k;q) &= p\left(-\frac{\langle q|\gamma^\mu|k]}{\sqrt{2}\langle qk\rangle}\right) = \frac{\langle q| - \not{p}|k]}{\sqrt{2}\langle qk\rangle} = \frac{\langle q|(|p\rangle [p| + |p]\langle p|)|k]}{\sqrt{2}\langle qk\rangle} && \text{(B.39)} \\
&= \frac{\langle q| |p\rangle [p||k]}{\sqrt{2}\langle qk\rangle} + \frac{\langle q| |p]\langle p| |k]}{\sqrt{2}\langle qk\rangle} && \text{(B.40)} \\
&= \frac{\langle qp\rangle [pk]}{\sqrt{2}\langle qk\rangle}. && \text{(B.41)}
\end{aligned}$$

To get (B.42) we used (B.8).

If the polarization has negative helicity:

$$p \cdot \varepsilon_-(k;q) = p\left(-\frac{[q|\gamma^\mu|k\rangle}{\sqrt{2}[qk]}\right) = \frac{[q|-\not{p}|k\rangle}{\sqrt{2}[qk]} = \frac{[q|(|p\rangle[p|+|p]\langle p|)|k\rangle}{\sqrt{2}[qk]} \quad \text{(B.42)}$$

$$= \frac{[q|\,|p\rangle\,[p|\,|k\rangle}{\sqrt{2}[qk]} + \frac{[q||p]\,\langle p|\,|k\rangle}{\sqrt{2}[qk]} \quad \text{(B.43)}$$

$$= \frac{[qp]\,\langle pk\rangle}{\sqrt{2}[qk]}. \quad \text{(B.44)}$$

By the antisymmetric property of the products (B.9) and (B.10) we will have:

$$k \cdot \varepsilon_\pm(k;q) = 0, \quad \text{(B.45)}$$

$$q \cdot \varepsilon_\pm(k;q) = 0. \quad \text{(B.46)}$$

The same argument can be used to note that the right-hand sides of (B.14) and (B.15) vanish if $q' = q$, and that the right-hand side of (B.16) vanish if $q = k'$ or $q' = k$.

B.1 Multi-photon amplitudes with spinor helicity technology.

In Chapter 2, we write the even powers of the total field strength tensor in terms of χ_+, χ_- and f_+f_- (Eq. 2.31). Here we present this derivation:

$$\begin{aligned}
F^{2n} &= (f^+ + f^-)^{2n} = \sum_{k \text{ even}} \binom{2n}{k} (f^+)^k (f^-)^{2n-k} + \sum_{k \text{ odd}} \binom{2n}{k} (f^+)^k (f^-)^{2n-k} \\
&= \sum_{k \text{ even}} \binom{2n}{k} (f^{+2})^{k/2} (f^{-2})^{(2n-k)/2} + \sum_{k \text{ odd}} \binom{2n}{k} (f^{+2})^{(k-1)/2} (f^{-2})^{(2n-k-1)/2} f^+ f^- \\
&= \sum_{k \text{ even}} \binom{2n}{k} (\chi_+)^{k/2} (\chi_-)^{(2n-k)/2} + \sum_{k \text{ odd}} \binom{2n}{k} (\chi_+)^{(k-1)/2} (\chi_-)^{(2n-k-1)/2} f^+ f^- \\
&= \sum_{k \text{ even}} \binom{2n}{k} \sqrt{\chi_+}^k \sqrt{\chi_-}^{2n-k} + \frac{f^+ f^-}{\sqrt{\chi_+ \chi_-}} \sum_{k \text{ odd}} \binom{2n}{k} \sqrt{\chi_+}^k \sqrt{\chi_-}^{2n-k} \\
&= \frac{(-1)^n}{2} \Bigg(\left[(\sqrt{\chi_+} + \sqrt{\chi_-})^{2n} + (\sqrt{\chi_+} - \sqrt{\chi_-})^{2n} \right] \eta^{\mu\nu} \\
&\quad + \frac{(f_+ f_-)^{\mu\nu}}{\sqrt{\chi_+ \chi_-}} \left[(\sqrt{\chi_+} - \sqrt{\chi_-})^{2n} - (\sqrt{\chi_+} + \sqrt{\chi_-})^{2n} \right] \Bigg).
\end{aligned} \tag{B.47}$$

As we studied in Chapter 2, the scalar propagator contains the information of the N-photon amplitudes in the low energy limit and to obtain this information, we have to expand the propagator, once we write the propagator in terms of χ_+, χ_- and f_+f_- (Eq. 2.36). The expansion can be preformed by the use of Mathematica. The following lines are dedicated to the Mathematica code of Eq. 2.36:

Mathematica code:

```
f[λp_, λm_]:=Sech[e * T(λp * Sqrt[χ+] + λm * Sqrt[χ-])]

Sech[e * T(λp * Sqrt[χ+] − λm * Sqrt[χ-])]

Exp[

−T * p^2/2

(Tanh[e * T(λp * Sqrt[χ+] + λm * Sqrt[χ-])]/

(e * T(λp * Sqrt[χ+] + λm * Sqrt[χ-]))+
```

$\mathrm{Tanh}[e*T(\lambda\mathrm{p}*\mathrm{Sqrt}[\chi_+]-\lambda\mathrm{m}*\mathrm{Sqrt}[\chi_-])]/$

$(e*T(\lambda\mathrm{p}*\mathrm{Sqrt}[\chi_+]-\lambda\mathrm{m}*\mathrm{Sqrt}[\chi_-]))-2)-$

$T*\lambda\mathrm{p}*\lambda\mathrm{m}*p.(f^+f^-).p/(2*\lambda\mathrm{p}*\lambda\mathrm{m}*\mathrm{Sqrt}[\chi_+]\mathrm{Sqrt}[\chi_-])$

$(\mathrm{Tanh}[e*T(\lambda\mathrm{p}*\mathrm{Sqrt}[\chi_+]-\lambda\mathrm{m}*\mathrm{Sqrt}[\chi_-])]/$

$(e*T(\lambda\mathrm{p}*\mathrm{Sqrt}[\chi_+]-\lambda\mathrm{m}*\mathrm{Sqrt}[\chi_-]))-$

$\mathrm{Tanh}[e*T(\lambda\mathrm{p}*\mathrm{Sqrt}[\chi_+]+\lambda\mathrm{m}*\mathrm{Sqrt}[\chi_-])]/$

$(e*T(\lambda\mathrm{p}*\mathrm{Sqrt}[\chi_+]+\lambda\mathrm{m}*\mathrm{Sqrt}[\chi_-])))]$

$A[\mathrm{K\_},\mathrm{L\_}]{:=}\mathrm{SeriesCoefficient}[f[\lambda\mathrm{p},\lambda\mathrm{m}],\{\lambda\mathrm{p},0,K\},\{\lambda\mathrm{m},0,L\}]$

(*Two legs*)

(*++ *)

$A[2,0]$

$-e^2T^2\chi_+ + \frac{1}{3}e^2p^2T^3\chi_+\mathrm{p}$

$\mathrm{Integrate}[A[2,0]\mathrm{Exp}[-T(m\char`^2+p\char`^2)],\{T,0,\mathrm{Infinity}\},$

$\mathrm{Assumptions}\rightarrow m>0\&\&p>0]$

$-\frac{2e^2m^2\chi_+}{(m^2+p^2)^4}$

(*-- *)

$A[0,2]$

$-e^2T^2\chi_- + \frac{1}{3}e^2p^2T^3\chi_-$

$\mathrm{Integrate}[A[0,2]\mathrm{Exp}[-T(m\char`^2+p\char`^2)],\{T,0,\mathrm{Infinity}\},$

$\mathrm{Assumptions}\rightarrow m>0\&\&p>0]$

$-\frac{2e^2m^2\chi_-}{(m^2+p^2)^4}$

(* +- *)

$A[1,1]$

$-\frac{2}{3}e^2T^3p.(f^+f^-).p$

Integrate[A[1, 1]Exp[$-T$(m^2 + p^2)], {T, 0, Infinity},

Assumptions → m > 0&&p > 0]

$-\frac{4e^2p.(f^+f^-).p}{(m^2+p^2)^4}$

(* Four legs *)

(* ++++ *)

$A[4,0]$

$\frac{1}{90}\left(60e^4T^4{\chi_+}^2 - 42e^4p^2T^5{\chi_+}^2 + 5e^4p^4T^6{\chi_+}^2\right)$

Integrate[A[4, 0]Exp[$-T$(m^2 + p^2)], {T, 0, Infinity},

Assumptions → m > 0&&p > 0]

$\frac{8e^4\left(2m^4-3m^2p^2\right){\chi_+}^2}{(m^2+p^2)^7}$

(*+++ −*)

$A[3,1]$

$\frac{6}{5}e^4T^5\chi_+p.(f^+f^-).p - \frac{2}{9}e^4p^2T^6\chi_+p.(f^+f^-).p$

Integrate[A[3, 1]Exp[$-T$(m^2 + p^2)], {T, 0, Infinity},

Assumptions → m > 0&&p > 0]

$-\frac{16e^4\left(-9m^2+p^2\right)\chi_+ p.(f^+f^-).p}{(m^2+p^2)^7}$

(* ++-- *)

A[2, 2]

$\frac{1}{45}e^4\left(90T^4\chi_-\chi_+ - 66p^2T^5\chi_-\chi_+ + 5p^4T^6\chi_-\chi_+ + 10T^6(p.(f^+f^-).p)^2\right)$

Integrate[A[2, 2]Exp[$-T(m$^2 + p^2)], {T, 0, Infinity},

Assumptions → $m > 0$&&$p > 0$]

$\frac{16e^4\left(\left(3m^4-5m^2p^2-3p^4\right)\chi_-\chi_+ + 10(p.(f^+f^-).p)^2\right)}{(m^2+p^2)^7}$

As we have learned throughout this work, multi-photon amplitudes can be expressed in terms of χ_+, χ_- and $p\cdot(f^+f^-)\cdot p$. In the following we will present the possibility of writing $p\cdot(f^+f^-)\cdot p$ in terms of brakets. First note that:

$$p\cdot(f^+f^-)\cdot p = p_\mu(f^+f^-)^{\mu\nu}p_\nu \tag{B.48}$$

Now we can use Eq. (1.4):

$$\begin{aligned}(f_i^+f_j^+)^{\mu\nu} &= (f_i^+)^{\mu\lambda}(f_j^+)_\lambda{}^\nu = (k_i^\mu\varepsilon_i^{+\lambda} - \varepsilon_i^{+\mu}k_i^\lambda)(k_{j\lambda}\varepsilon_j^{+\nu} - \varepsilon_{j\lambda}^{+}k_j^\nu) &\text{(B.49)}\\ &= -k_i\cdot k_j\varepsilon_i^{+\mu}\varepsilon_j^{-\nu} &\text{(B.50)}\end{aligned}$$

With the use of Eqs. (B.12), (B.13) and (B.17):

$$(f_i^+f_j^+)^{\mu\nu} = \frac{1}{4}[k_i|\gamma^\mu|k_j\rangle\,[k_i|\gamma^\nu|k_j\rangle \tag{B.51}$$

On the other hand, a massive momentum can be expressed in terms of 2 massless momenta as:

$$p^\mu = \hat{p}^\mu + \frac{m^2q^\mu}{2\hat{p}\cdot q} = \hat{p}^\mu + \lambda q^\mu \tag{B.52}$$

We will use the last ingredients to express Eq. (2.39) in a braket notation:

$$\begin{aligned}
p\cdot(f_1^+f_2^-)\cdot p &= (\hat{p}+\lambda q)\cdot(f_1^+f_2^-)\cdot(\hat{p}+\lambda q)\\
&= \hat{p}\cdot(f_1^+f_2^-)\cdot\hat{p}+2\lambda q\cdot(f_1^+f_2^-)\cdot\hat{p}+\lambda^2 q\cdot(f_1^+f_2^-)\cdot q\\
&= \frac{1}{4}[k_1|\hat{\not{p}}\,|k_2\rangle\,[k_1|\hat{\not{p}}\,|k_2\rangle+\frac{\lambda}{2}[k_1|\not{q}\,|k_2\rangle\,[k_1|\hat{\not{p}}\,|k_2\rangle+\frac{\lambda^2}{4}[k_1|\not{q}\,|k_2\rangle\,[k_1|\not{q}\,|k_2\rangle\\
&= \frac{1}{4}[k_1\hat{p}]^2\,\langle\hat{p}k_2\rangle^2+\frac{\lambda}{2}[k_1q]\,\langle qk_2\rangle\,[k_1\hat{p}]\,\langle\hat{p}k_2\rangle+\frac{\lambda^2}{4}[k_1q]^2\,\langle qk_2\rangle^2
\end{aligned}\tag{B.53}$$

To get the last line we used Eq. (B.11). One of the advantages of spinor helicity is that q is an arbitrary massless reference momentum and to simplify Eq. (B.54) we choose $q=k_1$:

$$p\cdot(f_1^+f_2^-)\cdot p = \frac{1}{4}[k_1\hat{p}]^2\,\langle\hat{p}k_2\rangle^2 \tag{B.54}$$

The previous calculation is just an example of how we can use spinor helicity to express $p\cdot(f^+f^-)\cdot p$ in the bracket notation.

Appendix C

$S[K, L]$ summation.

The principal object of study is:

$$S[K, L] = \sum_{k=0}^{K} \sum_{l=0}^{L} \frac{(-)^l \mathcal{B}_{k+l} \mathcal{B}_{N-k-l}}{k! l! (K-k)! (L-l)!}. \tag{C.1}$$

Preforming a change of variables $L = K - n$ and $k + l = x$ this double summation can be reduced to:

$$S[K, m] = \sum_{x=0}^{N} \mathcal{B}_x \mathcal{B}_{N-x} \sum_{k=0}^{K} \frac{(-)^{x-k}}{k!(x-k)!(K-k)!(K+k-m-x)}. \tag{C.2}$$

Since K and L are even numbers, this means that n muss be even too $n = 2m$:

$$S[K, m] = \sum_{x=0}^{N} \mathcal{B}_x \mathcal{B}_{N-x} \sum_{k=0}^{K} \frac{(-)^{x-k}}{k!(x-k)!(K-k)!(K+k-2m-x)}. \tag{C.3}$$

We used Mathematica to fix the value of m and see how this changed with the increase of x.

For $m = 0$:

```
S[m_, x_, K_]:=Sum[(−1)^(x − k)/(k! ∗ (x − k)! ∗ (K − k)! ∗ (K + k − (2m + x))!), {k, 0, K}];

TabX0mCero = Table[S[0, 0, 2K], {K, 1, 10}];

FullSimplify[FindSequenceFunction[TabX0mCero, n]]
```

$\frac{1}{\text{Gamma}[1+2n]^2}$

TabX2mCero = Table[S[0, 2, 2K], {K, 1, 16}];

FullSimplify[FindSequenceFunction[TabX2mCero, K]]

$-\frac{1}{\text{Gamma}[2K]\text{Gamma}[1+2K]}$

TabX4mCero = Table[S[0, 4, 2K], {K, 1, 16}];

FullSimplify[FindSequenceFunction[TabX4mCero, K]]

$\frac{1}{2\text{Gamma}[-1+2K]\text{Gamma}[1+2K]}$

TabX6mCero = Table[S[0, 6, 2K], {K, 2, 16}];

FullSimplify[FindSequenceFunction[TabX6mCero, $K - 1$]]

$-\frac{1}{6\text{Gamma}[-2+2K]\text{Gamma}[1+2K]}$

TabX8mCero = Table[S[0, 8, 2K], {K, 2, 16}];

FullSimplify[FindSequenceFunction[TabX8mCero, $K - 1$]]

$\frac{1}{24\text{Gamma}[-3+2K]\text{Gamma}[1+2K]}$

We can see a global factor:

$$\frac{1}{\Gamma[2K+1]}. \tag{C.4}$$

The other factors are very easy to deduce and we conclude that for $m = 0$:

$$\frac{(-1)^{x/2}}{(x/2)!\Gamma[2K+1]\Gamma[2K-\frac{x-2}{2}]}. \tag{C.5}$$

Now consider the case $m = 1$:

TabX0mUno = Table[S[1, 0, 2K], {K, 1, 16}];

FullSimplify[FindSequenceFunction[TabX0mUno, K]]

$\frac{1}{\text{Gamma}[-1+2K]\text{Gamma}[1+2K]}$

TabX2mUno = Table[S[1, 2, 2K], {K, 1, 15}];

FullSimplify[FindSequenceFunction[TabX2mUno, K]]

$-\frac{1}{2(-1+K)\text{Gamma}[-3+2K]\text{Gamma}[1+2K]}$

TabX4mUno = Table[S[1, 4, 2K], {K, 2, 16}];

FullSimplify[FindSequenceFunction[TabX4mUno, $K - 1$]]

$\frac{-5+2K}{2\text{Gamma}[-2+2K]\text{Gamma}[1+2K]}$

TabX6mUno = Table[S[1, 6, 2K], {K, 2, 15}];

FullSimplify[FindSequenceFunction[TabX6mUno, $K - 1$]]

$\frac{7-2K}{6\text{Gamma}[-3+2K]\text{Gamma}[1+2K]}$

TabX8mUno = Table[S[1, 8, 2K], {K, 2, 20}];

FullSimplify[FindSequenceFunction[TabX8mUno, $K - 1$]]

$\frac{-9+2K}{24\text{Gamma}[-4+2K]\text{Gamma}[1+2K]}$

Again we always get a global factor (C.4), with this observation we can infer that perhaps (C.5) is present for any value. After some manipulations for $m = 1$ we find that:

$$\frac{(-1)^{x/2}(2K-(x+1))}{(x/2)!\Gamma[2K+1]\Gamma[2K-(\frac{x-2}{2}+1)]} \tag{C.6}$$

We can proceed in this way to find expressions for different values. We present $m = 2$ and after that we write just the results for other values.

$\mathbf{TabX0mDos = Table[\mathit{S}[2,0,2\mathit{K}],\{\mathit{K},2,10\}];}$

$\mathbf{FullSimplify[FindSequenceFunction[TabX0mDos,\mathit{K}-1]]}$

$\frac{1}{\text{Gamma}[-3+2K]\text{Gamma}[1+2K]}$

$\mathbf{TabX2mDos = Table[\mathit{S}[2,2,2\mathit{K}],\{\mathit{K},2,24\}];}$

$\mathbf{FullSimplify[FindSequenceFunction[TabX2mDos,\mathit{K}-1]]}$

$-\frac{2(-5+K)}{\text{Gamma}[-3+2K]\text{Gamma}[1+2K]}$

$\mathbf{TabX2mDos = Table[\mathit{S}[2,2,2\mathit{K}],\{\mathit{K},2,24\}];}$

$\mathbf{FullSimplify[FindSequenceFunction[TabX2mDos,\mathit{K}-2]]}$

$-\frac{2(-6+K)}{\text{Gamma}[-5+2K]\text{Gamma}[-1+2K]}$

$\mathbf{TabX4mDos = Table[\mathit{S}[2,4,2\mathit{K}],\{\mathit{K},2,26\}];}$

$\mathbf{FullSimplify[FindSequenceFunction[TabX4mDos,\mathit{K}-1]]}$

$\frac{35+K(-21+2K)}{\text{Gamma}[-3+2K]\text{Gamma}[1+2K]}$

$\mathbf{TabX6mDos = Table[\mathit{S}[2,6,2\mathit{K}],\{\mathit{K},3,26\}];}$

$\mathbf{FullSimplify[FindSequenceFunction[TabX6mDos,\mathit{K}-2]]}$

$\frac{-63+(29-2K)K}{3\text{Gamma}[-4+2K]\text{Gamma}[1+2K]}$

$\mathbf{TabX8mDos = Table[\mathit{S}[2,8,2\mathit{K}],\{\mathit{K},3,26\}];}$

$\mathbf{FullSimplify[FindSequenceFunction[TabX8mDos,\mathit{K}-2]]}$

$\frac{99+K(-37+2K)}{12\text{Gamma}[-5+2K]\text{Gamma}[1+2K]}$

We found that for $m = 2$:

$$\frac{(-1)^{x/2}(4K^2 - 2K(4(x+1)+1) + 2(x+1)(x+3)}{(x/2)!\Gamma[2K+1]\Gamma[2K-(\frac{x-2}{2}+2)]}. \tag{C.7}$$

With this three cases we concluded that the summation has the structure:

$$\frac{(-1)^{x/2}P_m(K,x)}{(x/2)!\Gamma[2K+1]\Gamma[2K-\left(\frac{x-2}{2}+m\right)]}, \tag{C.8}$$

where:

$$\begin{aligned}
P_0(K,x) &= 1 && \text{(C.9)}\\
P_1(K,x) &= 2K-(x+1) && \text{(C.10)}\\
P_2(K,x) &= 4K^2-2K(4(x+1)+1)+2(x+1)(x+3) && \text{(C.11)}\\
P_3(K,x) &= 8K^3-4K^2(9(x+1)+3)+2K(12(x+1)^2\\
&\quad +33(x+1)+2)-4(x+1)(x+3)(x+5) && \text{(C.12)}\\
P_4(K,x) &= 16K^4-8K^3(16(x+1)+6)+4K^2(40(x+1)^2+128(x+1)+11)\\
&\quad -2K(32(x+1)^3+232(x+1)^2+368(x+1)+6)+8(x+1)(x+3)(x+5)(x+7) && \text{(C.13)}\\
P_6(K,x) &= 32K^5-16K^4(25(x+1)+10)+8K^3(100(x+1)^2+350(x+1)+35)\\
&\quad -4K^2(140(x+1)^3+1140(x+1)^2+1995(x+1)+50)+2K(80(x+1)^4+1100(x+1)^3\\
&\quad +4560(x+1)^2+5510(x+1)+24)-16(x+1)(x+3)(x+5)(x+7)(x+9) && \text{(C.14)}
\end{aligned}$$

In [22] they show that even though these polynomials looks very difficult they are represented by the next expression:

$$P_m(K,x) := \sum_{n=0}^{m}(-1)^{m-n}\binom{2m}{2n}(K-m-x/2-n+1)_n(x/2-m+n+1)_{m-n}, \tag{C.15}$$

.

where $(X)_n$ is the rising factorial function:

$$(X)_n := X*(X+1)*....*(X+n-1). \tag{C.16}$$

With this results we can express (C.2) as:

$$S[K,m]=\sum_{x=0}^{N}\mathcal{B}_x\mathcal{B}_{N-x}$$
$$\sum_{n=0}^{m}\frac{(-1)^{x/2+m-n}}{(x/2)!\Gamma[K+1]\Gamma[K-\left(\frac{x-2}{2}+m\right)]}\binom{2m}{2n}(K-m-x/2-n+1)_n(x/2-m+n+1)_{m-n} \tag{C.17}$$

C.1 Asymptotic limit

If we approximate the Bernoulli numbers as in Eq. (1.46) the double summation that appears in Eq. (1.39) can be solve analytically. The next two plots are related to the approximation of the Bernoulli numbers:

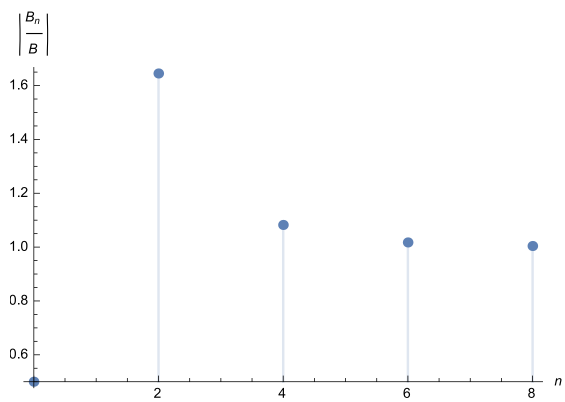

Figure C.1: Approximation for small values of n.

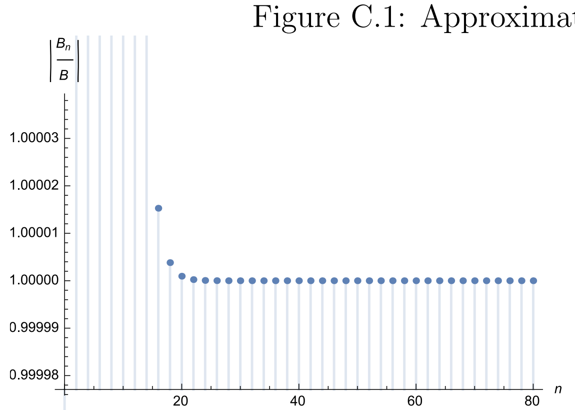

Figure C.2: As n increase the approximation is better.

Figure C.1 show us how this approximation is good for $n > 4$.

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