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Fractional operator in a quarter plane

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Resumen

En este documento buscamos realizar un estudio de un sistema ecuaciones diferenciales parciales con valores iniciales y de frontera, más específicamente, un problema de Dirichlet con derivada fraccionaria sobre el cuarto de plano superior derecho. Definiremos un derivada fraccionaria de manera conveniente para manipularla por medio del uso de la transformada de Laplace. Y de esta manera obtener los operadores de Green y de Frontera que describa una solución global del sistema, para luego llegar a ciertas estimaciones de estos operadores en espacios de Lebesgue y en otros espacios muy parecidos a los de Sobolev con peso. Para así concluir la buena colocación del problema, si ubicamos los valores iniciales y de frontera en dichos.

Palabras Clave: Problema de Dirichlet, Derivada Fraccionaria, Transformada de Laplace, Operador de Green, Solución global.

Abstract

In this document we seek to carry out a study of a system of partial differentials equations with initial and boundary values, specifically, a Dirichlet problem with fractional derivative on the upper right quarter plane. We will define a fractional derivative conveniently to manipulate it by means of the use of the Laplace transform. And so obtain the Green and Boundary operators that describe a global solution of the system, and then arrive at some estimates of these operators in Lebesgue spaces and in other highly similar to those of Sobolev with weight. In order to conclude the well possed of the problem, if we located the initial and boundary values in these.

Keywords: Dirichlet problem, Fractional derivate, Laplace transform, Green operator, Global solution.

Introduction

We consider the inhomogeneous Dirichlet initial-boundary value problem (IBV problem) for the partial fractional differential equations on a upper right quarter plane

$$\begin{cases} u_t(t,\mathbf{x}) - \nabla^{\beta} u(t,\mathbf{x}) = 0, \ \mathbf{x} \in \mathbf{D}, \ t > 0 \\ u(0,\mathbf{x}) = u_0(\mathbf{x}), \ \mathbf{x} \in \mathbf{D} \cup \bigcup_{j=1}^{2} \partial_j \mathbf{D}, \\ u \Big|_{\partial_j \mathbf{D}} = h_j(t, x_j), \ x_j \geqslant 0, \quad j = 1,2; \ t > 0, \end{cases} \tag{0.0.1}$$

where $\beta = 2 - \alpha$, where $\alpha \in \left(0, \frac{1}{2}\right]$, and ∇^{β} is a fractional derivative defined by

$$\nabla^{\beta} u = \nabla^{-\alpha} \triangle u = \sum_{j} \int_{\mathbf{0}}^{\mathbf{x}} \frac{u_{y_j y_j}}{(x_j - y_j)^{1-\alpha}} d\mathbf{y}. \tag{0.0.2}$$

with $u \in \mathbf{C}^2(\mathbb{R}_+^2)$.

Here $\mathbf{D}$ is upper right-quarter plane:

$$\mathbf{D} = \{\mathbf{x} = (x_1, x_2) : \ x_1 > 0, \ x_2 > 0\},$$

and $\partial \mathbf{D} = \bigcup_j \partial_j \mathbf{D}$ is boundary of $\mathbf{D}$:

$$\partial_1 \mathbf{D} := \{\mathbf{x}_1 = (x_1, 0) : \ x_1 \geqslant 0\},$$
$$\partial_2 \mathbf{D} := \{\mathbf{x}_2 = (0, x_2) : \ x_2 \geqslant 0\}.$$

The inhomogeneous problems are often called forced problems, when an external force is applied to a system. Frequently the forcing is putted as the inhomogeneous boundary condition. The initial-boundary problems have been much less studied than the Cauchy problems in spite of their importance. In the case of the initial-boundary value problems there appear new difficulties comparing with the Cauchy problems due to the boundary. For example, in the case of the initial-boundary value problems it is not clear the quantity of the boundary conditions that are required for the well possedness of the boundary problem. The answer to this question relies on the construction of the Green function for the linear fractional equation . Also it is necessary to take into account the boundary effects which affect the behavior of the solutions. Besides, in the case of the initial-boundary value problems, one needs to ask for as less regularity on the initial and boundary data as possible, because the regularity of the solution is related to the compatibility conditions on the initial and boundary data. Note that for the Cauchy problem there is no such complication. Usually one disposes of more regularity on the initial data in the study of the time behavior of the solutions. In spite of the importance and actuality as far as we know, there are no any results on the initial-boundary problem for the fractional partial equation on upper right quarter plane. In tesis we fill this gap, considering as example of evolution equation with fractional derivative of the Riemann-Liouville type

$$\nabla^{\beta} u = \sum_j \int_{\mathbf{0}}^{\mathbf{x}} \frac{u_{y_j y_j}}{(x_j - y_j)^{1-\alpha}} d\mathbf{y}$$

with a symbol $K(\mathbf{p}) = \mathbf{p}^{\beta} = \sum_j p_j^{\beta}$, where $\mathbf{p} = \{p_j\}_j$.

There are many natural open questions which should be studied. First we need to consider the following question: how many boundary data should be posed in problem (0.0.1) for its correct solvability? Also we study traditionally important problems of a theory of partial differential equations, such as existence and uniqueness of solution. Also we get the closed form of solution.

Note there are several ways to get Green operator for the multidimensional model. For example one can extend the boundary data into domain and homogenize the boundary conditions via suitable change of the dependent variable. However this approach requires a high regularity of the data. Another known approach is the method of separation of variables. It also can not be used in our case due to the nonhomogeneous boundary data. In this work we give general method to get the Green function based on the Riemann–Hilbert approach and theory of singular integro-differential equations with the Hilbert kernel and the discontinuous coefficients(see [16]). The integral formula is obtained by using the Laplace transform with respect to the space and time variable. We prove that the amount of the boundary data which we need to put in the problem for its well posedness is equal to $[\beta] + 1$. The remainder boundary values should be obtained as some necessary conditions for existence of solutions of the corresponding Riemann-Hilbert boundary value problem.

Chapter 1

Preliminaries

1.1 Fractional derivate

Due to the intensive development of the theory of fractional calculus itself as well its applications ([18], [19], [23]) , the initial-boundary value problem (0.0.1) plays an important role in the modern science. Apart from diverse areas of mathematics, evolution partial differential equations with fractional derivative arise in modern mathematical physics and many other branches of science, such as, for example, chemical physics and electrical networks (see for details, for example, [10], [9], [21], [22], [23], [25], [27]).

There are several definitions of a fractional derivative of order $\alpha > 0$ (see for details, for example, [18], [19], [24]). The two most commonly used definitions are the Riemann-Liouville and Caputo ([24]). Each definition uses Riemann-Liouville fractional integration and derivatives of an integer order. The Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined in one dimential case as

$$D^{\alpha}u = \frac{1}{\Gamma(1-\alpha+[\alpha])}\partial_x^{[\alpha a]+1}\int_0^x \frac{u(y)}{(x-y)^{\alpha-[\alpha]}}dy,\ x>0$$

and the Caputo derivative of order $a > 0$ is defined as

$$D_*^{\alpha}u = \frac{1}{\Gamma(1-\alpha+[\alpha])}\int_0^x \frac{\partial_y^{[\alpha]+1}u(y)}{(x-y)^{\alpha-[\alpha]}}dy, x>0.$$

There is a close connection between the Riemann-Liouville fractional derivative and Caputo derivative (see [24]). To reveal a relation between the Riemann-Liouville and Caputo fractional derivatives we rewrite all of them as pseudodifferential operators (see ([12])). For simplicity we consider the case of $\alpha \in (0,1)$, then we have

$$\begin{aligned} D^{\alpha}f(x) =& \mathscr{L}^{-1}\left(p^{\alpha}\widehat{f}(p)\right), \\ D_*^{\alpha}f(x) =& \mathscr{L}^{-1}\left(p^{\alpha}\left(\widehat{f}(p)-\frac{f(0)}{p}\right)\right) \end{aligned}$$

where $\mathscr{L}^{-1}$ denotes the inverse Laplace transform and $\widehat{f}(p) = \mathscr{L}f$ is the Laplace transform of f. Here and below p^{α} is the main branch of the complex analytic function in the complex half-plane $\mathrm{Re}\, p \geqslant 0$, so that $1^{\alpha} = 1$ (we make a cut along the negative real axis $(-\infty, 0)$). As we see the Riemann-Liouville and Caputo fractional derivatives have a symbol p^{α} which is analytic in the complex half-plane $\mathrm{Re} p \geqslant 0$. Some physical problems very often lead us naturally to evolution partial differential equation with the fractional derivative . So it is necessary to formulate correctly the initial - boundary value problem to such types of equation. It should be noted that most papers and books on fractional calculus are devoted to solvability of fractional ordinary differential equations (ODE). For the general theory of nonlinear equations on a half-line we refer to the book [12]. This book is the first attempt to develop systematically a general theory of the initial-boundary value problems for evolution equations with pseudodifferential operators on a half-line, where pseudodifferential operator $\mathbb{K}$ on a half-line was introduced by virtue of the inverse Laplace transformation of the product of the symbol $K(p) = O(p^{\alpha})$ which is analytic in the right complex half-line, and the Laplace transform of the derivative $\partial_x^{[\alpha]} u$. Thus, for example, in the case of $K(p) = p^{\alpha}$ we get the following definition of the fractional derivative ∂_x^{α},

$$\partial_x^{\alpha} = \mathscr{L}^{-1} \left\{ p^{\alpha} \left(\mathscr{L} - \sum_{j=1}^{[\alpha]} \frac{\lim_{x \to 0^+} \partial_x^{j-1}}{p^j} \right) \right\}. \tag{1.1.1}$$

Note that due to the analyticity of p^{α} for all $\mathrm{Re}\, p > 0$ the inverse Laplace transform gives us the function which is equal to 0 for all $x < 0$. To obtain an explicit form of the Green function it was used an approach based on the Laplace transformation with respect to the spatial variable contrary to the standard application of the Laplace transformation with respect to the time variable. It was proved that the amount of boundary data which we need to put in the problem for its well posedness is equal to the integer part of $\left[\frac{\alpha}{2}\right]$, where α is the order of the operator $\mathbb{K}$, which is not equal to an odd integer (in the case of odd integer order of operator $\mathbb{K}$ the amount of the boundary data depends also on the sign of the highest derivative).

For usually nonlinear partial differential equations local existence in some Sobolev spaces were obtained in papers [13] and [2]. R. Weder proved [28] that the initial–boundary value problem for the forced nonlinear Schrödinger equation with a potential on the half-line, is locally and (under stronger conditions) globally well posed. In [26], Q. Bu and W. Srauss proved the existence of global-in-time solution in the energy space for initial data in $\mathbf{H}^1$ and the boundary data from $\mathbf{C}^3$ with a compact support. In paper [7] it was considered the case of time-periodic boundary data. It was proved that the solution has different asymptotic behaviors in different regions. In paper [5] boundary value problems for the nonlinear Schrödinger equations on the half -line with homogeneous Robin boundary conditions were revisited using Bäcklund transformations. The results were illustrated by discussing several exact soliton solutions, which described the soliton reflection at the boundary. In paper [8] it was established the global existence in time of the solution of the nonlinear Schrödinger equation for a particular class of boundary conditions, called linearizable. In paper [15] it was introduced a method for analyzing IBV problems for dispersive evolution equations in one dimensional case, which is a generalization of the so-called factorization method. It was shown the existence of global in time solutions as well as the asymptotic behavior of solutions for a IBVP for NLS with boundary data. It was proved that in one dimension the long range scattering occurs (i.e. scattering to a nonlinear profile). The advantage of this approach is that it can also be applied to non-integrable equations with general inhomogeneous boundary data. By this method in paper [14] it was shown that long range scattering occurs in the 1D NLS equation with cubic nonlinearity and Dirichlet type of boundary data. In the case of the multidimensional IBV- problems there appear new difficulties comparing with the one dimensional case due to the boundary. The difficulty is explained by the fact that it is necessary to take into account the boundary effects which perturb the behavior of the solutions. A description of the large time asymptotic behavior of solutions of nonlinear evolution equations requires principally new approaches and the reorientation of points of view in the asymptotic methods. The difficulty is explained by the fact that they need not only a global existence of solutions, but also a number of additional a priori estimates of the difference between the solution and the approximate solution (usually in the weighted norms). Also the generalized solutions could not be acceptable. It is necessary to prove global existence of classical solutions. It is interesting to study the influence of the boundary data on the asymptotic behavior of solutions. In book [12] it was proved that in the case of the one dimensional problem for dissipative equations the solutions obtain more slow time decay comparing with the case of the Dirichlet problem.

The stochastic equations have also been studied in several works. In [4] and [3], Complex Landau -Ginzburg equation on a bounded domain with a multiplicative noise was studied. The global well-posedness of the $\mathbf{L}^p$-solutions and uniqueness of an invariant measure were shown under some additional assumptions. In [11], CLG-equation on the one-dimensional torus with a space–time white noise was studied and similar results were shown. Existence and properties of the solution to an evolution problem with boundary noise are investigated in the literature. Thus, for example, in paper [2], an existence and uniqueness theorem for stationary solutions of the Burgers equation on the segment with zero viscosity and random boundary conditions were established using a variational principle for entropy solutions of mixed problems. The paper [1] was devoted to one-dimensional quasilinear stochastic partial differential equations of parabolic type with homogeneous initial data and nonhomogeneous Dirichlet boundary conditions of white-noise type. The well-posedness of the IBVP with values in an appropriate weighted space were established using Malliavin calculus. Alternative approach was given by the semigroup techniques developed, among others, in [6] and [20]. The paper [17] is an early work dealing with dinamics of stochastic CLG-equation with random boundary data of mixed type. It was shown that a solution can be written in terms of the stochastic kernel related to the deterministic problem. By this formulation using Itô calculus the basic properties of the solution, such as the continuity and the boundary-layer behavior were studied.

1.2 Laplace transform

To state precisely the results of the tesis we give some notations.

The usual direct and inverse Laplace transformation we denote by $\mathscr{L}$ and $\mathscr{L}^{-1}$

respectively. The Fourier transform $\mathscr{F}$ and the inverse Fourier transform $\mathscr{F}^{-1}$ defined by

$$\mathscr{F}\phi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} \phi\left(x\right) dx,$$
$$\mathscr{F}^{-1}\phi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} \phi\left(\xi\right) d\xi.$$

Also we introduce Laplace transform denote by $\mathscr{L}_j$ and inverse Laplace transform by

$$\widehat{\phi} = \mathscr{L}_{x_j}\{\phi\} := \mathscr{L}_j \phi = \int_{\mathbb{R}^+} e^{-p_j x_j} \phi(x_j) dx_j$$

and the inverse Laplace transform by

$$\mathscr{L}_{x_j}^{-1}\{\widehat{\phi}\} := \mathscr{L}_j^{-1}\widehat{\phi} = \frac{1}{2\pi i} \int_{i\mathbb{R}} e^{p_j x_j} \widehat{\phi}(p_j) dp_j \ .$$

Let $\mathbf{x} = (x_1, x_2) = \{x_j\}^2$ and $\mathbf{p} = (p_1, p_2) = \{p_j\}^2$.

We define Laplace transform in upper right-quarter plane $\mathbf{D}$

$$\widehat{\phi}(\mathbf{p}) = \mathscr{L}\phi = \mathscr{L}_1\{\mathscr{L}_2\{\phi(\mathbf{x})\}\}$$
$$\phi(\mathbf{x}) = \mathscr{L}^{-1}\widehat{\phi} = \mathscr{L}_1^{-1}\{\mathscr{L}_2^{-1}\{\widehat{\phi}(\mathbf{x})\}\}.$$

1.3 Lebesgue weighted space

The usual Lebesgue space in $\mathbb{R}_2^+$:

$$\mathbf{L}^q(\mathbb{R}_2^+) := \{\phi \in \mathbf{S}' : \ \|\phi\|_{\mathbf{L}^q} < \infty\},$$

where the norm

$$\|\phi\|_{\mathbf{L}^q} := \left(\int_{\mathbf{D}} |\phi\left(\mathbf{x}\right)|^q \, d\mathbf{x}\right)^{1/q}$$

if $1 \leqslant q < \infty$ and

$$\|\phi\|_{\mathbf{L}^\infty} := \operatorname*{ess\,sup}_{\mathbf{x}\in\mathbf{D}} |\phi\left(\mathbf{x}\right)|$$

if $q = \infty$.

Weighted space

$$\mathbf{L}^{2,k}(\mathbb{R}_2^+) := \{\phi \in \mathbf{S}' : \ \|\phi\|_{\mathbf{L}^{2,k}} < \infty\},$$

where the norm

$$\|\phi\|_{\mathbf{L}^{2,k}} := \sum_j \left\|p_1^k p_2^k \phi\right\|_{\mathbf{L}^2},$$

with $k \in \mathbb{R}$.

Also we denote the norm

$$\|\phi\|_{\mathbf{H}^s} := \|\phi\|_{\mathbf{L}^2} + \|\phi\|_{\dot{\mathbf{H}}^s} < \infty, \quad \text{for } s < \frac{1}{2}$$
$$\|\phi\|_{\mathbf{H}^s} := \|\phi\|_{\mathbf{L}^2} + \|\phi\|_{\dot{\mathbf{H}}^{\frac{1}{2}-\gamma}} + \|\phi_{x_1x_2}\|_{\dot{\mathbf{H}}^{s-1}} < \infty, \text{ for } s \in \left(\frac{1}{2}, \frac{3}{2}\right],$$

with $\|\phi\|_{\dot{\mathbf{H}}^s} := \|\widehat{\phi}\|_{\mathbf{L}^{2,s}}$. Different positive constants we denote by the same letter C.

We define the functional spaces

$$\mathbf{Z} := \{\phi \in \mathbf{H}^s(\mathbf{D}) : \|\phi\|_{\mathbf{Z}} < \infty\}, \ s > 0$$

Let $\mathbf{B}$ be a Banach space; we then denote

$$\mathbf{C}^0([0,T], \mathbf{B}) := \left\{ f(t) \in \mathbf{B} : \lim_{\substack{t_1 \to t \\ t_1 \in [0,T]}} \|f(t_1) - f(t)\|_{\mathbf{B}} = 0, \ \forall t \in [0,T] \right\}.$$

Definition 1.3.1. *A solution $u(t, \mathbf{x})$ is stable with respect to the initial data $u_0(\mathbf{x})$ and boundary data $h_j(t, x_j)$, if for all $\varepsilon > 0$, exists $\delta > 0$ such that if*

$$\|u_0(\mathbf{x})\|_{M_1} + \sum_j \|h_j(t, x_j)\|_{M_2} < \delta,$$

implies $|u(t, \mathbf{x})| < \varepsilon$.

Now we define the well posedness of the problem (0.0.1).

Definition 1.3.2. *Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be Banach spaces. Problem (0.0.1) is called well posed in a semiclassical sense if the following two properties are fulfilled. Firstly, if there exists a unique solution $u(t, \mathbf{x})$ belonging to a metric space*

$$\mathbf{C}^0([0,T], \mathbf{M}_1) \cap \mathbf{C}^0((0,T], \mathbf{M}_2),$$

which satisfy equation $u_t + \nabla^\beta u = 0$ in the generalized sense. Boundary and initial conditions are fulfilled in the classical sense

$$\lim_{t \to 0} u(t, \mathbf{x}) = u_0(\mathbf{x}) \ \text{ in } \mathbf{M}_1 \text{ and}$$
$$\lim_{x_j \to 0} u(t, \mathbf{x}) = h_i(t, x_i) \ \text{in } \mathbf{M}_2 \text{ for all } j = 1, 2; \ t > 0.$$

Secondly, if solution $u(t, \mathbf{x})$ is stable with respect to the initial data $u_0(\mathbf{x})$, boundary data $h_j(t, x_j)$. The function $u(t, \mathbf{x})$ we call a semiclassical solution. If $T = +\infty$ the function $u(t, \mathbf{x})$ is called a global semiclassical solution.

We prove the following theorem

Theorem 1.3.1. *Let $u_0 \in \mathbf{H}^\varepsilon$, $0 < \varepsilon < \frac{1}{2}$, $0 < s < \frac{\beta}{2}$, $\frac{2}{2} < \beta < 2$ and $\mathbf{h} = 0$. Then there exists $T^* > 0$ such that for all $0 < T < T^*$ the initial-boundary value problem (0.0.1) has a unique local solution*

$$u(t) \in \mathbf{C}^0\left([0,T]; \mathbf{L}^2\right) \bigcap \mathbf{C}^0\left((0,T]; \mathbf{H}^s\right),$$

for any fixed $t > 0$, such that

$$\lim_{t \to 0} u = u_0 \ \text{in } \mathbf{L}^{\mathbf{2}}.$$

Chapter 2

Fractional operators on the quarter plane

2.1 Fractional Green and boundary operators

Define

$$\mathcal{G}(t)\phi := -\left(\frac{1}{2\pi i}\right)^2 \int_{i\mathbb{R}^2} d\mathbf{p}\ e^{K(\mathbf{p})t}\widehat{\phi}(\mathbf{p})\Phi(\mathbf{x},\mathbf{p}), \tag{2.1.1}$$

and

$$\mathcal{H}(t)\mathbf{h} := \left(\frac{1}{2\pi i}\right)^2 \sum_{j,i\neq j} \int_0^t d\tau \int_{i\mathbb{R}^2} d\mathbf{p}\ \Phi(\mathbf{x},\mathbf{p})e^{K(\mathbf{p})(t-\tau)}p_j^{\beta-1}\widetilde{\widehat{h}}_i(\tau,p_i) \tag{2.1.2}$$

with

$$\Phi(\mathbf{p},\mathbf{x}) = \prod_j \left(e^{p_j x_j} + \psi(p_j,x_j)\right),$$
$$\psi(p,x) = \frac{1}{2\pi i}\oint_\Gamma dp\ e^{qx}\frac{\beta p}{(q^\beta - p^\beta)\,q^\alpha},$$

where $\Gamma = \left\{q \in \mathbb{C} : \operatorname{Re} q < 0 \text{ such that } q^\beta - p^\beta \neq 0 \text{ for any fixed } \operatorname{Re} p = 0\right\}$, $\beta = 2-\alpha$ and

$$\widehat{\phi}(\mathbf{p}) = \mathscr{L}\phi = \int_{\mathbb{R}^2_+} d\mathbf{y}\ e^{-\mathbf{p}\mathbf{y}}\phi(\mathbf{y}),$$
$$\widehat{\widehat{h}}_i(\xi,p_i) = \int_{\mathbb{R}^2_+} dx_i\ dt\ e^{-\xi t - p_i x_i}h_i(t,x_i)$$

are Laplace transforms with respect to space and time variables. We prove the following proposition

Proposition 2.1.1. *Let $u_0, h \in \mathbf{L}^1(\mathbb{R}^+)$. Then the solution $u(t,\mathbf{x})$ of the initial-boundary value problem (0.0.1) has the following integral representation*

$$u(t,\mathbf{x}) = \mathcal{G}(t)u_0 + \mathcal{H}(t)\mathbf{h}. \tag{2.1.3}$$

Proof. We suppose that there exists some function $u(t,\mathbf{x})$, which satisfy equation (0.0.1). Then applying the Laplace transformation to problem (0.0.1) with respect to time and space variables and using

$$\mathscr{L}\nabla^{\beta}u = \sum_{j=1}^{2} p_j^{\beta}\left(\widehat{\hat{u}}(\xi,\mathbf{p}) - \frac{\hat{u}\Big|_{x_j=0}}{p_j} - \frac{\hat{u}_{x_j}\Big|_{x_j=0}}{p_j^2}\right)$$

we find

$$\widehat{\hat{u}}(\xi,\mathbf{p}) = \frac{1}{\xi - K(\mathbf{p})}\left(\hat{u}_0(\mathbf{p}) + \sum_j p_j^{-\alpha}\left(p_j\hat{u}(\xi,\mathbf{p}_j) + \hat{u}_{x_j}(\xi,\mathbf{p}_j)\right)\right) \tag{2.1.4}$$

for $p_j = (p_1,p_2)\left|_{p_i=0}\right.$, $i \neq j$, $\operatorname{Re} p_j > 0$, $\operatorname{Re}\xi > 0$, where

$$\widehat{\hat{u}}(\xi,\mathbf{p}) = \int_0^{\infty} dt \int_{\mathbb{R}_+^2} d\mathbf{x}\; e^{-\mathbf{p}\mathbf{x}-\xi t} u_0(t,\mathbf{x}),$$
$$\hat{u}_0(\mathbf{p}) = \int_{\mathbb{R}_+^2} d\mathbf{x}\; e^{-\mathbf{p}\mathbf{x}} u_0(\mathbf{x}),$$
$$\widehat{u}_{x_j}(\xi,\mathbf{p}_j) = \int_0^{\infty} dt \int_0^{\infty} dx_i\; e^{-p_i x_i - \xi t} u_{x_j}(t,\mathbf{x}_j).$$

Here $\mathbf{x}_j = \mathbf{x}\Big|_{x_j=0}$, $j = 1,2$ and $K(\mathbf{p}) = \sum_j p_j^{\beta}$, $\beta = 2-\alpha$ (to define p_j^{β} we make a cut along to negative axis). Thus we rewrite (2.1.4) in the form

$$\widehat{\hat{u}}(\xi,\mathbf{p}) = \frac{1}{(\xi - K(\mathbf{p}))\, p_1^{\alpha} p_2^{\alpha}}\left(\phi(\mathbf{p}) + p_1^{\alpha}\hat{u}_{x_2}(\xi,p_1) + p_2^{\alpha}\hat{u}_{x_1}(\xi,p_2)\right), \tag{2.1.5}$$

with

$$\phi(\mathbf{p}) = p_1^{\alpha}p_2^{\alpha}\hat{u}_0(\mathbf{p}) + \sum_j p_j^{1+\alpha}\hat{u}(\xi,p_i).$$

We fixed $p_j, \operatorname{Re} p_j > 0$ and ξ, $\operatorname{Re}\xi > 0$. There exists only one root $k_j = \left(\xi - p_j^{\beta}\right)^{\frac{1}{\beta}}$, $j = 1,2$ of equation $K(\mathbf{p}) = \xi$ such that $\operatorname{Re}\, k_j > 0$ for all fixed $\operatorname{Re}\,\xi > 0$ and $\operatorname{Re} p_j > 0$. Therefore in the expression for the function $\widehat{\hat{u}}(\xi,\mathbf{p})$ the factor $\dfrac{1}{\xi - K(\mathbf{p})}$ has a pole in the point $p_j = k_i$, $j = 1,2$. Note that $p_j = k_i(\xi, k_j(\xi,p_j))$. Also note that $k_j = \left(\xi - p_j^{\beta}\right)^{\frac{1}{\beta}}$ is analytic in the part of right half complex plane:

$$\left\{p_j \in \mathbb{C} : \operatorname{Arg}\, p_j \in \left(-\frac{\pi}{2\beta}, \frac{\pi}{2\beta}\right)\right\}$$

for any fixed ξ, $\operatorname{Re}\xi = 0$.

Since by construction $\widehat{\hat{u}}(\xi,\mathbf{p})$ must be analytical function in $\operatorname{Re}\mathbf{p} \in \mathbf{D}$, $\operatorname{Re}\xi = 0$ this implies that for the solubility of problem (0.0.1) it is necessary that the following conditions are satisfied

$$\left[\phi(\mathbf{p}) + p_1^{\alpha}\hat{u}_{x_2}(\xi,p_1) + p_2^{\alpha}\hat{u}_{x_1}(\xi,p_2)\right]\Big|_{p_j=k_i(\xi,p_i)} = 0, \quad j = 1,2. \tag{2.1.6}$$

Thus, if we consider Dirichlet boundary data $u(t,\mathbf{x})\Big|_{x_j=0} = h_i(t,x_i)$, $j=1,2$, the another unknown boundary data $u_{x_j}(t,\mathbf{x})\Big|_{x_j=0}$ we find from (2.1.6) as

$$\phi(\mathbf{p})\Big|_{p_2=k_1(\xi,p_1)} + p_1^\alpha \widehat{u}_{x_2}(\xi,p_1) + k_1^\alpha \widehat{u}_{x_1}(\xi,k_1) = 0,$$
$$\phi(\mathbf{p})\Big|_{p_1=k_2(\xi,p_2)} + k_2^\alpha \widehat{u}_{x_2}(\xi,k_2) + p_2^\alpha \widehat{u}_{x_1}(\xi,p_2) = 0,$$
$$\phi(k_2,k_1) + k_2^\alpha \widehat{u}_{x_2}(\xi,k_2) + k_1^\alpha \widehat{u}_{x_1}(\xi,k_1) = 0.$$

From which it follows

$$p_1^\alpha \widehat{u}_{x_2}(\xi,p_1)+p_2^\alpha \widehat{u}_{x_1}(\xi,p_2) = -\left(\phi(\mathbf{p})\Big|_{p_2=k_1(\xi,p_1)} + \phi(\mathbf{p})\Big|_{p_1=k_2(\xi,p_2)} - \phi(k_2,k_1)\right). \quad (2.1.7)$$

Substituting (2.1.7) into (2.1.5) we have

$$\widehat{\widehat{u}}(\xi,\mathbf{p}) = \widehat{\widehat{u}}_1(\xi,\mathbf{p}) + \widehat{\widehat{u}}_2(\xi,\mathbf{p}),$$

where

$$\widehat{\widehat{u}}_m(\xi,\mathbf{p}) = \frac{\widehat{\phi}_m(\mathbf{p}) - \sum_j \widehat{\phi}_m(\mathbf{p})\Big|_{p_j=k_i(\xi,p_i)} \left(\frac{k_i}{p_j}\right)^\alpha + \widehat{\phi}_m(k_2,k_1)\prod_j \left(\frac{k_i}{p_j}\right)^\alpha}{\xi - K(\mathbf{p})}, \quad (2.1.8)$$

with

$$\widehat{\phi}_1(\mathbf{p}) = \widehat{u}_0(\mathbf{p})$$

and

$$\widehat{\phi}_2(\mathbf{p}) = \sum_j p_j^{1-\alpha} h_i(\xi,p_i).$$

The fundamental importance of the proven fact, that the solution $\widehat{\widehat{u}}$ constitutes an analytic function in $\operatorname{Re}\mathbf{p}\in\mathbf{D}$, $\operatorname{Re}\xi>0$ and, as a consequence, taking inverse Laplace transform of (2.1.8) we obtain

$$u(t,\mathbf{x}) = \sum_m \mathcal{G}(t)\phi_m,$$

where operators $\mathcal{G}(t)$ given by

$$\mathcal{G}(t)\phi(t,\mathbf{x}) = \left(\frac{1}{2\pi i}\right)^3 \int_{-i\infty}^{i\infty} d\xi\ e^{\xi t} \int_{i\mathbb{R}^2} d\mathbf{p}\ e^{\mathbf{p}\mathbf{x}} \frac{\Psi(\xi,\mathbf{p})}{\xi - K(\mathbf{p})}, \quad (2.1.9)$$

and

$$\Psi(\xi,\mathbf{p}) = \left(\widehat{\phi}(\mathbf{p}) - \sum_j \widehat{\phi}(\mathbf{p})\Big|_{p_j=k_i(\xi,p_i)} \left(\frac{k_i}{p_j}\right)^\alpha + \widehat{\phi}(k_2,k_1)\prod_j \left(\frac{k_i}{p_j}\right)^\alpha\right).$$

Now we simplify $\mathcal{G}(t)\phi$ given by (2.1.9). We rewrite $\mathcal{G}(t)\phi$ in the following form

$$\mathcal{G}(t)\phi = \mathcal{G}_1(t)\phi + \mathcal{G}_2(t)\phi + \mathcal{G}_3(t)\phi,$$

where

$$\mathcal{G}_1(t)\phi = \left(\frac{1}{2\pi i}\right)^3 \int_{-i\infty}^{i\infty} d\xi\ e^{\xi t} \int_{i\mathbb{R}^2} d\mathbf{p} \frac{e^{\mathbf{px}}}{\xi - K(\mathbf{p})} \widehat{\phi}(\mathbf{p}),$$
$$\mathcal{G}_2(t)\phi = -\left(\frac{1}{2\pi i}\right)^3 \sum_j \int_{-i\infty}^{i\infty} d\xi\ e^{\xi t} \int_{i\mathbb{R}^2} d\mathbf{p} \frac{e^{\mathbf{px}}}{\xi - K(\mathbf{p})} \widehat{\phi}(\mathbf{p}) \Big|_{p_j = k(\xi, p_i)} \left(\frac{k(\xi, p_i)}{p_j}\right)^\alpha,$$
$$\mathcal{G}_3(t)\phi = \left(\frac{1}{2\pi i}\right)^3 \int_{-i\infty}^{i\infty} d\xi\ e^{\xi t} \int_{i\mathbb{R}^2} d\mathbf{p} \frac{e^{\mathbf{px}}}{\xi - K(\mathbf{p})} \widehat{\phi}(k_2, k_1) \prod_j \left(\frac{k_i}{p_j}\right)^\alpha.$$

Changing the order of the integration, applying the Jordan's lemma in the left half of the complex plane , taking the the residue at the point $\xi = K(\mathbf{p})$ we obtain

$$\begin{aligned}\mathcal{G}_1(t)\phi &= \left(\frac{1}{2\pi i}\right)^3 \int_{i\mathbb{R}^2} d\mathbf{p}\ e^{\mathbf{px}} \widehat{\phi}(\mathbf{p}) \int_{-i\infty}^{i\infty} d\xi \frac{e^{\xi t}}{\xi - K(\mathbf{p})} \\ &= \left(\frac{1}{2\pi i}\right)^2 \int_{i\mathbb{R}^2} d\mathbf{p}\ e^{\mathbf{px}} \widehat{\phi}(\mathbf{p}) \left(e^{\xi t}|_{\xi = K(\mathbf{p})}\right) \\ &= \left(\frac{1}{2\pi i}\right)^2 \int_{i\mathbb{R}^2} d\mathbf{p}\ e^{\mathbf{px} + K(\mathbf{p})t} \widehat{\phi}(\mathbf{p}).\end{aligned} \tag{2.1.10}$$

Recall that by the definition $\mathrm{Re}\, k(\xi, p_i) > 0$ for $\mathrm{Re}\, \xi = 0$, $\mathrm{Re}\, p_j = 0$. Therefore applying Jordan's lemma in the right-half of the complex plane and taking residuo at the point $q = k_i^\beta(\xi, p_i)$ we have

$$\begin{aligned}\frac{\beta}{2\pi i} \int_{-i\infty e^{i\varepsilon}}^{i\infty e^{-i\varepsilon}} dq \frac{q \widehat{\phi}(\mathbf{p}) \Big|_{p_j = q}}{(q^\beta - k^\beta(\xi, p_i))} &= -\left(q^{-\beta+2} \widehat{\phi}(\mathbf{p}) \Big|_{p_j = q}\right) \Big|_{q = k(\xi, p_i)} \\ &= -\left(q^\alpha \widehat{\phi}(\mathbf{p}) \Big|_{p_j = q}\right) \Big|_{q = k(\xi, p_i)} \\ &= -\widehat{\phi}(\mathbf{p}) \Big|_{p_j = k(\xi, p_i)} k^\alpha(\xi, p_i)\end{aligned} \tag{2.1.11}$$

for any fixed pure imaginary ξ, p_i and small enough $\varepsilon > 0$.

We use (2.1.11) in the definition of $\mathcal{G}_2(t)\phi$ to get

$$\mathcal{G}_2(t)\phi = -\sum_j \left(\frac{1}{2\pi i}\right)^3 \beta \int_{-i\infty}^{i\infty} d\xi e^{\xi t} \int_{i\mathbb{R}^2} d\mathbf{p} \frac{e^{\mathbf{px}} p_j^{-\alpha}}{\xi - K(\mathbf{p})} \int_{-i\infty}^{i\infty} dq \frac{q \widehat{\phi}(\mathbf{p}) \Big|_{p_j = q}}{q^\beta - k^\beta(\xi, p_i)}.$$

Note that $k^\beta(\xi, p_i) = p_i^\beta - \xi$ is analytic in the left-half complex plane. Therefore taking the residuo at the points $\xi = q^\beta + p_i^\beta = K(q) + K(p_i)$ and $\xi = K(\mathbf{p})$ we obtain

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\xi\ e^{\xi t} \frac{1}{\xi - K(\mathbf{p})} \frac{1}{q^\beta + p_i^\beta - \xi} = \frac{1}{q^\beta - p_j^\beta} \left(e^{K(\mathbf{p})t} - e^{\left(q^\beta + p_i^\beta\right)t}\right). \tag{2.1.12}$$

Using (2.1.12) and changing the order of the integration we rewrite $\mathscr{G}_2(t)\phi$ in the form

$$\mathcal{G}_2(t)\phi = I_1 + I_2,$$

where

$$I_1 = \sum_j \left(\frac{1}{2\pi i}\right)^3 \beta \int_{i\mathbb{R}^2} d\mathbf{p}\ e^{\mathbf{px}} e^{K(\mathbf{p})t} p_j^{-\alpha} \int_{-i\infty e^{i\varepsilon}}^{i\infty e^{-i\varepsilon}} dq\ \frac{q}{q^\beta - p_j^\beta} \widehat{\phi}(\mathbf{p}) \Big|_{p_j=q}\ ,$$
$$I_2 = -\sum_j \left(\frac{1}{2\pi i}\right)^3 \beta \int_{i\mathbb{R}^2} d\mathbf{p}\ e^{\mathbf{px}} p_j^{-\alpha} \int_{-i\infty e^{i\varepsilon}}^{i\infty e^{-i\varepsilon}} dq\ \frac{q e^{K(p_i)t + K(q_j)t}}{q^\beta - p_j^\beta} \widehat{\phi}(\mathbf{p}) \Big|_{p_j=q}\ .$$

Note that by Cauchy Theorem

$$\int_{-i\infty e^{i\varepsilon}}^{i\infty e^{-i\varepsilon}} dq \frac{q}{q^\beta - p_j^\beta} \widehat{\phi}(\mathbf{p}) \Big|_{p_j=q} = 0$$

since $\frac{q}{q^\beta - p_j^\beta} \widehat{\phi}(\mathbf{p}) \Big|_{p_j=q} = O(\langle q\rangle^{-1-\gamma})$ is analytic function in the right-half of the complex plane for an fixed pure imaganary p_j. As a result $I_1 = 0$. Making the change a variable $p_j \to q$ in the second integral we obtain

$$I_2 = \sum_j \left(\frac{1}{2\pi i}\right)^3 \int_{i\mathbb{R}^2} d\mathbf{p}\ e^{p_i x_i} e^{K(\mathbf{p})t} \widehat{\phi}(\mathbf{p}) \frac{p_j \beta}{2\pi i} \int_{-i\infty e^{i\varepsilon}}^{i\infty e^{-i\varepsilon}} dq \frac{e^{q x_j}}{\left(q^\beta - p_j^\beta\right) q^\alpha}.$$

Therefore recalling $\mathcal{G}_2 = I_1 + I_2$ we obtin

$$\mathcal{G}_2(t)\phi = \sum_j \left(\frac{1}{2\pi i}\right)^3 \int_{i\mathbb{R}^2} d\mathbf{p}\ e^{p_i x_i} e^{K(\mathbf{p})t} \widehat{\phi}(\mathbf{p}) \psi(p_j, x_j),$$
$$\psi(p,x) = \frac{1}{2\pi i} \int_{-i\infty e^{i\varepsilon}}^{i\infty e^{-i\varepsilon}} dq\ e^{qx} \frac{\beta p}{\left(q^\beta - p^\beta\right) q^\alpha}$$

where $\varepsilon > 0$, such that $q^\beta - p^\beta \neq 0$ for any fixed $\operatorname{Re} p = 0$.

Also again applying Jordan's lemma in the right-half plane we get

$$\left(\frac{1}{2\pi i}\right)^2 \beta^2 \int_{i\mathbb{R}^2+\varepsilon} d\mathbf{q} \frac{q_1 q_2 \widehat{\phi}(\mathbf{q})}{\left(q_1^\beta + p_2^\beta - \xi\right)} \frac{1}{(K(\mathbf{q}) - \xi)} \frac{1}{\left(q_2^\beta + p_1^\beta - \xi\right)}$$
$$= \left(\frac{1}{2\pi i}\right) \beta \int_{i\mathbb{R}} dq_1 \frac{q_1}{\left(q_1^\beta - k_2^\beta(p_2)\right)} \left(\frac{1}{2\pi i}\right) \beta \int_{i\mathbb{R}} dq_2 \frac{q_2 \widehat{\phi}(\mathbf{q})}{(K(\mathbf{q}) - \xi)\left(q_2^\beta - k_1^\beta(p_1)\right)}$$
$$= -\left(\frac{1}{2\pi i}\right) \frac{k_1^\alpha(p_1)}{(K(\mathbf{p}) - \xi)} \int_{i\mathbb{R}+\varepsilon} dq_1 \frac{q_1}{\left(q_1^\beta - k_2^\beta\right)} \left(\widehat{\phi}(\mathbf{q})\,|_{q_2=k_1}\right)$$
$$= -\frac{1}{\xi - K(\mathbf{p})} \widehat{\phi}(k_2, k_1) \prod_j (k_i)^\alpha$$

Here we choose $\varepsilon > 0$ such that $K(\mathbf{q}) - \xi \neq 0$ for all $\operatorname{Re} K(\mathbf{q}) > 0$. Therefore we can rewrite $\mathcal{G}_3(t)\phi$ as

$$\begin{aligned}\mathcal{G}_3(t)\phi &= \left(\frac{1}{2\pi i}\right)^3 \int_{-i\infty}^{i\infty} d\xi\ e^{\xi t} \int_{i\mathbb{R}^2} d\mathbf{p}\ e^{\mathbf{px}} p_1^{-\alpha} p_2^{-\alpha} \\ &\quad \times \left(\frac{1}{2\pi i}\right)^2 \beta^2 \int_{i\mathbb{R}^2+\varepsilon} d\mathbf{q} \frac{q_1 q_2 \widehat{\phi}(\mathbf{q})}{\left(q_1^\beta + p_2^\beta - \xi\right)} \frac{1}{(K(\mathbf{q}) - \xi)} \frac{1}{\left(q_2^\beta + p_1^\beta - \xi\right)} \\ &= \left(\frac{1}{2\pi i}\right)^4 \beta^2 p_1^{-\alpha} p_2^{-\alpha} \int_{i\mathbb{R}^2+\varepsilon} d\mathbf{q}\ e^{K(\mathbf{q})t} \frac{q_1 q_2 \widehat{\phi}(\mathbf{q})}{\left(p_2^\beta - q_2^\beta\right)} \frac{1}{\left(p_1^\beta - q_1^\beta\right)} \\ &= \left(\frac{1}{2\pi i}\right)^2 \int_{i\mathbb{R}^2} d\mathbf{p}\ e^{K(\mathbf{p})t} \widehat{\phi}(\mathbf{p}) \psi(x_i, p_i) \psi(x_2, p_2)\end{aligned}$$

Recalling $\mathcal{G}(t)\phi = \mathcal{G}_1(t)\phi + \mathcal{G}_2(t)\phi + \mathcal{G}_3(t)\phi$ from (2.1.10) finally we get

$$\begin{aligned}\mathcal{G}(t)\phi &= -\left(\frac{1}{2\pi i}\right)^2 \int_{i\mathbb{R}^2} d\mathbf{p}\ e^{\mathbf{px}} e^{K(\mathbf{p})t} \widehat{\phi}(\mathbf{p}) \\ &\quad \times \left(e^{\mathbf{px}} + \sum_{j,i\neq j} e^{p_i x_i} \psi(p_j, x_j) + \psi(x_i, p_i)\psi(x_2, p_2)\right) \\ &= -\left(\frac{1}{2\pi i}\right)^2 \int_{i\mathbb{R}^2} d\mathbf{p}\ e^{K(\mathbf{p})t} \widehat{\phi}(\mathbf{p}) \Phi(\mathbf{p}, \mathbf{x}) \\ &= \left(\frac{1}{2\pi}\right)^2 \int_{i\mathbb{R}^2} d\mathbf{p}\ e^{K(\mathbf{p})t} \widehat{\phi}(\mathbf{p}) \Phi(\mathbf{p}, \mathbf{x}),\end{aligned} \tag{2.1.13}$$

where $\Phi(\mathbf{p}, \mathbf{x}) = \prod_j \left(e^{p_j x_j} + \psi(p_j, x_j)\right)$.

Also substituting $\widehat{\phi}(\mathbf{p}) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \sum_j p_j^{\beta-1} h_i(\xi, p_i)$ into (2.1.13) we obtain

$$\mathcal{H}(t)\mathbf{h} = \sum_{j,i\neq j} \left(\frac{1}{2\pi i}\right)^2 \int_0^t d\tau \int_{i\mathbb{R}^2} d\mathbf{p}\ e^{K(\mathbf{p})(t-\tau)} p_j^{\beta-1} \Phi(\mathbf{p}, \mathbf{x}) h_i(\tau, p_i),$$

where $\widehat{h_j}(\tau, p_i) = \mathscr{L}\{h_j(\tau, x_i)\}$ is Laplace transform of $h_j(x_i)$ and

$$\Gamma_j = \left\{q \in \mathbb{C}\ :\ \operatorname{Re} q < 0,\ q^\beta - p_j^\beta \neq 0\right\},\ j = 1, 2;$$

for any fixed pure imaganary p_j^β. Proposition 2.1.1 is proved. ∎

2.2 Estimates a priori

Define

$$\mathbb{I}\phi = \left(\frac{1}{2\pi i}\right)^2 \int_{i\mathbb{R}^2} d\mathbf{p}\ \phi(\mathbf{p}) \Phi(\mathbf{p}, \mathbf{x}),$$

where

$$\Phi(\mathbf{p},\mathbf{x}) = \prod_j \Phi_j(p_j,x_j),$$
$$\Phi_j(p_j,x_j) = \left(e^{p_j x_j} + \psi(p_j,x_j)\right),$$
$$\psi(p,x) = \frac{1}{2\pi i}\oint_\Gamma dq\, e^{qx}\frac{\beta p}{(q^\beta - p^\beta)\, q^\alpha},$$

where $\Gamma = \Big\{q \in \mathbb{C} : \operatorname{Re} q < 0 \text{ such that } q^\beta - p^\beta \neq 0 \text{ for any fixed } \operatorname{Re} p = 0\Big\}$.

Lemma 2.2.1. *The following estimates are valid*

$$\|\mathbb{I}\phi\|_{\dot{\mathbf{H}}^s} \leqslant C\,\|\phi\|_{\dot{\mathbf{H}}^{s\pm\gamma}}, \quad s \in \left(0,\frac{1}{2}\right), \tag{2.2.1}$$

$$\left\|\mathbb{I}\widehat{\phi}\right\|_{\mathbf{L}^\infty_{\mathbf{X}}} \leqslant C\left(\|\phi\|_{\dot{\mathbf{H}}^{s\pm\gamma}} + \sum_j \left\|\mathbb{I}\widehat{\phi}\,\big|_{x_j=0}\right\|_{\mathbf{L}^\infty_{\mathbf{X}}} + \left|\mathbb{I}\widehat{\phi}\,|_{\mathbf{x}=0}\right|\right), \quad s \in \left[\frac{1}{2},\frac{3}{2}\right), \tag{2.2.2}$$

with C some constant and $\gamma > 0$ is small. Also if $\widehat{\phi}(\mathbf{p}) = \dfrac{c(p_i)}{p_j} + O(\langle p_j\rangle^{-2})$ then

$$\lim_{x_j\to 0}\left\|\mathbb{I}\widehat{\phi}(\mathbf{x}) - \mathbb{I}\widehat{\phi}(\mathbf{x})\,\big|_{x_j=0}\right\|_{H^\varepsilon_{\mathbf{X}_i}} \leqslant C\lim_{x_j\to 0} x_j^\delta \int_{i\mathbb{R}^2} d\mathbf{p}\left|\widehat{\phi}(\mathbf{p}) - \frac{c(p_i)}{p_j}\right| |p_i|^{\varepsilon-\frac{1}{2}}\,|p_j|^\delta, \tag{2.2.3}$$
$$\lim_{x_j\to 0}\left\|\mathbb{I}\widehat{\phi}(\mathbf{x}) - \mathbb{I}\widehat{\phi}(\mathbf{x})\,\big|_{x_j=0}\right\|_{L^\infty_{\mathbf{X}_i}} \leqslant C\lim_{x_j\to 0} x_j^\delta \int_{i\mathbb{R}^2} d\mathbf{p}\left|\widehat{\phi}(\mathbf{p}) - \frac{c(p_i)}{p_j}\right| |p_j|^\delta,$$

where $\varepsilon \in \left[0,\frac{1}{2}\right)$, $\delta \in [0,1)$.

Proof. Firstly we prove that $\mathbb{I}\phi = 0$ for $\mathbf{x} \notin \mathbf{D}$. Applying the Jordan's lemma in the right half of the complex plane, taking the the residue at the point $q = p$, where p is some fixed pure imaginary number , we obtain

$$\psi(p,x) = \frac{1}{2\pi i}\oint_\Gamma dq\; e^{qx}\;\frac{\beta p}{(q^\beta - p^\beta)\,q^\alpha} = e^{qx}\frac{-\beta p}{\beta q^{\beta-1+\alpha}}\,|_{q=p} = -e^{px}$$

for $x < 0$. Thus $\Phi(\mathbf{p},\mathbf{x}) = \prod_j \Phi_j(p_j,x_j) = 0$ for $\mathbf{x} \notin \mathbf{D}$ and as results

$$\mathbb{I}\phi = 0$$

for $\mathbf{x} \notin \mathbf{D}$. Therefore we get for Laplace transformation of $\mathbb{I}\widehat{\phi}$

$$\mathcal{L}_{\mathbf{X}}\left\{\mathbb{I}\widehat{\phi}\right\}(\mathbf{q}) = \left(\frac{1}{2\pi i}\right)^2\int_{i\mathbb{R}^2} d\mathbf{p}\;\widehat{\phi}(\mathbf{p})\prod_j \mathscr{L}\Phi_j(p_j,x_j) = \left(\frac{1}{2\pi i}\right)^2\int_{i\mathbb{R}^2} d\mathbf{p}\;\widehat{\phi}(\mathbf{p})\prod_j \frac{\beta p_j}{\left(q_j^\beta - p_j^\beta\right)q_j^\alpha}$$

where via Jordan's Lemma

$$
\begin{aligned}
\mathcal{L}\Phi_j(p_j,q_j) &= \int_0^\infty \; e^{-q_jx_j}(e^{p_jx_j}+\psi(p_j,x_j))dx_j \qquad (2.2.4)\\
&= \int_0^\infty dx_j\; e^{(p_j-q_j)x_j} + \int_0^\infty dx_j\; e^{-q_jx_j}\psi(p_j,x_j)\\
&= \frac{1}{p_j-q_j} + \int_0^\infty dx_j\; e^{-q_jx_j}\frac{1}{2\pi i}\oint_\Gamma ds_j\; e^{s_jx_j}\frac{\beta p_j}{\left(s_j^\beta-p_j^\beta\right)s_j^\alpha}\\
&= \frac{1}{p_j-q_j} + \frac{1}{2\pi i}\int_0^\infty dx_j\; e^{(s_j-q_j)x_j}\oint_\Gamma ds_j\;\frac{\beta p_j}{\left(s_j^\beta-p_j^\beta\right)s_j^\alpha}\\
&= \frac{1}{p_j-q_j} + \frac{1}{2\pi i}\oint_\Gamma ds_j\frac{1}{s_j-q_j}\frac{\beta p_j}{\left(s_j^\beta-p_j^\beta\right)s_j^\alpha}\\
&= \frac{1}{p_j-q_j} - \frac{\beta p_j}{\left(q_j^\beta-p_j^\beta\right)q_j^\alpha} - \frac{1}{p_j-q_j}\frac{\beta p_j}{\beta p_j^{\beta+\alpha-1}}\\
&= \frac{1}{p_j-q_j} - \frac{\beta p_j}{\left(q_j^\beta-p_j^\beta\right)q_j^\alpha} - \frac{1}{p_j-q_j}\\
&= -\frac{\beta p_j}{\left(q_j^\beta-p_j^\beta\right)q_j^\alpha}.
\end{aligned}
$$

Therefore via Minkowski inequality making the change of variables $q_j = |p_j|\,\widetilde{q}_j$ we get

$$
\begin{aligned}
\|\mathbb{I}\phi\|_{\dot{\mathbf{H}}^s} &= \|q_1^s q_2^s\;\mathcal{L}\{\mathbb{I}\phi\}\|_{\mathbf{L}^2} \qquad (2.2.5)\\
&= \left(\frac{1}{2\pi}\right)^2\sqrt{\int_{i\mathbb{R}^2+\varepsilon} d\mathbf{q}\;|q_1^{2s}||q_2^{2s}|\left|\int_{i\mathbb{R}^2} d\mathbf{p}\;\widehat{\phi}(\mathbf{p})\prod_j\frac{-\beta p_j}{\left(q_j^\beta-p_j^\beta\right)q_j^\alpha}\right|^2}\\
&\leqslant \frac{\beta}{(2\pi)^2}\int_{i\mathbb{R}^2} d\mathbf{p}\left|\widehat{\phi}(\mathbf{p})\right|\sqrt{\int_{i\mathbb{R}^2+\varepsilon} d\mathbf{q}\prod_j\frac{|q_j|^{2s}\,|p_j|^2}{\left|p_j^\beta-q_j^\beta\right|^2|q_j|^{2\alpha}}}\\
&= \frac{\beta}{(2\pi)^2}\int_{i\mathbb{R}^2} d\mathbf{p}\left|\widehat{\phi}(\mathbf{p})\right|\sqrt{\prod_j\int_{i\mathbb{R}+\varepsilon} dq_j\frac{|q_j|^{2s}\,|p_j|^2}{\left|q_j^\beta-p_j^\beta\right|^2|q_j|^{2\alpha}}}\\
&= \frac{\beta}{(2\pi)^2}\int_{i\mathbb{R}^2} d\mathbf{p}\left|\widehat{\phi}(\mathbf{p})\right|\sqrt{\prod_j\int_{i\mathbb{R}+\varepsilon} d\widetilde{q}_j|p_j|\frac{|p_j|^{2s}\,|\widetilde{q}_j|^{2s}\,|p_j|^2}{\left||p_j|^\beta\widetilde{q}_j^\beta-p_j^\beta\right|^2|p_j|^{2\alpha}\,|\widetilde{q}_j|^{2\alpha}}}\\
&= \frac{\beta}{(2\pi)^2}\int_{i\mathbb{R}^2} d\mathbf{p}\left|\widehat{\phi}(\mathbf{p})\right||p_1|^{s-\frac{1}{2}}|p_2|^{s-\frac{1}{2}}\sqrt{\prod_j\int_{i\mathbb{R}+\varepsilon} d\widetilde{q}_j\frac{|\widetilde{q}_j|^{2s-2\alpha}}{\left|\widetilde{q}_j^\beta-\dfrac{p_j^\beta}{|p_j|^\beta}\right|^2}}\\
&\leqslant \frac{\beta K}{(2\pi)^2}\int_{i\mathbb{R}^2} d\mathbf{p}\left|\widehat{\phi}(\mathbf{p})\right||p_1|^{s-\frac{1}{2}}\,|p_2|^{s-\frac{1}{2}} \leqslant C\,\|\phi\|_{\dot{\mathbf{H}}^{s\pm\gamma}}
\end{aligned}
$$

for $s \in \left(0, \frac{1}{2}\right)$. Let $\mathcal{L}\left\{\frac{\partial^2\,\phi}{\partial x_1 x_2}\right\}(\mathbf{p}) = \widehat{\phi}_{x_1 x_2}(\mathbf{p})$ and

$$\begin{aligned}\widehat{\phi}_{x_1x_2}(\mathbf{p}) &= p_1p_2\left(\widehat{\phi}(\mathbf{p}) - \frac{\widehat{\phi}^{x_1}(p_1,0)}{p_2} - \frac{\widehat{\phi}^{x_2}(0,p_2)}{p_1} + \frac{\phi(0,0)}{p_1p_2}\right) \\ &= p_1p_2\left(\frac{c_1(p_1)}{p_2} + \frac{c_2(p_2)}{p_1} + \frac{c_3}{p_2p_1} + O(\mathbf{p}^{-1-\gamma})\right).\end{aligned} \tag{2.2.6}$$

As a result,

$$\mathbb{I}\widehat{\phi}_{x_1x_2} = \left(\frac{1}{2\pi i}\right)^2 \int_{i\mathbb{R}^2} d\mathbf{p}\; p_1p_2\left(\widehat{\phi}(\mathbf{p}) - \left(\frac{c_1(p_1)}{p_2} + \frac{c_2(p_2)}{p_1}\right) + \frac{c_3}{p_2p_1}\right)\prod_j \partial_{x_j}\Phi_j(p_j,x_j).$$

Therefore in the case of $s \in \left[\frac{1}{2}, \frac{3}{2}\right)$

$$\begin{aligned}\|\mathbb{I}\phi\|_{\dot{\mathbf{H}}^s} &= \|\mathbb{I}\phi_{x_1x_2}\|_{\dot{\mathbf{H}}^{s-1}} \\ &\leqslant \int_{i\mathbb{R}^2} d\mathbf{p}\,|p_1p_2|^{s-1}\left|\widehat{\phi}(\mathbf{p}) - \frac{c_1(p_1)}{p_2} - \frac{c_2(p_2)}{p_1} + \frac{c_3}{p_2p_1}\right| \\ &\quad\times \prod_j \sqrt{\int_{i\mathbb{R}^2+\varepsilon} d\mathbf{q}\frac{|q_j|^{2s-1}}{\left|p_j^\beta - q_j^\beta\right|^2 |q_j|^{2\alpha}}} \\ &\leqslant C\int_{i\mathbb{R}^2} d\mathbf{p}\left|\widehat{\phi}(\mathbf{p}) - \left(\frac{c_1(p_1)}{p_2} + \frac{c_2(p_2)}{p_1}\right) + \frac{c_3}{p_2p_1}\right| |p_1|^{s-\frac{1}{2}}\,|p_2|^{s-\frac{1}{2}} \\ &\leqslant C\,\|\phi\|_{\dot{\mathbf{H}}^{s\pm\gamma}}\,.\end{aligned} \tag{2.2.7}$$

Since $\|\Phi(\mathbf{p},\mathbf{x})\|_{\mathbf{L}^\infty_\mathbf{X}} \leqslant C$ we have

$$\begin{aligned}\left\|\mathbb{I}\widehat{\phi}\right\|_{\mathbf{L}^\infty_\mathbf{X}} &= \left\|\int_{i\mathbb{R}^2} d\mathbf{p}\;\widehat{\phi}(\mathbf{p})\Phi(\mathbf{p},\mathbf{x})\right\|_{\mathbf{L}^\infty_\mathbf{X}} \\ &\leqslant C\int_{i\mathbb{R}^2} d\mathbf{p}\;\left|\widehat{\phi}(\mathbf{p}) - \left(\frac{c_1(p_1)}{p_2} + \frac{c_2(p_2)}{p_1}\right) + \frac{c_3}{p_1p_2}\right| \\ &\quad + \sum_j \left\|\int_{i\mathbb{R}^2} d\mathbf{p}\frac{c_i(p_i)}{p_j}\Phi(\mathbf{p},\mathbf{x})\right\|_{\mathbf{L}^\infty_\mathbf{X}} + \left\|\int_{i\mathbb{R}^2} d\mathbf{p}\frac{c_3}{p_1p_2}\Phi(\mathbf{p},\mathbf{x})\right\|_{\mathbf{L}^\infty_\mathbf{X}} \\ &\leqslant C\left(\|\phi\|_{\dot{\mathbf{H}}^{s\pm\gamma}} + \sum_j \left\|\mathbb{I}\widehat{\phi}\,\Big|_{x_j=0}\right\|_{\mathbf{L}^\infty_\mathbf{X}} + \left|\mathbb{I}\widehat{\phi}\,|_{\mathbf{x}=0}\right|\right),\;\; s > \frac{1}{2}.\end{aligned}$$

Thus from (2.2.5) ,(2.2.6) and (2.2.7) we obtain (2.2.1). Now we prove (2.2.3).

Using

$$\mathbb{I}\widehat{\phi}\Big|_{x_j=0} = \left(\frac{1}{2\pi i}\right)^2 \int_{i\mathbb{R}^2} d\mathbf{p}\ \widehat{\phi}(\mathbf{p})\Phi(\mathbf{p},\mathbf{x})\Big|_{x_j=0}$$

$$= \lim_{x_j\to 0}\left(\frac{1}{2\pi i}\right)^2 \int_{i\mathbb{R}^2} d\mathbf{p}\ \frac{c_i(p_i)}{p_j}\Phi(\mathbf{p},\mathbf{x})$$

$$+ \left(\frac{1}{2\pi i}\right)^2 \int_{i\mathbb{R}^2} d\mathbf{p}\left(\widehat{\phi}(\mathbf{p}) - \frac{\widehat{\phi\big|_{x_j=0}}(p_i)}{p_j}\right)\Phi_i(p_i,x_i)\Phi_j(p_j,0)$$

we get

$$\mathbb{I}\widehat{\phi}(\mathbf{x}) - \mathbb{I}\widehat{\phi}(\mathbf{x})\Big|_{x_j=0}$$

$$= \left(\frac{1}{2\pi i}\right)^2 \int_{i\mathbb{R}^2} d\mathbf{p}\left(\widehat{\phi}(\mathbf{p}) - \frac{c(p_i)}{p_j}\right)\Phi_i(p_i,x_i)\left(\Phi_j(p_j,x_j) - \Phi_j(p_j,0)\right).$$

Therefore since

$$\left|\Phi_j(p_j,x_j) - \Phi_j(p_j,0)\right| \leqslant |p_j|^\delta\, x_j^\delta,$$

for $\delta \in (0,1)$ we obtain

$$\lim_{x_j\to 0}\left\|\mathbb{I}\widehat{\phi}(\mathbf{x}) - \mathbb{I}\widehat{\phi}(\mathbf{x})\Big|_{x_j=0}\right\|_{H^\varepsilon_{\mathbf{x}_i}}$$

$$\leqslant \lim_{x_j\to 0}\left\|\int_{i\mathbb{R}^2} d\mathbf{p}\left|\widehat{\phi}(\mathbf{p}) - \frac{\widehat{\phi\big|_{x_j=0}}(p_i)}{p_j}\right|\left|\Phi_i(p_i,x_i)\right|\left|\left(\Phi_j(p_j,x_j) - \Phi_j(p_j,0)\right)\right|\right\|_{H^\varepsilon_{\mathbf{x}_i}}$$

$$\leqslant C\lim_{x_j\to 0} x_j^\delta \int_{i\mathbb{R}} d\mathbf{p}\left|\widehat{\phi}(\mathbf{p}) - \frac{\widehat{\phi\big|_{x_j=0}}(p_i)}{p_j}\right| |p_i|^{\varepsilon-\frac{1}{2}}\,|p_j|^\delta$$

where $\varepsilon \in \left[0,\frac{1}{2}\right)$. Also we have

$$\lim_{x_j\to 0}\left\|\mathbb{I}\widehat{\phi}(\mathbf{x}) - \mathbb{I}\widehat{\phi}(\mathbf{x})\Big|_{x_j=0}\right\|_{L^\infty_{\mathbf{x}_i}} \leqslant C\lim_{x_j\to 0} x_j^\delta \int_{i\mathbb{R}} d\mathbf{p}\left|\widehat{\phi}(\mathbf{p}) - \frac{\widehat{\phi\big|_{x_j=0}}(p_i)}{p_j}\right| |p_j|^\delta.$$

Lemma is proven. ■

2.3 Green operator estimates

Let $\widehat{\phi}(\mathbf{p})$ is the analytic function in the right quarter-complex plane. Define

$$\mathcal{G}(t)\phi = \left(\frac{1}{2\pi}\right)^2 \int_{i\mathbb{R}^2} d\mathbf{p}\ e^{K(\mathbf{p})t}\widehat{\phi}(\mathbf{p})\Phi(\mathbf{p},\mathbf{x}) = \mathbb{I}\left\{e^{K(\mathbf{p})t}\widehat{\phi}(\mathbf{p})\right\}.$$

Lemma 2.3.1. *The estimates*

$$\|\mathcal{G}(t)\phi\|_{\mathbf{L}^2} \leqslant C\{t\}^{-\gamma}\|\phi\|_{\mathbf{L}^2}, \tag{2.3.1}$$
$$\|\mathcal{G}(t)\phi\|_{\mathbf{L}^2} \leqslant Ct^{\gamma}\|\phi\|_{\mathbf{H}^{\varepsilon}} \tag{2.3.2}$$
$$\|\mathcal{G}(t)\phi\|_{\mathbf{L}^2} \leqslant Ct^{-\frac{1}{\beta}}\|\phi\|_{\mathbf{L}^1} \tag{2.3.3}$$
$$\|\mathcal{G}(t)\phi\|_{\dot{\mathbf{H}}^s} \leqslant t^{-\frac{1}{\beta}(1+2s\pm\gamma)}\|\phi\|_{\mathbf{L}^1}, \tag{2.3.4}$$
$$\|\mathcal{G}(t)\phi\|_{\dot{\mathbf{H}}^s} \leqslant Ct^{-\frac{1}{\beta}(\frac{1}{2}+2s\pm\gamma)}\|\phi\|_{\mathbf{L}^2}, \tag{2.3.5}$$
$$\|\mathcal{G}(t)\phi\|_{\dot{\mathbf{H}}^s} \leqslant Ct^{-\frac{2(s-\varepsilon)+\gamma}{\beta}}\|\phi\|_{\mathbf{H}^{\varepsilon}} \tag{2.3.6}$$

for $s \in \left(0, \frac{\beta}{2}\right)$ *and* $\gamma > 0$ *is small. Moreover*

$$\lim_{x_j\to 0}\|\mathcal{G}(t)\phi\|_{H^{\varepsilon}_{\mathbf{x}_i}} \leqslant Ct^{-\frac{1}{\beta}(\frac{1}{2}+\varepsilon+\delta\pm\gamma)}\left\|\widehat{\phi}(\mathbf{p})\right\|_{\mathbf{L}^2}\lim_{x_j\to 0}x_j^{\delta},$$
$$\lim_{x_j\to 0}\|\mathcal{G}(t)\phi\|_{H^{\varepsilon}_{\mathbf{x}_i}} \leqslant Ct^{-\frac{1}{\beta}(1-\delta\pm\gamma)}\left\|\widehat{\phi}(\mathbf{p})\right\|_{\mathbf{L}^2}\lim_{x_j\to 0}x_j^{\delta}$$

are fulfilled if the right-hand sides are bounded for $t > 0$ *and* $\delta < \frac{1}{2}$.

Proof. We use Lemma 2.2.1 with $\phi = e^{\operatorname{Re}K(\mathbf{p})t}\widehat{\phi}(\mathbf{p})$ to get

$$\|\mathcal{G}(t)\phi\|_{\dot{\mathbf{H}}^s} \leqslant C\left\|e^{\operatorname{Re}K(\mathbf{p})t}\widehat{\phi}(\mathbf{p})\right\|_{\dot{\mathbf{H}}^{s\pm\gamma}} \leqslant t^{-\frac{1}{\beta}(1+2s\pm\gamma)}\sup_{\mathbf{p}}\left|\widehat{\phi}(\mathbf{p})\right|,$$
$$\|\mathcal{G}(t)\phi\|_{\dot{\mathbf{H}}^s} \leqslant Ct^{-\frac{2s\pm\gamma}{\beta}}\|\phi\|_{\dot{\mathbf{H}}^{s\pm\gamma}}$$

for $s \in \left(0, \frac{\beta}{2}\right)$. Now we consider $\|\mathcal{G}(t)\phi\|_{\mathbf{L}^2}$. We rewrite $\mathcal{G}(t)\phi$ as the sum

$$\mathcal{G}(t)\phi = \left(\frac{1}{2\pi}\right)^2\sum_j\int_{\mathscr{C}_j}d\mathbf{p}\; e^{K(\mathbf{p})t}\widehat{\phi}(\mathbf{p})\Phi(\mathbf{p},\mathbf{x})$$
$$= I_1 + I_2 + I_3.$$

where

$$\mathscr{C}_1 = \left\{\mathbf{p} \in i\mathbb{R}^2 : \; |\mathbf{p}| > 1\right\}$$
$$\mathscr{C}_2 = \bigcup_j\left\{\mathbf{p} \in i\mathbb{R}^2 : \; |p_i| > 1, \; |p_j| < 1\right\}$$
$$\mathscr{C}_3 = \left\{\mathbf{p} \in i\mathbb{R}^2 : \; |\mathbf{p}| < 1\right\}$$

For first term I_1 we use (2.2.5) to get

$$\|I_1\|_{\mathbf{L}^2} \leqslant C\int_{\{\mathbf{p}\in i\mathbb{R}^2:\;|\mathbf{p}|>1\}}d\mathbf{p}\; e^{\operatorname{Re}K(\mathbf{p})t}\left|\widehat{\phi}(\mathbf{p})\right||p_i|^{-\frac{1}{2}}|p_j|^{-\frac{1}{2}} \leqslant t^{-\gamma}\|\phi\|_{\mathbf{L}^2}$$

Now we consider I_2. Denote

$$J = \int_{\{i\mathbb{R}:\;|p_i|>1\}}dp_i\; e^{\operatorname{Re}K(p_i)t}\Phi_i(p_i,x_i)\int_{-i}^{i}dp_j\;\widehat{\phi}(\mathbf{p})\Phi_j(p_j,x_j).$$

Since via (2.2.4) $\mathscr{L}\Phi_j(p_j,q_j) = \dfrac{\beta p_j}{\left(q_j^\beta - p_j^\beta\right) q_j^\alpha}$, we get

$$\mathscr{L}J = \int_{\{i\mathbb{R}:\ |p_i|>1\}} dp_i\ e^{\operatorname{Re} K(p_i)t} \frac{\beta p_i}{\left(q_i^\beta - p_i^\beta\right) q_i^\alpha} \times \int_{-i}^{i} dp_j\ \widehat{\phi}(\mathbf{p}) \frac{\beta p_j}{\left(q_j^\beta - p_j^\beta\right) q_j^\alpha},$$

where using $\widehat{\phi}(\mathbf{p})$ is analytic in $\operatorname{Re} p_j > 0$ changing the countour of integragion by

$$\left\{ p_j = e^{i\eta} :\ \eta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}$$

and taking residio in the point $p_j = q_j$ we obtain

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{-i}^{i} dp_j\ \widehat{\phi}(\mathbf{p}) \frac{\beta p_j}{\left(q_j^\beta - p_j^\beta\right) q_j^\alpha} \\
&= \widehat{\phi}(\mathbf{p})\left|_{p_j=q_j} + \frac{1}{2\pi i}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\eta \left(\widehat{\phi}(\mathbf{p}) \frac{\beta p_j}{\left(q_j^\beta - p_j^\beta\right) q_j^\alpha}\right)\right|_{p_j=e^{i\eta}}
\end{aligned}$$

Therefore using (2.2.5) we get

$$\begin{aligned}
\|J\|_{\mathbf{L}^2} &= \|\mathscr{L}J\|_{\mathbf{L}^2} \\
&\leqslant C \int_{\{i\mathbb{R}:\ |p_i|>1\}} e^{\operatorname{Re} K(p_i)t} |p_i|^{-\frac{1}{2}} \left\|\widehat{\phi}(\mathbf{p})\right\|_{\mathbf{L}^2_{\mathbf{X}_j}} dp_i \leqslant C t^{-\gamma} \left\|\widehat{\phi}(\mathbf{p})\right\|_{\mathbf{L}^2}
\end{aligned}$$

Here we use

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\eta \left|\widehat{\phi}(\mathbf{p})\right| \left|e^{i\eta}\right|^{-\frac{1}{2}} \Big|_{p_j=e^{i\eta}} \leqslant C \left\|\widehat{\phi}(\mathbf{p})\right\|_{\mathbf{L}^2_{\mathbf{X}_j}}.$$

Therefore we get

$$\|I_2\|_{\mathbf{L}^2} \leqslant \int_{\mathscr{C}_2} d\mathbf{p}\ e^{\operatorname{Re} K(p_i)t} \left|e^{K(p_j)t} - 1\right| \left|\widehat{\phi}(\mathbf{p})\right| + \frac{C}{t^\gamma} \left\|\widehat{\phi}(\mathbf{p})\right\|_{\mathbf{L}^2} \leqslant \frac{C}{\{t\}^\gamma} \left\|\widehat{\phi}(\mathbf{p})\right\|_{\mathbf{L}^2}$$

By the same way we can prove

$$\|I_3\|_{\mathbf{L}^2} \leqslant C\{t\}^{-\gamma} \left\|\widehat{\phi}(\mathbf{p})\right\|_{\mathbf{L}^2}.$$

Thus we get

$$\|\mathcal{G}(t)\phi\|_{\mathbf{L}^2} \leqslant C\{t\}^{-\gamma} \left\|\widehat{\phi}(\mathbf{p})\right\|_{\mathbf{L}^2}.$$

On another hand to estimate $\mathcal{G}\phi$ in $\mathbf{L}^2$-norm, we use the Jordan's Lemma to get

$$\mathcal{G}\phi = \mathbb{I}\left(e^{K(\mathbf{p})t} - 1\right)\widehat{\phi} + \widehat{\phi}$$

Also we have

$$\int_{i\mathbb{R}^2} d\mathbf{p} \left|e^{K(\mathbf{p})t} - 1\right| \left|\widehat{\phi}(\mathbf{p})\right| p_i^{-\frac{1}{2}} p_j^{-\frac{1}{2}} \leqslant t^\gamma \|\phi\|_{\mathbf{H}^\varepsilon}, \varepsilon > 0 \text{ is a small.}$$

Therefore

$$\|\mathcal{G}\phi\|_{\mathbf{L}^2} \leqslant Ct^{\gamma}\,\|\phi\|_{\mathbf{H}^{\varepsilon}}\,, \quad \gamma > 0 \text{ is small .}$$

Using $e^{K(\mathbf{p})t}\widehat{\phi}(\mathbf{p}) = t^{-\gamma}O(\mathbf{p}^{-1-\gamma\beta})$ and

$$\int_{i\mathbb{R}^2} d\mathbf{p}\; e^{\operatorname{Re}K(\mathbf{p})t}\left|\widehat{\phi}(\mathbf{p})\right| |p_i|^{\varepsilon-\frac{1}{2}}\,|p_j|^{\delta} \leqslant Ct^{-\frac{1}{\beta}(\frac{1}{2}+\varepsilon+\delta\pm\gamma)}\left\|\widehat{\phi}(\mathbf{p})\right\|_{\mathbf{L}^2};$$
$$\int_{i\mathbb{R}^2} d\mathbf{p}\; e^{\operatorname{Re}K(\mathbf{p})t}\left|\widehat{\phi}(\mathbf{p})\right| |p_i|^{\varepsilon-\frac{1}{2}}\,|p_j|^{\delta} \leqslant Ct^{-\frac{1}{\beta}(\frac{1}{2}+\varepsilon-\delta\pm\gamma)}\left\|\widehat{\phi}(\mathbf{p})\right\|_{\mathbf{H}^{\delta\pm\gamma}},$$
$$\int_{i\mathbb{R}^2} d\mathbf{p}\; e^{\operatorname{Re}K(\mathbf{p})t}\left|\widehat{\phi}(\mathbf{p})\right| |p_j|^{\delta} \leqslant Ct^{-\frac{1}{\beta}(1-\delta\pm\gamma)}\left\|\widehat{\phi}(\mathbf{p})\right\|_{\mathbf{H}^{\delta\pm\gamma}}$$

by virtue of Lemma 2.2.1 we have

$$\lim_{x_j\to 0}\left\|\mathcal{G}(t)\phi - \mathcal{G}(t)\phi\,\Big|_{x_j=0}\right\|_{H^{\varepsilon}_{\mathbf{X}_i}} \leqslant Ct^{-\frac{1}{\beta}(\frac{1}{2}+\varepsilon+\delta\pm\gamma)}\left\|\widehat{\phi}(\mathbf{p})\right\|_{\mathbf{L}^2}\lim_{x_j\to 0} x_j^{\delta},$$
$$\lim_{x_j\to 0}\left\|\mathcal{G}(t)\phi - \mathcal{G}(t)\phi\,\Big|_{x_j=0}\right\|_{H^{\varepsilon}_{\mathbf{X}_i}} \leqslant Ct^{-\frac{1}{\beta}(1-\delta\pm\gamma)}\left\|\widehat{\phi}(\mathbf{p})\right\|_{\mathbf{L}^2}\lim_{x_j\to 0} x_j^{\delta}$$

for $t > 0$, $\varepsilon < \frac{1}{2}$ and $\delta < \frac{1}{2}$. Lemma 2.3.1 is proved. ■

Chapter 3

Proof of the local existence of solutions theorem

We will conclude this thesis with the proof of the Theorem 1.3.1.

Proof. (Theorem 1.3.1) Via Proposition 2.1.1, we know that

$$u(t,\mathbf{x}) = \mathcal{G}(t)u_0 + \mathcal{H}(t)\mathbf{h},$$

since $\mathbf{h} = 0$, then

$$u = u(t,\mathbf{x}) = \mathcal{G}(t)u_0.$$

From the inequalities (2.3.2) and (2.3.6) of the Lemma 2.3.1, we have

$$\begin{aligned} \|u\|_{\mathbf{L}^2} &= \|\mathcal{G}(t)u_0\|_{\mathbf{L}^2} \leqslant Ct^{\gamma}\|u_0\|_{\mathbf{H}^{\varepsilon}}, \\ \|u\|_{\dot{\mathbf{H}}^s} &= \|\mathcal{G}(t)u_0\|_{\dot{\mathbf{H}}^s} \leqslant Ct^{-\frac{2(s-\varepsilon)+\gamma}{\beta}}\|u_0\|_{\mathbf{H}^{\varepsilon}}. \end{aligned}$$

where $\gamma > 0$ is small enough. Therefore,

$$\|u\|_{\mathbf{H}^s} = \|u\|_{\mathbf{L}^2} + \|u\|_{\dot{\mathbf{H}}^s} \leqslant C(t^{\gamma} + t^{-\frac{2(s-\varepsilon)+\gamma}{\beta}})\|u_0\|_{\mathbf{H}^{\varepsilon}}$$

with $s \in \left(0, \frac{\beta}{2}\right)$. We conclude that there exists $T^* > 0$ such that for all $0 < T < T^*$ the problem (0.0.1) has a unique local solution

$$u\left(t\right) \in \mathbf{C}^0\left(\left[0,T\right];\mathbf{L}^2\right)\bigcap \mathbf{C}^0\left(\left(0,T\right];\mathbf{H}^s\right).$$

Also, we have

$$\begin{aligned} \|u - u_0\|_{\mathbf{L}^2} = \|\mathcal{G}(t)u_0 - u_0\|_{\mathbf{L}^2} &= \left\|\mathbb{I}\{e^{\operatorname{Re}K(\mathbf{p})t}\widehat{u}_0\} - \mathbb{I}\{\widehat{u}_0\}\right\|_{\mathbf{L}^2} \\ &= \left\|\mathbb{I}\{(e^{\operatorname{Re}K(\mathbf{p})t} - 1)\widehat{u}_0\}\right\|_{\mathbf{L}^2} \leqslant Ct^{\gamma}\|u_0\|_{\mathbf{H}^{\varepsilon}}. \end{aligned}$$

Then, using the Dominated Convergence Theorem

$$\lim_{t\to 0}\|u - u_0\|_{\mathbf{L}^2} = \|\lim_{t\to 0}u(t,\mathbf{x}) - u_0\|_{\mathbf{L}^2} = \|u_0 - u_0\|_{\mathbf{L}^2} = 0.$$

So, the Theorem 1.3.1 is proved.

■

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