Circular Integration in The Worldline Formalism.

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To entropy, and to the memory of Ampelio Mata Aguila. A farmer whose efforts in the fields created opportunities for new journeys in life.

Contents

Resumen

Este trabajo explora el formalismo de la línea de mundo como un enfoque alternativo a los cálculos estándar de diagramas de Feynman en teoría cuántica de campos, particularmente en electrodinámica cuántica (QED). Tras una revisión histórica de los primeros desarrollos en QED, incluyendo la birrefringencia del vacío, la producción de pares y la dispersión fotón-fotón, esta tesis introduce el marco matemático del formalismo de la línea de mundo para calcular amplitudes de N-fotones a un lazo tanto en QED escalar como espinorial. Como se demuestra en esta investigación, una vez establecidos los cálculos para QED escalar, la extensión al caso espinorial resulta notablemente sencilla. Para lectores no familiarizados con este formalismo, destacamos su capacidad para preservar la invariancia gauge y manifestar simetrías mediante integración por partes.
En esta tesis, exploramos técnicas novedosas de integración para abordar integrales complejas sobre el lazo. Mostramos cómo estas integrales encapsulan diagramas de Feynman, verificando nuestros resultados al compararlos con los obtenidos mediante otros formalismos. Entre estas nuevas técnicas, presentamos un método innovador que destaca por evitar la necesidad de dividir las integrales de línea de mundo en sectores. Esta división por sectores dentro del formalismo de la línea de mundo es análoga a la proliferación de diagramas de Feynman en los cálculos dentro de la teoría cuántica de campos; nuestro avance se logra mediante una expansión en derivadas inversas del integrando de la línea de mundo, que incluye polinomios de Hermite y números de Bernoulli.
Finalmente, estudiamos exhaustivamente el caso de cuatro fotones, presentando cálculos detallados para QED escalar y espinorial, considerando todas las configuraciones posibles usando helicidad espinorial, y analizando el límite de baja energía. Los resultados aquí presentados no solo avanzan las metodologías de línea de mundo para procesos en QED, sino que también establecen una base para futuras aplicaciones en escenarios teóricos más complejos.

Palabras clave: Electrodinámica Cuántica, amplitudes de fotones, un lazo, integrales de camino, números de Bernoulli .

Abstract

This work explores the worldline formalism as an alternative approach to standard Feynman diagram calculations in quantum field theory, particularly in quantum electrodynamics (QED). Following a historical overview of early developments in QED, including vacuum birefringence, pair production, and photon-photon scattering, this thesis introduces the mathematical framework of the worldline for calculating N-photon amplitudes at one-loop order for systems described by either scalar or spinor QED. As demonstrated in this research, once the calculations are established for scalar QED, extending to the spinor case becomes remarkably straightforward. For readers unfamiliar with this formalism, we highlight its capacity to maintain gauge invariance and manifest symmetries through integration by parts.

In this thesis, we explore novel integration techniques to tackle challenging integrals over the loop. We show how these integrals encapsulate Feynman diagrams, verifying our results by comparing them with those obtained using other formalisms. Among these new techniques, we present an innovative method that stands out for avoiding the need to partition worldline integrals into sectors. This sector division within the worldline formalism is analogous to the proliferation of Feynman diagrams in calculations within quantum field theory; our advancement is achieved through an inverse derivative expansion of the worldline integrand, which includes Hermite polynomials and Bernoulli numbers.

Finally, we comprehensively study the four-photon case, presenting detailed calculations for scalar and spinor QED, considering all possible configurations using spinor helicity, and analyzing the low-energy limit. The results presented here not only advance worldline methodologies for QED processes but also establish a foundation for future applications in more complex theoretical scenarios.

Key words: Quantum Electrodynamics, photon amplitudes, one-loop, path integrals, Bernoulli numbers.

Introduction

The Lagrangian[*] is a fundamental concept and has wide applications in various fields of physics. In classical mechanics, the equations of motion and the symmetries of systems can be obtained from the Lagrangian. The function of the Lagrangian in quantum field theory QFT is to specify the dynamics of the fields, different types of fields, such as scalar,vector, and spinor fields, are introduced to represent various particles and their properties. The Lagrangian in QFT typically includes terms representing the kinetic and potential energies, and the interactions are often represented by terms involving derivatives of the fields. In many cases, fundamental theories, such as the Standard Model of particle physics, can be highly intricate and involve a large number of particles and interactions. Calculating and understanding the behavior of such systems in all possible energy regimes is computationally and conceptually challenging. However, the concept of effective Lagrangian allows you to study physical phenomena for specific energy scales or in certain conditions where the full complexity of the underlying theory is not necessary or is difficult to handle.
This work is primarily associated with the study of a master formula designed to compute one-loop N-point amplitudes in Quantum Electrodynamics (QED)[†]. This master formula, derived from the worldline formalism, offers an efficient method for computing one-loop N-point amplitudes in scalar and spinor QED. The underlying calculation involves a summation over all possible orderings of the external photon legs attached to the loop.
At the classical level, electrodynamics is guided by Maxwell's equations, which maintain linearity in the fields. However, upon delving into quantum effects, nonlinear corrections to Maxwell's theory come to the forefront. Upon considering these nonlinear corrections, a diverse array of captivating effects emerges.
These nonlinear interactions include the process of “vacuum polarization” or “self-energy correction”. Vacuum polarization involves the creation of virtual electron-positron pairs in the presence of an external electromagnetic field, which then interact with the original field, leading to corrections in the behavior of charged particles and

[*] The Italian physicist Joseph-Louis Lagrange played a crucial role in establishing the foundation of analytical mechanics with his publication ”Mécanique Analytique” in 1788.

[†] QED distinguishes itself as among the most successful quantum field theories to date.

modifying the properties of the electromagnetic field itself.
Although the concept of virtual particles can be contentious, the phenomenon of vacuum polarization is a significant aspect of the quantum behavior of electromagnetic fields and particles. It has been extensively studied and calculated in various theoretical and experimental contexts [1][2, 3, 4]. One notable example that greatly enhanced the reputation of QED is the experimental measurement and theoretical prediction of the electron's anomalous magnetic moment [5, 6].
The progression that led theoretical physics to the notion that virtual particles could induce vacuum polarization starts with Paul Dirac who presented a demonstration in 1928 that highlighted the inadequacy of previous theories in describing the atomic structure with electrons treated as point-like charged particles [7]. This insufficiency stemmed from the fact that existing models exhibited discrepancies both with the theory of relativity and with the theory of transformations in quantum mechanics. In his paper, Dirac began by considering a relativistic Hamiltonian approach and attempted to construct the wave equation similarly to how it was done in non-relativistic quantum mechanics. However, he encountered two difficulties in carrying out this process. The first difficulty is related to the interpretation of the wave function. The theory aimed to answer the question: "What is the probability that any dynamic variable at a specific time has a value between certain limits, when the system is represented by a given wave function?" This question can be answered regarding the electron's position but not concerning other dynamic variables. The second difficulty, in my opinion, marks a crucial point in theoretical physics. Dirac realized that the complex conjugate part of the wave equation is equivalent to replacing -e with e in the equation. Additionally, the existence of solutions with negative energy was observed. At the classical level, such solutions are arbitrarily discarded; however, in the quantum context, a perturbation can lead from a state with positive energy to one with negative energy. This necessitates the conclusion that the wave equation in relativistic quantum theory must be structured in such a way that its solutions are divided into two distinct and non-combinable sets, corresponding to the charges -e and e. A key reinterpretation here is that solutions to the Dirac equation with negative energy can be viewed as equivalent to solutions with positive energy but with the opposite charge. This perspective shifts our understanding of negative energy solutions, not as anomalies to be discarded, but as manifestations of particles with opposite charges within the framework of quantum theory.
Shortly thereafter, Dirac proposed the existence of a new particle with the same mass as the electron but with opposite charge [8]. This is how the idea of the positron was introduced, which was later experimentally verified [9].
After Dirac's work, Heisenberg began in 1934 to provide a formal treatment of quantum vacuum fluctuations in QED. In 1936, together with his PhD student Euler, they defined the concept of critical field [10]. Although the probability of pair creation persists for small amplitudes, this probability experiences exponential suppression.

This concept refers to a specific strength[‡] of an external electromagnetic field where quantum vacuum effects become significantly prominent and non-linear. At this critical field strength, the interaction between virtual electron-positron pairs and the external field becomes substantial enough to induce observable and measurable effects, such as changes in particle properties and interactions. The production of an electron-positron pair is known as Sauter-Schwinger pair production, as Sauter first explored the concept in 1931 [11], and Schwinger further developed it in 1951 [12].
Another nonlinear interaction studied in QED is the "photon-photon scattering" process, where two photons interact and can potentially produce electron-positron pairs. This process occurs at higher orders of the electromagnetic interaction and involves complex calculations and theoretical considerations due to the inherent challenges of dealing with photon-photon interactions. In 1951 [13], R. Karplus and M. Neuman were the first to calculate the differential cross-section for this scattering.
Other direct consequences of the existence of virtual electron-positron pairs include vacuum birefringence and Delbrück scattering. Vacuum birefringence refers to the phenomenon where the vacuum, as described in QED, behaves like a birefringent medium in the presence of a strong magnetic field, causing the polarization of light to change as it travels through the field. See [14] for comprehensive research that delves into exploring the measurement of vacuum birefringence, utilizing solely an x-ray free-electron laser (XFEL).
Delbrück scattering describes the elastic scattering of a photon from a nucleus's electromagnetic field, a process mediated by the transient creation and subsequent annihilation in the vacuum of virtual electron-positron pairs. An experimental account of this effect can be found in [15].
These nonlinear interactions are essential for understanding the intricate quantum behavior of charged particles and electromagnetic fields and are subjects of ongoing research in the field of theoretical physics, particularly through the use of Feynman diagrams, which are a useful way of expressing perturbation theory.
Feynman, a leading physicist in the latter half of the 20th century, achieved a breakthrough in depicting particle interactions in spacetime through diagrams. These diagrams consist of propagators, vertices, and loops. The lines can take different forms depending on the particle they describe. For example, as seen in the diagrams in Figure 2, photons are represented by wavy lines. The vertices denote the points of interaction between particles, the propagators depict the temporal evolution of a particle from one vertex to another, and the loops represent virtual particles that arise as quantum corrections in certain processes.
In QFT, this tool is known as Feynman diagrams. These diagrams provided an intuitive and powerful way to visualize and calculate quantum processes, greatly fa-

[‡]The critical field strength is $\mathcal{E}_{cr} = \frac{m^2c^3}{|e|\hbar} \approx 1.3 \times 10^{16}$ V/cm, is inaccessible in experiments but may occur near compact astrophysical objects or in early-universe conditions.

cilitating the understanding of the complexities of the Quantum world, for a more accessible exposition of quantum processes employing Feynman diagrams, please consult reference [16]. For a more academic introduction, consider reviewing Chapter 10 of Srednicki's book [17].
Experimentally, the interaction of particles is described through scattering processes, which mathematically involves assigning a function known as the wave function to describe the state of the field, imposing that it satisfies the Schrödinger equation.
For a more detailed description of the scattering processes, consult Chapter 4 in Peskin-Schroeder's work [18]. In a broader context, the theorization of these scattering processes relies on the assumption that at an initial time t_i, your system is described by $\psi(t_i)$.To determine its state after scattering at a time t_f we introduce a unitary operator S called the scattering matrix:

$$\psi(t_f) = S\psi(t_i) \tag{0.1}$$

An approach to investigating this problem using axioms (such as Lorentz invariance, the vacuum state, and creation and annihilation operators) involves the application of perturbation theory. Within this framework, when the scattering process does not modify the state of the particles, we can postulate that the elements of the S-matrix are composed of the identity matrix plus components arising from potential interactions. These components are represented in the form of Feynman diagrams with a specific number of external lines.
In standard QED calculations, one typically begins by constructing all possible contributing Feynman diagrams and then applying the corresponding Feynman rules. This approach, however, can become computationally intensive, especially for processes requiring higher-order corrections. The anomalous magnetic moment (g-2) serves as a paradigmatic example to demonstrate the computational superiority of the worldline formalism relative to conventional perturbative approaches in quantum field theory. At the one-loop level, both methodologies yield identical results through equivalent single-diagram contributions. However, their computational demands diverge substantially at higher orders: while the standard formalism necessitates evaluation of 7 distinct Feynman diagrams at two-loop order, the worldline formalism accomplishes the same calculation through merely 2 worldline diagrams. See for instance, Figure 1. This disparity becomes particularly pronounced at five-loop order, where traditional methods require analysis of 12,672 Feynman diagrams compared to only 32 worldline configurations.
This work is based on the worldline formalism. In this approach, a single master formula is employed, which inherently encompasses all possible diagrams. As we will explore in Chapter 1, where the necessary tools for solving these integrals will be presented and utilized throughout subsequent chapters, tackling these types of integrals still presents a significant mathematical challenge. It is important to note that

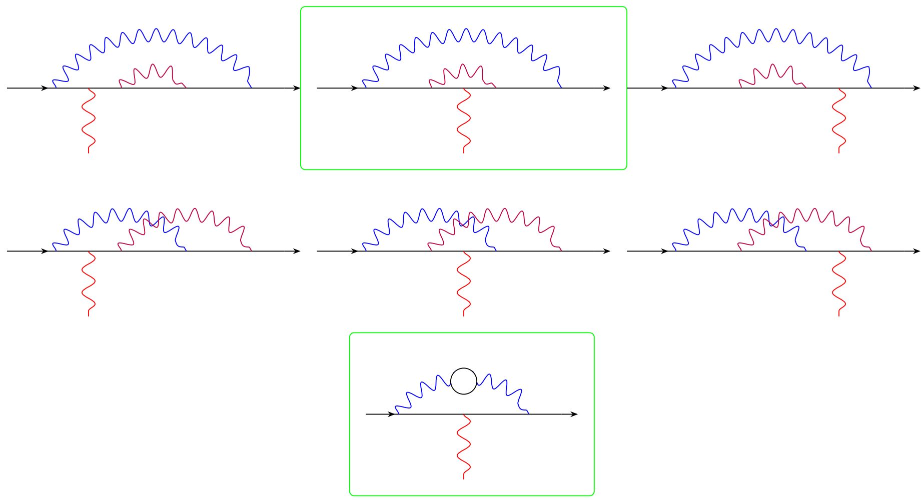

Figure 1: The seven Feynman diagrams contributing to the two-loop QED correction to the anomalous magnetic moment (g-2). The highlighted diagrams correspond to the worldline formalism topologies.

this alternative formalism, distinct from the conventional Feynman diagram method, is also referred to as "string-inspired." This nomenclature stems from the fact that its techniques bear similarities to computations in string perturbation theory, and because it is fundamentally a path integral formalism that is built upon relativistic first-quantized particle path integrals.

The development of the Worldline formalism begins with Feynman who, despite having created a diagrammatic technique for performing calculations in QFT, also introduced in the appendix A of [19] a representation of the S-matrix for scalar QED in terms of relativistic particle path integral. Furthermore, he extended this representation to the Spinor QED scenario [20].

Despite Feynman's initial work, the worldline formalism remained relatively underutilized for several decades, with Feynman diagrams becoming the dominant tool in QFT. However, it was not until the early 1990s that Bern and Kosower [21], through their work, demonstrated the potential of the worldline formalism to surpass standard field theory methods for certain computations. They derived a set of "string-inspired" Feynman rules for one-loop gluon amplitudes by considering the point-particle limit of string scattering amplitudes.

These rules significantly simplified the calculation of scattering amplitudes in gauge theories.

A pivotal moment in the development of the worldline formalism came in 1992 when

Strassler provided a purely field-theoretic derivation of the Bern-Kosower rules [22]. Strassler's work demonstrated that the worldline formalism could be a powerful tool within QFT itself, independent of string theory. This led to a resurgence of interest in the formalism and its applications in various areas of theoretical physics.

The worldline formalism has proven to be a versatile tool in quantum field theory, finding applications across a wide spectrum of problems. Its initial success in perturbative Quantum Chromodynamics (QCD) for calculating gluon amplitudes [23, 24, 25] and effective actions [26, 27] paved the way for its extension into gravity. This extension has enabled the computation of graviton scattering amplitudes [28, 29, 30, 31] and the exploration of classical gravitational bremsstrahlung and gravitational wave physics relevant to astrophysics [32, 33, 34, 35]. The formalism also provides a robust framework for multi-loop calculations [36, 37, 38] and numerical applications in non-trivial backgrounds, such as Casimir geometries [39]. Furthermore, it has been employed for non-perturbative analyses in QFT [40, 41, 42] and in QED for deriving all-order S-matrix formalisms and amplitudes in background fields [43, 44].

The present work adopts a more focused approach, concentrating on the investigation of compact master formulas for scalar and spinor QED N-photon amplitudes in vacuum, valid both on-shell and off-shell. It is important to acknowledge the significant advancements of this formalism in scenarios involving constant fields and plane-wave fields, as detailed in previous reviews and studies [45] and the references therein. Building upon our prior research [46], where we presented the numerical solution for the 4-photon dispersion amplitude, this study aims to maximize the potential of the worldline formalism. Our objective is to extend and refine our understanding of these intricate quantum interactions within this specific context.

We aim to draw the attention of the community by highlighting some advantages offered by this formalism.

The first advantage we recognize is that at the one-loop level, there is no need to solve momentum integrals, nor do we encounter the Dirac algebra for closed-loop processes. Another significant advantage lies in the ability to homogenize the integrands through integration by parts. This process respects gauge invariance and makes underlying symmetries manifest, often streamlining subsequent calculations.

At the multi-loop level, the worldline formalism offers a powerful perspective on the relationship between loop diagrams and more fundamental building blocks. Over time, our understanding of how this formalism connects to standard Feynman diagrams has deepened [47]. This connection is evident in multi-loop amplitudes, which in the Feynman diagram approach are represented by a multitude of distinct Feynman graphs. In contrast, the worldline formalism provides a more unified picture, where these multi-loop amplitudes can be represented by a single integral over closed loops in spacetime – akin to a "worldline diagram."

Furthermore, the Feynman parameter expressions, which can appear disparate for different Feynman graphs, can be reformulated as components of a singular mas-

ter expression within the worldline framework. Crucially, the one-loop amplitude formula in this formalism is valid both on-shell and off-shell. This feature allows for the derivation of multi-loop amplitudes by effectively ”sewing” pairs of photons together, offering a constructive approach to higher-order calculations. A clear illustration of this is found in chapter 6 of [48]. To calculate the one-loop correction to the graviton-scalar vertex (analogous to the gss vertex in QED), the authors first re-derive the off-shell amplitude for two-photon and one-graviton interactions and subsequently apply the sewing procedure to determine the radiative correction.
A key aspect of the worldline formalism is that the loop integrals are performed over the proper time of the virtual particles. This integration over proper time inherently introduces a proper-time ordering, which corresponds to dividing the loop integrals into different time-ordered sectors. For instance, in the four-photon scenario within the worldline formalism, 6 different ordering sectors need to be considered and summed over to obtain the complete amplitude.
In a conventional Feynman diagram calculation, these sectors would align with the six distinct spinor QED diagrams illustrated in Figure 2.

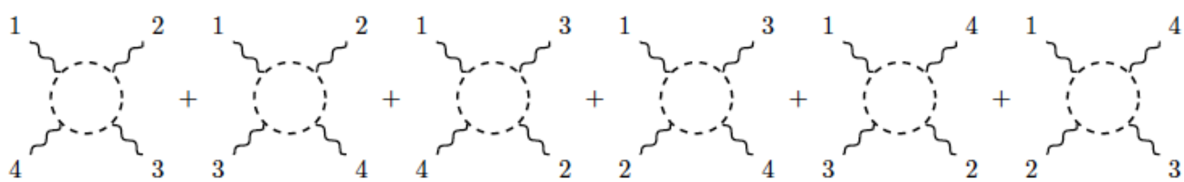

Figure 2: Feyman diagrams for photon-photon scattering.

For scalar QED, there are additional diagrams that include seagull vertices. We find this aspect of the master formula intriguing since there exist methods to resolve amplitudes without the explicit need for such a division into sectors.
Our work delves into the implications of integrating over all potential orderings of the photon leg insertion points, represented by the proper-time parameters τ_i, around the loop within the worldline formalism. This approach inherently encompasses the summation over inequivalent crossed diagrams that arises naturally in the standard Feynman diagrammatic expansion.
Concurrently, we endeavor to further develop and enhance the applicability of the worldline formalism across a broader spectrum of physical scenarios. By rigorously investigating these intrinsic advantages and actively expanding their scope, we aspire to open new avenues for exploration and contribute to a deeper understanding of quantum field theoretic phenomena.
In our analysis, we focus on the one-loop effective action, characterized by particles moving in a closed loop. This means we will be addressing integrals over the unit circle. Addressing these types of integrals presents a significant mathematical challenge

because our integrands rely on Worldline Green's functions and their first and second derivatives, which in turn explicitly depend on absolute values, sign functions, and delta functions.

A historically challenging and significant aspect of evaluating one-loop Feynman diagrams lies in their associated loop integrals, which frequently lead to divergent expressions. These infinities primarily stem from two sources: ultraviolet (UV) divergences, occurring at the high-momentum (short-distance) end of the integration range, indicating potential issues with the theory's behavior at very small scales; and infrared (IR) divergences, arising from the low-momentum (long-distance) behavior of the integrand, often associated with massless particles in the loop or the emission of soft/collinear particles.

The presence of these divergences necessitates sophisticated mathematical techniques. It's crucial to note that dimensional regularization, a standard tool for handling these divergences, is also fully applicable within the worldline formalism. This allows us to maintain consistency when addressing the infinities inherent in the theory. However, the renormalization process, essential for extracting finite, physically meaningful results from the theory, is (still) performed using the standard quantum field theory formalism. One-loop diagrams—including bubbles, triangles, and boxes—are the simplest components of quantum corrections in quantum field theories. Understanding their properties and calculation methods is an essential prerequisite for tackling the complexity of higher-loop amplitudes, effectively serving as the fundamental "building blocks" for multi-loop calculations.

There are several ways to study the circular integration problem in the worldline formalism. The first method involves decomposing our integrands completely into sectors and then summing the results for the different sectors, which is very similar to calculations using Feynman diagrams. The second approach, which we refer to as the "moderately ambitious" method, consists of fixing an order each time we perform an integral and then rewriting the result in terms of the worldline Green's functions involving the remaining integration variables. This eliminates the dependence on a specific order of the remaining variables to be integrated. The last approach, which we will call the "super ambitious" method, involves solving the integrals without ever fixing an order.

In Chapter 1, we provide an overview of some concepts underlying the integral representation of the one-loop effective action in ϕ^3 theory, scalar QED, and spinor QED. Additionally, we introduce the state of the art of our novel integration techniques.

One of the significant benefits of the worldline formulation in QED lies in its transferability between different forms of QED. Once the method for calculating N-photon amplitudes at the one-loop level is mastered in scalar QED, it becomes straightforward to adapt this knowledge to spinor QED. This is achieved simply by applying a specific replacement rule originally found by Bern-Kosower in the non-abelian context. Therefore, it is common within the worldline formalism to first resolve the scalar case, a characteristic that notably contrasts with the first-order formalism where spinor QED is typically the natural starting point, and scalar QED can introduce additional complications.

In Chapter 2, we present a new approach that takes advantage of the fact that in the fact that in the Hilbert space composed of periodic functions which are orthogonal to constant functions, the integral kernel of the inverse of the nth derivative operator in this space is associated with the nth Bernoulli polynomial. Upon integration, these polynomials naturally become Bernoulli numbers, which in turn help us to form hypergeometric functions.

In Chapter 3, we construct a novel representation for the master formula from Chapter 1, referred to as the "Gaussian linearization" method. We then address a previously identified gap concerning this formula by studying and solving the case where all photons possess the same helicity, utilizing the N-point master formula in the low-energy limit.

In Chapter 4, we introduce the most recent approach within the worldline formalism for computing the four-photon amplitudes, beginning with a ϕ^3 theory and subsequently performing the calculation in scalar and spinor QED. QED provides the foundational framework for understanding light-matter interactions, among which light-by-light (LBL) scattering mediated by virtual electron-positron loops—stands out as a remarkable prediction. The theoretical exploration of this phenomenon traces back to pioneering work by Euler and Kockel [49, 50], who first derived the low-energy cross section. Shortly thereafter, Akhiezer et al. [51] [52] tackled the opposite high-energy regime, where computational complexity grows significantly.

In 1950, Karplus and Neuman [53, 13] accomplished the first full treatment of the four-photon amplitude with arbitrary on-shell photons. Their analysis established the amplitude's gauge invariance and finiteness, expressing it via three-parameter integrals over rational functions of the external momenta. Later, De Tollis [54, 55] streamlined these results using dispersion relations, yielding a more compact representation. (For a comprehensive review of these foundational developments, see [56].) Notably, in the low-energy limit, the cross section can be derived as a textbook exercise in the center-of-mass frame (see Section 7 of [57]), where it exhibits the characteristic ω^6 dependence at low frequencies. This strong suppression has historically hindered optical-range detection [58, 59]. However, advances in high-intensity laser technology [60, 61, 62, 63, 64, 65] and novel experimental designs—such as those exploiting relativistic heavy-ion collisions [66, 67, 68] —have revitalized prospects for direct observation. A landmark achievement came in 2017 when the ATLAS collaboration measured LBL scattering at GeV energies using ultra-peripheral heavy-ion collisions at the LHC [69, 70, 71, 72, 73, 74]. These breakthroughs underscore the enduring interplay between theoretical QED and cutting-edge experimental techniques, particularly in extreme electromagnetic regimes [75].

The theoretical investigation of the four-photon amplitude becomes significantly more complex when the participating photons are"off-shell" meaning they do not satisfy the usual energy-momentum relation for free photons. The groundbreaking work initiated by Ahmadiniaz et al. in their series of papers, starting with "The QED four-photon amplitudes off-shell: part 1," tackles this challenging scenario using the worldline formalism [76]. This initial part primarily introduces the formalism and presents the calculation of the amplitude specifically in the low-energy limit for all four photon legs. The subsequent "part 2" [77] then delves into the more involved case where the amplitude is calculated off-shell but with two of the photon legs restricted to the low-energy regime. It is not until the fourth installment of this comprehensive study that the fully general kinematic case, with all four photons being off-shell and having arbitrary energies, is addressed. This systematic approach, leveraging the power of the worldline formalism to maintain crucial physical properties like gauge invariance and UV finiteness from the outset, paves the way for the exploration of various physical limits and applications, ranging from fundamental scattering processes to contributions to precision measurements.

Following the conclusions, this work includes several appendices providing supplementary information: Appendix A outlines our conventions. Appendix B is dedicated to demonstrating the inverse derivative expansion of the worldline formalism, which forms the basis of our novel method for integrating while avoiding sector division within the formalism. In Appendix C, we present a different approach to the box integral in Scalar QED, based on a generating function. Finally, Appendix D contains relevant formulas for the worldline formalism derived during this investigation, which may be useful for future research.

The theoretical framework and key findings presented in this thesis have been further developed and published in the following four articles: "New Techniques for Worldline Integration" [78]; "One-loop amplitudes in the worldline formalism" [79]; "Summing Feynman diagrams in the worldline formalism" [47] and "Worldline integration of photon amplitudes" [80]. In addition, a further article is currently in preparation "Four-photon amplitudes in Scalar and Spinor QED" [81].

Chapter 1

Mathematical Preliminaries

1.1 A master formula

In quantum mechanics, there are two mathematical approaches to describe quantum processes: the operator formalism and the path integral method. The former employs linear operators that act in a Hilbert space, while the latter utilizes integrals over a space of functions. It is important to understand both approaches since the choice between them may depend on the specific problem at hand. In this context, we will use the latter approach i.e the integral representation of the effective action at the one-loop level for a Maxwell field. The fundamental concept behind path integrals is to consider all possible trajectories or paths that a particle can take in its journey from one point to another in spacetime and assign a partial amplitude to each path. Additionally, Feynman proposes that these partial amplitudes have the same absolute value but vary in phase[20]. Consequently, the total amplitude A is determined by the equation:

$$A = \sum_n \mathrm{e}^{\frac{i}{\hbar}S(\gamma_n)} \equiv \int \mathcal{D}x(t)\mathrm{e}^{\frac{i}{\hbar}S[x(t)]}, \tag{1.1}$$

where $S(\gamma_n)$ denotes the action of the particle as calculated along the path γ_n. For the remainder of this work, we will adopt natural units ($\bar{h} = 1$), as established in Appendix A.

The final expression indicates that the summation is essentially an integral across the space of paths.

Let's consider, for example, the transition amplitude in non-relativistic quantum mechanics for a time-independent Hamiltonian:

$$K(x', x; t) = \langle x'|e^{-it\mathbb{H}}|x\rangle = \langle x', t \mid x, 0\rangle \tag{1.2}$$

$$K(x', x; 0) = \delta(x' - x) \tag{1.3}$$

Transition, in this context, refers to the movement of our quantum system. At the initial time $t = 0$, it occupies position x, and at a subsequent time $t = t$, it will have shifted to position x' thanks to the time evolution operator e^{-iHt}. This conceptual framework was instrumental in Feynman's derivation of the one-loop effective action for scalar QED, introduced in the appendix of [19]. Foundational treatments of the path integral in quantum mechanics are provided by [82], with a more exhaustive resource covering various contexts offered by [83]. For a specialized discussion within the worldline formalism, see [84]. In this formalism, the one-loop action is expressed as:

$$\Gamma_{scalar}[A] = \int_0^\infty \frac{dT}{T} e^{-Tm^2} \int_{x(0)=x(T)} \mathcal{D}x e^{-\int_0^T d\tau (\frac{1}{4}\dot{x}^2 + ieA_\mu \dot{x}^\mu)}. \tag{1.4}$$

In this worldline representation, several components have a deep physical meaning. The integral over the proper time T, ranging from 0 to ∞, arises from the Schwinger parameterization of the particle's propagator. This integration effectively corresponds to a sum over the contributions from all possible loop durations; short times ($T \to 0$) probe the high-energy (UV) behavior, while long times probe the low-energy (IR) aspects of the theory. The parameter m is the rest mass of the scalar particle, and its role in the exponent, $e^{-m^2 T}$, is to provide a natural infrared cutoff, exponentially suppressing the contributions from infinitely long loops.

The entire expression is then averaged over all possible closed spacetime loops $x(\tau)$ via the functional path integral $\int \mathcal{D}x$. This is the mathematical implementation of Feynman's "sum over histories" principle. By integrating over every conceivable trajectory, we account for all quantum fluctuations of the particle.

The integrand's most important interactive component is the Wilson loop factor, $\exp\left(ie\int_0^T d\tau\, \dot{x}^\mu A_\mu\right)$. This gauge-invariant phase measures the total Aharonov-Bohm-like phase shift accumulated by the charged particle as it traverses a *specific* closed path through the background potential A_μ. By averaging this phase factor over the ensemble of all possible paths, the path integral calculates the full one-loop quantum response of the vacuum to the presence of the external field, effectively "probing" the field's configuration along every possible trajectory.

Our objective is to investigate N photon amplitudes by setting the background field A_μ as a combination of N plane waves of define momentum k_i and polarization ε_i that represent their asymptotic states.

$$A^\mu(x(\tau)) = \sum_{i=1}^{N} \varepsilon_i^\mu e^{ik_i \cdot x(\tau)}\,. \tag{1.5}$$

Once the background is fixed, we can expand the exponential part that contains the interaction term:

$$e^{-\int_0^T d\tau ieA_\mu \dot{x}^\mu} = \sum_{n=0}^{\infty} \frac{(-ie)^n}{n!} \left(\int_0^T d\tau \sum_{i=1}^{N} \varepsilon_i \cdot \dot{x}(\tau) e^{ik_i \cdot x(\tau)} \right)^n \tag{1.6}$$

Each term in this series corresponds to a distinct physical process. The specific term relevant for an N-photon interaction is for $n = N$, as higher-order terms correspond to processes with additional photons and lower-order terms vanish when taking functional derivatives with respect to the external fields. When expanding the term inside the parenthesis to the N-th power, we select the part containing each distinct photon k_j, ε_j once. The $N!$ possible permutations of these photons cancel the $\frac{1}{N!}$ factor in front.

Therefore, taking into account the considerations discussed in the preceding paragraphs regarding $\mathrm{e}^{-\int_0^T d\tau \left(\frac{\dot{x}^2}{4} + ieA_\mu \dot{x}^\mu\right)}$, we can extract:

$$(-ie)^N \mathrm{e}^{-\int_0^T d\tau \frac{\dot{x}^2}{4}} \int_0^T d\tau_1 \varepsilon_1 \cdot \dot{x}(\tau_1) e^{ik_1 \cdot x(\tau_1)} \cdots \int_0^T d\tau_N \varepsilon_N \cdot \dot{x}(\tau_N) e^{ik_N \cdot x(\tau_N)} \tag{1.7}$$

In this formalism, the interaction of a photon with the particle loop is described by the following vertex operator:

$$V_{scal}[k_i, \varepsilon_i] = \int_0^T d\tau_i \varepsilon_i \cdot \dot{x}(\tau_i) e^{ik_i \cdot x(\tau_i)} \tag{1.8}$$

Once we have introduced the vertex, we can write:

$$\begin{aligned} \Gamma_{\mathrm{scal}}(k_1, \varepsilon_1; \ldots; k_N, \varepsilon_N) \quad = \quad & (-ie)^N \int_0^\infty \frac{dT}{T} \mathrm{e}^{-m^2 T} \int_{x(0)=x(T)} \mathcal{D}x \mathrm{e}^{-\int_0^T d\tau \frac{1}{4} \dot{x}^2} \\ & \times V_{scal}[k_1, \varepsilon_1] \cdots V_{scal}[k_N, \varepsilon_N] \end{aligned} \tag{1.9}$$

All mathematical manipulations and results presented in this work stem from the Gaussian form of the path integral, allowing us to express it in terms of Green's function, its first and second derivatives. For more details on the construction of the Green function, please refer to [46] and a full derivation of the following "Bern-Kosower master formula" can be found in Appendix C of [85]. The one-loop N-photon amplitude for scalar QED is given by the following expression.

$$\begin{aligned}\Gamma_{\text{scal}}(k_1,\varepsilon_1;\ldots;k_N,\varepsilon_N) &= (-ie)^N\int_0^\infty \frac{dT}{T}(4\pi T)^{-D/2}e^{-m^2T}\prod_{i=1}^N\int_0^T d\tau_i \\ &\times\exp\left\{\sum_{i,j=1}^N\left(\frac{1}{2}G_{Bij}k_i\cdot k_j - i\dot{G}_{Bij}\varepsilon_i\cdot k_j+\frac{1}{2}\ddot{G}_{Bij}\varepsilon_i\cdot\varepsilon_j\right)\right\}\Bigg|_{\varepsilon_1\ldots\varepsilon_N}\end{aligned} \tag{1.10}$$

For readers not familiar with the wordline formalism, it is important to make some comments on the previous expression. The notation $|_{\varepsilon_1\varepsilon_2\ldots\varepsilon_N}$ signifies the necessity to carry out an expansion of the exponential. Subsequently, only those terms should be retained that exhibit a linear dependence on each polarization vector. On the other hand, the dots in the Green function indicate that the derivative operates on the first variable of said function. The Green function, along with its first and second derivatives, explicitly depends on the proper time parameters T and τ, as illustrated below

$$G(\tau_i,\tau_j) = |\tau_i-\tau_j| - \frac{(\tau_i-\tau_j)^2}{T} \tag{1.11}$$

$$\dot{G}(\tau_i,\tau_j) = \text{sgn}(\tau_i-\tau_j) - 2\frac{\tau_i-\tau_j}{T} \tag{1.12}$$

$$\ddot{G}(\tau_i,\tau_j) = 2\delta(\tau_i-\tau_j) - \frac{2}{T} \tag{1.13}$$

As is customary, we scale to the unit circle by setting $\tau_i = Tu_i$:

$$\begin{aligned} G(\tau_i,\tau_j) &\rightarrow G_{ij} = |u_i-u_j| - (u_i-u_j)^2 \\ \dot{G}(\tau_i,\tau_j) &\rightarrow \dot{G}_{ij} = \text{sgn}(u_i-u_j) - 2(u_i-u_j) \\ \ddot{G}(\tau_i,\tau_j) &\rightarrow \ddot{G}_{ij} = 2(\delta(u_i-u_j)-1)\end{aligned} \tag{1.14}$$

The above formulas hold true even in off-shell conditions.
A major advantage of the worldline formalism is its ability to smoothly transition from scalar QED to spinor QED. This is because the effective action for spinor QED has a very similar integral representation to that of scalar QED (1.4), differing only

by the addition of a global factor of -2[*] and the inclusion of a spin factor, S$pin[x, A]$

$$\Gamma_{spin}[A] = -\frac{1}{2}(-ie)^N \int_0^\infty \frac{dT}{T} e^{-Tm^2} \int_{x(0)=x(T)} \mathcal{D}x e^{-\int_0^T d\tau\left(\frac{1}{4}\dot{x}^2 + ieA_\mu \dot{x}^\mu\right)} \mathrm{S}pin[x, A]. \tag{1.15}$$

Where

$$\mathrm{S}pin[x, A] = \mathrm{tr}_\Gamma \mathcal{P} \exp\left[\frac{i}{4} e\left[\gamma^\mu, \gamma^\nu\right] \int_0^T d\tau F_{\mu\nu}(x(\tau))\right]. \tag{1.16}$$

Here, tr_Γ denotes the spin trace (or Dirac trace), an operation that sums over the internal spinor indices of gamma-matrix products to yield a Lorentz-invariant scalar. This is distinct from the standard matrix trace, tr, which is the sum of the diagonal elements of a single operator in a given vector space.

The symbol $\mathcal{P}$ denotes path-ordering, a necessity due to the non-commutativity of exponents at different proper times. Consequently, the exponential function, in general, does not adhere to the usual commutative properties.

As mentioned in the introduction, the integral representation of scalar and spinor QED was first introduced by Feynman [19, 20]. Many years later, the anticommutation properties of Grassmann variables became a valuable tool in the study of quantum relativistic theories involving spin, as demonstrated in works such as Berezin's *The Method of Second Quantization* [86], along with several other foundational studies [87, 88, 89] and beyond. Building on Fradkin's work [90], the spin factor can alternatively be expressed through a Grassmann path integral.

$$\mathrm{S}pin[x, A] \to \int \mathcal{D}\psi(\tau) \mathrm{exp}\left[-\int_0^T d\tau \left(\frac{1}{2}\psi \cdot \dot{\psi} - ie\psi^\mu F_{\mu\nu}\psi^\nu\right)\right] \tag{1.17}$$

Employing Grassmann variables, specifically by considering the anticommutation and antiperiodicity properties of the Lorentz vectors $\psi(\tau)$with respect to proper time, the ordering operator is removed, thereby yielding a standard exponential function.

[*]The negative sign accounts for the fermionic nature of the fields in Spinor QED, which dictates the behavior of particles obeying the Pauli exclusion principle. The factor of 2 is a consequence of the renormalization performed in the path integral over ψ.

In terms of the vertex operator, the effective action in spinor QED is expressed as:

$$\begin{aligned}\Gamma_{\text{spin}}(k_1,\varepsilon_1;\ldots;k_N,\varepsilon_N) \;=\;& -\frac{1}{2}(-ie)^N\int_0^\infty \frac{dT}{T}\mathrm{e}^{-m^2T}\int_{x(0)=x(T)} Dx\mathrm{e}^{-\int_0^T d\tau\frac{1}{4}\dot{x}^2}\\ &\times\int_A \mathcal{D}\psi(\tau)\mathrm{e}^{-\int_0^\infty d\tau\frac{1}{2}\psi\cdot\dot{\psi}}V^\gamma_{spin}[k_1,\varepsilon_1]\cdots V^\gamma_{spin}[k_N,\varepsilon_N]\end{aligned} \tag{1.18}$$

In this context, a spinorial vertex operator is defined, representing the process of photon emission or absorption by a spinor loop.

$$V^\gamma_{spin}[k,\varepsilon] \equiv \int_0^T d\tau\left[\varepsilon\cdot\dot{x}(\tau)+2i\varepsilon\cdot\psi(\tau)k\cdot\psi(\tau)\right]\mathrm{e}^{ik\cdot x(\tau)} \tag{1.19}$$

Moreover, as we will explore later on, to transform the parameter integrals for the amplitudes in scalar QED (1.10) into their respective expressions for spinor QED, one can employ the "cycle replacement rule", which was first observed in [21, 22]. This rule consists of replacing every closed cycle $\dot{G}_{i_1i_2}\dot{G}_{i_2i_3}\cdots\dot{G}_{i_ki_1}$ by:

$$\dot{G}_{i_1i_2}\dot{G}_{i_2i_3}\cdots\dot{G}_{i_ki_1}\to\dot{G}_{i_1i_2}\dot{G}_{i_2i_3}\cdots\dot{G}_{i_ki_1}-G_{Fi_1i_2}G_{Fi_2i_3}\cdots G_{Fi_ki_1} \tag{1.20}$$

Where G_{Fij} acts as the inverse of the first derivative operator within the Hilbert space of anti-periodic boundary conditions:

$$G_{Fij}\equiv\text{sign}(u_i-u_j). \tag{1.21}$$

1.2 Circular and Feynman integrals

By expanding the master formula (1.10), we obtain polynomials P that depend on both the first and second derivatives of the Green function. These derivatives as mentioned in the introduction explicitly involve sign and delta functions and this expansion is known as the "P-representation". In the process of integration by parts with respect to variables $\tau_1,\ldots,\tau_N$, the polynomial P is transformed into a different polynomial, denoted as Q. This transformation is crucial, especially when dealing with the absolute value introduced by the Green function within the exponential term. It is through this transformation that we obtain what is known as the "Q -representation".

$$e^{\{\}}\Big|_{\varepsilon_1\ldots\varepsilon_N}=(-i)^NP_N(\dot{G}_{Bij},\ddot{G}_{Bij})\mathrm{e}^{\frac{1}{2}\sum_{i,j=1}^NG_{Bij}k_i\cdot k_j}\longrightarrow Q_N(\dot{G}_{Bij})\mathrm{e}^{\frac{1}{2}\sum_{i,j=1}^NG_{Bij}k_i\cdot k_j}. \tag{1.22}$$

Perturbative QFT calculations are often complicated by the multitude of terms involved in different processes. Although the worldline formalism's sector decomposition mirrors many of these terms, we will demonstrate how calculations within this framework enable us to fix a single sector and explicitly solve one of the integrals. This yields a result expressed in terms of the first derivative of the Green's function at the polynomial level, a technique we refer to as partial sector decomposition. Before detailing how to completely avoid this sector decomposition, we will present a couple of examples. As a first example, we will consider the u_i integral of $\dot{G}_{i1}\dot{G}_{i2}\dot{G}_{i3}$. Despite the simplicity of this example, it serves as crucial evidence that it's important to rethink how we approach integrals, as even straightforward cases can present significant difficulties when divided by sectors.

$$\int_0^1 du_i \dot{G}_{i1}\dot{G}_{i2}\dot{G}_{i3} = \int_0^1 du_i(\mathrm{sgn}(u_i - u_1) - 2(u_i - u_1)) \cdots (\mathrm{sgn}(u_i - u_3) - 2(u_i - u_3)) \tag{1.23}$$

Out of the 6 possible sectors:

$$\begin{aligned}
&1 > u_1 > u_2 > u_3 > 0 \\
&1 > u_1 > u_3 > u_2 > 0 \\
&1 > u_2 > u_3 > u_1 > 0 \\
&1 > u_2 > u_1 > u_3 > 0 \\
&1 > u_3 > u_2 > u_1 > 0 \\
&1 > u_3 > u_1 > u_2 > 0
\end{aligned} \tag{1.24}$$

We choose $1 > u_1 > u_2 > u_3 > 0$:

$$\begin{aligned}
\int_0^1 du_i \dot{G}_{i1}\dot{G}_{i2}\dot{G}_{i3} \;=\; & \int_0^{u_3} du_i(-1 - 2u_i + 2u_1)(1 + 2u_i - 2u_2)(1 + 2u_i - 2u_3) \\
& + \int_{u_3}^{u_2} du_i(-1 - 2u_i + 2u_1)(-1 - 2u_i + 2u_2)(1 - 2u_i + 2u_3) \\
& + \int_{u_2}^{u_1} du_i(-1 - 2u_i + 2u_1)(1 - 2u_i + 2u_2)(1 - 2u_i + 2u_3) \\
& + \int_{u_1}^{1} du_i(1 - 2u_i + 2u_1)(1 - 2u_i + 2u_2)(1 - 2u_i + 2u_3)
\end{aligned} \tag{1.25}$$

After solving the integrals and grouping the terms, it is straightforward to see that:

$$\int_0^1 du_i \dot{G}_{i1}\dot{G}_{i2}\dot{G}_{i3} = -\frac{1}{6}(\dot{G}_{12}-\dot{G}_{23})(\dot{G}_{23}-\dot{G}_{31})(\dot{G}_{31}-\dot{G}_{12}) \tag{1.26}$$

A second, more sophisticated example considers the cycle-integrals that appear in the calculation of the N-photon amplitude in the low-energy limit. For the scalar and spinor cases, we have, respectively:

$$\int_0^1 du_1 \int_0^1 du_2 \cdots \int_0^1 du_N (\dot{G}_{12}\dot{G}_{23}\cdots\dot{G}_{N1}) \tag{1.27}$$

$$\int_0^1 du_1 \int_0^1 du_2 \cdots \int_0^1 du_N (\dot{G}_{12}\dot{G}_{23}\cdots\dot{G}_{N1} - G_{F12}G_{F23}\cdots G_{FN1}) \tag{1.28}$$

Our initial example provided an indication of the complexities inherent in sector decomposition. Given the presence of N integrals, the imperative to develop novel methods for solving these cycle integrals within the worldline formalism becomes apparent. As will be demonstrated in chapter 2, the new method developed in this research remarkably simplifies these types of calculations, showing that these integrals can be solved in a closed form, yielding Bernoulli numbers.

$$\int_0^1 du_1 \int_0^1 du_2 \cdots \int_0^1 du_N (\dot{G}_{12}\dot{G}_{23}\cdots\dot{G}_{N1}) = -2^N \frac{B_N}{N!} \tag{1.29}$$

Meanwhile, for the spinor case:

$$\int_0^1 du_1 \int_0^1 du_2 \cdots \int_0^1 du_N (\dot{G}_{12}\dot{G}_{23}\cdots\dot{G}_{N1} - G_{F12}G_{F23}\cdots G_{FN1}) = 2^N(2^N-2)\frac{B_N}{N!} \tag{1.30}$$

Observing how this difficulty also manifests when dealing with exponential functions involving the first derivative of the Green's function, we proceed to a more sophisticated example: a calculation performed within the worldline formalism for the incorporation of a constant external field. The efficiency of incorporating a constant external field in the worldline formalism lies in its minimal requirements: only the path-integral determinant must be modified, and generalized worldline Green's functions need to be introduced.

In the low-energy regime of scalar QED, the coefficients of the generalized Green's function can be rewritten in terms of a function H, which we define in the following. This requires selecting a Lorentz frame where both the magnetic and electric fields point in the z-direction. For more details on this modfications, refer to [26, 91, 92, 93].

$$H^B_{ij}(z) \equiv \frac{\mathrm{e}^{z\dot{G}_{ij}}}{\sinh(z)} - \frac{1}{z} \tag{1.31}$$

Reference [93] shows that function $H^B_{ij}(z)$ exhibits the notable characteristic of self-replication under folding.

$$\begin{aligned}
H^{B(2)}_{ik}(z,z') &\equiv \int_0^T d\tau_j H^B_{ij}(z) H^B_{jk}(z') = \frac{H^B_{ik}(z)}{z'-z} + \frac{H^B_{ik}(z')}{z-z'}\,, \qquad (1.32)\\
H^{B(3)}_{il}(z,z',z'') &\equiv \int_0^T d\tau_j \int_0^T d\tau_k H^B_{ij}(z) H^B_{jk}(z') H^B_{kl}(z'')\\
&= \frac{H^B_{il}(z)}{(z'-z)(z''-z)} + \frac{H^B_{il}(z')}{(z-z')(z''-z')} + \frac{H^B_{il}(z'')}{(z-z'')(z'-z'')}\,,\\
&\vdots\\
H^{B(n)}_{i_1 i_{n+1}}(z_1,\dots,z_n) &= \sum_{k=1}^{n} \frac{H^B_{i_1 i_{n+1}}(z_k)}{\prod_{l\neq k}(z_l - z_k)}\,. \qquad (1.33)
\end{aligned}$$

Remarkably, (1.33) possesses full invariance under permutations. Once more, the result is considerably more beautiful and compact than that derived from individual sectors.

The presence of products of the first derivative of worldline Green's functions makes it challenging to solve circular integrals. We've made significant progress in understanding how to integrate worldline Green's functions where the integration variable appears at most twice in the integrand (as exemplified in Equation 1.12). However, due to the nature of the master formula, it's necessary to generalize this integration to cases where the integration variable appears more than twice. For example, when the variable appears three times, Equation (1.13) is obtained:

$$\begin{aligned}\int_0^1 du H^B_{uu_i}(z_i)H^B_{uu_j}(z_j)H^B_{uu_k}(z_k) &= \frac{\sinh(z_i)\mathrm{e}^{z_j\dot{G}_{ij}+z_k\dot{G}_{ik}}}{\sinh(z_i)\sinh(z_j)\sinh(z_k)(z_i+z_j+z_k)} \\ &\quad +\{i\leftrightarrow j, i\leftrightarrow k\} \\ &\quad -\frac{\sinh(z_i)\mathrm{e}^{z_j\dot{G}_{ij}}+\sinh z_j\mathrm{e}^{z_i\dot{G}_{ji}}}{\sinh(z_i)\sinh(z_j)z_k(z_i+z_j)} \\ &\quad -\{k\leftrightarrow i, k\leftrightarrow j\}+\frac{2}{z_iz_jz_k} \\ &= \frac{H^B_{u_iu_j}(z_j)H^B_{u_iu_k}(z_k)}{(z_i+z_j+z_k)}+\sum\text{Perm.} \\ &\quad -\frac{H^B_{u_iu_j}(z_j)+H^B_{u_ju_i}(z_i)}{(z_i+z_j)}-\sum\text{Perm.}\end{aligned} \tag{1.34}$$

The foregoing examples provide compelling evidence for the imperative of reconsidering integral evaluation methodologies within the worldline formalism.
To circumvent sector decomposition in solving circular integrals, we utilize the structure of the aforementioned Hilbert space. Within the worldline formalism, the effective action for a background field (e.g., the electromagnetic field A_μ) due to a virtual particle loop is expressed as a path integral over the particle's spacetime trajectories $x^\mu(\tau)$, satisfying periodic boundary conditions $x^\mu(T) = x^\mu(0)$. The periodicity requirement reflects the loop's closure, while orthogonality to constant functions stems from the zero-mode problem intrinsic to worldline path integrals in QED effective actions. Below, we analyze the challenges posed by this zero mode and demonstrate how its removal yields a mathematically consistent framework. The action typically includes a kinetic term of the form:

$$S_{\text{kin}} \propto \int_0^T d\tau\, \frac{1}{4}(\dot{x}^\mu)^2. \tag{1.35}$$

The zero-mode problem originates from the kinetic operator $\mathcal{O} = -\frac{d^2}{d\tau^2}$ acting on periodic functions. This operator has:

- A **zero eigenvalue** with corresponding **constant eigenfunction**:

$$-\frac{d^2}{d\tau^2}(\text{const}) = 0 \tag{1.36}$$

In Gaussian path integrals, such zero modes cause divergences because spacetime translations of the entire loop (which leave the kinetic energy unchanged) produce an infinite integration volume.
The conventional solution decomposes the trajectory:

$$x^\mu(\tau) = x_0^\mu + q^\mu(\tau) \tag{1.37}$$

where $x_0^\mu = \frac{1}{T}\int_0^T d\tau\, x^\mu(\tau)$ is the center-of-mass and $q^\mu(\tau)$ represents the fluctuation variable.
After implementing the trajectory decomposition into

$$\begin{aligned}\Gamma_{\text{scal}}(k_1,\varepsilon_1;\ldots;k_N,\varepsilon_N) &= (-ie)^N \int_0^\infty \frac{dT}{T}\mathrm{e}^{-m^2T}\int_{x(0)=x(T)} \mathcal{D}x\mathrm{e}^{-\int_0^T d\tau \frac{1}{4}\dot{x}^2} \\ &\times \int_0^T d\tau_1 \varepsilon_1\cdot\dot{x}(\tau_1)e^{ik_1\cdot x(\tau_1)}\cdots\int_0^T d\tau_N\varepsilon_N\cdot\dot{x}(\tau_N)e^{ik_N\cdot x(\tau_N)},\end{aligned} \tag{1.38}$$

we observe that:

- A global energy momentum conservation factor:

$$\int_{\text{PBC}} d^D x_0 \mathrm{e}^{\sum k_i\cdot x_0} = (2\pi)^D\delta\left(\sum k_i\right) \tag{1.39}$$

- A **gaussian** integral over $q^\mu(\tau)$:

$$\int_{\text{PBC}} Dq(\tau)\mathrm{e}^{-\int_0^T d\tau\frac{1}{4}\dot{q}^2} = (4\pi T)^{-\frac{D}{2}} \tag{1.40}$$

While a full derivation is beyond the scope of this discussion, a pedagogical treatment is provided by Bastianelli and Schubert [84]. As soon as the zero-mode problem has been resolved through the decomposition $x^\mu(\tau) = x_0^\mu + q^\mu(\tau)$, the effective action can be systematically expressed in terms of worldline Green's functions. This formulation ultimately leads to the master formula (1.10). A key consequence of this Hilbert space's structure is that the nth derivative operator, ∂_P^n, becomes invertible. The integral kernel of its inverse is directly associated with the nth Bernoulli polynomial $B_n(x)$ [27]:

$$\begin{aligned}\langle u_i|\partial_P^{-n}|u_j\rangle &= -\frac{1}{n!}B_n(|u_i-u_j|)sign^n(u_i-u_j). \\ \langle u_i|\partial_P^0|u_j\rangle &= \delta(u_i-u_j)-1\end{aligned} \tag{1.41}$$

Our Green's functions in this basis have the following representation:

$$\frac{1}{2}G_{ij} = \langle u_i|\partial_P^{-2}|u_j\rangle + \frac{1}{12}. \tag{1.42}$$

$$\frac{1}{2}\dot{G}_{ij} = \langle u_i|\partial_P^{-1}|u_j\rangle. \tag{1.43}$$

$$\frac{1}{2}\ddot{G}_{ij} = \langle u_i|\partial_P^{0}|u_j\rangle = \delta(u_i - u_j) - 1 \tag{1.44}$$

Considering the polynomial level, as exemplified by our first example (1.23) and more generally, the following properties of integration within this space are crucial:

$$\int |u\rangle\langle u| = 1 \tag{1.45}$$

$$\int du_i du_j \langle u_i|\partial_P^{-n}|u_j\rangle = 0 \tag{1.46}$$

$$\langle u|\partial_P^{-2n}|u\rangle = -\frac{B_{2n}}{(2n)!} \tag{1.47}$$

$$\langle u|\partial_P^{-(2n-1)}|u\rangle = 0 \tag{1.48}$$

$$\langle u_i|\partial_P^{-n}|u_j\rangle = (-1)^n\langle u_j|\partial^{-n}|u_i\rangle \tag{1.49}$$

Using (1.43)†:

$$\int_0^1 du_i \dot{G}_{i1}\dot{G}_{i2}\dot{G}_{i3} = 2^3\int_0^1 du_i \langle u_i|\partial_P^{-1}|u_1\rangle\langle u_i|\partial_P^{-1}|u_2\rangle\langle u_i|\partial_P^{-1}|u_3\rangle \tag{1.50}$$

After integration by parts:

$$\int_0^1 du_i \dot{G}_{i1}\dot{G}_{i2}\dot{G}_{i3} = -2^3\int_0^1 du_i\Big(\langle u_i|\partial_P^{0}|u_1\rangle\langle u_i|\partial_P^{-1}|u_2\rangle + \langle u_i|\partial_P^{-1}|u_1\rangle\langle u_i|\partial_P^{0}|u_2\rangle\Big) \times\langle u_i|\partial_P^{-2}|u_3\rangle \tag{1.51}$$

†Note that according to the definition of the Green's function $i \neq 1, 2, 3$.

Now we apply (1.44)

$$\begin{aligned}\int_0^1 du_i \dot{G}_{i1}\dot{G}_{i2}\dot{G}_{i3} &= 2^3\int_0^1 du_i(1-\delta_{i1})\langle u_i|\partial_P^{-1}|u_2\rangle\langle u_i|\partial_P^{-2}|u_3\rangle \\ &\quad +2^3\int_0^1 du_i(1-\delta_{i2})\langle u_i|\partial_P^{-1}|u_1\rangle\langle u_i|\partial_P^{-2}|u_3\rangle \\ &= 2^3\Bigg(\int_0^1 du_i\langle u_i|\partial_P^{-1}|u_2\rangle\langle u_i|\partial_P^{-2}|u_3\rangle - \langle u_1|\partial_P^{-1}|u_2\rangle\langle u_1|\partial_P^{-2}|u_3\rangle \\ &\quad +\int_0^1 du_i\langle u_i|\partial_P^{-1}|u_1\rangle\langle u_i|\partial_P^{-2}|u_3\rangle - \langle u_2|\partial_P^{-1}|u_1\rangle\langle u_2|\partial_P^{-2}|u_3\rangle\Bigg)\end{aligned} \tag{1.52}$$

Ultimately, the application of(1.49) results in the emergence of a negative sign, and it merely entails the application of (1.45) to obtain:

$$\begin{aligned}\int_0^1 du_i \dot{G}_{i1}\dot{G}_{i2}\dot{G}_{i3} &= -2^3\langle u_2|\partial_P^{-3}|u_3\rangle - 2^3\langle u_1|\partial_P^{-3}|u_3\rangle - 2^3\langle u_1|\partial_P^{-1}|u_2\rangle\langle u_1|\partial_P^{-2}|u_3\rangle \\ &\quad -2^3\langle u_2|\partial_P^{-1}|u_1\rangle\langle u_2|\partial_P^{-2}|u_3\rangle.\end{aligned} \tag{1.53}$$

As illustrated in the preceding example, this method holds promise for solving integrals that arise within the worldline formalism, avoiding sector decomposition. Now, by utilizing the following relationships:

$$\begin{aligned}\langle u_i|\partial_P^{-3}|u_j\rangle &= -\frac{1}{12}\dot{G}_{ij}G_{ij} \\ \dot{G}_{ij}^2 &= 1-4G_{ij},\end{aligned} \tag{1.54}$$

the result can be expressed in terms of $\dot{G}_{ij}$ as follows:

$$\int_0^1 du_i \dot{G}_{i1}\dot{G}_{i2}\dot{G}_{i3} = \frac{1}{6}\left(\dot{G}_{12}^2(\dot{G}_{23}-\dot{G}_{31}) + \dot{G}_{23}^2(\dot{G}_{31}-\dot{G}_{12}) + \dot{G}_{31}^2(\dot{G}_{12}-\dot{G}_{23})\right) \tag{1.55}$$

In the following chapter, we will further elaborate on how the circular integrals in more sophisticated cases, such as our second example above, can be solved using this new method. However, to tackle these types of calculations, it is necessary to utilize the following expansion:

$$\mathrm{e}^{k_i \cdot k_j G_{ij}} = 1 + 2\sum_{a=1}^{\infty}(k_i \cdot k_j)^{a-1/2}\mathrm{H}_{2a-1}\left[\frac{\sqrt{k_i \cdot k_j}}{2}\right]\left(\langle u_i|\partial^{-2a}|u_j\rangle + \frac{B_{2a}}{(2a)!}\right), \quad (1.56)$$

The proof of the preceding mathematical identity is given in Appendix B.
Building on the method's development, we will solve the N integrals of (2.7) without using sector decomposition. Subsequently, we present its application to the simplest non-trivial interactions: the 2- and 3-point functions in ϕ^3 theory.
The N-point amplitude calculation in this framework involves integrals that include the following form:

$$\int_0^1 du_1 \cdots du_N \mathrm{e}^{k_1 \cdot k_2 G_{12}} \mathrm{e}^{k_1 \cdot k_3 G_{13}} \cdots \mathrm{e}^{k_{N-1} \cdot k_N G_{N-1N}} \quad (1.57)$$

Because it is prudent to start solving the problem for ϕ^3 because in the worldline formalism once you understand how things work with ϕ^3, transitioning to scalar QED is straightforward.
The mathematical difficulties encountered in the computation of one-loop amplitudes in QED have spurred significant research efforts, resulting in various techniques to address these challenges. Notably, Bollini and Giambiagi in [94], and 't Hooft and Veltman in [95], pioneered the method of dimensional regularization, which involves analytically continuing the number of spacetime dimensions to regulate ultraviolet and infrared divergences.
Building upon this crucial development, Passarino and Veltman [96] established a systematic tensorial reduction procedure for one-loop integrals, decomposing them into a basis of scalar integral functions. Substantial progress in the evaluation of these fundamental scalar integrals was made by Boos and Davydychev [97]. The method of differential equations for loop integrals was further advanced by Kotikov [98] in the early 1990s. Further advancing the field, particularly for multi-loop computations, the work of Johannes Henn and collaborators significantly propelled the method of differential equations [99] [100]. Subsequent investigations by Riemann [101] have yielded new analytical results and revisited the treatment of scalar one-loop integrals in arbitrary dimensions.
For a detailed discussion of the integration challenges at the one-loop level, see references [102, 103] and [104].
Despite these significant advancements, the underlying mathematical structure of general n-point amplitudes at one-loop remains an important object of ongoing research.
Our interest in this state-of-the-art approach for solving one-loop integrals within the worldline formalism stems from the fact that, as we will demonstrate in the follow-

ing chapter when calculating the two- and three-point functions, these integrals can not only be solved very easily, but it also appears that hypergeometric functions can be generated within the worldline framework using this novel approach. Indeed, the emergence of hypergeometric and special functions at the one- and two-loop levels has been observed since the late last century (see references [105, 106, 107, 108, 109]).

Chapter 2

Bernoulli numbers and polynomials in the Worldline Formalism

As shown in the previous chapter, the master formula for the N-photon amplitude in scalar QED inherently contains the worldline Green's function, its first derivative, and its second derivative. However, through integration by parts, the second-derivative terms can be systematically eliminated. This simplification reduces the expression to a polynomial in $\dot{G}_{ij}$, coupled with the exponential factor $\mathrm{e}^{k_i k_j G_{ij}}$. Consequently, the integrals one must evaluate in practice take the general form:

$$\int_0^1 du_1 \int_0^1 du_2 \cdots \int_0^1 du_N \mathrm{Pol}(\dot{G}_{ij}) \exp\left(\sum_{i<j=1}^{N} \lambda_{ij} G_{ij}\right). \tag{2.1}$$

The aforementioned representation can be approached via a decomposition into ordered sectors, as explored in the preceding section. One might naturally assume this decomposition is imperative, given that $G_{ij} = |u_i - u_j| - (u_i - u_j)^2$ and $\dot{G}_{ij} = \mathrm{sgn}(u_i - u_j) - 2(u_i - u_j)$.

In this section, however, we present the method designed to address such integrals without necessitating the decomposition into ordered sectors.

The calculations presented here are exclusively applied to the vacuum. For a comprehensive understanding of how the worldline formalism operates in the presence of a background field, refer to the foundational work by Reuter, Schmidt, and Schubert [110], which directly applies the worldline formalism to calculate one-loop effective actions for spin 0, 1/2, and 1 particles in constant external fields. Additionally, a recent and excellent review by C. Schubert [111] provides an updated overview of the worldline formalism's applications, particularly in strong-field QED, which often involves constant or plane-wave external fields.

As mentioned in Chapter 1, the worldline integration can be connected with Bernoulli polynomials and numbers,taking into account that the execution of the coordinate

path integral took place in the Hilbert space H'_P, which is defined by periodic functions orthogonal to those that are constant.
Building on the invertibility of the ∂_P^n operator in this space, we can now describe its inverse. The integral kernel is essentially represented by the nth Bernoulli polynomial [27]:

$$\langle u_i|\partial_P^{-n}|u_j\rangle = -\frac{1}{n!}B_n(|u_i-u_j|)sign^n(u_i-u_j). \tag{2.2}$$

The preceding chapter demonstrated that working within this space at the polynomial level greatly simplifies the resolution of a single circular integral. We now extend this approach to the calculation of N circular integrals.

2.1 Cycle-Integral

In this section, we present the calculation of a cycle-integral using the brackets, which, as anticipated in the previous chapter, yields a result expressed in Bernoulli numbers. To understand this result more easily, it's simpler to first calculate the following integral:

$$\int_0^1 du_2 \int_0^1 du_3 \cdots \int_0^1 du_n \dot{G}_{12}\dot{G}_{23}\cdots\dot{G}_{n(n+1)} \tag{2.3}$$

Solving the integral above only requires the following identities:

$$\dot{G}_{ij} = 2\langle u_i|\partial_P^{-1}|u_j\rangle \tag{2.4}$$

$$\int_0^1 du_j \langle u_i|\partial_P^{-1}|u_j\rangle\langle u_j|\partial_P^{-1}|u_k\rangle = \langle u_i|\partial_P^{-2}|u_k\rangle \tag{2.5}$$

So,

$$
\begin{aligned}
&\int_0^1 du_2\cdots\int_0^1 du_n\dot{G}_{12}\dot{G}_{23}\cdots\dot{G}_{n(n+1)}\\
&= 2^n\int_0^1 du_2\cdots\int_0^1 du_n\langle u_1|\partial_P^{-1}|u_2\rangle\langle u_2|\partial_P^{-1}|u_3\rangle\cdots\langle u_n|\partial_P^{-1}|u_{n+1}\rangle\\
&= 2^n\int_0^1 du_3\cdots\int_0^1 du_n\langle u_1|\partial_P^{-2}|u_3\rangle\langle u_3|\partial_P^{-1}|u_4\rangle\cdots\langle u_n|\partial_P^{-1}|u_{n+1}\rangle\\
&.\\
&.\\
&.\\
&= 2^n\langle u_1|\partial_P^{-n}|u_{n+1}\rangle = -\frac{2^n}{n!}B_n(|u_1-u_{n+1}|)\mathrm{sign}^n(u_1-u_{n+1})
\end{aligned}
\tag{2.6}
$$

In the previous calculation, we evaluated $n-1$ integrals. By imposing $u_{n+1}=u_1$, the integral over u_1 becomes trivial, yielding the following result without the need for a sector decomposition to compute the "cycle integral."

$$
\begin{aligned}
\int_0^1 du_1\int_0^1 du_2\cdots\int_0^1 du_n\dot{G}_{12}\dot{G}_{23}\cdots\dot{G}_{n1} &= -\int_0^1 du_1\frac{2^n}{n!}B_n(|u_1-u_1|)\mathrm{sign}^n(|u_1-u_1|)\\
&= -2^n\frac{B_n}{n!}
\end{aligned}
\tag{2.7}
$$

Notice that, due to the presence of Bernoulli numbers $B_n(0)=B_n$, cases where n is odd will be zero. This vanishing for odd n is directly attributable to the antisymmetric nature of the derivative operator.

Recall that within this formalism, it is sufficient to apply the replacement rule (1.20) to determine the results for the spinorial loop once those for the scalar loop are known. Following our recent calculation of the "cycle integral", the application of this rule leads us directly to the evaluation of the "super cycle integral", which we define as follows:

$$
\int_0^1 du_1\int_0^1 du_2\cdots\int_0^1 du_n(\dot{G}_{12}\dot{G}_{23}\cdots\dot{G}_{n1}-\dot{G}_{F12}\dot{G}_{F23}\cdots\dot{G}_{Fn1}). \tag{2.8}
$$

Having previously calculated the scalar component, our attention will now shift to determining the fermionic contribution. Prior to this, we will examine the definition of the fermionic Green's function in terms of the braket notation:

$$
G_{Fij} = 2\langle u_i|\partial_A^{-1}|u_j\rangle \tag{2.9}
$$

In this case, we have the subscript A due to the anti-periodic property of the fermionic loop. The calculation of the n-1 integrals is analogous to the scalar case, resulting in the following:

$$\int_0^1 du_2 \cdots \int_0^1 du_n \dot{G}_{F12}\dot{G}_{F23}\cdots\dot{G}_{Fn(n+1)} = 2^n \langle u_1|\partial_A^{-n}|u_{n+1}\rangle \tag{2.10}$$

The derivative acting on anti-periodic functions is also invertible, and the previous expression can be written in terms of Euler polynomials.

$$\begin{aligned}\int_0^1 du_2 \cdots \int_0^1 du_n \dot{G}_{F12}\dot{G}_{F23}\cdots\dot{G}_{Fn(n+1)} &= 2^n \langle u_1|\partial_A^{-n}|u_{n+1}\rangle \\ &= \frac{2^{n-1}}{(n-1)!} E_{n-1}(|u_1-u_{n+1}|)\mathrm{sgn}^n(u_1-u_{n+1})\end{aligned} \tag{2.11}$$

Once again, we close the cycle by setting n+1=1, and we arrive at:

$$\begin{aligned}\int_0^1 du_1 du_2 \cdots du_n \dot{G}_{F12}\dot{G}_{F23}\cdots\dot{G}_{Fn1} &= \frac{2^{n-1}}{(n-1)!} E_{n-1}(0) \\ &= \frac{2^{n-1}}{(n-1)!}\frac{2^n}{n}\left(B_n\left(\frac{1}{2}\right) - B_n\right)\end{aligned} \tag{2.12}$$

However,

$$B_n\left(\frac{1}{2}\right) = \begin{cases} \left(2^{1-n}-1\right) B_n & n \text{ even}, \\ 0 & n \quad \text{odd}\ , \end{cases} \tag{2.13}$$

Consequently, it is found that:

$$\int_0^1 du_1 \int_0^1 du_2 \cdots \int_0^1 du_n \dot{G}_{F12}\dot{G}_{F23}\cdots\dot{G}_{Fn1} = 2^n(1-2^n)\frac{B_n}{n!} \tag{2.14}$$

By combining the previous findings, we determine that:

$$\int_0^1 du_1 \cdots \int_0^1 du_n (\dot{G}_{12}\dot{G}_{23}\cdots\dot{G}_{n1} - \dot{G}_{F12}\dot{G}_{F23}\cdots\dot{G}_{Fn1}) = 2^n(2^n-2)\frac{B_n}{n!} \tag{2.15}$$

2.2 One-Loop Two- and Three-Point Functions in ϕ^3 Theory

We commence by advancing the state of the art, tackling the solution of off-shell, one-loop, N-point amplitudes in the simplest scenario: the scalar ϕ^3 theory in D dimensions, where the master formula is expressed as follows:

$$I_N(p_1,\dots p_N) = \frac{1}{2}(4\pi)^{-D/2}(2\pi)^D\delta\left(\sum_{i=1}^N p_i\right)\hat{I}_N(p_1,\dots,p_N), \tag{2.16}$$

$$\hat{I}_N(p_1,\dots,p_N) = \int_0^\infty \frac{dT}{T}T^{N-D/2}e^{-m^2T}\int_0^1 du_1\cdots\int_0^1 du_N e^{T\sum_{i<j=1}^N \lambda_{ij}G_{ij}}, \tag{2.17}$$

where $\lambda_{ij} = p_i \cdot p_j$, we will solve (2.17) by employing the expansion of the exponential in terms of Hermite polynomials H_{2n-1} and the kernel elements.
Although we introduced this expansion in Section 2, we revisit and elaborate on it here, as it is crucial for the computation

$$\mathrm{e}^{T\lambda_{ij}G_{Bij}} = 1 + 2\sum_{n=1}^\infty (T\lambda_{ij})^{n-1/2}\mathrm{H}_{2n-1}\left(\frac{\sqrt{T\lambda_{ij}}}{2}\right)\overline{\langle u_i|\partial^{-2n}|u_j\rangle}, \tag{2.18}$$

where we have abbreviated

$$\overline{\langle u_i|\partial^{-2n}|u_j\rangle} \equiv \langle u_i|\partial^{-2n}|u_j\rangle - \langle u_i|\partial^{-2n}|u_i\rangle. \tag{2.19}$$

Utilizing (1.47) and introducing the notation $\frac{B_{2n}}{(2n)!} = \hat{B}_{2n}$

$$\overline{\langle u_i|\partial^{-2n}|u_j\rangle} = \langle u_i|\partial^{-2n}|u_j\rangle + \hat{B}_{2n}. \tag{2.20}$$

2.2.1 2-point

Although the two-point amplitude in D dimensions is mathematically simple in any formalism, within this context, it is significantly more straightforward owing to (1.46). Here, the integrals over u_1 and u_2 are rendered trivial, leaving us solely with the task of solving the T integral

$$\begin{aligned}
\hat{I}_2(p_1,p_2) &= \int_0^\infty \frac{dT}{T} T^{2-D/2} e^{-m^2T} \int_0^1 du_1 \int_0^1 du_2 e^{T\lambda_{12}G_{12}}. \\
&= \int_0^\infty dT T^{1-D/2} \mathrm{e}^{-m^2T} \left(1 + 2\sum_{n=1}^{\infty} (T\lambda_{12})^{n-1/2} \mathrm{H}_{2n-1}\left(\frac{\sqrt{T\lambda_{12}}}{2}\right) \hat{B}_{2n}\right)
\end{aligned} \tag{2.21}$$

At first glance, the T integral may seem challenging, given its appearance as an argument in the Hermite polynomials. However, upon considering the explicit formula for the Hermite polynomials:

$$H_n(x) = n! \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^m \frac{(2x)^{n-2m}}{m!(n-2m)!} \tag{2.22}$$

Reveals,

$$2\sum_{n=1}^{\infty} (T\lambda)^{n-\frac{1}{2}} H_{2n-1}\left(\frac{\sqrt{T\lambda}}{2}\right) \hat{B}_{2n} = \sum_{i=1}^{\infty} (T\lambda)^i \sum_{n=1}^{i} h_n^i \hat{B}_{2n}, \tag{2.23}$$

where the coefficients h_n^i are given by:

$$h_n^i = (-1)^{i+1} \frac{2(2n-1)!}{(2n-i-1)!(2i-2n+1)!}. \tag{2.24}$$

So the integrals over T are:

$$\int_0^\infty dT T^{1-D/2+i} \mathrm{e}^{-m^2T} \quad = \quad \frac{\Gamma\left(2 - \frac{D}{2} + i\right)}{m^{4-D+2i}}. \tag{2.25}$$

Utilizing the expansion (2.18), we can determine the general integral for the variable T, applicable across any number N and dimension D, as shown below:

$$\int_0^\infty \frac{dT}{T} T^{N-D/2+i} \mathrm{e}^{-m^2T} = \frac{\Gamma\left(\frac{2N+2i-D}{2}\right)}{m^{2N+2i-D}}. \tag{2.26}$$

With the previous results:

$$\hat{I}_2(p_1,p_2) \quad = \quad \frac{\Gamma\left(2-\frac{D}{2}\right)}{m^{4-D}} + \sum_{i=1}^{\infty} \lambda_{12}^i \frac{\Gamma\left(2-\frac{D}{2}+i\right)}{m^{4-D+2i}} \sum_{n=1}^{i} h_n^i \hat{B}_{2n}. \tag{2.27}$$

This technique yielded two sums as a result. The sum that includes the Bernoulli numbers is easy to solve and proves to be

$$\sum_{a=1}^{i} h_a^i \hat{B}_{2a} = \frac{\Gamma\left(\frac{3}{2}\right)}{4^i\Gamma\left(i+\frac{3}{2}\right)}. \tag{2.28}$$

Utilizing the previous result:

$$\hat{I}_2(p_1,p_2) = \frac{\Gamma\left(2-\frac{D}{2}\right)}{m^{4-D}} + \frac{\lambda_{12}B_2\Gamma\left(\frac{6-D}{2}\right)}{m^{6-D}}\,{}_2F_1\left(1,\frac{6-D}{2},\frac{5}{2},\frac{\lambda_{12}}{4m^2}\right) \tag{2.29}$$

Our result is in complete agreement with Eq. (59) of [106], which was derived therein using methods involving Gram determinants for the evaluation of loop integrals. Furthermore, this agreement was cross-checked against the work of [112] for dimensions D=2,3, and 4, where the corresponding results are obtained via geometrical techniques based on properties of N-dimensional simplices, yielding full consistency. When considering this amplitude in scalar QED, it is important to remember that the underlying process is linked to vacuum polarization. The worldline formalism provides the following representation for this:

$$\Gamma_{\text{scal}}(k_1,\varepsilon_1;k_2,\varepsilon_2) \equiv (2\pi)^D\delta(k_1+k_2)\varepsilon_1\cdot\Pi_{\text{scal}}\cdot\varepsilon_2 \tag{2.30}$$

Within this Q-representation, the usual transversal projector $\delta^{\mu\nu}k^2 - k^\mu k^\nu$ is extracted directly at the integral level, as seen in the expression below. Using momentum conservation $k_1 = -k_2 =: k$ we arrive at the following integral:

$$\begin{aligned}\Pi^{\mu\nu}_{\text{scal}}(k) &= \frac{e^2}{(4\pi)^{D/2}}\left(\delta^{\mu\nu}k^2 - k^\mu k^\nu\right)\int_0^\infty \frac{dT}{T}\,T^{2-D/2}\,e^{-m^2T}\\ &\quad\times\int_0^1 du_1\,du_2\,\dot{G}_{12}^2\,e^{-G_{12}k^2}\\ &= \frac{e^2\left(\delta^{\mu\nu}k^2 - k^\mu k^\nu\right)}{(4\pi)^{D/2}}I_2\end{aligned} \tag{2.31}$$

$$\begin{aligned}I_2 &= \int_0^\infty dT\mathrm{e}^{-m^2T}T^{1-D/2}\int_0^1 du_1du_1\mathrm{e}^{-TkG_{12}}\dot{G}_{12}^2\\ &= \frac{2B_2}{m^{4-D}}\left(\Gamma\left[\frac{4-D}{2}\right] + 12B_4\Gamma\left[\frac{6-D}{2}\right]\frac{k}{4m^2}{}_2F_1\left[1,\frac{6-D}{2},\frac{7}{2},\frac{-k}{4m^2}\right]\right)\end{aligned} \tag{2.32}$$

At this point, we observe that the ultraviolet divergence persists in D=4. However, we are more interested in simplifying the integration methods within the formalism, so we proceed directly to the three-point case.

2.2.2 3-point

As discussed throughout this chapter, the calculation of amplitudes in the worldline formalism can be approached in various ways. This flexibility arises from the fact that integration by parts allows one to work either with only the first derivative and the Green's function itself, or with the Green's function alongside its first and second derivatives. In our quest to determine the most efficient method for solving such integrals, we have found that the optimal approach depends on the specific problem at hand. In this section, we will compute the one-loop 3-point amplitude in ϕ^3 theory. We employed the same method used in the previous section for the two-point amplitude.
We shall now advance to calculate the three-point case expanding $\mathrm{e}^{\lambda_{ij}G_{ij}}$, starting with the integration over the variables u_i, postponing the integration of T until the end. This approach leads us to apply the expansion (2.18):

$$\begin{aligned}\mathrm{e}^{T(\lambda_{12}G_{12}+\lambda_{23}G_{23}+\lambda_{31}G_{31})} = &\left\{1+2\sum_{i=1}^{\infty}(T\lambda_{12})^{i-1/2}\mathrm{H}_{2i-1}\left(\frac{\sqrt{T\lambda_{12}}}{2}\right)\left[\langle u_1|\partial^{-2i}|u_2\rangle+\hat{B}_{2i}\right]\right\}\\ &\times\left\{1+2\sum_{j=1}^{\infty}(T\lambda_{23})^{j-1/2}\mathrm{H}_{2j-1}\left(\frac{\sqrt{T\lambda_{23}}}{2}\right)\left[\langle u_2|\partial^{-2j}|u_3\rangle+\hat{B}_{2j}\right]\right\}\\ &\times\left\{1+2\sum_{k=1}^{\infty}(T\lambda_{31})^{k-1/2}\mathrm{H}_{2k-1}\left(\frac{\sqrt{T\lambda_{31}}}{2}\right)\left[\langle u_3|\partial^{-2k}|u_1\rangle+\hat{B}_{2k}\right]\right\}.\end{aligned} \tag{2.33}$$

Building on the previous expansion, we identify terms that are dependent on u_1, u_2 and u_3. Consequently, the specific integrals requiring solution, while ignoring the sums and the Bernoulli numbers, are composed of the following elements

$$\int_0^1 du_1\langle u_1|\partial^{-2i}|u_2\rangle + 2\text{ Perms.} \tag{2.34}$$

$$\int_0^1 du_1du_2\langle u_1|\partial^{-2i}|u_2\rangle\langle u_2|\partial^{-2k}|u_3\rangle + 2\text{ Perms.} \tag{2.35}$$

$$\int_0^1 du_1du_2du_3\langle u_1|\partial^{-2i}|u_2\rangle\langle u_2|\partial^{-2j}|u_3\rangle\langle u_3|\partial^{-2k}|u_1\rangle \tag{2.36}$$

Due to (1.46), the three integrals corresponding to types (2.34) and (2.35) are tivially zero.
The only non-zero integral is given by [79]:

$$\begin{aligned}&\int_0^1 du_1du_2du_3\langle u_1|\partial^{-2i}|u_2\rangle\langle u_2|\partial^{-2j}|u_3\rangle\langle u_3|\partial^{-2k}|u_1\rangle\\&=\int_0^1 du_1du_2\langle u_1|\partial^{-2i}|u_2\rangle\langle u_2|\partial^{-2(j+k)}|u_1\rangle\\&=\int_0^1 du_1\langle u_1|\partial^{-2(i+j+k)}|u_1\rangle=-\hat{B}_{2(i+j+k)}.\end{aligned} \tag{2.37}$$

Using (2.23) we found:

$$\begin{aligned}&\int_0^1 du_1du_2du_3e^{T(\lambda_{12}G_{12}+\lambda_{23}G_{23}+\lambda_{31}G_{31})}\\&=1+\sum_{a=1}^{\infty}(T\lambda_{12})^a\sum_{i=1}^{a}h_i^a\hat{B}_{2i}+2\text{ Perms.}\\&+\sum_{a,b=1}^{\infty}T^{a+b}\lambda_{12}^a\lambda_{23}^b\sum_{i=1}^{a}\sum_{j=1}^{b}h_i^ah_j^b\hat{B}_{2i}\hat{B}_{2j}+2\text{ Perms.}\\&+\sum_{a,b,c=1}^{\infty}T^{a+b+c}\lambda_{12}^a\lambda_{23}^b\lambda_{31}^c\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}h_i^ah_j^bh_k^c\hat{B}_{2i}\hat{B}_{2j}\hat{B}_{2k}\\&-\sum_{a,b,c=1}^{\infty}T^{a+b+c}\lambda_{12}^a\lambda_{23}^b\lambda_{31}^c\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}h_i^ah_j^bh_k^c\hat{B}_{2(i+j+k)}\end{aligned} \tag{2.38}$$

Given that we have set $D = 4$ and $N = 3$ the integration over T leave us with:

$$
\begin{aligned}
&\int_0^\infty dT \mathrm{e}^{-m^2T} \int_0^1 du_1 du_2 du_3 e^{T(\lambda_{12}G_{12}+\lambda_{23}G_{23}+\lambda_{31}G_{31})} \\
&= \frac{1}{m^2} + \sum_{a=1}^{\infty} \frac{\Gamma(a+1)\lambda_{12}^a}{m^{2(a+1)}} \sum_{i=1}^{a} h_i^a \hat{B}_{2i} + 2 \text{ Perms.} \\
&+ \sum_{a,b=1}^{\infty} \frac{\Gamma(a+b+1)\lambda_{12}^a\lambda_{23}^b}{m^{2(a+b+1)}} \sum_{i=1}^{a}\sum_{j=1}^{b} h_i^a h_j^b \hat{B}_{2i}\hat{B}_{2j} + 2 \text{ Perms.} \\
&+ \sum_{a,b,c=1}^{\infty} \frac{\Gamma(a+b+c+1)\lambda_{12}^a\lambda_{23}^b\lambda_{31}^c}{m^{2(a+b+c+1)}} \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} h_i^a h_j^b h_k^c \left(\hat{B}_{2i}\hat{B}_{2j}\hat{B}_{2k} - \hat{B}_{2(i+j+k)}\right)
\end{aligned}
\tag{2.39}
$$

Utilizing (2.28), the emergence of multiple sums facilitates the generation of distinct hypergeometric functions. It is essential to recognize that our sums initiate at 1, in contrast to the definitions of hypergeometric functions, which incorporate sums commencing at zero. This discrepancy necessitates the adaptation of our sums for the construction of hypergeometric functions, leading to specific combinations of these functions. For instance, terms incorporating two sums give rise to the function ${}_2F_1$, as identified in the two-point calculation. In cases involving four sums, both the Appell F_2 and the ${}_2F_1$ functions are generated. Furthermore, a term with six sums yields the previously mentioned functions in addition to the Lauricella $F_{\mathrm{A}}^{(3)}$ function.*

$$\begin{aligned}\sum_{a=1}^{\infty} \quad & \frac{\Gamma(a+1)\lambda_{12}^{a}}{m^{2(a+1)}} \sum_{i=1}^{a} h_i^a \hat{B}_{2i} \\ &= \frac{1}{m^2} \sum_{a=1}^{\infty} \frac{\sqrt{\pi}}{2} \left(\frac{\lambda_{12}}{4m^2}\right)^a \frac{\Gamma(a+1)}{\Gamma\left(a+\frac{3}{2}\right)} \\ &= \frac{1}{m^2}\left({}_2F_1\left[1,1,\frac{3}{2},\frac{\lambda_{12}}{4m^2}\right]-1\right)\end{aligned} \tag{2.40}$$

$$\begin{aligned}\sum_{a,b=1}^{\infty} \quad & \frac{\Gamma(a+b+1)\lambda_{12}^{a}\lambda_{23}^{b}}{m^{2(a+b+1)}} \sum_{i=1}^{a}\sum_{j=1}^{b} h_i^a h_j^b \hat{B}_{2i}\hat{B}_{2j} \\ &= \frac{1}{m^2} \sum_{a,b=1}^{\infty} \frac{\pi}{4} \left(\frac{\lambda_{12}}{4m^2}\right)^a \left(\frac{\lambda_{23}}{4m^2}\right)^b \frac{\Gamma(a+b+1)}{\Gamma\left(a+\frac{3}{2}\right)\Gamma\left(b+\frac{3}{2}\right)} \\ &= \frac{1}{m^2}\Bigg(F_2\left[1;1,1;\frac{3}{2},\frac{3}{2};\frac{\lambda_{12}}{4m^2},\frac{\lambda_{23}}{4m^2}\right] - {}_2F_1\left[1,1,\frac{3}{2},\frac{\lambda_{12}}{4m^2}\right] \\ & -{}_2F_1\left[1,1,\frac{3}{2},\frac{\lambda_{23}}{4m^2}\right]+1\Bigg)\end{aligned} \tag{2.41}$$

---

*The Lauricella function $F_A^{(3)}$ is a generalization of the Appell hypergeometric series to three variables. It is defined for $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{C}^3$ with $|x_i| < 1$ $(i = 1, 2, 3)$ by the series:

$$F_A^{(3)}(a, b_1, b_2, b_3; c_1, c_2, c_3; x_1, x_2, x_3) = \sum_{n_1,n_2,n_3=0}^{\infty} \frac{(a)_{n_1+n_2+n_3}(b_1)_{n_1}(b_2)_{n_2}(b_3)_{n_3}}{(c_1)_{n_1}(c_2)_{n_2}(c_3)_{n_3}} \frac{x_1^{n_1}x_2^{n_2}x_3^{n_3}}{n_1!\,n_2!\,n_3!},$$

where:

- $(a)_n$ is the Pochhammer symbol, defined as $(a)_n = \Gamma(a+n)/\Gamma(a)$.
- a, b_1, b_2, b_3 are the function parameters.
- c_1, c_2, c_3 are parameters which must not be zero or negative integers.

$$
\begin{aligned}
&\sum_{a,b,c=1}^{\infty} \frac{\Gamma(a+b+c+1)\lambda_{12}^{a}\lambda_{23}^{b}\lambda_{31}^{c}}{m^{2(a+b+c+1)}} \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} h_i^a h_j^b h_k^c \hat{B}_{2i}\hat{B}_{2j}\hat{B}_{2k} \\
&= \frac{1}{m^2} \sum_{a,b,c=1}^{\infty} \frac{\pi^{3/2}}{8} \left(\frac{\lambda_{12}}{4m^2}\right)^a \left(\frac{\lambda_{23}}{4m^2}\right)^b \left(\frac{\lambda_{31}}{4m^2}\right)^c \frac{\Gamma\left(a+b+c+1\right)}{\Gamma\left(a+\frac{3}{2}\right)\Gamma\left(b+\frac{3}{2}\right)\Gamma\left(c+\frac{3}{2}\right)} \\
&= \frac{1}{m^2} \quad F_A^{(3)}\left[1,1,1,1;\frac{3}{2},\frac{3}{2},\frac{3}{2};\frac{\lambda_{12}}{4m^2},\frac{\lambda_{23}}{4m^2},\frac{\lambda_{31}}{4m^2}\right] \\
&- F_2\left[1;1,1;\frac{3}{2},\frac{3}{2};\frac{\lambda_{12}}{4m^2},\frac{\lambda_{23}}{4m^2}\right] - F_2\left[1;1,1;\frac{3}{2},\frac{3}{2};\frac{\lambda_{12}}{4m^2},\frac{\lambda_{31}}{4m^2}\right] \\
&- F_2\left[1;1,1;\frac{3}{2},\frac{3}{2};\frac{\lambda_{23}}{4m^2},\frac{\lambda_{31}}{4m^2}\right] + {}_2F_1\left[1,1,\frac{3}{2},\frac{\lambda_{12}}{4m^2}\right] \\
&+ {}_2F_1\left[1,1,\frac{3}{2},\frac{\lambda_{23}}{4m^2}\right] + {}_2F_1\left[1,1,\frac{3}{2},\frac{\lambda_{31}}{4m^2}\right] - 1 \Bigg)
\end{aligned}
\tag{2.42}
$$

This adaptation leads to the elimination of the Hypergeometric ${}_2F_1$ and Appell F_2 functions, culminating in the emergence of the Lauricella function as a key component of the result, along with an additional term. This is because the coefficients h_i^a combined with the Bernoulli numbers $\hat{B}_{2i}$ generate Gamma functions. Upon integrating over T, the expansion 2.18 provides a straightforward method for generating hypergeometric functions. However, equation 2.39 includes a sum involving $\hat{B}_{2i+2j+2k}$ which currently poses a challenge. This difficulty stems not from a fundamental limitation of the formalism itself, but rather from the fact that the specific identities required to express this term entirely in therms of hypergeometric functions are not yet known in the mathematical literature. In fact, considering that at the three-point level only one term remains to be identified, it should be possible to infer the missing identity using the known result. These current complexities motivate the exploration of alternative approaches to derive these identities, as discussed in the subsequent section, rather than abandoning this promising path.

Thus, the representation of the three-point case in four dimensions, using new integration techniques within the world-line formalism, is as follows:

$$\int_0^\infty dT \mathrm{e}^{-m^2T} \int_0^1 du_1 du_2 du_3 e^{T(\lambda_{12}G_{12}+\lambda_{23}G_{23}+\lambda_{31}G_{31})}$$
$$= \frac{1}{m^2}\Bigg(F_A^{(3)}\left[1,1,1,1;\frac{3}{2},\frac{3}{2},\frac{3}{2};\frac{\lambda_{12}}{4m^2},\frac{\lambda_{23}}{4m^2},\frac{\lambda_{31}}{4m^2}\right]$$
$$- \sum_{a,b,c=1}^{\infty} \frac{\Gamma(a+b+c+1)\lambda_{12}^a\lambda_{23}^b\lambda_{31}^c}{m^{2(a+b+c+1)}} \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} h_i^a h_j^b h_k^c \hat{B}_{2(i+j+k)} \qquad (2.43)$$

When applying this novel method to the 3-point case, we find ourselves limited to expressing the resulting sums involving Bernoulli numbers in terms of hypergeometric functions.

However, this method has proven effective for calculating the coefficients of the heat kernel expansion of the effective action for more details on this expansion within this formalism, see [113, 114, 115]), where this expansion reduces to the following family of integrals:

$$I_N(n_{12}, n_{13}, \ldots, n_{(N-1)N}) \equiv \int_{12\ldots N} \prod_{i<j=1}^{N} G_{ij}^{n_{ij}} \qquad (2.44)$$

We shall now examine the first nontrivial scenario, which occurs at N = 3:

$$\int_{1,2,3} G_{12}^a G_{13}^b G_{23}^c \quad = \quad a!b!c! \sum_{\substack{i=1\\j=1\\k=1}}^{a,b,c} h_i^a h_j^b h_k^c \left(\hat{B}_{2i}\hat{B}_{2j}\hat{B}_{2k} - \hat{B}_{2(i+j+k)}\right) \qquad (2.45)$$

Having established the results for $N = 3$, we proceed to examine the $N = 4$ scenario at which the cubic worldline vertex appears, defined as:

$$V_3^{ijk}(u_a, u_b, u_c) \equiv \int_0^1 du \left\langle u\middle|\partial^{-i}\middle|u_a\right\rangle \left\langle u\middle|\partial^{-j}\middle|u_b\right\rangle \left\langle u\middle|\partial^{-k}\middle|u_c\right\rangle \qquad (2.46)$$

By employing partial integration, this vertex can be computed:

$$\begin{aligned}
V_3^{ijk}(u_a,u_b,u_c) &= -\int_0^1 du(\left\langle u\middle|\partial^{-(i-1)}\middle|u_a\right\rangle\left\langle u\middle|\partial^{-j}\middle|u_b\right\rangle+\left\langle u\middle|\partial^{-i}\middle|u_a\right\rangle\left\langle u\middle|\partial^{-(j-1)}\middle|u_b\right\rangle) \\
&\quad \times\left\langle u\middle|\partial^{-(k+1)}\middle|u_c\right\rangle \\
&= \int_0^1 du(\left\langle u\middle|\partial^{-(i-2)}\middle|u_a\right\rangle\left\langle u\middle|\partial^{-j}\middle|u_b\right\rangle+\left\langle u\middle|\partial^{-i}\middle|u_a\right\rangle\left\langle u\middle|\partial^{-(j-2)}\middle|u_b\right\rangle \\
&\quad +2\left\langle u\middle|\partial^{-(i-1)}\middle|u_a\right\rangle\left\langle u\middle|\partial^{-(j-1)}\middle|u_b\right\rangle)\left\langle u\middle|\partial^{-(k+2)}\middle|u_c\right\rangle \\
&\quad . \\
&\quad . \\
&\quad . \\
&= \sum_{\alpha=i}^{i+j-1} C_i^{\alpha}\left(\left\langle u_a\middle|\partial^{-(i+j-\alpha)}\middle|u_b\right\rangle\left\langle u_a\middle|\partial^{-(k+\alpha)}\middle|u_c\right\rangle-(-1)^{i+j-\alpha}\left\langle u_b\middle|\partial^{-(i+j+k)}\middle|u_c\right\rangle\right) \\
&\quad +\sum_{\beta=j}^{i+j-1} C_j^{\beta}\left(\left\langle u_b\middle|\partial^{-(i+j-\beta)}\middle|u_a\right\rangle\left\langle u_b\middle|\partial^{-(k+\beta)}\middle|u_c\right\rangle-(-1)^{i+j-\beta}\left\langle u_a\middle|\partial^{-(i+j+k)}\middle|u_c\right\rangle\right) \\
&= \sum_{\alpha=i}^{i+j-1} C_i^{\alpha}\left(\left\langle u_a\middle|\partial^{-(i+j-\alpha)}\middle|u_b\right\rangle\left\langle u_a\middle|\partial^{-(k+\alpha)}\middle|u_c\right\rangle-(-1)^{i+j-\alpha}\left\langle u_b\middle|\partial^{-(i+j+k)}\middle|u_c\right\rangle\right) \\
&\quad +\{i\leftrightarrow j, a\leftrightarrow b\}
\end{aligned} \tag{2.47}$$

Where

$$C_i^{\alpha} := (-1)^{\alpha}\binom{\alpha-1}{i-1} \tag{2.48}$$

The algorithm for solving these integrals for general N can be extended by defining the worldline n-vertex

$$V_n^{i_1 i_2 \ldots i_n}(u_{a_1},u_{a_2},\ldots,u_{a_n}) \equiv \int_0^1 du\left\langle u\middle|\partial^{-i_1}\middle|u_{a_1}\right\rangle\left\langle u\middle|\partial^{-i_2}\middle|u_{a_2}\right\rangle\cdots\left\langle u\middle|\partial^{-i_n}\middle|u_{a_n}\right\rangle \tag{2.49}$$

The previous vertex can be solved by partial integration.
In the next chapter, we introduce an alternative method to study these integrals for the N-point formula.

Chapter 3

N-photon

Having established the foundational principles of the worldline formalism in the introduction, we now turn our attention to one of its most central applications: the computation of the one-loop N-photon amplitude in QED. In Section 3.1 of this chapter, we will present a new representation for scalar QED's N-photon amplitude. Following this, in Section 3.2, we will explore the low-energy limit for both scalar and spinor QED. It is worth noting that while the general calculation of multi-photon amplitudes at loop level remains a formidable task within both conventional field theory and even the full worldline framework, a remarkable simplification occurs when all external photon energies are taken to be much smaller than the electron mass.

3.1 New Representations for Circular Integrals with N Variables

As stated in the introduction, the master formula for the one-loop amplitude of an N-photon process within scalar QED is given by [27]:

$$\Gamma_{\text{scal}}(k_1,\varepsilon_1;\ldots;k_N,\varepsilon_N) = (-ie)^N \int_0^\infty \frac{dT}{T}(4\pi T)^{-D/2} e^{-m^2 T} \prod_{i=1}^{N} \int_0^T d\tau_i \times \exp\left\{ \sum_{i,j=1}^{N} \left(\frac{1}{2} G_{Bij} k_i \cdot k_j - i\dot{G}_{Bij}\varepsilon_i \cdot k_j + \frac{1}{2}\ddot{G}_{Bij}\varepsilon_i \cdot \varepsilon_j \right) \right\} \Bigg|_{\varepsilon_1 \ldots \varepsilon_N} \tag{3.1}$$

When we rescale the integrals over τ to the unit circle using the change of variable $\tau_i = u_i T$ and work in the Q_N* representation, we obtain that:

$$\Gamma_{\text{scal}}(k_1, \varepsilon_1; \ldots; k_N, \varepsilon_N) = (-ie)^N \int_0^\infty \frac{dT}{T} (4\pi T)^{N-D/2} e^{-m^2 T} \times \prod_{i=1}^{N} \int_0^1 du_i Q_N(\dot{G}_{Bij}) \mathrm{e}^{\frac{T}{2}\sum_{i,j=1}^{N} G_{Bij} k_i \cdot k_j} \tag{3.2}$$

In the context of QED within the worldline formalism, Green's functions play a fundamental role in describing the propagation of charged particles and their interactions with the electromagnetic field. These functions are essential tools for computing probability amplitudes in various physical processes, as they encapsulate the behavior of particles as they traverse spacetime and exchange photons.
Appendix F of [27] provides useful formulas for evaluating loop integrals, and in particular, we will make use of Equation F.25.

$$\dot{G}_{ij}^2 = 1 - 4G_{ij}. \tag{3.3}$$

Such relations within the worldline formalism provide alternative strategies to traditional methods for evaluating such integrals, particularly in contrast to the sector decomposition technique. However, our goal is to develop new methods that allow us to solve these integrals while avoiding sector decomposition. Specifically, using the aforementioned equation, we transition from studying:

$$\int_0^1 du_1 \cdots \int_0^1 du_n \, \mathrm{e}^{\frac{1}{2}\sum_{i,j=1}^{n} c_{ij} G_{ij}} \tag{3.4}$$

to analyzing

$$\int_0^1 du_1 \cdots \int_0^1 du_n \, \mathrm{e}^{\frac{1}{2}\sum_{i,j=1}^{n} c_{ij} \dot{G}_{ij}} \tag{3.5}$$

This is made possible by Gaussian integrals, particularly

$$\mathrm{e}^{-\frac{T}{4}\dot{G}_{ij}^2 p_{ij}} = \sqrt{\frac{T}{\pi}} \int_{-\infty}^{\infty} dx_{ij} \mathrm{e}^{-Tx_{ij}^2 + Tx_{ij} r_{ij} \dot{G}_{ij}}, \tag{3.6}$$

*Q_N depends on the different $\dot{G}_{ij}$ terms as well as the kinematic invariantes

where $r_{ij} \equiv \sqrt{-p_{ij}}$. Another useful formula from [27] is

$$\int_0^1 du \mathrm{e}^{\sum_{i=1}^n c_i \dot{G}(u,u_i)} = \frac{\sum_{i=1}^n \sinh(c_i) \mathrm{e}^{\sum_{j=1}^n c_j \dot{G}_{ij}}}{\sum_{i=1}^n c_i}, \tag{3.7}$$

which provides a solution for a single integral within an n-integral expression. This formula is particularly useful as it simplifies the evaluation of one of the integrals, serving as a stepping stone toward solving the complete set. More generally, for these n-integrals, we find that

$$I_n(\{c_{im}\}) = \int_0^1 du_1 ... du_n \mathrm{e}^{\sum_{i<j=1}^n c_{ij} \dot{G}_{ij}} \quad = \quad \sum_{\pi(2,...,n)} \prod_{m=2}^n \frac{\sinh\left(\sum_{i=1}^{m-1} c_{im}\right)}{\sum_{j=1}^{m-1} C_j} \tag{3.8}$$

We have also defined $C_i \equiv \sum_{j=1}^n c_{ij}$. Additionally, the coefficients c_{ij}, initially specified solely for $i < j$, have been extended to the case where $i > j$ by employing the antisymmetry property $c_{ij} = -c_{ji}$. The symbol $\sum_{\pi(2,...,n)}$ represents the complete symmetrization operation with respect to the indices $2, \ldots, n$. As examples, let us examine the cases for $n = 2, 3$, and 4.

$$I_2(\{c_{12}\}) = \frac{\sinh(c_{12})}{c_{12}} \tag{3.9}$$

$$I_3(\{c_{12}, c_{13}, c_{23}\}) = \frac{\sinh(c_{12})}{C_1} \frac{\sinh(c_{13}+c_{23})}{C_1+C_2} + \frac{\sinh(c_{13})}{C_1} \frac{\sinh(c_{12}-c_{23})}{C_1+C_3} \tag{3.10}$$

$$\begin{aligned} I_4(\{c_{12}, \ldots, c_{34}\}) = & \frac{\sinh(c_{12})}{C_1} \frac{\sinh(c_{13}+c_{23})}{C_1+C_2} \frac{\sinh(c_{14}+c_{24}+c_{34})}{C_1+C_2+C_3} \\ & + \frac{\sinh(c_{12})}{C_1} \frac{\sinh(c_{14}+c_{24})}{C_1+C_2} \frac{\sinh(c_{13}+c_{23}+c_{43})}{C_1+C_2+C_4} \\ & + \frac{\sinh(c_{13})}{C_1} \frac{\sinh(c_{12}+c_{32})}{C_1+C_3} \frac{\sinh(c_{14}+c_{24}+c_{34})}{C_1+C_2+C_3} \\ & + \frac{\sinh(c_{13})}{C_1} \frac{\sinh(c_{14}+c_{34})}{C_1+C_3} \frac{\sinh(c_{12}+c_{42}+c_{32})}{C_1+C_4+C_3} \\ & + \frac{\sinh(c_{14})}{C_1} \frac{\sinh(c_{12}+c_{42})}{C_1+C_4} \frac{\sinh(c_{13}+c_{43}+c_{23})}{C_1+C_4+C_2} \\ & + \frac{\sinh(c_{14})}{C_1} \frac{\sinh(c_{13}+c_{43})}{C_1+C_4} \frac{\sinh(c_{12}+c_{42}+c_{32})}{C_1+C_4+C_3} \end{aligned} \tag{3.11}$$

We call this new technique the **Gaussian linearization method** and combining (3.6) and (3.8) we get

$$\hat{I}_N(p_1,\dots,p_N) = \int_0^\infty dT T^{-D/2}\left(\frac{T}{\pi}\right)^{\frac{N(N-1)}{4}} e^{-M^2T} \times \prod_{i<j=1}^{N}\int_{-\infty}^{\infty} dx_{ij}\, \mathrm{e}^{-TX} \sum_{\pi(2,\dots,N)}\prod_{m=2}^{N}\frac{\sinh(TY_m)}{Z_m} \quad (3.12)$$

where:

$$\begin{aligned} M^2 &\equiv m^2 + \frac{1}{8}\sum_{i=1}^{N} p_i^2 \\ X &\equiv \sum_{i<j=1}^{N} x_{ij}^2 \\ Y_m &\equiv \sum_{i=1}^{m-1} r_{im}x_{im} \\ Z_m &\equiv \sum_{j=1}^{m-1}\sum_{k=m}^{N} r_{jk}x_{jk} \end{aligned} \quad (3.13)$$

To demonstrate the application of this representation, we will calculate the most well-established case, namely $N = 2$. We begin with ϕ^3 theory, as it serves as an ideal pedagogical model for understanding the structure of scattering amplitudes.
The key simplification in ϕ^3 theory arises from its interaction vertex being independent of the worldline velocity. In the worldline formalism, this means the final expression for the amplitude does not generate terms with first and second derivatives of the Green's function. Consequently, we ignore the Q-term in (3.2)

$$\hat{I}_2(p_1,p_2) = \int_0^\infty dT T^{-D/2}\left(\frac{T}{\pi}\right)^{\frac{1}{2}} e^{-M^2T}\int_{-\infty}^{\infty} dx_{12}\, \mathrm{e}^{-Tx_{12}^2}\frac{\sinh(TY_2)}{Z_2} \quad (3.14)$$

While it might initially seem more straightforward to solve the integral with respect to T, this approach introduces complexities in the subsequent integration over x_{12}. Consequently, we proceed by first evaluating the integral over x_{12}. To facilitate this, we represent the term $\frac{1}{Z_2}$ using Schwinger parameters and express $\sinh(TY_2)$ using its exponential representation, as demonstrated below.

$$\int_{-\infty}^{\infty} dx_{12}\, \mathrm{e}^{-Tx_{12}^2} \frac{\sinh(TY_2)}{Z_2} = \frac{1}{2}\int_0^{\infty} d\alpha \int_{-\infty}^{\infty} dx_{12} \mathrm{e}^{-Tx_{12}^2-\alpha Z_2}(\mathrm{e}^{TY_2} - \mathrm{e}^{-TY_2}) \quad (3.15)$$

Given that $Z_2 = r_{12}x_{12} = Y_2$:

$$\frac{1}{2}\int_0^{\infty} d\alpha \int_{-\infty}^{\infty} dx_{12} \mathrm{e}^{-Tx_{12}^2-\alpha Z_2}(\mathrm{e}^{TY_2} - \mathrm{e}^{-TY_2}) = \frac{\sqrt{\pi}}{2\sqrt{T}} \int_0^{\infty} d\alpha \left(\mathrm{e}^{\frac{r_{12}^2}{4T}(\alpha+T)^2} - \mathrm{e}^{\frac{r_{12}^2}{4T}(\alpha-T)^2} \right) \quad (3.16)$$

We obtained the following new integral representation for the two point function:

$$\hat{I}_2(p_1,p_2) = \frac{1}{2}\int_0^{\infty} d\alpha \int_0^{\infty} dT T^{-D/2} e^{-M^2 T} \left(\mathrm{e}^{\frac{r_{12}^2}{4T}(\alpha+T)^2} - \mathrm{e}^{\frac{r_{12}^2}{4T}(\alpha-T)^2} \right) \quad (3.17)$$

A change of variables $\bar{\alpha} = \alpha - T$ allows us to rewrite the integration limits, and a rescaling $\bar{\alpha} = aT$ helps us transform the integral over T into a purely Gaussian integral, which will yield the well-known Gamma term of the two-point function.

$$\begin{aligned} \hat{I}_2(p_1,p_2) &= \frac{1}{2}\int_{-1}^{1} da \int_0^{\infty} dT T^{1-D/2} e^{-T\left(M^2-\left(\frac{ar_{12}}{2}\right)^2\right)} \\ &= \frac{\Gamma\left(\frac{4-D}{2}\right)}{2} \int_{-1}^{1} \frac{da}{\left(M^2 - \frac{ar_{12}}{2}^2\right)} \end{aligned} \quad (3.18)$$

The previous example suggests that this new representation requires, in the first instance, exponentiating the Z_i factors to bring the integral of x_i to the following form:

$$\int_{-\infty}^{\infty} dx_i \mathrm{e}^{-Tx_i^2 + T\sum_{m=2}^{N} S_m Y_m - \sum_{m=2}^{N} \alpha_m z_m} \quad (3.19)$$

This approach effectively resolves all x_{ij} integrals. Subsequently, through a series of variable changes, the integration of T can also be performed, leading to a representation of parametric integrals incorporated into the exponentiation of the Z_i. For a structured presentation, the three-point case was thoroughly examined in the preceding section, and the four-photon case will be specifically addressed in the upcoming chapter. In the next section, we will outline the computational procedures within the worldline formalism specifically in the low-energy limit.

3.2 Low Energy Limit

In existing literature, the study of the on-shell N-photon amplitude in the low-energy limit has been approached using spinor helicity techniques applied to the effective action of QED, as detailed in [116]. In [117], we explored the behavior of these results, and a comprehensive analysis of this approximation in the photon scattering scenario can be found in [118]. This section presents a more modern derivation for studying this limit within the worldline formalism.
This low-energy regime signifies that all photon energies are considerably smaller than the scalar mass.

$$\omega_i \ll m, \quad i = 1, \ldots N. \tag{3.20}$$

The crucial question now becomes: how does this limit impact our mathematical derivations? To begin, let's recall the definition of the photon vertex operator (1.8) :

$$V_{scal}[k_i, \varepsilon_i] = \int_0^T d\tau_i \varepsilon_i \cdot \dot{x}(\tau_i) e^{ik_i \cdot x(\tau_i)} \tag{3.21}$$

In this low-energy regime, we approximate the photon vertex operators by retaining only their linear momentum terms.

$$V_{scal}^{(\mathrm{LE})}[k, \varepsilon] \;\; = \;\; \int_0^T d\tau \, \varepsilon \cdot \dot{x}(\tau) ik \cdot x(\tau) \, . \tag{3.22}$$

At this point, we need a mathematical trick. To express the vertex operator by utilizing the photon field strength tensor $f^{\mu\nu} \equiv k^\mu \varepsilon^\nu - \varepsilon^\mu k^\nu$, we must introduce a total derivative,

$$\begin{aligned} V_{scal}^{(\mathrm{LE})}[f] &= V_{scal}^{(\mathrm{LE})}[k, \varepsilon] - \frac{i}{2} \int_0^T d\tau \frac{d}{d\tau} (\varepsilon \cdot x(\tau) k \cdot x(\tau)) \\ &= \frac{i}{2} \int_0^T d\tau x(\tau) \cdot f \cdot \dot{x}(\tau). \end{aligned} \tag{3.23}$$

The previous vertex operator serves as the primary ingredient for performing Wick contractions. However, as is common in the worldline formalism, applying integration by parts will allow us to express the results in terms of "τ-cycles". These cycles are formed from closed chains of the Green's function derivative:

$$\dot{G}_{B_{i_1 i_2}} \dot{G}_{B_{i_2 i_3}} \cdots \dot{G}_{B_{i_n i_1}} \, . \tag{3.24}$$

Nevertheless, these τ-cycles are then used to construct the "bicycles", which are essentially the τ-cycles accompanied by traces of the photon field strength tensors.

$$\int_0^T d\tau_{i_1}\cdots\int_0^T d\tau_{i_n}\dot{G}_{i_1i_2}\dot{G}_{i_2i_3}\cdots\dot{G}_{i_ni_1}\mathrm{tr}(f_{i_1}f_{i_2}\cdots f_{i_n})\,. \tag{3.25}$$

Such a bicycle in the one-dimensional worldline QFT corresponds to the one-loop n-point Feynman diagram depicted in Fig. 3.1:

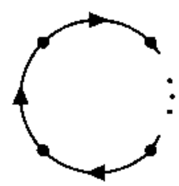

Figure 3.1: Worldline representation of an integrated bicycle diagram contribution.

We now introduce the Lorentz cycles Z_n:

$$Z_2(ij) \equiv \varepsilon_i\cdot k_j\varepsilon_j\cdot k_i-\varepsilon_i\cdot\varepsilon_jk_i\cdot k_j=\frac{1}{2}\mathrm{tr}(f_if_j), \tag{3.26}$$

$$Z_n(i_1i_2\dots i_n) \equiv \mathrm{tr}(f_{i_1}f_{i_2}\cdots f_{i_n}),\quad n\geq 3. \tag{3.27}$$

We can express the contraction of N vertices in the low-energy limit using simple combinatorics as:

$$\left\langle V_{scal}^{\gamma(\mathrm{LE})}[f_1]\dots V_{scal}^{\gamma(\mathrm{LE})}[f_n]\right\rangle = i^nT^n exp\left\{\sum_{m=1}^{\infty}b_{2m}\sum_{\{i_1\dots i_{2m}\}}Z_{2m}^{\mathrm{dist}}(\{i_1i_2\dots i_{2m}\})\right\}\Bigg|_{f_1\dots f_N}, \tag{3.28}$$

The term $Z_{2m}^{\mathrm{dist}}(\{i_1i_2\dots i_{2m}\})$ refers to the sum across every distinct Lorentz cycle that can be created from the provided set of infices. Meanwhile, b_n stands for the foundamental "bosonic cycle integral."

$$b_n\equiv\int_0^1 du_1du_2\dots du_n\dot{G}_{B12}\dot{G}_{B23}\cdots\dot{G}_{Bn1}\,. \tag{3.29}$$

As shown in the previous chapter, the preceding integral can be expressed in terms of Bernoulli numbers (2.7):

$$b_n = \begin{cases} -2^n \frac{B_n}{n!} & n \text{ even}, \\ 0 & n \text{ odd} \end{cases} \tag{3.30}$$

Now we can introduce the total field strength tensor, f_{tot}, which is essentially the sum of each individual photon's field strength tensor $f_{tot} \equiv \sum_{i=1}^{n} f_i$. With this, and by utilizing the combinatorial fact that

$$tr[(f_1 + \cdots + f_N)^n]|_{\text{all}\, different} = 2n \sum_{\{i_1 \ldots i_n\}} Z_n^{\text{dist}}(\{i_1 i_2 \ldots i_n\}). \tag{3.31}$$

After setting the dimension to 4, Equation (1.9) yields the following formula for the one-loop amplitude for an N-photon process, when taken at the low-energy limit:

$$\Gamma_{\text{scal}}^{(LE)}(k_1, \varepsilon_1; \ldots; k_N, \varepsilon_N) = \frac{e^N \Gamma(N-2)}{(4\pi)^2 m^{2N-4}} \exp\left\{ \sum_{m=1}^{\infty} \frac{b_{2m}}{4m} \text{tr}(f_{\text{tot}}^{2m}) \right\} \Bigg|_{f_1 \ldots f_N}. \tag{3.32}$$

To proceed, we will consider the general case for the helicity configuration of the N-photons. Let us assume that among these N photons, K have positive helicity '+' and the remaining $N-K$ have negative helicity '-'. The conventions and definitions for our use of the spinor-helicity formalism are detailed in Chapter 4. This assumption allows the total constant field strength to be mathematically represented by the following sum:

$$f_{tot} = \sum_{i=1}^{K} f_i^+ + \sum_{j=K+1}^{K+L} f_j^- = f^+ + f^-. \tag{3.33}$$

At the heart of the spinor-helicity formalism lies the idea of replacing 4-momentum vectors with more fundamental objects: two-component Weyl spinors[†].Given a massless 4-momentum k_i^μ, it is decomposed into two spinors: a negative helicity spinor, $|k_i\rangle$, and a positive helicity spinor, $|k_i]$. These objects are not merely abstract notation; they are directly related to the standard Minkowski dot product through the fundamental identity:

$$-2k_i \cdot k_j = \langle k_i k_j \rangle [k_j k_i] \tag{3.34}$$

Thanks to this correspondence, any scattering amplitude can be expressed as a rational function of these brackets. This representation reveals a structural simplicity that is hidden when using 4-vector notation, vastly simplifying the calculations.

[†]For a more detailed description, see Chapter 50 of [17].

The steps to derive the following identity are detailed in [117], but here we will only present the final result:

$$\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \chi_+ + \chi_-, \tag{3.35}$$

$$\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} = -i(\chi_+ - \chi_-). \tag{3.36}$$

In the previous equations, we have introduced the notation:

$$\chi_+ \equiv \frac{1}{2}\sum_{1\leq i<j\leq N}[k_ik_j]^2, \quad \chi_- \equiv \frac{1}{2}\sum_{1\leq i<j\leq N}\langle k_ik_j\rangle^2. \tag{3.37}$$

A significant simplification of the previous result (3.32) occurs in the special case where all photons possess positive helicity. Using this new notation, it can be shown that:

$$f_{\rm tot}^2 = -\chi_+ \tag{3.38}$$

Moreover:

$$\mathrm{tr}(f_{\rm tot}^{2m}) = 4(-1)^m(\chi_+)^m. \tag{3.39}$$

This allows us to rewrite

$$exp\left\{\sum_{m=1}^{\infty}\frac{b_{2m}}{4m}\mathrm{tr}(f_{\rm tot}^{2m})\right\}\Bigg|_{f_1\dots f_N} = exp\left\{\sum_{m=1}^{\infty}(-1)^m\frac{b_{2m}}{m}\chi_+^m\right\}\Bigg|_{all\ different}. \tag{3.40}$$

When we employ the series expansion,

$$\frac{\chi_+}{sin^2\sqrt{\chi_+}} = -\sum_{k=0}^{\infty}(-1)^k\frac{2k-1}{(2k)!}2^{2k}B_{2k}\chi_+^k, \tag{3.41}$$

it becomes evident that the exponent corresponds to the series expansion of $-2\ln\frac{sin\sqrt{\chi_+}}{\sqrt{\chi_+}}$ This observation allows us to finally express the low-energy limit of the N-photon "all +" amplitudes as [119]:

$$\Gamma_{\rm scal}^{(\rm LE)}(k_1,\varepsilon_1;...;k_N,\varepsilon_N) = -\frac{m^4}{(4\pi)^2}\left(\frac{2ie}{m^2}\right)^N\frac{B_N}{N(N-2)}\chi_N^+ \tag{3.42}$$

in which we additionally define:

$$\chi_N^+ \equiv \chi_+^{N/2}|_{\text{all different}} = \frac{(N/2)!}{2^{N/2}} \left\{[k_1k_2]^2[k_3k_4]^2 \cdots [k_{N-1}k_N]^2 + \text{all distinct permutations}\right. . \tag{3.43}$$

As has been demonstrated throughout this work and other publications [21, 22, 27, 103], the transition from the scalar to the spinorial case primarily involves a substitution. The low-energy limit is no exception; in this instance, we simply replace the chain integral (3.29) with the "super chain integral"

$$\int_0^1 du_1...du_n(\dot{G}_{B12}\dot{G}_{B23}\cdots\dot{G}_{Bn1} - G_{F_{12}}G_{F_{23}}\cdots G_{F_{n1}}) = (2-2^n)b_n. \tag{3.44}$$

The only remaining modification is the inclusion of a global factor of (-2), accounting for statistics and degrees of freedom. Consequently:

$$\Gamma_{\text{spin}}^{(LE)}(k_1,\varepsilon_1;...;k_N,\varepsilon_N) = -2\frac{e^N\Gamma(N-2)}{(4\pi)^2m^{2N-D}}\exp\left\{\sum_{m=1}^{\infty}(1-2^{2m-1})\frac{b_{2m}}{2m}\text{tr}(f_{\text{tot}}^{2m})\right\}\Bigg|_{f_1...f_N} . \tag{3.45}$$

Here, we'll delve into the structure of the amplitude coefficients, as presented in [116], assuming that these N photons are distributed such that K exhibit "+" helicity and the remaining ones exhibit "-" helicity:

$$\begin{aligned}
\Gamma_{scalar}^{(1)(EH)}[\varepsilon_1^+;...;\varepsilon_K^+;\varepsilon_{K+1}^-;...;\varepsilon_N^-] &= \frac{m^4}{16\pi^2}\left(\frac{2e}{m^2}\right)^N c_{\text{scal}}^{(1)}\left(\frac{K}{2},\frac{N-K}{2}\right)\chi_K^+\chi_{N-K}^-,\\
\Gamma_{spin}^{(1)(EH)}[\varepsilon_1^+;...;\varepsilon_K^+;\varepsilon_{K+1}^-;...;\varepsilon_N^- &= -\frac{m^4}{8\pi^2}\left(\frac{2e}{m^2}\right)^N c_{spin}^{(1)}\left(\frac{K}{2},\frac{N-K}{2}\right)\chi_K^+\chi_{N-K}^-.
\end{aligned} \tag{3.46}$$

Where,

$$\begin{aligned}
\chi_K^+ &\equiv (\chi_+)^{\frac{K}{2}}|_{\text{all different}}\\
&= \frac{\left(\frac{K}{2}\right)!}{2^{K/2}}\left\{[k_1k_2]^2[k_3k_4]^2\cdots[k_{K-1}k_K]^2 + \text{all permutations}\right. ,
\end{aligned} \tag{3.47}$$

$$\begin{aligned}
\chi_{N-K}^- &\equiv (\chi_-)^{\frac{N-K}{2}}|_{\text{all different}}\\
&= \frac{\left(\frac{N-K}{2}\right)!}{2^{\frac{N-K}{2}}}\left\{\langle k_{K+1}k_{K+2}\rangle^2\langle k_{K+3}k_{K+4}\rangle^2\cdots\langle k_{N-1}k_N\rangle^2 + \text{all permutations}\right.
\end{aligned} \tag{3.48}$$

The coefficients for the scalar and spinorial cases are, respectively, defined as:

$$\begin{aligned} c^{(1)}_{scal}\Big(\frac{K}{2},\frac{N-K}{2}\Big) &= (-1)^{\frac{N}{2}}(N-3)!\sum_{k=0}^{K}\sum_{l=0}^{N-K}(-1)^l \\ &\quad\times\frac{(1-2^{1-k-l})(1-2^{1-N+k+l})\mathcal{B}_{k+l}\mathcal{B}_{N-k-l}}{k!l!(K-k)!(N-K-l)!}. \\ c^{(1)}_{spin}\Big(\frac{K}{2},\frac{N-K}{2}\Big) &= (-1)^{\frac{N}{2}}(N-3)!\sum_{k=0}^{K}\sum_{l=0}^{N-K}(-1)^l\frac{\mathcal{B}_{k+l}\mathcal{B}_{N-k-l}}{k!l!(K-k)!(N-K-l)!}. \end{aligned} \tag{3.49}$$

In the following, we will recap the study of these sums and finally present a new representation, accompanied by some specific cases for the amplitude of N photons at the one-loop level in the low-energy limit using spinor helicity techniques.
The first step in manipulating the double summation

$$S_{\text{spin}}[K,L]=\sum_{k=0}^{K}\sum_{l=0}^{L}\frac{(-)^l B_{k+l}B_{N-k-l}}{k!l!(K-k)!(L-l)!}, \tag{3.50}$$

is to apply the change of variables $L=N-K$ and $k+l=n$:

$$S_{\text{spin}}[K,L]\rightarrow S_{\text{scal}}[N,K] \tag{3.51}$$

This change of variables allows us to write :

$$S_{\text{spin}}[N,K]=\sum_{n=0}^{N}(-1)^n B_n B_{N-n}\frac{1}{K!(N-K)!}\sum_{k=0}^{K}(-1)^k\binom{K}{k}\binom{N-K}{n-k} \tag{3.52}$$

Before moving on to the most general case, we first study one particular case. Recall that we consider a system with N photons in the low-energy limit, where K of them have positive helicity and L have negative helicity. The case in consideration corresponds to a system in which all photons share the same helicity, i.e., $K=N$. For this particular case, we have:

$$S_{\text{spin}}[N,N]=\sum_{n=0}^{N}(-1)^n B_n B_{N-n}\frac{1}{N!0!}\sum_{k=0}^{K}(-1)^k\binom{K}{k}\binom{0}{n-k} \tag{3.53}$$

The binomial can be written in terms of the rising factorial, also known as the Pochhammer symbol, as follows:

$$\binom{N-K}{n-k} = \frac{(N-K)_{n-k}}{(n-k)!} \tag{3.54}$$

In this case, when $N-K=0$, we have:

$$(0)_{n-k} = 0. \tag{3.55}$$

However, it is possible that $n-k=0$, in which case:

$$\binom{0}{n-k} = \begin{cases} 1, & \text{if } n-k=0, \\ 0, & \text{if } n-k>0. \end{cases} \tag{3.56}$$

If the system has the same helicity, the sum over k has only one contribution for $k=n$, leading to:

$$\sum_{k=0}^{K}(-1)^k\binom{K}{k}\binom{0}{n-k} = (-1)^n\binom{N}{n}. \tag{3.57}$$

Thus, we obtain Euler's well-known identity:

$$\begin{aligned} S_{\text{spin}}[N,N] &= \sum_{n=0}^{N}\frac{B_n}{n!}\frac{B_{N-n}}{(N-n)!} \\ &= -(N-1)\frac{B_N}{N!}. \end{aligned} \tag{3.58}$$

For the scalar case:

$$S_{\text{scal}}[K,L] = \sum_{k=0}^{K}\sum_{l=0}^{L}(1-2^{1-k-l})(1-2^{1-N+k+l})\frac{(-)^l B_{k+l}B_{N-k-l}}{k!l!(K-k)!(L-l)!}. \tag{3.59}$$

Following the same steps as in the spinor case, we find that the factor distinguishing the spinor and scalar cases does not affect the final result.:

$$\begin{aligned} S_{\text{scal}}[N,N] &= \sum_{n=0}^{N}(1-2^{1-n})(1-2^{1-N+n})\frac{B_n}{n!}\frac{B_{N-n}}{(N-n)!} \\ &= -(N-1)\frac{B_N}{N!}. \end{aligned} \tag{3.60}$$

For the general case the sum over k :

$$p_n(K,N-K) = \sum_{k=0}^{K}(-1)^k\binom{K}{k}\binom{N-K}{n-k} \tag{3.61}$$

After multiple evaluations for different values of N, n and K, we found another way to express $p_n(K,N-K)$:

$$p_n(K,N-K) = \frac{(-1)^{n/2}\Gamma[N-K+1]P_{K-\frac{N}{2}}(K,n)}{\left(\frac{n}{2}\right)!\Gamma\left[\frac{N}{2}-\frac{n}{2}+1\right]}. \tag{3.62}$$

Where:

$$P_{K-\frac{N}{2}}(K,n) := \sum_{\mu=0}^{K-\frac{N}{2}}(-1)^{\mu}\binom{2K-N}{2\mu}\left(\frac{N}{2}-\frac{n}{2}\right)_{K-\frac{N}{2}-\mu}\left(\frac{n}{2}\right)_{\mu}, \tag{3.63}$$

In this equation, the notation for the raising factorial, also known as the Pochhammer symbol, appears for the first time. The raising factorial is defined as follows:

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{3.64}$$

Moreover, it is found that:

$$P_{K-\frac{N}{2}}(K,n) = \frac{\Gamma\left[K-\frac{n}{2}\right]}{\Gamma\left[\frac{N-n}{2}\right]}{}_3F_2\left(\frac{n}{2},\frac{N}{2}-K,\frac{N}{2}-K+\frac{1}{2};\frac{1}{2},1-K+\frac{n}{2};1\right) \tag{3.65}$$

With all the above:

$$\begin{aligned} p_n(K,N-K) \;&=\; \frac{(-1)^{\frac{n}{2}}\Gamma[N-K+1]\Gamma\left[K-\frac{n}{2}\right]}{\left(\frac{n}{2}\right)!\Gamma\left[\frac{N}{2}-\frac{n}{2}+1\right]\Gamma\left[\frac{N-n}{2}\right]} \\ &\quad {}_3F_2\left(\frac{n}{2},\frac{N}{2}-K,\frac{N}{2}-K+\frac{1}{2};\frac{1}{2},1-K+\frac{n}{2};1\right) \end{aligned} \tag{3.66}$$

Despite the existence of many relations for the hypergeometric function ${}_3F_2(a_1,a_2,a_3;b_1,b_2;z)$ when its argument $z=1$ (see https://functions.wolfram.com/PDF/Hypergeometric3F2.pdf), none of them help us further simplify our result. . This is why our new representations for $S_{\text{spin}}[N,K]$ and $S_{\text{scal}}[N,K]$ are:

$$
\begin{aligned}
S_{\text{spin}}[N,K] &= \frac{1}{K!}\sum_{n=0}^{N}(-1)^{\frac{n}{2}}\frac{B_n}{\left(\frac{n}{2}\right)!}\frac{B_{N-n}}{\left(\frac{N-n}{2}\right)!}P_{K-\frac{N}{2}}(K,n)\\
S_{\text{scal}}[N,K] &= \frac{1}{K!}\sum_{n=0}^{N}(1-2^{1-n})(1-2^{1-N+n})(-1)^{\frac{n}{2}}\frac{B_n}{\left(\frac{n}{2}\right)!}\frac{B_{N-n}}{\left(\frac{N-n}{2}\right)!}P_{K-\frac{N}{2}}(K,n)
\end{aligned}
\tag{3.67}
$$

This concludes our study of the N-photon amplitude in the low-energy limit. In the next chapter, dedicated to the four-photon amplitude, we will present the corresponding calculations for this limit.

Chapter 4

4-photon Amplitude in the Wordline Formalism

In this chapter, we will employ the potential and advantages of the worldline formalism to recalculate the on-shell 4-photon amplitude with the aim of unifying its calculation in both scalar and spinor QED. Taking advantage of this formalism led to our publication [81], where we also included the study of the infinite mass limit. Our calculations, which are more technical, remain valid both on and off shell; however, prior to initiating the most basic calculation, 'the basic box integral,' we shall introduce the Q-representation for the 4-photon amplitude in scalar QED, which was initially developed within the worldline formalism in [27]:

$$\begin{aligned}\Gamma_{\rm scal}[k_1,\varepsilon_1;\dots;k_4,\varepsilon_4] \quad &= \frac{e^4}{(4\pi)^{\frac{D}{2}}}\int\frac{dT}{T}T^{4-D/2}\\ &\times\int_0^1 du_1du_2du_3du_4Q_4(\dot G_{ij})\exp\left\{\frac{T}{2}\sum_{i,j=1}^4 G_{ij}k_i\cdot k_j\right\}\end{aligned} \tag{4.1}$$

where $Q_4(\dot G_{ij}) = Q_4^4+Q_4^3+Q_4^2+Q_4^{22}$.
With:

$$\begin{aligned}Q_4 &= Q_4^4+Q_4^3+Q_4^2+Q_4^{22}\\ Q_4^4 &= \dot G(1234)+\dot G(1243)+\dot G(1324)\\ Q_4^3 &= \dot G(123)T(4)+\dot G(234)T(1)+\dot G(341)T(2)+\dot G(412)T(3)\\ Q_4^2 &= \dot G(12)T(34)+\dot G(13)T(24)+\dot G(14)T(23)+\dot G(23)T(14)+\dot G(24)T(13)+\dot G(34)T(12)\\ Q_4^{22} &= \dot G(12)\dot G(34)+\dot G(13)\dot G(24)+\dot G(14)\dot G(23)\end{aligned} \tag{4.2}$$

The previous representation is expressed in terms of bi-clycle, defined as

$$\dot{G}(i_1 i_2 \dots i_n) \equiv \dot{G}_{i_1 i_2} \dot{G}_{i_2 i_3} \cdots \dot{G}_{i_n i_1} Z_n(i_1 i_2 \dots i_n) \tag{4.3}$$

These bi-cycle encompass what is known as the Lorentz-cycle Z_n:

$$\begin{aligned} Z_2(ij) &\equiv \frac{1}{2}\mathrm{tr}(f_i f_j) = \varepsilon_i \cdot k_j \varepsilon_j \cdot k_i - \varepsilon_i \cdot \varepsilon_j k_i \cdot k_j \\ Z_n(i_1 i_2 \cdots i_n) &\equiv \mathrm{tr}\left(\prod_{j=1}^{n} f_{i_j}\right) \quad (n \geq 3) \end{aligned} \tag{4.4}$$

It is within this context that the dependence on the field intensity tensor $f_i^{\mu\nu} \equiv k_i^\mu \varepsilon_i^\nu - \varepsilon_i^\mu k_i^\nu$, associated with each photon, emerges.
In addition to the aforementioned building blocks, it is necessary to define the one- and two-tails $T(a), T(ab)$:

$$\begin{aligned} T(a) &\equiv \sum_{i \neq a} \dot{G}_{ai} \varepsilon_a \cdot k_i \\ T(ab) &\equiv \sum_{\substack{i \neq a, j \neq b \\ (i,j) \neq (b,a)}} \dot{G}_{ai} \varepsilon_a \cdot k_i \dot{G}_{bj} \varepsilon_b \cdot k_j + \frac{1}{2} \dot{G}_{ab} \varepsilon_a \cdot \varepsilon_b \Big[\sum_{i \neq a,b} \dot{G}_{ai} k_a \cdot k_i - \sum_{j \neq b,a} \dot{G}_{bj} k_b \cdot k_j \Big]. \end{aligned} \tag{4.5}$$

The previous Q-representation for the 4-photon amplitude in Scalar QED is already concise. However, within these structures, polarization vectors emerge in the tails. Interestingly, these vectors do not form a field-strength tensor on their own. Yet, a significant aspect is that these polarization vectors can be absorbed into field-strength tensors, a process made through integration by parts and the introduction of an arbitrary "reference vector" r_i, which obeys the condition $r_i \cdot k_i \neq 0$. This incorporation significantly enhances the Q-representational efficiency and allows us to write the one- and two-tail as

$$\begin{aligned} T_R(i) &= \sum_{j \neq i} \dot{G}_{ij} \, \frac{r_i \cdot f_i \cdot k_j}{r_i \cdot k_i}, \\ T_{sh}(ab) &= \sum_{r,s \neq a,b} \dot{G}_{ra} \dot{G}_{bs} \, \frac{k_r \cdot f_a \cdot f_b \cdot k_s}{k_a \cdot k_b}. \end{aligned} \tag{4.6}$$

The subscript 'sh' denotes 'short', referring to the definition of this short tail where the indices r, s are limited to values in the cycle variables. This restriction results in only four terms, as opposed to the eight terms derived from 4.5.
At this point, we shall write the four-photon amplitude in a minimal tensor basis as:

$$\Gamma_{scal} = \Gamma^{(1)}_{scal} + \Gamma^{(2)}_{scal} + \Gamma^{(3)}_{scal} + \Gamma^{(4)}_{scal} + \Gamma^{(5)}_{scal} \tag{4.7}$$

$$\begin{aligned}
\Gamma^{(1)}_{scal} &= \Gamma^{scal}_{(1234)} T^{(1)}_{(1234)} + \Gamma^{scal}_{(1243)} T^{(1)}_{(1243)} + \Gamma^{scal}_{(1324)} T^{(1)}_{(1324)}, \\
\Gamma^{(2)}_{scal} &= \Gamma^{scal}_{(12)(34)} T^{(2)}_{(12)(34)} + \Gamma^{scal}_{(13)(24)} T^{(2)}_{(13)(24)} + \Gamma^{scal}_{(14)(23)} T^{(2)}_{(14)(23)}, \\
\Gamma^{(3)}_{scal} &= \sum_{i=1,2,3} \Gamma^{scal}_{(123)i} T^{(3)r_4}_{(123)i} + \sum_{i=2,3,4} \Gamma^{scal}_{(234)i} T^{(3)r_1}_{(234)i} + \sum_{i=3,4,1} \Gamma^{scal}_{(341)i} T^{(3)r_2}_{(341)i} + \sum_{i=4,1,2} \Gamma^{scal}_{(412)i} T^{(3)r_3}_{(412)i}, \\
\Gamma^{(4)}_{scal} &= \sum_{i<j} \Gamma^{scal}_{(ij)ii} T^{(4)}_{(ij)ii} + \sum_{i<j} \Gamma^{scal}_{(ij)jj} T^{(4)}_{(ij)jj}, \\
\Gamma^{(5)}_{scal} &= \sum_{i<j} \Gamma^{scal}_{(ij)ij} T^{(5)}_{(ij)ij} + \sum_{i<j} \Gamma^{scal}_{(ij)ji} T^{(5)}_{(ij)ji},
\end{aligned} \tag{4.8}$$

where,

$$\begin{aligned}
T^{(1)}_{(1234)} &\equiv Z(1234)\,, \\
T^{(2)}_{(12)(34)} &\equiv Z(12)Z(34)\,, \\
T^{(3)r_4}_{(123)i} &\equiv Z(123)\frac{r_4 \cdot f_4 \cdot k_i}{r_4 \cdot k_4} \quad (i = 1, 2, 3)\,, \\
T^{(4)}_{(12)ii} &\equiv Z(12)\,\frac{k_i \cdot f_3 \cdot f_4 \cdot k_i}{k_3 \cdot k_4} \quad (i = 1, 2)\,, \\
T^{(5)}_{(12)ij} &\equiv Z(12)\,\frac{k_i \cdot f_3 \cdot f_4 \cdot k_j}{k_3 \cdot k_4} \quad ((i, j) = (1, 2), (2, 1))\,.
\end{aligned} \tag{4.9}$$

It is important to mention that the amplitude's worldline representation displayed manifest permutative symmetry from the beginning, and this symmetry was consistently preserved during all the integrations by parts. Additionally, it's noteworthy that, through a distinct procedure and up to normalisation, along with different notation, the exact same tensorial basis for this amplitude was also identified in [120].
Our interest lies in solving the integrals of the coefficient functions Γ, which, with the current choice of tails, appear as:

$$\Gamma^{scal}_{\ldots} = \frac{e^4}{(4\pi)^{\frac{D}{2}}} \int_0^\infty \frac{dT}{T} T^{4-\frac{D}{2}} \mathrm{e}^{-m^2 T} \int_0^1 \prod_{i=1}^{4} du_i\, \gamma^{scal}_{\ldots}(\dot{G}_{ij})\, \mathrm{e}^{(\cdot)}, \tag{4.10}$$

where we introduced the abbreviation

$$\mathrm{e}^{(\cdot)} \equiv \exp\left[\frac{1}{2}\sum_{i,j=1}^{N} G_{ij} k_i \cdot k_j\right] . \tag{4.11}$$

Having executed the standard rescaling to the unit interval the $\gamma^{scal}_{\dots}$ are given by:

$$\begin{aligned}
\gamma^{scal}_{(1234)} &= \dot{G}_{12}\dot{G}_{23}\dot{G}_{34}\dot{G}_{41}\\
\gamma^{scal}_{(12)(34)} &= \dot{G}_{12}\dot{G}_{21}\dot{G}_{34}\dot{G}_{43}\\
\gamma^{scal}_{(123)i} &= \dot{G}_{12}\dot{G}_{23}\dot{G}_{31}\dot{G}_{4i}\\
\gamma^{scal}_{(12)ii} &= \dot{G}_{12}\dot{G}_{21}\dot{G}_{i3}\dot{G}_{4i}\\
\gamma^{scal}_{(12)ij} &= \dot{G}_{12}\dot{G}_{21}\dot{G}_{i3}\dot{G}_{4j} \, .
\end{aligned} \tag{4.12}$$

The aforementioned result is suitable for applying the replacement rule, thereby facilitating the derivation of our outcome for spinor QED. Upon applying this rule, it is found that:

$$\Gamma^{spin}_{\dots} = (-2)\frac{e^4}{(4\pi)^{\frac{D}{2}}}\int_0^\infty \frac{dT}{T} T^{4-\frac{D}{2}} \mathrm{e}^{-m^2T} \int_0^1 \prod_{i=1}^{4} du_i \, \gamma^{spin}_{\dots}(\dot{G}_{ij}) \, \mathrm{e}^{(\cdot)} \tag{4.13}$$

where now

$$\gamma^{(1)spin}_{(1234)} = \dot{G}_{12}\dot{G}_{23}\dot{G}_{34}\dot{G}_{41} - G_{F12}G_{F23}G_{F34}G_{F41} \tag{4.14}$$

$$\begin{aligned}
\gamma^{(2)spin}_{(12)(34)} &= \big(\dot{G}_{12}\dot{G}_{21} - G_{F12}G_{F21}\big)\big(\dot{G}_{34}\dot{G}_{43} - G_{F34}G_{F43}\big)\\
\gamma^{(3)spin}_{(123)i} &= \big(\dot{G}_{12}\dot{G}_{23}\dot{G}_{31} - G_{F12}G_{F23}G_{F31}\big)\dot{G}_{4i}\\
\gamma^{(4)spin}_{(12)ii} &= \big(\dot{G}_{12}\dot{G}_{21} - G_{F12}G_{F21}\big)\dot{G}_{i3}\dot{G}_{4i}\\
\gamma^{(5)spin}_{(12)ij} &= \big(\dot{G}_{12}\dot{G}_{21} - G_{F12}G_{F21}\big)\dot{G}_{i3}\dot{G}_{4j}
\end{aligned} \tag{4.15}$$

In the previous expressions, we have made use of the replacement rule (1.20). These rules essentially provide a prescription for leveraging the known calculation in the scalar case to obtain the corresponding spinor result. In the case of QED, every closed cycle for example:

$$\dot{G}_{12}\dot{G}_{23}\dot{G}_{34}\dot{G}_{41} \tag{4.16}$$

is replaced by

$$\dot{G}_{12}\dot{G}_{23}\dot{G}_{34}\dot{G}_{41} - G_{F12}G_{F23}G_{F34}G_{F41} \tag{4.17}$$

where the fermionic Green's function, defined by

$$G_{F_{ij}} = \text{sign}(u_i - u_j) \tag{4.18}$$

is introduced. Additionally, a factor of -2 is included to account for Fermi-Dirac statistics.

4.1 ϕ^3 theory

To understand the structure of such integrals, we will begin with the simplest case by setting $\gamma^{\text{scal}}_{\dots} = 1$. In other words, we will focus on the basic box integral for the ϕ^3 theory.
Studying amplitudes in Scalar QED within the worldline formalism is greatly simplified by starting with a toy model: the (ϕ^3) theory. The reason for this simplification lies in the nature of the interaction. Unlike in QED, where the photon-current interaction depends on the particle's velocity in the loop (represented by the $\dot{x}^\mu$ term in the vertex), the interaction in the ϕ^3 model is purely scalar and lacks this velocity dependence.
In the language of the formalism, this difference is crucial: the absence of the velocity term means that the amplitude calculation does not generate the first or second derivative of the Green's function ($\dot{G}_{ij}$ and $\ddot{G}_{ij}$). This allows us to isolate and understand the fundamental structure of the amplitude, which depends solely on the Green's function (G_{ij}), before introducing the additional algebraic complexity of its derivatives - a characteristic feature of full QED.
In order to proceed, we will operate within a four-dimensional framework, where divergences are naturally absent, while omitting the prefactor $\frac{e^4}{(4\pi)^4}$. Consistent with the conventional methodology applied in a four-point on-shell computation, we introduce the Mandelstam variables

$$\begin{aligned} s &= -(k_1 + k_2)^2 = -2k_1 \cdot k_2 = -2k_3 \cdot k_4 & (4.19)\\ t &= -(k_1 + k_3)^2 = -2k_1 \cdot k_3 = -2k_2 \cdot k_4 & (4.20)\\ u &= -(k_1 + k_4)^2 = -2k_1 \cdot k_4 = -2k_2 \cdot k_3 & (4.21) \end{aligned}$$

which fulfill

$$s + t + u = 0 \tag{4.22}$$

Additionally, we will consider the s-channel kinematics, characterized by the conditions $s > 0$, $t < 0$, $u < 0$. Furthermore, we will operate below the threshold, maintaining that $s \leq 4m^4$. Consequently, the master formula for a four-point scenario is presented as follows:

$$
\begin{aligned}
I_4 &\equiv \int_0^\infty dT\, T\, \mathrm{e}^{-m^2 T} \int_0^1 du_1 ... du_4\, \mathrm{e}^{-\frac{T}{2}\left[(G_{12}+G_{34})s+(G_{13}+G_{24})t+(G_{14}+G_{23})u\right]} \\
&= \int_0^\infty dT\, T\, \mathrm{e}^{-m^2 T} \int_0^1 du_1 ... du_4\, \mathrm{e}^{T\Lambda} \,. \qquad (4.23)
\end{aligned}
$$

By substituting the Green's function and introducing the abbreviation $u_{ij} \equiv u_i - u_j$, the exponent Λ is represented as:

$$
\begin{aligned}
\Lambda &= -\frac{1}{2}\Big[(|u_{12}| + 2u_1u_2 + |u_{34}| + 2u_3u_4)s + (|u_{13}| + 2u_1u_3 + |u_{24}| + 2u_2u_4)t \\
&\quad +(|u_{14}| + 2u_1u_4 + |u_{23}| + 2u_2u_3)u\Big] \qquad (4.24)
\end{aligned}
$$

As previously mentioned, our initial method involves sector decomposition of such integrals. For this particular case, we will begin by addressing the integral over u_4. To accomplish this, it is necessary to define an order for the remaining legs and we choose $u_1 > u_2 > u_3$. Therefore, the integral over u_4 is expressed as follows

$$
\int_0^1 du_4 = \int_0^{u_3} du_4 + \int_{u_3}^{u_2} du_4 + \int_{u_2}^{u_1} du_4 + \int_{u_1}^1 du_4 \qquad (4.25)
$$

Prior to executing the computation, it is expected that the lower limit of the first integral ($u_4 = 0$) should coincide with the upper limit of the final integral ($u_4 = 1$). This requirement stems from the context of operating on the unit circle, wherein the start and end points are essentially indiscernible.

Consequently, the outcome derived from the four preceding sectors is

$$
\begin{aligned}
\int_0^{u_3} du_4\, \mathrm{e}^{T\Lambda} &= T^{-1}(u_{23}s - u_{12}u)^{-1}\Big[\mathrm{e}^{u_{23}(1-u_{13})sT} - \mathrm{e}^{u_{23}(1-u_1)sT+u_{12}u_3uT}\Big] \\
\int_{u_3}^{u_2} du_4\, \mathrm{e}^{T\Lambda} &= T^{-1}\big((-1+u_{23})s - u_{12}u\big)^{-1}\Big[\mathrm{e}^{u_{12}u_{23}tT} - \mathrm{e}^{u_{23}(1-u_{13})sT}\Big] \\
\int_{u_2}^{u_1} du_4\, \mathrm{e}^{T\Lambda} &= T^{-1}\big(u_{23}s + (1-u_{12})u\big)^{-1}\Big[\mathrm{e}^{u_{12}(1-u_{13})uT} - \mathrm{e}^{u_{12}u_{23}tT}\Big] \\
\int_{u_1}^{1} du_4\, \mathrm{e}^{T\Lambda} &= T^{-1}(u_{23}s - u_{12}u)^{-1}\Big[\mathrm{e}^{u_{23}(1-u_1)sT+u_{12}u_3uT} - \mathrm{e}^{u_{12}(1-u_{13})uT}\Big] \,.
\end{aligned} \qquad (4.26)
$$

We express the last result as

$$T\int_0^1 du_4\,\mathrm{e}^{T\Lambda} = \left[\frac{1}{u_{23}s+(1-u_{12})u} - \frac{1}{u_{23}s-u_{12}u}\right]\mathrm{e}^{u_{12}(1-u_{13})uT} + \left[\frac{1}{(-1+u_{23})s-u_{12}u} - \frac{1}{u_{23}s+(1-u_{12})u}\right]\mathrm{e}^{u_{12}u_{23}tT} + \left[\frac{1}{u_{23}s-u_{12}u} - \frac{1}{(-1+u_{23})s-u_{12}u}\right]\mathrm{e}^{u_{23}(1-u_{13})sT}\,. \tag{4.27}$$

Recall that in order to solve the integral over u_4, we employed a specific ordering for the remaining legs. However, we can disregard this sequence if we express our result in terms of Green's functions, thereby obtaining

$$T\int_0^1 du_4\,\mathrm{e}^{T\Lambda} = \left[\frac{2}{u+\dot{G}_{12}t+\dot{G}_{13}s} + \frac{2}{u-\dot{G}_{12}t-\dot{G}_{13}s}\right]\mathrm{e}^{\frac{1}{2}(G_{12}+G_{13}-G_{23})uT} + \left[\frac{2}{t+\dot{G}_{23}s+\dot{G}_{21}u} + \frac{2}{t-\dot{G}_{23}s-\dot{G}_{21}u}\right]\mathrm{e}^{\frac{1}{2}(G_{12}+G_{23}-G_{13})tT} + \left[\frac{2}{s+\dot{G}_{31}u+\dot{G}_{32}t} + \frac{2}{s-\dot{G}_{31}u-\dot{G}_{32}t}\right]\mathrm{e}^{\frac{1}{2}(G_{13}+G_{23}-G_{12})sT}\,. \tag{4.28}$$

Initially, the sector decomposition results in a greater number of integrals, thereby leading to more terms to resolve. However, as evident in the preceding expression, the second and third lines are merely permutations of the first. Consequently, it suffices to solve the first line and then apply the substitutions $(s,t,u)\to(u,s,t)$ ($(s,t,u)\to(t,u,s)$) and the change of integration variables $(u_1,u_2,u_3)\to(u_2,u_3,u_1)$ ($(u_1,u_2,u_3)\to(u_3,u_1,u_2)$).

Focusing solely on resolving the first line, we will apply our expertise in the worldline formalism, which teaches us that the orientation-reversing transformation $u_i\to 1-u_i\,,(i=1,2,3)$, leaves G_{ij} invariant, but alters the sign in each $\dot{G}_{ij}$. Thus, it will suffice to solve only one of the two terms in each line.

We decided to proceed with the integral over u_3. For this purpose, we are going to establish the order $u_1>u_2$ so :

$$\int_0^1 du_3 = \int_0^{u_2} du_3 + \int_{u_2}^{u_1} du_3 + \int_{u_1}^1 du_3\,, \tag{4.29}$$

This approach falls within the "moderately ambitious" strategy, as it allows us to utilize sector decomposition for solving the integrals. This distinguishes it from more ambitious strategies that would avoid such decomposition entirely.

Similarly to the integral of u_4, in this case, we also anticipate that the term arising from the lower limit of the first integral will be cancelled by the result of the upper limit of the final integral. The remaining terms can be evaluated using a standard integral identity. We apply the following formula from Abramowitz and Stegun [121]:

$$\int_{x_1}^{x_2} dx\, \frac{e^{-ax}}{x+b} = e^{ab} \left[\mathrm{Ei}\left(-a(b+x_2)\right) - \mathrm{Ei}\left(-a(b+x_1)\right)\right], \tag{4.30}$$

where $\mathrm{Ei}(x)$ is the exponential integral function, defined for real, non-zero values of x as [121]:

$$\mathrm{Ei}(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t}\, dt. \tag{4.31}$$

Given our selection of the ordering $u_1 > u_2$, it is necessary to similarly address the integral for the alternate ordering $u_1 < u_2$. Interestingly, we observe that in the context of the $u_1 > u_2$ ordering, the terms emerging from the $u_2 < u_3 < u_1$ sector are equivalent to those in the $u_1 < u_2$ ordering, which are attributed to the combined sectors $0 < u_3 < u_1$ and $u_2 < u_3 < 1$.
Similarly, the terms derived under the first ordering from the sectors $0 < u_3 < u_2$ and $u_1 < u_3 < 1$ correspond to those under the second ordering from the $u_2 < u_3 < u_1$ sector. This allows us to consolidate both results, expressing them in terms of G_{12} and $\dot{G}_{12}$, thereby obtaining

$$\begin{aligned}\int_0^1 du_3 \frac{2\,\mathrm{e}^{\frac{1}{2}(G_{12}+G_{13}-G_{23})uT}}{u+\dot{G}_{12}t+\dot{G}_{13}s} \quad &= \quad \frac{1}{s}\mathrm{e}^{-A\frac{T}{s}}\Big[Ei\Big(A\frac{T}{s}\Big) - Ei\Big(-B\frac{T}{s}\Big)\Big] \\ &-\frac{1}{s}\,\mathrm{e}^{-(A+C)\frac{T}{s}}\Big[Ei\Big((A+C)\frac{T}{s}\Big) - Ei\Big((C-B)\frac{T}{s}\Big)\Big],\end{aligned} \tag{4.32}$$

where

$$\begin{aligned} A &= G_{12}tu, \\ B &= G_{12}u^2, \\ C &= \frac{1}{2}(1-\dot{G}_{12})u^2. \end{aligned} \tag{4.33}$$

We are now at a stage where a decision can be made to continue with the integration of either u_1 or u_2. Nevertheless, the incorporation of the Ei function offers a more streamlined approach for the integration of T, given the existence of the subsequent formula.

$$\int_0^\infty dT\, \mathrm{e}^{-aT}\mathrm{E}i(\pm bT) = -\frac{\ln\frac{b\mp a}{b}}{a} \tag{4.34}$$

Utilizing the aforementioned formula, the integration over T results in the following expression.

$$\begin{aligned}\int_0^\infty \frac{dT}{s}\mathrm{e}^{-m^2T}\Bigg(\mathrm{e}^{-\frac{TA}{s}}\Big[Ei\Big(\frac{TA}{s}\Big) - Ei\Big(-\frac{TB}{s}\Big)\Big] - \mathrm{e}^{-\frac{T(A+C)}{s}}\Big[Ei\Big(\frac{T(A+C)}{s}\Big) - Ei\Big(\frac{T(C-B)}{s}\Big)\Big]\Bigg)\\ = \frac{\ln\left(1+\frac{A+B}{m^2s}\right)+\ln\left(-\frac{A}{B}\right)}{m^2s+A} - \frac{\ln\left(1+\frac{A+B}{m^2s}\right)+\ln\left(-\frac{A+C}{B-C}\right)}{m^2s+A+C}\end{aligned} \tag{4.35}$$

To conclude the calculation of the integrals, we can employ time translational invariance in proper time. This allows us to set any variable u_i to either 0 or 1. We utilize this freedom to fix $u_1 = 0$. Furthermore, it suffices to solve only the first term of the preceding equation, as the second term will yield an identical result, albeit with s and t interchanged. This is a direct consequence of dividing our calculations into sectors, as was demonstrated previously. Thus, the integral over u_2 can be written as*

$$\int_0^1 du_2 \frac{\ln\Big(-\frac{t}{u}\Big) + \ln\Big[1-u_2(1-u_2)\frac{u}{m^2}\Big]}{m^2s+u_2(1-u_2)tu} \equiv I^{(s,t,u)}\,. \tag{4.36}$$

The variable of integration appears as a squared power, which allows us to linearize these quadratic forms

$$\begin{aligned}1-u_2(1-u_2)\frac{u}{m^2} &= 4\hat{u}\Big(u_2-\frac{1}{2}+\frac{\beta_{\hat{u}}}{2}\Big)\Big(u_2-\frac{1}{2}-\frac{\beta_{\hat{u}}}{2}\Big)\\ m^2s+u_2(1-u_2)tu &= -16m^4\hat{t}\hat{u}\Big(u_2-\frac{1}{2}+\frac{\beta_{\hat{t}\hat{u}}}{2}\Big)\Big(u_2-\frac{1}{2}-\frac{\beta_{\hat{t}\hat{u}}}{2}\Big)\end{aligned} \tag{4.37}$$

Wherein we have introduced $\hat{s} \equiv \frac{s}{4m^2}, \hat{t} \equiv \frac{t}{4m^2}, \hat{u} \equiv \frac{u}{4m^2}$ and, following [122],

$$\beta_{\hat{u}} \equiv \sqrt{1-\frac{1}{\hat{u}}}, \quad \beta_{\hat{t}\hat{u}} \equiv \sqrt{1-\frac{1}{\hat{t}}-\frac{1}{\hat{u}}} \tag{4.38}$$

*Since we ultimately have to sum all the permutations, the term $\ln\left(-\frac{t}{u}\right)$ will reappear with the opposite sign in permutation $I^{s,u,t}$.

In this manner, we have reduced our integral 4.36 to the following basic integral:

$$\int dx\, \frac{\log(x+d)}{x+c} = \log(x+d)\log\left(\tfrac{x+c}{c-d}\right) + \mathrm{Li}_2\left(\tfrac{x+d}{d-c}\right), \tag{4.39}$$

with $Li_2(z)$ being the dilogarithm, defined in [123] as:

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-t)}{t}\, dt. \tag{4.40}$$

After applying the previously basic integral, the outcome can be expressed in the form of:

$$I^{(s,t,u)} = \frac{1}{8m^4\hat{t}\hat{u}\beta_{\hat{t}\hat{u}}}\left\{\frac{1}{2}\ln\big(-4\hat{t}\big)\ln\Big(\frac{1+\beta_{\hat{t}\hat{u}}}{1-\beta_{\hat{t}\hat{u}}}\Big)^2 + \mathrm{b}i_0\Big(\frac{\beta_{\hat{t}\hat{u}}}{2},\frac{\beta_{\hat{u}}}{2}\Big) - \mathrm{b}i_0\Big(-\frac{\beta_{\hat{t}\hat{u}}}{2},\frac{\beta_{\hat{u}}}{2}\Big)\right\}. \tag{4.41}$$

Where

$$\begin{aligned}\mathrm{b}i_0(c,d) \equiv \int_0^1 du_2 \frac{\ln\big(u_2-\frac{1}{2}+d\big)}{u_2-\frac{1}{2}+c} &= \log\left(d+\frac{1}{2}\right)\log\left(\frac{c+\frac{1}{2}}{c-d}\right) + \mathrm{Li}_2\left(\frac{d+\frac{1}{2}}{d-c}\right)\\ &- \log\left(d-\frac{1}{2}\right)\log\left(\frac{c-\frac{1}{2}}{c-d}\right) - \mathrm{Li}_2\left(\frac{d-\frac{1}{2}}{d-c}\right)\end{aligned} \tag{4.42}$$

Without necessitating additional calculations, the result for the full amplitude can now be stated.

$$I_4 = 2\big(I^{(s,t,u)} + I^{(t,s,u)} + I^{(u,s,t)} + I^{(s,u,t)} + I^{(t,u,s)} + I^{(u,t,s)}\big) \tag{4.43}$$

This follows from the observation that the second and third lines of equation 4.28 are permutations of the first, and each line's pair of terms integrates to an identical outcome.
To compare our result with [122], first, let us note that in their Equation (2), Davydychev defines the Mandelstam variables as $s \equiv (k_1+k_2)^2$, $t \equiv (k_2-k_3)^2$ and $u \equiv (k_1-k_3)^2$. Taking this into account, along with the different conventions, we will

ignore the $i\pi$ factor and adapt their Equation (4) to our notation as follows[†]

$$\begin{aligned} J(\hat{t},\hat{u};m) &= \frac{1}{6m^4}F_3\left(1,1,1,1;5/2|\hat{t},\hat{u}\right) \\ &= \frac{1}{8m^4\hat{t}\hat{u}\beta_{\hat{t}\hat{u}}}\Bigg\{2\ln^2\left(\frac{\beta_{\hat{t}\hat{u}}+\beta_{\hat{t}}}{\beta_{\hat{t}\hat{u}}+\beta_{\hat{u}}}\right)+\ln\left(\frac{\beta_{\hat{t}\hat{u}}-\beta_{\hat{t}}}{\beta_{\hat{t}\hat{u}}+\beta_{\hat{t}}}\right)\ln\left(\frac{\beta_{\hat{t}\hat{u}}-\beta_{\hat{u}}}{\beta_{\hat{t}\hat{u}}+\beta_{\hat{u}}}\right)-\frac{\pi^2}{2} \\ &\quad+\sum_{i=\hat{t},\hat{u}}\left[2\mathrm{Li}_2\left(\frac{\beta_i-1}{\beta_{\hat{t}\hat{u}}+\beta_i}\right)-2\mathrm{Li}_2\left(-\frac{\beta_{\hat{t}\hat{u}}-\beta_i}{\beta_i+1}\right)-\ln^2\left(\frac{\beta_i+1}{\beta_{\hat{t}\hat{u}}+\beta_i}\right)\right]\Bigg\} \end{aligned} \tag{4.44}$$

By applying various dilogarithm identities, it has been demonstrated that:

$$J(\hat{t},\hat{u};m) = I^{(s,t,u)} + I^{(s,u,t)} \tag{4.45}$$

The foregoing demonstrates the equivalence between our result and that presented in [122]. After studying these types of functions, it was found that the various permutations of s, t, and u can be reduced to just three terms by defining a function $\bar{B}$ as follows:

$$\bar{B}(s,t) = \frac{4}{st\beta_{\hat{s}\hat{t}}}\int_0^1 dx\left[\frac{d}{dx}\ln([m^2-s(1-x)x][m^2-t(1-x)x])\right]\ln\left(x-\frac{1+\beta_{\hat{s}\hat{t}}}{2}\right) \tag{4.46}$$

In terms of this function $\bar{B}$, our I_4 turns out to be:

$$I_4(s,t,u) = \bar{B}(s,t) + \bar{B}(s,u) + \bar{B}(t,u) \tag{4.47}$$

In the following section, we will apply what we have learned to compute the four-photon integrals in scalar QED.

4.2 Scalar QED

Integration by parts is a fundamental technique in QFT calculations, especially within the worldline formalism. This formalism reformulates QFT problems as path integrals over point particle trajectories, often simplifying complex computations.
In the worldline formalism, scattering amplitudes are expressed as functional integrals over these trajectories. Integration by parts is employed to simplify expressions.

[†]The "difference" in their definition of Mandelstam variables is not a true conceptual difference, but rather stems from their use of a different metric convention.

For example, when computing Feynman diagrams using the worldline approach, integration by parts helps reduce intricate integrals into more manageable forms.
The worldline formalism is particularly useful for studying non-perturbative effects, such as pair production in strong electric fields. Integration by parts is employed to simplify integrals that arise in the computation of effective actions and transition probabilities.
In the calculation of radiative corrections in QFT, the worldline formalism expresses these corrections in terms of path integrals. Integration by parts is beneficial for simplifying terms that involve field derivatives or nonlinear interactions. In [124] Bastianelli and van Nieuwenhuizen cover the use of the worldline formalism in curved space and discuss how integration by parts is applied in computing anomalies and radiative corrections.
Schubert's work [27] establishes the worldline formalism's utility for amplitude computations and renormalization. A key focus is how partial integration methods shape the integrand structure for general cases of both N-photon and N-gluon amplitudes.
In this part we will apply integration by parts to simplify the calculation of the four-photon amplitude at the one-loop level in scalar QED.
First of all, the difference between the integrals appearing in the toy model of the box integral and the case of scalar QED in the worldline formalism lies in the presence of four factors of the first derivative of $\dot{G}$. In addition to integration by parts, an attempt is made to generalize master formulas by generating these $\dot{G}$ through the introduction of derivatives in Appendix C.
As we saw previously, the integrals that need to be solved to determine the 4-photon amplitude in scalar QED in four dimensions ($D = 4$) using the worldline formalism are the following:

$$\Gamma^{scal}_{\dots} = \frac{e^4}{(4\pi)^2}\int_0^\infty dT T \mathrm{e}^{-m^2T}\int_0^1 \prod_{i=1}^4 du_i\, \gamma^{scal}_{\dots}(\dot{G}_{ij})\, \mathrm{e}^{\Lambda T} \tag{4.48}$$

where:

$$\begin{aligned}
\gamma^{scal}_{(1234)} &= \dot{G}_{12}\dot{G}_{23}\dot{G}_{34}\dot{G}_{41} \\
\gamma^{scal}_{(12)(34)} &= \dot{G}_{12}\dot{G}_{21}\dot{G}_{34}\dot{G}_{43} \\
\gamma^{scal}_{(123)i} &= \dot{G}_{12}\dot{G}_{23}\dot{G}_{31}\dot{G}_{4i} \\
\gamma^{scal}_{(12)ii} &= \dot{G}_{12}\dot{G}_{21}\dot{G}_{i3}\dot{G}_{4i} \\
\gamma^{scal}_{(12)ij} &= \dot{G}_{12}\dot{G}_{21}\dot{G}_{i3}\dot{G}_{4j}\,.
\end{aligned} \tag{4.49}$$

To ilustrate how we will apply integration by parts in this problem, we consider Γ^{scal}_{1234}.

$$\Gamma^{scal}_{1234} = \frac{e^4}{(4\pi)^2}\int_0^\infty dT T \mathrm{e}^{-m^2 T}\int_0^1 \prod_{i=1}^4 du_i\, \dot{G}_{12}\dot{G}_{23}\dot{G}_{34}\dot{G}_{41}\,\mathrm{e}^{\Lambda T}. \tag{4.50}$$

In this context, the application of integration by parts on the variable u_1 serves a highly specific purpose: the maximal elimination of the variable u_4. It is crucial to underscore that this operation fundamentally differs from the "standard" integration by parts commonly employed for transitioning from a P-representation to a Q-representation. While they share the same mathematical foundation, our objective here is not a transformation between representations, but rather a targeted manipulation aimed at simplifying and ultimately eradicating terms involving u_4.
Let us consider:

$$\int_0^1 du_1 ... du_4\, \ddot{G}_{12}\dot{G}_{23}\dot{G}_{34}\,\mathrm{e}^{T\Lambda}, \tag{4.51}$$

where $\Lambda = -\frac{1}{2}\left(s(G_{12}+G_{34}) + t(G_{13}+G_{24}) + u(G_{14}+G_{23})\right)$.
Integrating by parts with respect to the variable u_1, we find:

$$\int_0^1 du_1 ... du_4\, \ddot{G}_{12}\dot{G}_{23}\dot{G}_{34}\,\mathrm{e}^{T\Lambda} = \frac{T}{2}\int_0^1 du_1 ... du_4\, \dot{G}_{12}\dot{G}_{23}\dot{G}_{34}\,\mathrm{e}^{T\Lambda}\Big(s\dot{G}_{12} + t\dot{G}_{13} + u\dot{G}_{14}\Big). \tag{4.52}$$

From the previous expression, we can obtain another representation of the integrals over u_i, which appears in $\Gamma^{scal}_{(1234)}$:

$$\begin{aligned}\int_0^1 du_1 ... du_4\, \dot{G}_{12}\dot{G}_{23}\dot{G}_{34}\dot{G}_{41}\,\mathrm{e}^{T\Lambda} &= \int_0^1 du_1 ... du_4\,\mathrm{e}^{T\Lambda}\Big(-\frac{2}{uT}\ddot{G}_{12}\dot{G}_{23}\dot{G}_{34} + \frac{s}{u}\dot{G}^2_{12}\dot{G}_{23}\dot{G}_{34} \\ &\quad + \frac{t}{u}\dot{G}_{12}\dot{G}_{23}\dot{G}_{34}\dot{G}_{13}\Big).\end{aligned} \tag{4.53}$$

The main idea is to apply integration by parts to reduce the number of $\dot{G}$ factors in the chain and, in doing so, eliminate the dependence on the variable u_4 . However, after a single integration by parts, two terms still contain a u_4 dependance. Therefore, from the second and third terms , we find that:

$$\begin{aligned}
\int_0^1 du_1...du_4\, \dot{G}_{12}^2 \ddot{G}_{32}\, \mathrm{e}^{T\Lambda} &= \frac{T}{2}\int_0^1 du_1...du_4\, \dot{G}_{12}^2 \dot{G}_{32}\, \mathrm{e}^{T\Lambda}\Big(s\dot{G}_{34}+t\dot{G}_{31}+u\dot{G}_{32}\Big).\\
\int_0^1 du_1...du_4\, \dot{G}_{12}\ddot{G}_{31}\dot{G}_{32}\, \mathrm{e}^{T\Lambda} &= -\int_0^1 du_1...du_4\, \dot{G}_{12}\dot{G}_{31}\ddot{G}_{32}\, \mathrm{e}^{T\Lambda}\\
&\quad +\frac{T}{2}\int_0^1 du_1...du_4\, \dot{G}_{12}\dot{G}_{31}\dot{G}_{32}\, \mathrm{e}^{T\Lambda}\Big(s\dot{G}_{34}+t\dot{G}_{31}+u\dot{G}_{32}\Big).
\end{aligned} \tag{4.54}$$

Combining the previous results:

$$\begin{aligned}
\int_0^1 du_1...du_4\, \dot{G}_{12}\dot{G}_{23}\dot{G}_{34}\dot{G}_{41}\, \mathrm{e}^{T\Lambda} &= \int_0^1 du_1...du_4\, \mathrm{e}^{T\Lambda}\Big(-\frac{2}{uT}\Big[\ddot{G}_{12}\dot{G}_{23}\dot{G}_{34}+\ddot{G}_{32}\dot{G}_{12}^2\Big]\\
&\quad +\frac{2t}{Tus}\Big[\ddot{G}_{31}\dot{G}_{12}\dot{G}_{32}+\ddot{G}_{32}\dot{G}_{12}\dot{G}_{31}\Big]+\frac{t}{u}\,\dot{G}_{12}^2\dot{G}_{32}\dot{G}_{31}\\
&\quad +\dot{G}_{12}^2\dot{G}_{32}^2-\frac{t^2}{us}\,\dot{G}_{12}\dot{G}_{32}\dot{G}_{31}^2-\frac{t}{s}\,\dot{G}_{12}\dot{G}_{32}^2\dot{G}_{31}\Big).
\end{aligned} \tag{4.55}$$

In this way, through integration by parts, we have successfully reduced the dependence on the variable u_4 , which now appears only in the exponential, with the exception of the first term on the right-hand side of the previous equation. In this term, we will integrate over u_1 since applying integration by parts again would introduce a $1/T$ factor, leading to a divergence. Thus, the previous equation is now ready to be integrated over T

This is important because, from the box integral in ϕ^3 theory, we learned that for any function $f(a+b\Lambda)$, it turns out that

$$\begin{aligned}
b\int_0^1 du_4\, f(a+b\Lambda) = &\left[\frac{2}{u+\dot{G}_{12}t+\dot{G}_{13}s}+\frac{2}{u-\dot{G}_{12}t-\dot{G}_{13}s}\right]F(\lambda_u)\\
&+\left[\frac{2}{t+\dot{G}_{23}s+\dot{G}_{21}u}+\frac{2}{t-\dot{G}_{23}s-\dot{G}_{21}u}\right]F(\lambda_t)\\
&+\left[\frac{2}{s+\dot{G}_{31}u+\dot{G}_{32}t}+\frac{2}{s-\dot{G}_{31}u-\dot{G}_{32}t}\right]F(\lambda_s)\,.
\end{aligned} \tag{4.56}$$

where $F'(z) = f(z)$ and we have defined the functions λ_x as

$$\begin{aligned}
\lambda_u &= a + \frac{1}{2}(G_{12} + G_{13} - G_{23})ub, \\
\lambda_t &= a + \frac{1}{2}(G_{12} + G_{23} - G_{13})tb, \\
\lambda_s &= a + \frac{1}{2}(G_{13} + G_{23} - G_{12})sb. \qquad (4.57)
\end{aligned}$$

With the previous considerations, we will have two types of cases in the integration over T:

$$\int_0^\infty dT \mathrm{e}^{-T(m^2-\Lambda)} = \frac{1}{m^2 - \Lambda} \qquad (4.58)$$

$$\int_0^\infty dT T \mathrm{e}^{-T(m^2-\Lambda)} = \frac{1}{(m^2 - \Lambda)^2} \qquad (4.59)$$

As an illustrative case, we consider the first term of (4.55) :

$$\begin{aligned}
\int_0^\infty dT T \mathrm{e}^{-m^2 T} \int_0^1 du_1 ... du_4 \, \mathrm{e}^{T\Lambda}\Big(-\frac{2}{uT}\Big[\ddot{G}_{12}\dot{G}_{23}\dot{G}_{34}\Big]\Big) &= -\frac{2}{u}\int_0^1 du_1 ... du_4 \, \frac{\ddot{G}_{12}\dot{G}_{23}\dot{G}_{34}}{m^2 - \Lambda} \\
&= -\frac{4}{u}\int_0^1 du_1 ... du_4 \, \frac{\delta(u_1 - u_2)\dot{G}_{23}\dot{G}_{34}}{m^2 - \Lambda} + \frac{4}{u}\int_0^1 du_1 ... du_4 \, \frac{\dot{G}_{23}\dot{G}_{34}}{m^2 - \Lambda} \qquad (4.60)
\end{aligned}$$

To perform the integrals over u_1, let us recall the definition of Λ:

$$\Lambda = -\frac{1}{2}\big[(G_{12} + G_{34})s + (G_{13} + G_{24})t + (G_{14} + G_{23})u\big]. \qquad (4.61)$$

Therefore, for the term containing the delta function, the integral over u_1 can be performed directly:

$$\int_0^1 du_1 . \, \frac{(\delta(u_1 - u_2))\dot{G}_{23}\dot{G}_{34}}{m^2 - \Lambda} = \frac{\dot{G}_{23}\dot{G}_{34}}{m^2 + \frac{1}{2}\big[(G_{22} + G_{34})s + (G_{23} + G_{24})t + (G_{24} + G_{23})u\big]}. \qquad (4.62)$$

It is at this stage of the calculation that we use translational invariance to fix the value of one of the integration variables ($u_3 = 0$), thereby reducing the problem to only two remaining integrals. With these two integrals, we can select a specific sector ($1 > u_4 > u_2 > 0$) and at the end include a factor of 2 to account for the contribution of the complementary sector. Additionally, we use $s + t + u = 0$ to simplify the expressions, leading us to integrals of the form:

$$\int_0^1 du_2 du_3 du_4 \frac{\dot{G}_{23}\dot{G}_{34}}{m^2 + (G_{22}+G_{34})\frac{s}{2} + (G_{23}+G_{24})\frac{t}{2} + (G_{24}+G_{23})\frac{u}{2}}$$

$$\begin{aligned} &= \int_0^1 du_4 \int_0^{u_4} du_2 \frac{(2-4u_4)u_2 + 2u_4 - 1}{su_2(u_2-u_4)+m^2} \\ &= -2\left(-\frac{10}{s} + \frac{4\beta_{\hat{s}}}{s}\ln\left(\frac{\beta_{\hat{s}}+1}{\beta_{\hat{s}}-1}\right) - \frac{1}{2s}\left(\frac{4m^2}{s}+1\right)\ln^2\left(\frac{\beta_{\hat{s}}+1}{\beta_{\hat{s}}-1}\right)\right) \end{aligned} \tag{4.63}$$

Let's recall that the function $\beta_{\hat{s}}$ is defined as: $\beta_{\hat{s}} = \sqrt{1-\frac{4m^2}{s}}$.
Significantly, this integration method is expected to produce a result formulated as rational functions of the Mandelstam variables s, t, and u, which will be confirmed by the final outcomes of this calculation
For the other term in (4.60), we must use equation (4.56), but with the substitution of u_4 by u_1, wich implies that in (4.56), we must also interchange s and t:

$$\begin{aligned} \int_0^1 du_1 \frac{\dot{G}_{23}\dot{G}_{34}}{m^2-\Lambda} = &\left[\frac{2}{u+\dot{G}_{42}s+\dot{G}_{43}t} + \frac{2}{u-\dot{G}_{42}s-\dot{G}_{43}t}\right]\ln\left(m^2 + \frac{u}{2}(G_{42}+G_{43}-G_{23})\right) \\ &+ \left[\frac{2}{s+\dot{G}_{23}t+\dot{G}_{24}u} + \frac{2}{s-\dot{G}_{23}t-\dot{G}_{24}u}\right]\ln\left(m^2 + \frac{s}{2}(G_{42}+G_{23}-G_{43})\right) \\ &+ \left[\frac{2}{t+\dot{G}_{34}u+\dot{G}_{32}s} + \frac{2}{t-\dot{G}_{34}u-\dot{G}_{32}s}\right]\ln\left(m^2 + \frac{t}{2}(G_{43}+G_{23}-G_{42})\right). \end{aligned} \tag{4.64}$$

At this stage, we notice that our integrals and integrands recursively produce logarithmic and dilogarithmic functions. This is not surprising, as the integral definitions of these functions are given by:

$$\begin{aligned} \ln(1-x) &= \int_0^x \frac{dy}{y-1} \\ \mathrm{Li}_2(x) &= -\int_0^x \frac{dy}{y}\ln(1-y) = \int_0^x \frac{dy}{y}\int_0^x \frac{dz}{1-z} \end{aligned} \tag{4.65}$$

Furthermore, in more general cases, the polylogarithm is defined as:

$$\mathrm{Li}_n(x) = \int_0^x \frac{dt}{t}\mathrm{Li}_{n-1}(t) \tag{4.66}$$

Moreover, in these previous definitions, all singularities are located at zero. However, in general, one encounters the so-called multiple polylogarithms, which are defined as

$$G(a_1, a_2, \ldots, a_n; x) = \int_0^x \frac{dt}{t-a_1} G(a_2, a_3, \ldots, a_n; t) \tag{4.67}$$

For our calculation at the one-loop level, we only require:

$$G(a; z) = \ln\left(1 - \frac{z}{a}\right), \tag{4.68}$$

$$G(a, b; z) = \mathrm{Li}_2\left(\frac{b-z}{b-a}\right) - \mathrm{Li}_2\left(\frac{b}{b-a}\right) + \ln\left(1-\frac{z}{a}\right)\ln\left(\frac{z-a}{b-a}\right). \tag{4.69}$$

Typically, the integrals we obtain can be manipulated using the following formula:

$$\begin{aligned}\int_0^1 dx \int_0^x dy \, \frac{p(x,y)}{bx+dy} \, \ln\left[1 - a(1-x)y\right] \;=\; & \frac{1}{d}\int_0^1 dx \Bigg(p(x,x)\, G\left(-cx, \frac{1}{a(1-x)}; x\right) \\ & - \int_0^x dy \frac{\partial p(x,y)}{\partial y} G\left(-cx, \frac{1}{a(1-x)}; y\right)\Bigg)\end{aligned} \tag{4.70}$$

with $p(x, y)$ representing a polynomial in x and y , and $c = \frac{b}{d}$.
To simplify the results, we consider:

$$\int_0^1 dx p(x,x)\, G\left(-cx, \frac{1}{a(1-x)}; x\right) = \int_0^1 dx P(x) g(a, b, c; x), \tag{4.71}$$

where

$$
\begin{aligned}
P(x) &= \int_0^x dx\, p(x,x), \\
g(a,b,c;x) &= \frac{2x-1}{x(x-1)[m^2c+x(1-x)ab]}\left[\frac{ab}{c}x(x-1)\ln\left(1+\frac{c}{b}\right)\right. \\
&\left. -m^2\ln\left(1-x(1-x)\frac{a}{m^2}\right)\right].
\end{aligned} \tag{4.72}
$$

As a result of this approach, Γ^{scal}_{1234} can be written as:

$$
\begin{aligned}
\Gamma^{\text{scal}}_{(1234)} &= \frac{e^4}{(4\pi)^2}\Big\{r^{(1)}_{(1234)} + r^{(2)}_{(1234)}\ln\left(\frac{\beta_{\hat{s}}-1}{\beta_{\hat{s}}+1}\right) + r^{(3)}_{(1234)}\ln\left(\frac{\beta_{\hat{t}}-1}{\beta_{\hat{t}}+1}\right) + r^{(4)}_{(1234)}\ln\left(\frac{\beta_{\hat{u}}-1}{\beta_{\hat{u}}+1}\right) \\
&+r^{(5)}_{(1234)}\left[\ln\left(\frac{\beta_{\hat{s}}-1}{\beta_{\hat{s}}+1}\right)\right]^2 + r^{(6)}_{(1234)}\left[\ln\left(\frac{\beta_{\hat{t}}-1}{\beta_{\hat{t}}+1}\right)\right]^2 + r^{(7)}_{(1234)}\left[\ln\left(\frac{\beta_{\hat{u}}-1}{\beta_{\hat{u}}+1}\right)\right]^2 \\
&+r^{(8)}_{(1234)}\bar{B}(s,t) + r^{(9)}_{(1234)}\bar{B}(s,u) + r^{(10)}_{(1234)}\bar{B}(t,u)\Big\}
\end{aligned} \tag{4.73}
$$

where

$$
\begin{aligned}
r^{(1)}_{(1234)} &= -\frac{16}{3}\left(\frac{3u}{st^2}+\frac{3}{st}+\frac{29}{su}\right), \\
r^{(2)}_{(1234)} &= -\frac{16}{3s^2}\left(\frac{2t^2}{u^2}-\frac{7u}{t}-\frac{15u^2}{t^2}-\frac{6u^3}{t^3}\right)\beta_{\hat{s}}, \\
r^{(3)}_{(1234)} &= -\frac{16}{3su}\left(3+\frac{2t}{s}+\frac{2t}{u}\right)\beta_{\hat{t}}, \\
r^{(4)}_{(1234)} &= \frac{16(3s+u)}{3t^3}\left(8+\frac{3s}{u}+\frac{4u}{s}+\frac{2u^2}{s^2}\right)\beta_{\hat{u}}, \\
r^{(5)}_{(1234)} &= -\frac{4}{3}\left[\frac{12m^2}{s^2}\left(\frac{1}{t}-\frac{t}{u^2}+\frac{4u}{t^2}+\frac{2u^2}{t^3}\right)+\frac{4}{t^2}+\frac{4t}{u^3}-\frac{3}{u^2}-\frac{12u}{t^3}-\frac{12u^2}{t^4}\right], \\
r^{(6)}_{(1234)} &= \frac{4}{3s^3}\left(\frac{12m^2st}{u^2}-12t+\frac{4t^4}{u^3}+\frac{9t^3}{u^2}+\frac{3t^2}{u}-6u\right), \\
r^{(7)}_{(1234)} &= \frac{4}{3s^2}\left[12m^2\left(-\frac{1}{t}+\frac{2t}{u^2}+\frac{2}{u}-\frac{2u^2}{t^3}-\frac{4u}{t^2}\right)\right. \\
&\left.+\frac{1}{s}\left(-\frac{12u^5}{t^4}-\frac{48u^4}{t^3}-\frac{68u^3}{t^2}-\frac{36u^2}{t}-3t-3u\right)\right],
\end{aligned} \tag{4.74}
$$

$$
\begin{aligned}
r^{(8)}_{(1234)} &= -\frac{\beta^2_{\hat{s}\hat{t}}}{u^2}\left[m^2\left(\frac{4s^2}{t}+4s-\frac{8t}{3}\right)+\frac{1}{u}\left(s^3+s^2t-\frac{5st^2}{3}+t^3\right)\right],\\
r^{(9)}_{(1234)} &= \frac{16m^4}{s}\left(-\frac{u}{t^2}-\frac{1}{t}+\frac{2}{3u}\right)+\frac{4m^2}{s}\left(\frac{8u^3}{t^3}+\frac{16u^2}{t^2}+\frac{14u}{3t}-\frac{t}{u}-\frac{10}{3}\right)\\
&\quad +\frac{8u^4}{t^4}+\frac{16u^3}{t^3}+\frac{16u^2}{3t^2}-\frac{8u}{3t}-1,\\
r^{(10)}_{(1234)} &= -\frac{\beta^2_{\hat{t}\hat{u}}}{3s^2}\left[m^2\left(\frac{12u^2}{t}-8t+12u\right)+\frac{1}{s}\left(3t^3-5t^2u+3tu^2+3u^3\right)\right].
\end{aligned} \tag{4.75}
$$

Interestingly, for scalar QED, all $\Gamma^{\text{scal}}_{\dots}$ functions exhibit the following form,

$$
\begin{aligned}
\Gamma^{\text{scal}}_{\dots} &= \Bigg\{r^{(1)}_{\dots}+r^{(2)}_{\dots}\ln\left(\frac{\beta_{\hat{s}}-1}{\beta_{\hat{s}}+1}\right)+r^{(3)}_{\dots}\ln\left(\frac{\beta_{\hat{t}}-1}{\beta_{\hat{t}}+1}\right)+r^{(4)}_{\dots}\ln\left(\frac{\beta_{\hat{u}}-1}{\beta_{\hat{u}}+1}\right)\\
&\quad +r^{(5)}_{\dots}\left[\ln\left(\frac{\beta_{\hat{s}}-1}{\beta_{\hat{s}}+1}\right)\right]^2+r^{(6)}_{\dots}\left[\ln\left(\frac{\beta_{\hat{t}}-1}{\beta_{\hat{t}}+1}\right)\right]^2+r^{(7)}_{\dots}\left[\ln\left(\frac{\beta_{\hat{u}}-1}{\beta_{\hat{u}}+1}\right)\right]^2\\
&\quad +r^{(8)}_{\dots}\bar{B}(s,t)+r^{(9)}_{\dots}\bar{B}(s,u)+r^{(10)}_{\dots}\bar{B}(t,u)\Bigg\},
\end{aligned} \tag{4.76}
$$

with the rational functions $r^{(i)}_{\dots}$ defined in equation (4.75) for $r_{(1234)}$, while the remaining cases are presented below.

$$
\begin{aligned}
r^{(1)}_{(12)(34)} &= \frac{16}{3}\left(\frac{3}{t^2}-\frac{4}{tu}+\frac{3}{u^2}\right), & r^{(1)}_{(123)1} &= -\frac{112}{3}\left(\frac{1}{st}+\frac{2}{su}\right),\\
r^{(1)}_{(12)11} &= -\frac{112}{3tu}, & r^{(1)}_{(12)12} &= -\frac{16}{u}\left(\frac{1}{3t}+\frac{1}{u}\right),
\end{aligned} \tag{4.77}
$$

$$
\begin{aligned}
r^{(2)}_{(12)(34)} &= \frac{16\beta_{\hat{s}}}{3t^3}\left(\frac{6s^4}{u^3}+\frac{29s^3}{u^2}+\frac{53s^2}{u}+48s+24u\right),\\
r^{(2)}_{(123)1} &= -\frac{8\beta_{\hat{s}}}{3}\left(\frac{3u}{st^2}+\frac{3}{su}+\frac{5}{t^2}+\frac{4}{u^2}\right),\\
r^{(2)}_{(12)11} &= \frac{16s^2\beta_{\hat{s}}}{3t^2u^2},\\
r^{(2)}_{(12)12} &= -\frac{16\beta_{\hat{s}}}{3}\left(\frac{2}{t^2}+\frac{6t}{u^3}-\frac{2}{tu}+\frac{5}{u^2}\right),
\end{aligned} \tag{4.78}
$$

$$r^{(3)}_{(12)(34)} = \frac{16\beta_{\hat{t}}}{3u}\left(\frac{6s}{u^2}+\frac{2}{s}+\frac{11}{u}\right), \qquad r^{(3)}_{(123)1} = \frac{8\beta_{\hat{t}}}{3}\left(\frac{2}{s^2}-\frac{6}{st}-\frac{3}{su}+\frac{4}{u^2}\right),$$
$$r^{(3)}_{(12)11} = \frac{16\beta_{\hat{t}}}{3u}\left(\frac{s}{tu}+\frac{u}{st}-\frac{1}{t}\right), \qquad r^{(3)}_{(12)12} = -\frac{16\beta_{\hat{t}}}{3s}\left(\frac{6t^2}{u^3}+\frac{11t}{u^2}-\frac{3}{t}+\frac{3}{u}\right), \quad (4.79)$$

$$r^{(4)}_{(12)(34)} = \frac{16\beta_{\hat{u}}}{3t^3}\left(\frac{2u^2}{s}-3s-7u\right), \qquad r^{(4)}_{(123)1} = -\frac{8\beta_{\hat{u}}}{3t^2}\left(\frac{2u^2}{s^2}+\frac{7u}{s}+\frac{6s}{u}+9\right),$$
$$r^{(4)}_{(12)11} = \frac{16\beta_{\hat{u}}}{3u}\left(\frac{s}{t^2}+\frac{1}{s}-\frac{1}{t}\right), \qquad r^{(4)}_{(12)12} = \frac{16(3s+u)\beta_{\hat{u}}}{3st^2}, \quad (4.80)$$

$$r^{(5)}_{(12)(34)} = -\frac{8}{3}\left[\frac{12m^2}{s}\left(-\frac{u}{t^3}-\frac{t}{u^3}\right)-\frac{6u^2}{t^4}+\frac{2u}{t^3}-\frac{6t^2}{u^4}-\frac{1}{t^2}+\frac{2t}{u^3}-\frac{1}{u^2}\right],$$
$$r^{(5)}_{(123)1} = \frac{m^2}{s^2}\left(\frac{8u}{t^2}+\frac{16t}{u^2}+\frac{24}{t}+\frac{24}{u}\right)+\frac{8u^2}{3st^3}+\frac{16t^2}{3su^3}+\frac{8u}{st^2}+\frac{8t}{su^2}+\frac{22}{3st}+\frac{8}{3su},$$
$$r^{(5)}_{(12)11} = \frac{8m^2s}{t^2u^2}+\frac{8u}{3t^3}+\frac{20}{3t^2}+\frac{8t}{3u^3}-\frac{2}{tu}+\frac{20}{3u^2},$$
$$r^{(5)}_{(12)12} = \frac{m^2}{t^3}\left(-\frac{32s^3}{u^3}-\frac{80s^2}{u^2}-\frac{64s}{u}+\frac{16u}{s}\right)+\frac{1}{t^3}\left(\frac{16s^5}{u^4}+\frac{176s^4}{3u^3}\right.$$
$$\left.+\frac{80s^3}{u^2}+\frac{44s^2}{u}+4s-8u\right), \quad (4.81)$$

$$r^{(6)}_{(12)(34)} = \frac{8}{3s}\left[\frac{12m^2t}{u^3}+\frac{1}{s}\left(\frac{6t^4}{u^4}+\frac{10t^3}{u^3}+\frac{3t^2}{u^2}+3\right)\right],$$
$$r^{(6)}_{(123)1} = \frac{m^2}{s^2}\left(\frac{8u}{t^2}+\frac{16t}{u^2}+\frac{24}{t}+\frac{24}{u}\right)+\frac{16t^4}{3s^3u^3}+\frac{56t^3}{3s^3u^2}+\frac{24t^2}{s^3u}-\frac{2u^2}{s^3t}+\frac{6t}{s^3}-\frac{4u}{s^3},$$
$$r^{(6)}_{(12)11} = \frac{8m^2s}{t^2u^2}+\frac{1}{s^2}\left(\frac{8t^3}{3u^3}+\frac{12t^2}{u^2}+\frac{14t}{u}-\frac{2u}{t}+4\right),$$
$$r^{(6)}_{(12)12} = \frac{m^2\left(-\frac{32s}{u}-16\right)}{tu^2}+\frac{1}{t}\left(\frac{16s^3}{u^4}+\frac{80s^2}{3u^3}-\frac{4u}{3s^2}+\frac{32s}{3u^2}+\frac{8}{3s}-\frac{4}{u}\right), \quad (4.82)$$

$$\begin{aligned}
r^{(7)}_{(12)(34)} &= \frac{32m^2u}{st^3} + \frac{8}{3}\left(\frac{6s^2}{t^4} + \frac{2}{s^2} + \frac{14s}{t^3} + \frac{9}{t^2}\right), \\
r^{(7)}_{(123)1} &= \frac{m^2}{s^2}\left(\frac{8u}{t^2} + \frac{16t}{u^2} + \frac{24}{t} + \frac{24}{u}\right) + \frac{8u^4}{3s^3t^3} + \frac{40u^3}{3s^3t^2} + \frac{4t^2}{s^3u} + \frac{26u^2}{s^3t} + \frac{10t}{s^3} + \frac{24u}{s^3}, \\
r^{(7)}_{(12)11} &= \frac{8m^2s}{t^2u^2} + \frac{1}{s^2}\left(\frac{8u^3}{3t^3} + \frac{12u^2}{t^2} + \frac{14u}{t} - \frac{2t}{u} + 4\right), \\
r^{(7)}_{(12)12} &= \frac{1}{t^2}\left(\frac{16m^2}{u} - \frac{4u^3}{3s^2t} + \frac{4s}{t}\right),
\end{aligned} \tag{4.83}$$

$$\begin{aligned}
r^{(8)}_{(12)(34)} &= m^4\left(\frac{16}{t^2} - \frac{32}{3tu} + \frac{16}{u^2}\right) + m^2\left(-\frac{32t^2}{u^3} + \frac{16t}{3u^2} - \frac{8}{t} - \frac{8}{3u}\right) \\
&\quad + \frac{8t^4}{u^4} + \frac{16t^3}{3u^3} - \frac{4t^2}{3u^2} + \frac{4t}{3u} + 1, \\
r^{(8)}_{(123)1} &= \frac{4(t-s)\beta^2_{\hat{s}\hat{t}}}{3u^2}\left(m^2 + \frac{st}{u}\right), \\
r^{(8)}_{(12)11} &= \frac{\beta^2_{\hat{s}\hat{t}}}{3u^2}\left[4m^2s + \frac{t\left(7s^2 - 3t^2\right)}{u}\right], \\
r^{(8)}_{(12)12} &= \frac{16m^4}{3u^2}\left(\frac{2s}{t} - 1\right) + \frac{2m^2}{u^3}\left(-\frac{2s^3}{t} + \frac{8s^2}{3} + \frac{2t^3}{s} - \frac{28st}{3} + 4t^2\right) \\
&\quad + \frac{1}{u^4}\left(-s^4 + \frac{2s^3t}{3} - \frac{16s^2t^2}{3} + 2st^3 + t^4\right),
\end{aligned} \tag{4.84}$$

$$\begin{aligned}
r^{(9)}_{(12)(34)} &= m^4\left(\frac{16}{t^2} - \frac{32}{3tu} + \frac{16}{u^2}\right) + m^2\left(-\frac{32u^2}{t^3} + \frac{16u}{3t^2} - \frac{8}{3t} - \frac{8}{u}\right) \\
&\quad + \frac{8u^4}{t^4} + \frac{16u^3}{3t^3} - \frac{4u^2}{3t^2} + \frac{4u}{3t} + 1, \\
r^{(9)}_{(123)1} &= \frac{4}{3}\left[\frac{4m^4}{u}\left(\frac{1}{s} - \frac{1}{t}\right) + \frac{m^2}{u}\left(\frac{5s^2}{t^2} - 2\right) + \frac{s}{t}\left(\frac{s^2}{t^2} - \frac{1}{4}\right)\right], \\
r^{(9)}_{(12)11} &= \frac{\beta^2_{\hat{s}\hat{u}}}{3t}\left[\frac{4m^2s}{t} + u\left(\frac{4s^2}{t^2} - \frac{6s}{t} - 3\right)\right], \\
r^{(9)}_{(12)12} &= \frac{16m^4}{3u^2}\left(\frac{2s}{t} - 1\right) + \frac{4m^2}{u}\left(\frac{10s^2}{3t^2} + \frac{13s}{3t} + \frac{t}{s} + 4\right) + \frac{8s^3}{3t^3} + \frac{14s^2}{3t^2} + \frac{4s}{t} + 1,
\end{aligned} \tag{4.85}$$

$$
\begin{aligned}
r^{(10)}_{(12)(34)} &= \frac{1}{s^2}\Bigg[16m^4\left(\frac{t^2}{u^2}+\frac{u^2}{t^2}+\frac{4t}{3u}+\frac{4u}{3t}+\frac{2}{3}\right)-8m^2\left(\frac{t^2}{u}+\frac{u^2}{t}+\frac{t}{3}+\frac{u}{3}\right) \\
&\quad + t^2-\frac{2tu}{3}+u^2\Bigg], \\
r^{(10)}_{(123)1} &= \frac{4}{3}\left[\frac{4m^4}{u}\left(\frac{1}{s}-\frac{1}{t}\right)+\frac{m^2}{u}\left(2-\frac{5t^2}{s^2}\right)+\frac{t}{s}\left(\frac{1}{4}-\frac{t^2}{s^2}\right)\right], \\
r^{(10)}_{(12)11} &= \frac{\beta^2_{\hat{t}\hat{u}}}{3s}\left(4m^2-\frac{2tu}{s}\right), \\
r^{(10)}_{(12)12} &= \frac{16m^4}{3u^2}\left(\frac{2s}{t}-1\right)+\frac{4m^2}{u}\left(\frac{s}{t}+\frac{t}{3s}+\frac{10}{3}\right)+\frac{2t^2}{3s^2}+\frac{8t}{3s}+1.
\end{aligned} \tag{4.86}
$$

4.3 Spinor QED.

As previously discussed, one of the well-established advantages of the worldline formalism is its efficiency in extending from the scalar case to the spinorial case. In the context of the box integral, the following substitutions will be required $\gamma^{scal} \to \gamma^{spin}$. Where:

$$
\gamma^{(1)spin}_{(1234)} = \dot{G}_{12}\dot{G}_{23}\dot{G}_{34}\dot{G}_{41} - G_{F12}G_{F23}G_{F34}G_{F41} \tag{4.87}
$$

$$
\begin{aligned}
\gamma^{(2)spin}_{(12)(34)} &= \big(\dot{G}_{12}\dot{G}_{21} - G_{F12}G_{F21}\big)\big(\dot{G}_{34}\dot{G}_{43} - G_{F34}G_{F43}\big) \\
\gamma^{(3)spin}_{(123)i} &= \big(\dot{G}_{12}\dot{G}_{23}\dot{G}_{31} - G_{F12}G_{F23}G_{F31}\big)\dot{G}_{4i} \\
\gamma^{(4)spin}_{(12)ii} &= \big(\dot{G}_{12}\dot{G}_{21} - G_{F12}G_{F21}\big)\dot{G}_{i3}\dot{G}_{4i} \\
\gamma^{(5)spin}_{(12)ij} &= \big(\dot{G}_{12}\dot{G}_{21} - G_{F12}G_{F21}\big)\dot{G}_{i3}\dot{G}_{4j}
\end{aligned} \tag{4.88}
$$

As we did in the scalar case, we will solve Γ^{spin}_{1234}. As an illustrative example of the necessary manipulations

$$
\Gamma^{spin}_{1234} = \frac{e^4}{(4\pi)^2}\int_0^\infty dT T \mathrm{e}^{-m^2T}\int_0^1\prod_{i=1}^4 du_i \left(\dot{G}_{12}\dot{G}_{23}\dot{G}_{34}\dot{G}_{41} - G_{F12}G_{F23}G_{F34}G_{F41}\right)\mathrm{e}^{\Lambda T} \tag{4.89}
$$

From the scalar calculation, we already know the partial result of the previous integral. For the $G_{F12}G_{F23}G_{F34}G_{F41}$ cycle, we note that the Worldline methods also offer the advantage of incorporating identities that emerge from the chains of Green's functions, as well as those arising from their structure. In this example, we use the following identities:

$$G_{Fij}^2 = 1 \tag{4.90}$$
$$G_{Fij}G_{Fjk}G_{Fki} = -(\dot{G}_{ij} + \dot{G}_{jk} + \dot{G}_{ki}) \tag{4.91}$$

to express the fermionic part in terms of the first derivative or the Green function :

$$G_{F12}G_{F23}G_{F34}G_{F41} = G_{F12}G_{F23}G_{F34}G_{F41}G_{F42}^2 \tag{4.92}$$

$$= -(\dot{G}_{12} + \dot{G}_{24} + \dot{G}_{41})(\dot{G}_{23} + \dot{G}_{34} + \dot{G}_{42}) \tag{4.93}$$

So,

$$\begin{aligned}&\int_0^\infty dT T \mathrm{e}^{-m^2 T} \int_0^1 \prod_{i=1}^4 du_i \left(G_{F12}G_{F23}G_{F34}G_{F41}\right) \mathrm{e}^{\Lambda T} \\ &= \int_0^\infty dT T \mathrm{e}^{-m^2 T} \int_0^1 \prod_{i=1}^4 du_i \left(\dot{G}_{12} + \dot{G}_{24} + \dot{G}_{41})(\dot{G}_{23} + \dot{G}_{34} + \dot{G}_{42}\right) \mathrm{e}^{\Lambda T}\end{aligned} \tag{4.94}$$

From the previous product, we can identify three types of integrals:

$$\int_0^\infty dT\, T\, \mathrm{e}^{-m^2 T} \int_0^1 du_1 \dots du_4\, \mathrm{e}^{T\Lambda}\, \dot{G}_{ij}\dot{G}_{jk}, \tag{4.95}$$

$$\int_0^\infty dT\, T\, \mathrm{e}^{-m^2 T} \int_0^1 du_1 \dots du_4\, \mathrm{e}^{T\Lambda}\, \dot{G}_{ij}^2 \tag{4.96}$$

$$\int_0^\infty dT\, T\, \mathrm{e}^{-m^2 T} \int_0^1 du_1 \dots du_4\, \mathrm{e}^{T\Lambda}\, \dot{G}_{ij}\dot{G}_{lk}. \tag{4.97}$$

In equation (4.92), we introduced $1 = G_{F24}^2$. In this case (4.96) will have $i = 2$ and $j = 4$. On the other hand, for (4.95) we have $j = 2, 4$, $i \neq j \neq k$ and $i < k$. The third type of integrals, given by (4.97), can be expressed in terms of the other two types. Moreover, due to the symmetry of the problema, it is sufficient to compute only one case, as the remaining cases will be obtained through permutations.
There permutations, applied to the indices of the Green´s functions, must also be carried out for the Mandelstam variables. For example, after performing the corresponding manipulations, we find that:

$$
\begin{aligned}
\Gamma_1(s,t,u) &= \int_0^\infty dT\, T\, \mathrm{e}^{-m^2T} \int_0^1 du_1 \ldots du_4\, \mathrm{e}^{T\Lambda}\, \dot{G}_{12}\dot{G}_{23}, \\
&= \frac{2}{su}\left\{\left[\ln\left(\frac{\beta_{\hat{t}}-1}{\beta_{\hat{t}}+1}\right)\right]^2 + \frac{s^2+u^2}{t^2}\left(\left[\ln\left(\frac{\beta_{\hat{s}}-1}{\beta_{\hat{s}}+1}\right)\right]^2 + \left[\ln\left(\frac{\beta_{\hat{u}}-1}{\beta_{\hat{u}}+1}\right)\right]^2\right)\right\} \\
&\quad + \frac{s\beta_{\hat{s}\hat{t}}^2}{u}\bar{B}(s,t) + \frac{2(2m^2t+su)}{t^2}\bar{B}(s,u) + \frac{u\beta_{\hat{t}\hat{u}}^2}{s}\bar{B}(t,u).
\end{aligned}
\tag{4.98}
$$

$$
\begin{aligned}
\Gamma_1(s,t,u) &= \int_0^\infty dT\, T\, \mathrm{e}^{-m^2T} \int_0^1 du_1 \ldots du_4\, \mathrm{e}^{T\Lambda}\, \dot{G}_{12}\dot{G}_{23}, \\
&= \frac{2}{su}\left\{\left[\ln\left(\frac{\beta_{\hat{t}}-1}{\beta_{\hat{t}}+1}\right)\right]^2 + \frac{s^2+u^2}{t^2}\left(\left[\ln\left(\frac{\beta_{\hat{s}}-1}{\beta_{\hat{s}}+1}\right)\right]^2 + \left[\ln\left(\frac{\beta_{\hat{u}}-1}{\beta_{\hat{u}}+1}\right)\right]^2\right)\right\} \\
&\quad + \frac{s\beta_{\hat{s}\hat{t}}^2}{u}\bar{B}(s,t) + \frac{2(2m^2t+su)}{t^2}\bar{B}(s,u) + \frac{u\beta_{\hat{t}\hat{u}}^2}{s}\bar{B}(t,u).
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_2(s,t,u) &= \int_0^\infty dT\, T\, \mathrm{e}^{-m^2T} \int_0^1 du_1 \ldots du_4\, \mathrm{e}^{T\Lambda}\, \dot{G}_{12}^2 \\
&= 4\left(\frac{1}{u^2}+\frac{1}{t^2}\right)\left[\ln\left(\frac{\beta_{\hat{s}}-1}{\beta_{\hat{s}}+1}\right)\right]^2 + \frac{4}{u^2}\left[\ln\left(\frac{\beta_{\hat{t}}-1}{\beta_{\hat{t}}+1}\right)\right]^2 + \frac{4}{t^2}\left[\ln\left(\frac{\beta_{\hat{u}}-1}{\beta_{\hat{u}}+1}\right)\right]^2 \\
&\quad + \left(\frac{s^2+t^2}{u^2}+\frac{4m^2s}{tu}\right)\bar{B}(s,t) + \left(\frac{s^2+u^2}{t^2}+\frac{4m^2s}{tu}\right)\bar{B}(s,u) \\
&\quad + \beta_{\hat{t}\hat{u}}^2\bar{B}(t,u).
\end{aligned}
\tag{4.99}
$$

Given the previous result, if we wanted to solve the integral with $\dot{G}_{42}\dot{G}_{23}$, this would be require the permutation of 1 with 4, which is equivalent to permuting t with s.

$$
\int_0^\infty dT\, T\, \mathrm{e}^{-m^2T} \int_0^1 du_1 \ldots du_4\, \mathrm{e}^{T\Lambda}\, \dot{G}_{42}\dot{G}_{23} = \Gamma_1(t,s,u). \tag{4.100}
$$

The same applies to the first type of integrals:

$$\begin{aligned}\Gamma_2(s,t,u) &= \int_0^\infty dT\, T\, \mathrm{e}^{-m^2T} \int_0^1 du_1 \ldots du_4\, \mathrm{e}^{T\Lambda}\, \dot{G}_{12}^2 \\ &= 4\left(\frac{1}{u^2}+\frac{1}{t^2}\right)\left[\ln\left(\frac{\beta_{\hat{s}}-1}{\beta_{\hat{s}}+1}\right)\right]^2 + \frac{4}{u^2}\left[\ln\left(\frac{\beta_{\hat{t}}-1}{\beta_{\hat{t}}+1}\right)\right]^2 + \frac{4}{t^2}\left[\ln\left(\frac{\beta_{\hat{u}}-1}{\beta_{\hat{u}}+1}\right)\right]^2 \\ &\quad + \left(\frac{s^2+t^2}{u^2}+\frac{4m^2s}{tu}\right)\bar{B}(s,t) + \left(\frac{s^2+u^2}{t^2}+\frac{4m^2s}{tu}\right)\bar{B}(s,u) \\ &\quad + \beta_{\hat{t}\hat{u}}^2 \bar{B}(t,u).\end{aligned} \tag{4.101}$$

Thus, if we want to determine the result for $\dot{G}_{42}^2$, the necessary permutations are the same as in the previous case, that is,

$$\int_0^\infty dT\, T\, \mathrm{e}^{-m^2T} \int_0^1 du_1 \ldots du_4\, \mathrm{e}^{T\Lambda}\, \dot{G}_{42}^2 = \Gamma_2(t,s,u) \tag{4.102}$$

For the last type of integrals, we only have two possible cases [‡]:

$$\begin{aligned}&\int_0^1 du_1 \ldots du_4\, \mathrm{e}^{T\Lambda}\, \dot{G}_{41}\dot{G}_{23} \\ &\int_0^1 du_1 \ldots du_4\, \mathrm{e}^{T\Lambda}\, \dot{G}_{43}\dot{G}_{21}\end{aligned} \tag{4.103}$$

In turns out that the last type of integrals can be represented in terms of Γ_1:

$$\int_0^\infty dT\, T\, \mathrm{e}^{-m^2T} \int_0^1 du_1 \ldots du_4\, \mathrm{e}^{T\Lambda}\, \dot{G}_{41}\dot{G}_{23} = \frac{1}{u}\left(s\Gamma_1(u,t,s) - t\,\Gamma_1(u,s,t)\right), \tag{4.104}$$

$$\int_0^\infty dT\, T\, \mathrm{e}^{-m^2T} \int_0^1 du_1 \ldots du_4\, \mathrm{e}^{T\Lambda}\, \dot{G}_{43}\dot{G}_{21} = \frac{1}{s}\left(u\Gamma_1(s,t,u) - t\,\Gamma_1(s,u,t)\right), \tag{4.105}$$

[‡]Note that when multiplying the original integral by, $1 = G_{F42}$ neither of the two integrals contains a $\dot{G}_{42}$.

Using the result from the scalar case and the previous analysis, we find that

$$\Gamma^{\text{spin}}_{(1234)}(s,t,u) = -2\Bigg[\Gamma^{\text{scal}}_{(1234)}(s,t,u) + \left(1+\frac{u}{s}\right)\Gamma_1(s,t,u) - \left(\frac{t}{s}+2\right)\Gamma_1(s,u,t) - 2\Gamma_1(t,s,u) + \left(1+\frac{s}{u}\right)\Gamma_1(u,t,s) - \frac{t}{u}\Gamma_1(u,s,t) - \Gamma_2(t,s,u)\Bigg], \quad (4.106)$$

which, once (4.101) is introduced, takes the form:

$$\Gamma^{\text{spin}}_{(1234)}(s,t,u) = -2\left[\Gamma^{\text{scal}}_{(1234)}(s,t,u) - \bar{B}(s,t) + \bar{B}(s,u) - \bar{B}(t,u)\right]. \quad (4.107)$$

Similarly, all $\Gamma^{\text{spin}}_{\cdots}$ functions can be computed in terms of the scalar result, the function $\bar{B}$, and certain logarithmic terms, which are given by

$$\begin{aligned}
\Gamma^{\text{spin}}_{(12)(34)}(s,t,u) &= -2\Bigg[\Gamma^{\text{scal}}_{(12)(34)}(s,t,u) - \left(\frac{8m^2 s}{tu} + \frac{4t^2}{u^2} + \frac{4t}{u} + 1\right)\bar{B}(s,t) \\
&\quad - \left(\frac{8m^2 s}{tu} + \frac{4u^2}{t^2} + \frac{4u}{t} + 1\right)\bar{B}(s,u) - \left(\frac{8m^2 s}{tu} + 1\right)\bar{B}(t,u) \\
&\quad - \left(\frac{8}{t^2} + \frac{8}{u^2}\right)\ln^2\left(\frac{\beta_{\hat{s}}-1}{\beta_{\hat{s}}+1}\right) - \frac{8}{u^2}\ln^2\left(\frac{\beta_{\hat{t}}-1}{\beta_{\hat{t}}+1}\right) - \frac{8}{t^2}\ln^2\left(\frac{\beta_{\hat{u}}-1}{\beta_{\hat{u}}+1}\right)\Bigg], \\
\Gamma^{\text{spin}}_{(123)1}(s,t,u) &= -2\Bigg[\Gamma^{\text{scal}}_{(123)1}(s,t,u) - \frac{s}{t}\bar{B}(s,u) + \frac{t}{s}\bar{B}(t,u) + \frac{2}{t^2+tu}\ln^2\left(\frac{\beta_{\hat{s}}-1}{\beta_{\hat{s}}+1}\right) \\
&\quad - \frac{2}{t^2+tu}\ln^2\left(\frac{\beta_{\hat{t}}-1}{\beta_{\hat{t}}+1}\right) - \frac{2(2t+u)}{stu}\ln^2\left(\frac{\beta_{\hat{u}}-1}{\beta_{\hat{u}}+1}\right)\Bigg],
\end{aligned} \quad (4.108)$$

$$\begin{aligned}
\Gamma^{\text{spin}}_{(12)11}(s,t,u) &= -2\Bigg[\Gamma^{\text{scal}}_{(12)11}(s,t,u) + \frac{2}{tu}\Bigg\{\ln^2\left(\frac{\beta_{\hat{s}}-1}{\beta_{\hat{s}}+1}\right) + \frac{t^2+u^2}{s^2}\Bigg[\ln^2\left(\frac{\beta_{\hat{t}}-1}{\beta_{\hat{t}}+1}\right) + \\
&\quad \ln^2\left(\frac{\beta_{\hat{u}}-1}{\beta_{\hat{u}}+1}\right)\Bigg]\Bigg\} + \frac{t\beta^2_{\hat{s}\hat{t}}}{u}\bar{B}(s,t) + \frac{u\beta^2_{\hat{s}\hat{u}}}{t}\bar{B}(s,u) + \frac{2(2m^2 s+tu)}{s^2}\bar{B}(t,u)\Bigg], \\
\Gamma^{\text{spin}}_{(12)12}(s,t,u) &= -2\Bigg[\Gamma^{\text{scal}}_{(12)12}(s,t,u) + \left(\frac{4m^2(t+2u)}{su} + \frac{2s^2}{u^2} - 1\right)\bar{B}(s,t) + \frac{(t+2u)\beta^2_{\hat{s}\hat{u}}}{t}\bar{B}(s,u) \\
&\quad + \left(\frac{4m^2(u-s)}{su} + \frac{t^2+2tu-u^2}{s^2}\right)\bar{B}(t,u) - \frac{4s}{tu^2}\ln^2\left(\frac{\beta_{\hat{s}}-1}{\beta_{\hat{s}}+1}\right) \\
&\quad + \frac{4(t+2u)\left(t^2+tu+u^2\right)}{s^2tu^2}\ln^2\left(\frac{\beta_{\hat{t}}-1}{\beta_{\hat{t}}+1}\right) + \frac{4u}{s^2t}\ln^2\left(\frac{\beta_{\hat{u}}-1}{\beta_{\hat{u}}+1}\right)\Bigg].
\end{aligned} \quad (4.109)$$

A new representation for the $\Gamma^{\text{spin/scal}}_{\cdots}$ functions has been obtained.

4.4 Spinor Helicity.

Spinor helicity technology is a mathematical and computational technique used to simplify and calculate scattering amplitudes of particles, particularly in gauge theories such as QED and quantum chromodynamics (QCD) [125],[126],[127]. Helicity refers to the projection of a particle's spin along its direction of motion, and for spin one particles like photons, there are two helicity states: $'+'$ (right-handed helicity) and $'-'$ (left-handed helicity).

As we will see one of the main advantages of spinor helicity techniques is their ability to significantly simplify the expressions for scattering amplitudes, especially when dealing with many photons (N-photons). This simplification leads to increased computational efficiency, reducing the complexity of calculations and making the amplitudes more manageable and less computationally intensive.

The spinor-helicity formalism is based on bracket notation, which includes square brackets and angle brackets. Each type of bracket is associated with "+" helicity and "-" helicity, respectively. Using this notation, the slashed four-momentum and the photon's polarization vector can be expressed as follows:

$$
\begin{aligned}
-\not{p} &= |p\rangle[p| + |p]\langle p|, &(4.110)\\
\varepsilon_+^\mu(k,q) &= -\frac{\langle q|\gamma^\mu|k]}{\sqrt{2}\,\langle qk\rangle}, &(4.111)\\
\varepsilon_-^\mu(k,q) &= -\frac{[q|\gamma^\mu|k\rangle}{\sqrt{2}\,[qk]}. &(4.112)
\end{aligned}
$$

The Schouten identity will help us simplify certain results.

$$\langle p\,q\rangle\langle r\,s\rangle + \langle p\,r\rangle\langle s\,q\rangle + \langle p\,s\rangle\langle q\,r\rangle = 0 \qquad (4.113)$$

In this context, the polarization vector depends on an arbitrary reference vector q, but the results must be independent of this vector. In the worldline formalism each photon enters through its field-strength tensor $f_{\mu\nu} = k_\mu\varepsilon_\nu - k_\nu\varepsilon_\mu$. In the spinor-helicity formalism, this tensor can be expressed as follows:

$$
\begin{aligned}
f^{-\mu\nu} &= -\frac{1}{4\sqrt{2}}\langle k|[\gamma^\mu,\gamma^\nu]|k\rangle &(4.114)\\
f^{+\mu\nu} &= -\frac{1}{4\sqrt{2}}[k|[\gamma^\mu,\gamma^\nu]|k] &(4.115)
\end{aligned}
$$

These brackets possess the following properties

$$
\begin{aligned}
[k||p] &= [kp]\,, && (4.116)\\
\langle k||p\rangle &= \langle kp\rangle, && (4.117)\\
[kp] &= -[pk]\,, && (4.118)\\
\langle kp\rangle &= -\langle pk\rangle, && (4.119)\\
[k|p\rangle &= 0, \\
\langle k|p] &= 0. && (4.120)\\
& && (4.121)
\end{aligned}
$$

From the aforementioned equations, we can easily derive the following properties of $f^{\pm}$:

- Products:

$$
\begin{aligned}
(f_1^+ f_2^+)_{\mu\nu} &= \frac{1}{4}[12][1|\gamma^\mu\gamma^\nu|2], && (4.122)\\
(f_1^- f_2^-)_{\mu\nu} &= \frac{1}{4}\langle 12\rangle\langle 1|\gamma^\mu\gamma^\nu|2\rangle, && (4.123)\\
(f_1^+ f_2^-)_{\mu\nu} &= (f_2^- f_1^+)_{\mu\nu} = \frac{1}{4}[1|\gamma^\mu|2\rangle[1|\gamma^\nu|2\rangle. && (4.124)
\end{aligned}
$$

- Anticommutators:

$$
\begin{aligned}
\{f_1^+, f_2^+\}_{\mu\nu} &= -\frac{1}{2}[12]^2\eta_{\mu\nu}, && (4.125)\\
\{f_1^-, f_2^-\}_{\mu\nu} &= -\frac{1}{2}\langle 12\rangle^2\eta_{\mu\nu}. && (4.126)
\end{aligned}
$$

- Same-helicity traces:

$$
\begin{aligned}
\mathrm{tr}(f_{i_1}^+ \cdots f_{i_N}^+) &= \frac{(-1)^N}{\sqrt{2^{N-2}}}[i_1 i_2]\ldots[i_{N-1}i_N][i_N i_1], \\
\mathrm{tr}(f_{i_1}^- \cdots f_{i_N}^{)} &= \frac{(-1)^N}{\sqrt{2^{N-2}}}\langle i_1 i_2\rangle\ldots\langle i_{N-1}i_N\rangle\langle i_N i_1\rangle. && (4.127)
\end{aligned}
$$

In the spinor-helicity formalism, the product of momentum and polarization vectors for positive and negative helicities takes the following form:

$$\begin{aligned} p\cdot\varepsilon_+(k;q) &= p\cdot\left(-\frac{\langle q|\gamma^\mu|k]}{\sqrt{2}\langle qk\rangle}\right)=\frac{\langle q|-\not{p}|k]}{\sqrt{2}\langle qk\rangle}=\frac{\langle q|(|p\rangle[p|+|p]\langle p|)|k]}{\sqrt{2}\langle qk\rangle} \\ &= \frac{\langle q|p\rangle[p|k]}{\sqrt{2}\langle qk\rangle}+\frac{\langle q|p]\langle p|k]}{\sqrt{2}\langle qk\rangle} \\ &= \frac{\langle qp\rangle[pk]}{\sqrt{2}\langle qk\rangle}. \end{aligned} \tag{4.128}$$

$$\begin{aligned} p\cdot\varepsilon_-(k;q) &= p\cdot\left(-\frac{[q|\gamma^\mu|k\rangle}{\sqrt{2}[qk]}\right)=\frac{[q|-\not{p}|k\rangle}{\sqrt{2}[qk]}=\frac{[q|(|p\rangle[p|+|p]\langle p|)|k\rangle}{\sqrt{2}[qk]} \\ &= \frac{[q|p\rangle[p|k\rangle}{\sqrt{2}[qk]}+\frac{[q|p]\langle p|k\rangle}{\sqrt{2}[qk]} \\ &= \frac{[qp]\langle pk\rangle}{\sqrt{2}[qk]}. \end{aligned} \tag{4.129}$$

After introducing some definitions and fundamental concepts of the spinor-helicity formalism, we emphasize that all the material discussed so far will be applied to the four-photon one-loop amplitude in QED. The potential of this formalism to simplify amplitude calculations in gauge theories will thus become evident.

4.5 4-photon on-shell with Spinor Helicity.

4.5.1 $(++++)$ Scalar and Spinor

In this part, we present the calculation of the three possible configurations for the four-photon amplitude within the spinor-helicity formalism. We begin with the case where all four photons have positive helicity $(++++)$, as it provides a straightforward starting point to demonstrate the application of this formalism. The other configurations, involving mixed helicities, will be addressed later in the section.
Once the helicities are determined, tails can be calculated with remarkable simplicity by taking advantage of the freedom to choose reference momenta. We decided to fix $r_1 = k_4$ and $r_2 = r_3 = r_4 = k_1$[§].
The first element to calculate using the spinor helicity formalism of the minimal tensor

[§]This choice of reference momenta will also be applied to the other cases, namely $(-+++)$ and $(--++)$

basis corresponds to calculating $T^{(1)}_{(1234)}$ and its permutations.

$$T^{(1)}_{(1234)} = Z(1234) = \mathrm{tr}(f_1^+ f_2^+ f_3^+ f_4^+) = \frac{[12][23][34][41]}{2}. \quad (4.130)$$

As previously introduced, a fundamental component for constructing the four-photon amplitude in the minimal tensor basis (4.7) is the Lorentz cycle defined in (4.4). However, with the application of (4.127), these cycles become trivial. In addition, for the subsequent tail $T^{(2)}_{(12)(34)}$, property (4.118) is employed to simplify the result.

$$\begin{aligned} T^{(2)}_{(12)(34)} &= Z(12)Z(34) = \frac{1}{2^2}\mathrm{tr}(f_1^+ f_2^+)\mathrm{tr}(f_3^+ f_4^+) = \frac{[12][21][34][43]}{4} \\ &= \frac{[12]^2[34]^2}{4}. \end{aligned} \quad (4.131)$$

Fot tail $T^{(3)r_4}_{(123)i}$ eq. (4.115) is used.

$$\begin{aligned} T^{(3)r_4}_{(123)i} &= Z(123)\frac{r_4 \cdot f_4 \cdot k_i}{r_4 \cdot k_4} = \frac{1}{\sqrt{2}}\mathrm{tr}(f_1^+ f_2^+ f_3^+)\frac{k_1^\mu f_4^{\mu\nu} k_i^\nu}{k_1 \cdot k_4} \\ &= -\frac{[12][23][31]}{\sqrt{2}}\frac{[41]\langle 1i\rangle[i1]}{2^{3/2}k_1 \cdot k_4} = \frac{[12][23][31][41]\langle 1i\rangle[i1]}{4k_1 \cdot k_4} \\ &= -\frac{[12][23][31][41]\langle 1i\rangle[i1]}{2u}, (i = 1, 2, 3) \end{aligned} \quad (4.132)$$

At this point, we can introduce the Mandelstam variables; in the previous case, we used $u = -2k_1 \cdot k_4$. Finally, for the tails $T^{(4)}_{(12)ii}$ and $T^{(5)}_{(12)ij}$, (4.122) will need to be applied. But, the property (4.120) of the product of brakets and the representation of $\not{k}$(4.110) in spinor helicity render T^4 to zero.

$$\begin{aligned} T^{(4)}_{(12)ii} &= Z(12)\frac{k_i \cdot f_3 \cdot f_4 \cdot k_i}{k_3 \cdot k_4} = \frac{\mathrm{tr}(f_1^+ f_2^+)}{2}\frac{k_i^\mu (f_3^+ f_4^+)_{\mu\nu} k_i^\nu}{k_3 \cdot k_4} \\ &= \frac{[12][21]}{2}\frac{[34][3|\not{k}_i \not{k}_i|4]}{k_3 \cdot k_4} = 0. \end{aligned} \quad (4.133)$$

$$\begin{aligned} T^{(5)}_{(12)ij} &= Z(12)\frac{k_i \cdot f_3 \cdot f_4 \cdot k_j}{k_3 \cdot k_4} = \frac{\mathrm{tr}(f_1^+ f_2^+)}{2}\frac{k_i^\mu (f_3^+ f_4^+)_{\mu\nu} k_j^\nu}{k_3 \cdot k_4} \\ &= \frac{[12][21]}{2}\frac{[34][3|\not{k}_i \not{k}_j|4]}{4k_3 \cdot k_4} = \frac{[12]^2[3i]\langle ij\rangle[j4]}{8k_3 \cdot k_4} \\ &= -\frac{[12]^2[3i]\langle ij\rangle[j4]}{s} \quad ((i,j) = (1,2),(2,1)). \end{aligned} \quad (4.134)$$

The results obtained can be further simplified by using momentum conservation and the Schouten identity (4.113).

$$T^{(1)}_{(1234)} = -2\kappa_1 \frac{u}{s}. \tag{4.135}$$

$$T^{(2)}_{(12)(34)} = \kappa_1 s^2 \tag{4.136}$$

$$T^{(3)r_4}_{(123)3} = 2\kappa_1 st \tag{4.137}$$

$$T^{(4)}_{(12)ii} = 0 \tag{4.138}$$

$$T^{(5)}_{(12)ij} = -\kappa_1 st \tag{4.139}$$

Where:

$$\kappa_1 := \frac{[k_1k_2]^2[k_3k_4]^2}{4s^2} \tag{4.140}$$

Thus, fot the $(++++)$ configuration, our minimal tensor basis is as follows:

$$\begin{aligned}
\Gamma^{(1)}_{scal} &= -2\kappa_1 \left(su\,\Gamma^{scal}_{(1234)} + st\,\Gamma^{scal}_{(1243)} + tu\,\Gamma^{scal}_{(1324)} \right), \\
\Gamma^{(2)}_{scal} &= \kappa_1 \left(s^2\,\Gamma^{scal}_{(12)(34)} + t^2\,\Gamma^{scal}_{(13)(24)} + u^2\,\Gamma^{scal}_{(14)(23)} \right), \\
\Gamma^{(3)}_{scal} &= 2\kappa_1 \Bigg[st\left(\Gamma^{scal}_{(123)3} + \Gamma^{scal}_{(234)3} - \Gamma^{scal}_{(234)2} - \Gamma^{scal}_{(123)2} \right) + tu\left(\Gamma^{scal}_{(341)4} - \Gamma^{scal}_{(341)3} \right) \\
&\quad + us\left(\Gamma^{scal}_{(412)4} - \Gamma^{scal}_{(412)2} \right) \Bigg], \\
\Gamma^{(4)}_{scal} &= 0, \\
\Gamma^{(5)}_{scal} &= -\kappa_1 \Bigg[ts\left(\Gamma^{scal}_{(12)12} + \Gamma^{scal}_{(13)13} + \Gamma^{scal}_{(24)24} + \Gamma^{scal}_{(34)34} \right) \\
&\quad + tu\left(\Gamma^{scal}_{(14)41} + \Gamma^{scal}_{(13)31} + \Gamma^{scal}_{(23)32} + \Gamma^{scal}_{(24)42} \right) \\
&\quad + us\left(\Gamma^{scal}_{(12)21} + \Gamma^{scal}_{(14)14} + \Gamma^{scal}_{(23)23} + \Gamma^{scal}_{(34)43} \right) \Bigg].
\end{aligned} \tag{4.141}$$

At this stage, it is important to highlight how the formalism naturally facilitates the unification of scalar and spinorial calculations, so we write the previous result as:

$$
\begin{aligned}
\Gamma^{(1)}_{scal/spin} &= -2\kappa_1 \left(su\,\Gamma^{scal/spin}_{(1234)} + st\,\Gamma^{scal/spin}_{(1243)} + tu\,\Gamma^{scal/spin}_{(1324)} \right), \\
\Gamma^{(2)}_{scal/spin} &= \kappa_1 \left(s^2\,\Gamma^{scal/spin}_{(12)(34)} + t^2\,\Gamma^{scal/spin}_{(13)(24)} + u^2\,\Gamma^{scal/spin}_{(14)(23)} \right), \\
\Gamma^{(3)}_{scal/spin} &= 2\kappa_1 \Bigg[st \left(\Gamma^{scal/spin}_{(123)3} + \Gamma^{scal/spin}_{(234)3} - \Gamma^{scal/spin}_{(234)2} - \Gamma^{scal/spin}_{(123)2} \right) + tu \left(\Gamma^{scal/spin}_{(341)4} - \Gamma^{scal/spin}_{(341)3} \right) \\
&\quad + us \left(\Gamma^{scal/spin}_{(412)4} - \Gamma^{scal/spin}_{(412)2} \right) \Bigg], \\
\Gamma^{(4)}_{scal/spin} &= 0, \\
\Gamma^{(5)}_{scal/spin} &= -\kappa_1 \Bigg[ts \left(\Gamma^{scal/spin}_{(12)12} + \Gamma^{scal/spin}_{(13)13} + \Gamma^{scal/spin}_{(24)24} + \Gamma^{scal/spin}_{(34)34} \right) \\
&\quad + tu \left(\Gamma^{scal/spin}_{(14)41} + \Gamma^{scal/spin}_{(13)31} + \Gamma^{scal/spin}_{(23)32} + \Gamma^{scal/spin}_{(24)42} \right) \\
&\quad + us \left(\Gamma^{scal/spin}_{(12)21} + \Gamma^{scal/spin}_{(14)14} + \Gamma^{scal/spin}_{(23)23} + \Gamma^{scal/spin}_{(34)43} \right) \Bigg].
\end{aligned}
\tag{4.142}
$$

Based on the previous equation and the results from chapter 4,we find that the scalar and spinor amplitudes $\Gamma^{++++}_{\text{scal/spin}}$, associated with the "all +" helicity case,remarkably simplify to:

$$
\begin{aligned}
\Gamma^{++++}_{\text{scal}} &= -16\,\kappa_1 \left[1 - m^4 I_4(s,t,u) \right], \\
\Gamma^{++++}_{\text{spin}} &= -2\,\Gamma^{++++}_{\text{scal}},
\end{aligned}
\tag{4.143}
$$

where $I_4(s,t,u)$ is defined in the equation (4.47). For easier reading, we rewrite it below:

$$
I_4(s,t,u) = \bar{B}(s,t) + \bar{B}(s,u) + \bar{B}(t,u) \tag{4.144}
$$

4.5.2 $(-+++)$ Scalar and Spinor

In this case, we consider a photon with negative helicity, which significantly simplifies the calculations due to the properties of the spinor-helicity formalism introduced earlier. Moreover, it is straightforward to verify that, in this scenario,

$$\begin{aligned}
\Gamma^{(1)}_{scal/spin} &= 0 \\
\Gamma^{(2)}_{scal/spin} &= 0, \\
\Gamma^{(3)}_{scal/spin} &= 2st\kappa_2\left(\Gamma^{scal/spin}_{(234)2} - \Gamma^{scal/spin}_{(234)3}\right), \\
\Gamma^{(4)}_{scal/spin} &= \kappa_2\left[st\left(\Gamma^{scal/spin}_{(23)22} + \Gamma^{scal/spin}_{(23)33}\right) + su\left(\Gamma^{scal/spin}_{(24)22} + \Gamma^{scal/spin}_{(24)44}\right)\right. \\
&\quad \left. + tu\left(\Gamma^{scal/spin}_{(34)33} + \Gamma^{scal/spin}_{(34)44}\right)\right], \\
\Gamma^{(5)}_{scal/spin} &= -\kappa_2\left[st\left(\Gamma^{scal/spin}_{(23)23} + \Gamma^{scal/spin}_{(23)32}\right) + su\left(\Gamma^{scal/spin}_{(24)24} + \Gamma^{scal/spin}_{(24)42}\right)\right. \\
&\quad \left. + tu\left(\Gamma^{scal/spin}_{(34)34} + \Gamma^{scal/spin}_{(34)43}\right)\right],
\end{aligned} \tag{4.145}$$

$$\kappa_2 = \frac{1}{4stu}\langle k_1 k_2\rangle^2 [k_2 k_4]^2 [k_2 k_3]^2 \tag{4.146}$$

where $\kappa_2 = \frac{1}{4stu}\langle 12\rangle^2[24]^2[23]^2$.
Using the results of the equation (4.145), and after some algebraic manipulation, we obtain that:

$$\begin{aligned}
\Gamma^{-+++}_{\text{scal}} &= -16\,\kappa_2\left\{1 - m^4 I_4(s,t,u) - \frac{m^2}{2stu}\left[s^2t^2\bar{B}(s,t) + s^2u^2\bar{B}(s,u)\right.\right. \\
&\quad \left.\left. + t^2u^2\bar{B}(t,u)\right] - \frac{m^2(t^2-su)}{stu}\left[\ln^2\left(\frac{\beta_{\hat{s}}-1}{\beta_{\hat{s}}+1}\right) + \ln^2\left(\frac{\beta_{\hat{t}}-1}{\beta_{\hat{t}}+1}\right) + \ln^2\left(\frac{\beta_{\hat{u}}-1}{\beta_{\hat{u}}+1}\right)\right]\right\}, \\
\Gamma^{-+++}_{\text{spin}} &= -2\,\Gamma^{-+++}_{\text{scal}}.
\end{aligned}$$

It is a well-established fact that the relationships $\Gamma_{\text{spin}} = -2\Gamma_{\text{scal}}$ hold for the $++++$ and $-+++$ configurations. Specifically, for the $++++$ case, this relationship stems from the inherent self-duality of a "all-plus" photon field [37].

4.5.3 $(--++)$ Scalar and Spinor

For the final case, considering two photons with negative helicity and two with positive helicity, the calculations remain straightforward. Once again, due to the spinor-helicity formalism, certain terms cancel directly, leading to the result:

$$\begin{aligned}
\Gamma^{(1)}_{scal/spin} &= s^2\kappa_3\left(\Gamma^{scal/spin}_{(1234)}+\Gamma^{scal/spin}_{(1243)}+\Gamma^{scal/spin}_{(1324)}\right),\\
\Gamma^{(2)}_{scal/spin} &= s^2\kappa_3\,\Gamma^{scal/spin}_{(12)(34)},\\
\Gamma^{(3)}_{scal/spin} &= 0,\\
\Gamma^{(4)}_{scal/spin} &= 0,\\
\Gamma^{(5)}_{scal/spin} &= -\kappa_3\left[t\left(\Gamma^{scal/spin}_{(12)12}+\Gamma^{scal/spin}_{(34)34}\right)+u\left(\Gamma^{scal/spin}_{(12)21}+\Gamma^{scal/spin}_{(34)43}\right)\right],
\end{aligned} \tag{4.147}$$

where $\kappa_3 = \frac{1}{4s^2}\langle k_1k_2\rangle^2[k_3k_4]^2$. Finally, substituting the known results, we obtain:

$$\begin{aligned}
\Gamma^{--++}_{\rm scal} &= 16\kappa_3\left\{1+m^4I_4(s,t,u)-\frac{2m^2s+tu}{s^2}\left[\ln^2\left(\frac{\beta_{\hat t}-1}{\beta_{\hat t}+1}\right)+\ln^2\left(\frac{\beta_{\hat u}-1}{\beta_{\hat u}+1}\right)\right]\right.\\
&\left.+\frac{t-u}{s}\left[\beta_{\hat u}\ln\left(\frac{\beta_{\hat u}-1}{\beta_{\hat u}+1}\right)-\beta_{\hat t}\ln\left(\frac{\beta_{\hat t}-1}{\beta_{\hat t}+1}\right)\right]+\frac{t^2u^2}{2s^2}\beta^2_{\hat t\hat u}\bar B(t,u)\right\},\\
\Gamma^{--++}_{\rm spin} &= -32\kappa_3\left\{1+m^4I_4(s,t,u)-\frac{4m^2s-t^2-u^2}{2s^2}\left[\ln^2\left(\frac{\beta_{\hat t}-1}{\beta_{\hat t}+1}\right)+\ln^2\left(\frac{\beta_{\hat u}-1}{\beta_{\hat u}+1}\right)\right]\right.\\
&+\frac{t-u}{s}\left[\beta_{\hat u}\ln\left(\frac{\beta_{\hat u}-1}{\beta_{\hat u}+1}\right)-\beta_{\hat t}\ln\left(\frac{\beta_{\hat t}-1}{\beta_{\hat t}+1}\right)\right]-\frac{tu(t^2+u^2)}{4s^2}\bar B(t,u)\\
&\left.-\frac{m^2}{2}\left[s\bar B(s,t)+s\bar B(s,u)+\frac{(t-u)^2}{s}\bar B(t,u)\right]\right\}.
\end{aligned} \tag{4.148}$$

Contrary to the $(++++)$ and $(-+++)$ cases, the relationship between $\Gamma^{--++}_{\rm scal}$ and $\Gamma^{--++}_{\rm spin}$ requires the introduction of a $\Delta\Gamma(\bar B,\beta)$:

$$\begin{aligned}
\Gamma^{--++}_{\rm scal} &= 16\kappa_3\left\{1+m^4I_4(s,t,u)-\frac{2m^2s+tu}{s^2}\left[\ln^2\left(\frac{\beta_{\hat t}-1}{\beta_{\hat t}+1}\right)+\ln^2\left(\frac{\beta_{\hat u}-1}{\beta_{\hat u}+1}\right)\right]\right.\\
&\left.+\frac{t-u}{s}\left[\beta_{\hat u}\ln\left(\frac{\beta_{\hat u}-1}{\beta_{\hat u}+1}\right)-\beta_{\hat t}\ln\left(\frac{\beta_{\hat t}-1}{\beta_{\hat t}+1}\right)\right]+\frac{t^2u^2}{2s^2}\beta^2_{\hat t\hat u}\bar B(t,u)\right\},\\
\Gamma^{--++}_{\rm spin} &= -2\Gamma^{--++}_{\rm scal}+\Delta\Gamma(\bar B,\beta).
\end{aligned}$$

with $\Delta\Gamma(\bar{B},\beta)$ defined as:

$$\begin{aligned}\Delta\Gamma(\bar{B},\beta) &= 8m^2\kappa_3\Bigg(2sI_4(s,t,u) - \frac{tu}{s^2}\bar{B}(t,u)(t^2+u^2-8s+2tu\beta^2_{\hat{t}\hat{u}}) \\ &\quad -2\left[\ln^2\left(\frac{\beta_{\hat{t}}-1}{\beta_{\hat{t}}+1}\right)+\ln^2\left(\frac{\beta_{\hat{u}}-1}{\beta_{\hat{u}}+1}\right)\right]\Bigg)\end{aligned} \tag{4.149}$$

Conclusions and Outlook

In the first quarter of the 21st century, the worldline formalism has emerged as a powerful and significant tool for addressing a diverse range of challenges in contemporary theoretical physics, from amplitudes in Abelian and non-Abelian theories to, more recently, the scattering of black holes. This dissertation has focused specifically on investigating various integration techniques for vacuum photon amplitudes within both scalar and spinor QED.

Our interest in pursuing simpler representations for these calculations stems directly from a key advantage of the worldline formalism : its remarkable capacity to combine a multitude of individual Feynman diagrams into a single, unified worldline diagram. This consolidation of countless Feynman contributions into a single expression not only visually simplifies complex problems but also ushers us into uncharted mathematical territory. We intend to use this powerful property in future work to systematically construct efficient multi-loop amplitudes, effectively using one-loop results as fundamental building blocks within this novel and promising mathematical landscape.

Beyond this specific application, the worldline formalism offers profound conceptual and computational advantages over conventional QFT calculations based on Feynman diagrams. As we have demonstrated, it drastically reduces the number of contributions required in perturbative calculations. While the standard Feynman diagram approach suffers from a combinatorial explosion of diagrams at higher loop orders—for instance, requiring 12,672 Feynman diagrams for a five-loop electron g-2 calculation—the worldline formalism remarkably encapsulates these within just 32 worldline contributions. This simplification significantly streamlines computations, making previously intractable calculations feasible.

Furthermore, the formalism provides a more unified and intuitive framework. Instead of assembling amplitudes from individual Feynman diagrams, it describes QFT processes as path integrals over particle trajectories in background fields. This first-quantized perspective not only clarifies the underlying physics but also automatically sums over distinct diagram topologies, eliminating the risk of missing contributions and reducing manual effort. Its particular suitability for multi-loop calculations and systems in external fields (e.g., Schwinger pair production or light-by-light scattering) allows for more elegant and tractable solutions through compact integral representations. Finally, its close mathematical resemblance to string theory—especially one-loop string amplitudes—facilitates a valuable cross-pollination of techniques, revealing deeper structural connections between QFT and string theory.

Despite these compelling advantages, the practical application of the worldline formalism, particularly to the multi-point functions we have explored, presents its own set of mathematical challenges. The presence of sign and absolute value functions within the integrals often necessitates a re-evaluation of standard integration methods. While a division into sectors can manage these functions, it risks complicating the handling and proliferation of terms in quantum correction calculations. Our research indicates that the inverse derivative expansion of the worldline master formula, which utilizes Hermite polynomials and Bernoulli numbers, provides the most straightforward method for resolving these involved integrals and circumventing sector decomposition.

However, a critical challenge arises as the number of points within the loop increases: this method yields summations that, even in the simplest scenarios (e.g., the two-point function), reduce to known hypergeometric functions. More notably, for higher-point functions like the three-point function, this expansion gives rise to novel sums for which we were unable to identify a corresponding hypergeometric representation in the existing literature. Consequently, a thorough characterization of these previously undocumented sums presents a compelling and crucial avenue for future research, promising to further unlock the full potential of the worldline formalism for advanced QFT calculations.

Another promising approach explored in this investigation is the Gaussian linearization method. This technique uses a specific relation between the Worldline Green's function and the square of its first derivative, effectively linearizing the problem. A significant advantage of this method is its ability to resolve all integrals over the unit circle, thereby completely avoiding the need for sector decomposition. This represents a notable extension of a previously known result within the formalism, expanding its application from single integrals to N-point integrals. While this investigation led to the introduction of a new representation of the worldline master formula, this alternative representation, though promising for simplifying the integration process, contains integrals that introduce their own set of complexities. Further study is warranted to investigate if the resulting "sinh" functions are indeed related to the well-established results obtained from connecting the one-loop N-point Feynman diagram to its geometrical representation based on the N-dimensional simplex [112]. This potential connection offers another exciting direction for future research, aiming to bridge different powerful methodologies within the field.

Our investigation significantly advanced the understanding of scattering amplitudes in QED whithin the worldline formalism. After introducing a novel representation of the master formula, we extended this analysis to the general N-point case, focusing

on the low-energy limit. This yielded a compact representation for N-photon amplitudes, whose conciseness proved beneficial for calculating specific scenarios, such as the all-positive helicity configuration. Furthermore, in our pursuit of identifying hypergeometric functions within these amplitudes, we successfully presented a ${}_2F_1$ hypergeometric function representation for the coefficients accompanying this amplitude.
Following the comprehensive study of N-photon amplitudes, we delved into the detailed calculation of on-shell 4-photon amplitudes for both scalar and spinor QED. This included a thorough examination of all possible helicity configurations using spinor helicity formalism: all photons with the same helicity, one photon with different helicity, and the two-positive, two-negative helicity case, for both scalar and spinor QED. Reinforcing this development, it was recently shown that this representation can also be derived via a novel and more general approach, the "Discontinuity Method" [128], for calculating one-loop worldline integrals. However, despite this progress, developing a technique to bypass sector decomposition within the Worldline formalism remains a key objective for future research.

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Appendix A

Conventions

We adopt the notation introduced by Mark Srednicki [17] throughout this work. The natural unit system, where $\hbar = c = 1$, is employed, and four-dimensional spacetime indices are represented by lower case Greek letters $(\alpha, \beta, \dots)$. The Minkowski space metric is expressed as $\eta^{\mu\nu} = \text{diag}(-,+,+,+)$. Throughout this work, the Einstein summation convention is assumed, and the metric is applied to raise and lower Lorentz indices, such that $x_\mu = \eta_{\mu\nu} x^\nu$. The completely antisymmetric tensor is denoted by $\varepsilon^{\mu\nu\alpha\beta}$, with $\varepsilon^{0123} = -\varepsilon_{0123} = 1$. The dot product of two four-vectors is written as $x \cdot y = x^\mu y_\mu = x^\mu \eta_{\mu\nu} y^\nu$.
The values for the Bernoulli numbers are those provided by Wolfram 14.1, that is, $B_1 = -\frac{1}{2}$.

Appendix B

Demonstration of the inverse derivative expansion of the worldline exponential

The following is the demonstration of the mathematical identity presented in Equation (1.56), which was originally published in [78]. It is included here for the sake of completeness and to provide the reader with all necessary details within this document.

The central identity of this work is given by:

$$e^{pG_{ab}} = 1 + 2\sum_{n=1}^{\infty} p^{n-\frac{1}{2}} H_{2n-1}\left(\frac{\sqrt{p}}{2}\right) \overline{\langle u_a|\partial^{-2n}|u_b\rangle}, \tag{B.1}$$

where p is a parameter and the other components are defined as follows:

- The *Green's function* G_{ab} encodes the geometric relationship between points:

$$G_{ab} = |u_a - u_b| - (u_a - u_b)^2 \quad \text{for} \quad |u_a|, |u_b| < 1 \tag{B.2}$$

- The *averaged matrix elements* regularize the inverse derivatives:

$$\overline{\langle u_a|\partial^{-2n}|u_b\rangle} := \langle u_a|\partial^{-2n}|u_b\rangle - \langle u_a|\partial^{-2n}|u_a\rangle \tag{B.3}$$

- $H_n(x)$ denote the *Hermite polynomials*

Geometric Framework

The identity (B.1) is intrinsically tied to the geometry of the unit circle:

1. **1-Periodic Coordinates**: The variables u_a, u_b represent angular positions on S^1 with periodicity $u \sim u+1$, making all quantities invariant under full rotations.

2. **Coordinate Constraints**: The condition $|u_a|, |u_b| < 1$ ensures:

   - Uniqueness of representation (avoiding redundant copies)
   - Convergence of the series expansion

3. **Translation Symmetry**: The system's invariance under $u \mapsto u + c$ permits expressing all quantities through the difference variable $x = u_a - u_b \in (-1, 1)$. This leads to the natural emergence of *periodic Bernoulli polynomials*:

$$\bar{B}_n(x) := B_n(x - \lfloor x \rfloor) \tag{B.4}$$

   which inherit the circle's 1-periodicity while maintaining the polynomial structure locally.

The inverse derivative matrix elements $\langle u_a|\partial^{-n}|u_b\rangle$ admit an explicit representation through the periodic Bernoulli polynomials:

$$\langle u_a|\partial^{-n}|u_b\rangle = \delta_{n,0} \sum_{k=-\infty}^{\infty} \delta(x-k) - \frac{1}{n!}\bar{B}_n(x), \tag{B.5}$$

where $x = u_a - u_b \in (-1, 1)$ and $\bar{B}_n(x)$ denotes the n-th periodically continued Bernoulli polynomial.

Key Properties:

- *Interval Restriction*: On $[0, 1)$, these polynomials coincide with classical Bernoulli polynomials:
$$\bar{B}_n(x) = B_n(x) \quad \text{for} \quad x \in [0, 1). \tag{B.6}$$

- *Periodic Continuation*: For all $x \in \mathbb{R} \setminus \mathbb{Z}$, the continuation is defined by:
$$\bar{B}_n(x) = B_n(x - \lfloor x \rfloor), \tag{B.7}$$
  with special values at integers $k \in \mathbb{Z}$:
$$\bar{B}_n(k) = \begin{cases} B_n(0) & \text{for } n \neq 1, \\ -\frac{1}{2} & \text{for } n = 1. \end{cases} \tag{B.8}$$

- *Continuity*: For $n \geq 2$, $\bar{B}_n(x)$ is C^{n-2} across integer points, while $\bar{B}_1(x)$ has a jump discontinuity of -1 at integers.

Substituting (B.5) into (B.1) and restricting to $x \in [0,1)$, we obtain the reduced identity:

$$e^{px(1-x)} = 1 + 2\sum_{n=1}^{\infty} p^{n-\frac{1}{2}} H_{2n-1}\left(\frac{\sqrt{p}}{2}\right)\left(B_{2n}(x) - B_{2n}(0)\right), \tag{B.9}$$

where the dependence on u_a, u_b now appears solely through their difference $x = u_a - u_b$.

The proof of Identity (B.21) hinges on the following non-trivial relationship between Bernoulli numbers and polynomial coefficients:

Lemma 1 (Bernoulli Cancellation Identity)**.** *For any odd positive integer r and indeterminate x,*

$$\sum_{n=0}^{r} \binom{r}{n} (2x - r + n)_n (x)_{r-n} B_n = 0, \tag{B.10}$$

where $(a)_k$ denotes the falling factorial and $B_n = B_n(0)$ are Bernoulli numbers.

Proof. We establish this result through a sequence of analytic and combinatorial manipulations:

Step 1: Antisymmetry Setup

For odd r, Bernoulli polynomials satisfy $B_r(1-x) = -B_r(x)$. This implies that for any even function $f(x) = f(1-x)$:

$$\int_0^1 f(x) B_r(x)\, dx = 0. \tag{B.11}$$

Step 2: Strategic Integral Choice

Take $f(x) = x^{u-1}(1-x)^{u-1}$ (clearly even) and expand $B_r(x)$:

$$B_r(x) = \sum_{n=0}^{r} \binom{r}{n} B_n x^{r-n}. \tag{B.12}$$

Substituting into (B.11) gives:

$$\sum_{n=0}^{r} \binom{r}{n} B_n \underbrace{\int_0^1 x^{u+r-n-1}(1-x)^{u-1}\, dx}_{I(n)} = 0. \tag{B.13}$$

Step 3: Beta Integral Evaluation

Each $I(n)$ is a Beta function:

$$I(n) = B(u + r - n, u) = \frac{\Gamma(u + r - n)\Gamma(u)}{\Gamma(2u + r - n)}. \tag{B.14}$$

Step 4: Gamma Function Expansion
Using $\Gamma(z+1) = z\Gamma(z)$, we express:

$$\Gamma(u+r-n) = (u+r-n-1)_{r-n}\Gamma(u) \quad \text{(falling factorial)}, \tag{B.15}$$

$$\Gamma(2u+r-n) = \frac{\Gamma(2u+r)}{(2u+r-1)_n}. \tag{B.16}$$

Step 5: Pochhammer Rewriting
Substituting back into $I(n)$:

$$I(n) = \frac{(u+r-n-1)_{r-n}\Gamma(u)\cdot\Gamma(u)}{\Gamma(2u+r)/(2u+r-1)_n} \tag{B.17}$$

$$= (u+r-n-1)_{r-n}(2u+r-1)_n \frac{\Gamma(u)^2}{\Gamma(2u+r)}. \tag{B.18}$$

Detailed Gamma Cancellation. After Step 5, we have:

$$\frac{\Gamma(u)^2}{\Gamma(2u+r)} \sum_{n=0}^{r} \binom{r}{n} B_n (u+r-n-1)_{r-n}(2u+r-1)_n = 0. \tag{B.19}$$

Critical Observations:

1. The prefactor $\frac{\Gamma(u)^2}{\Gamma(2u+r)}$ is *independent* of the summation index n
2. For $\mathrm{Re}(u) > 0$, the gamma functions never vanish ($\Gamma(z) \neq 0$ for all $z \in \mathbb{C}$)
3. Thus, we may safely divide both sides by this non-zero factor

This leaves the fundamental combinatorial identity:

$$\sum_{n=0}^{r} \binom{r}{n} B_n (u+r-n-1)_{r-n}(2u+r-1)_n = 0. \tag{B.20}$$

The final polynomial form (B.10) follows through the substitution $x = -u + \frac{r+1}{2}$ and properties of falling factorials. □

Remark 1. *The lemma's cancellation mechanism works because:*

- *Odd-indexed Bernoulli numbers B_{2k+1} (except B_1) vanish*
- *The Pochhammer terms enforce precise combinatorial weights*
- *The symmetry $x \leftrightarrow 1-x$ propagates through all steps*

Detailed Proof of Reduced Identity

$$e^{px(1-x)} = 1 + 2\sum_{n=1}^{\infty} p^{n-\frac{1}{2}} H_{2n-1}\left(\frac{\sqrt{p}}{2}\right)(B_{2n}(x) - B_{2n}(0)) \tag{B.21}$$

Step 1: Series Expansion of Both Sides

Left side expansion:

$$e^{px(1-x)} = \sum_{m=0}^{\infty} \frac{p^m x^m (1-x)^m}{m!} = 1 + \sum_{m=1}^{\infty} \frac{p^m x^m (1-x)^m}{m!} \tag{B.22}$$

Right side expansion (using Hermite polynomials):

$$H_{2n-1}(y) = \sum_{k=0}^{n-1} \frac{(2n-1)!}{(2n-2k-1)!k!}(-1)^k (2y)^{2n-2k-1} \tag{B.23}$$

Step 2: Index Transformation

For the double sum $\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}$, we set $m = 2n - k - 1$ with:

- **Lower bound**: $n \geq \lceil \frac{m+1}{2} \rceil$ (from $k \geq 0$)
- **Upper bound**: $n \leq m$ (from $k \leq n - 1$)
- **Implied**: $k = 2n - m - 1$,

where $\lceil \cdot \rceil$ denotes the **ceiling function** (smallest integer $\geq$ the argument).

Step 3: Coefficient Matching

For each $m \geq 1$, equate coefficients of p^m:

$$\frac{x^m (1-x)^m}{m!} = \sum_{n=\lceil \frac{m+1}{2} \rceil}^{m} \frac{2(B_{2n}(x) - B_{2n}(0))}{(2n-m-1)!(2m-2n+1)!2n} \tag{B.24}$$

Step 4: Bernoulli Polynomial Analysis

Expand $B_{2n}(x) = \sum_{\ell=0}^{2n} \binom{2n}{\ell} B_\ell x^{2n-\ell}$ and compare x^k terms:

$$\sum_{\substack{n=\lceil k/2\rceil \\ n \text{ even}}}^{m} \frac{2\binom{2n}{k}B_{2n-k}}{2n}\binom{m}{2n-m-1} = \begin{cases}(-1)^k\binom{m}{k-m} & k \text{ odd} \\ 0 & k \text{ even (by Corollary 1)}\end{cases} \tag{B.25}$$

Example Verification for $m = 2$

$$\begin{aligned} \text{LHS} &= \frac{x^2(1-x)^2}{2} = \frac{x^2}{2} - x^3 + \frac{x^4}{2} \\ \text{RHS} &= \frac{2(B_4(x) - B_4(0))}{1!1!4} = \frac{2(x^4 - 2x^3 + x^2)}{4} = \frac{x^4}{2} - x^3 + \frac{x^2}{2} \end{aligned}$$

□

Appendix C

Generating Function Approach to the Box Integral in Scalar QED

In 4.1, we set $\gamma^{scal}_{...} = 1$ to calculate a four-point amplitude at the level of ϕ^3 theory. Now, to develop an alternative approach for computing the box integral in scalar QED, we will utilize the elegant property of the derivative of the exponential function by adding a source term in the exponent,

$$JT \equiv T \sum_{i<j} J_{ij} \dot{G}_{ij} \tag{C.1}$$

Where it can be assumed that $i_n < j_n$ for $n = 1, \ldots, 4$. However, in instances where $i_n > j_n$, it should be understood that $J_{i_n j_n} \equiv -J_{j_n i_n}$ and $J_{i_n i_n} = 0$.
This approach will allow us to generate $\gamma^{scal}_{...}$ (see 4.49) through derivatives of these sources, enhancing our understanding of the underlying processes.
The representation of the box integral in scalar QED using the worldline formalism can be expressed as follows:

$$I_4(i_1, j_1; \ldots; i_4, j_4) \;=\; \int_0^\infty dT\, T\, \mathrm{e}^{-m^2 T} \frac{1}{T^4} \prod_{n=1}^{4} \frac{\partial}{\partial J_{i_n j_n}} \int_0^1 du_1 \ldots du_4\, \mathrm{e}^{(\Lambda + J)T} \Big|_{J_{12} = \ldots = J_{34} = 0} \tag{C.2}$$

This approach is employed to make the calculations in scalar QED analogous to those of the basic scalar box integral in ϕ^3.Equation (4.28) can be generalized to

$$
\begin{aligned}
I_{du_4} = T\int_0^1 du_4\,\mathrm{e}^{(\Lambda+J)T} = & \left[\frac{2\mathrm{e}^{TJ_{41}}}{u+\dot{G}_{12}t+\dot{G}_{13}s+4J_4}+\frac{2\mathrm{e}^{-TJ_{41}}}{u-\dot{G}_{12}t-\dot{G}_{13}s-4J_4}\right]\mathrm{e}^{(u\Lambda_u+\lambda_u)T} \\
& +\left[\frac{2\mathrm{e}^{TJ_{42}}}{t+\dot{G}_{23}s+\dot{G}_{21}u+4J_4}+\frac{2\mathrm{e}^{-TJ_{42}}}{t-\dot{G}_{23}s-\dot{G}_{21}u-4J_4}\right]\mathrm{e}^{(t\Lambda_t+\lambda_t)T} \\
& +\left[\frac{2\mathrm{e}^{TJ_{43}}}{s+\dot{G}_{31}u+\dot{G}_{32}t+4J_4}+\frac{2\mathrm{e}^{-TJ_{43}}}{s-\dot{G}_{31}u-\dot{G}_{32}t-4J_4}\right]\mathrm{e}^{(s\Lambda_s+\lambda_s)T}
\end{aligned}
\tag{C.3}
$$

To simplify our result, we have introduced the following notation:

$$\Lambda_u \equiv \frac{1}{2}(G_{12}+G_{13}-G_{23}) \tag{C.4}$$

$$\Lambda_t \equiv \frac{1}{2}(G_{12}+G_{23}-G_{13}) \tag{C.5}$$

$$\Lambda_s \equiv \frac{1}{2}(G_{13}+G_{23}-G_{12}) \tag{C.6}$$

$$\lambda_u = (J_{12}-J_{24})\dot{G}_{12}+(J_{13}-J_{34})\dot{G}_{13}+J_{23}\dot{G}_{23} \tag{C.7}$$

$$\lambda_t = (J_{23}-J_{34})\dot{G}_{23}+(J_{21}-J_{14})\dot{G}_{21}+J_{31}\dot{G}_{31} \tag{C.8}$$

$$\lambda_s = (J_{31}-J_{14})\dot{G}_{31}+(J_{32}-J_{24})\dot{G}_{32}+J_{12}\dot{G}_{12} \tag{C.9}$$

Furthermore, the integration of u_4 highlighted the feasibility of introducing a $J_4 = J_{41}+J_{42}+J_{43}$, prompting the general introduction of

$$J_i \equiv \sum_{j=1}^{4} J_{ij}. \tag{C.10}$$

Let us note that equation C.3 simplifies the computation of the integrals presented in [129], as it provides a general formula to calculate any integral of the type:

$$\int_0^1 du_4\dot{G}_{4i}^n\mathrm{e}^{\Lambda T} = \frac{1}{T^{n+1}}\frac{\partial^n}{\partial J_{4i}^n} I_{du_4}|_{J_{ij}=0} \tag{C.11}$$

Similar to ϕ^3 calculations, the second and third line on the right side of (C.3) correspond to permutations of the Mandelstam variables from the first line, along with their respective changes in the integrations variables $(u_1,u_2,u_3)\to(u_2,u_3,u_1)$ and $(u_1,u_2,u_3)\to(u_3,u_1,u_2)$. These permutations are also reflected in the J_{ij} terms, which allows us to focus solely on calculating the first line. However, in contrast to the ϕ^3 case, the introduction of the J's leds to a spurious divergence in T. As this divergence is spurious, it can be handled as demostrated at the end of this appendix, and the integration over T yields:

$$2\int_0^\infty \frac{dT}{T^4}\mathrm{e}^{-Tm^2}\mathrm{e}^{(u\Lambda_u+\lambda_u+J_{41})T} = \frac{(m^2-(u\Lambda_u+\lambda_u+J_{41}))^3}{3}\ln\big(m^2-(u\Lambda_u+\lambda_u+J_{41})\big) \tag{C.12}$$

So, equation (C.2) can be written as:

$$I_4(i_1,j_1;\ldots;i_4,j_4) = \prod_{n=1}^{4}\frac{\partial}{\partial J_{i_n j_n}}(I^{(s,t,u)}_{1234+}+I^{(u,s,t)}_{2314+}+I^{(t,u,s)}_{3124+}+I^{(s,t,u)}_{1234-}+I^{(u,s,t)}_{2314-}+I^{(t,u,s)}_{3124-}) \tag{C.13}$$

Where,

$$I^{(s,t,u)}_{1234\pm} = \int_0^1 du_1du_2du_3\left[\frac{(m^2-(u\Lambda_u+\lambda_u\pm J_{41}))^3}{3(u\pm\dot{G}_{12}t\pm\dot{G}_{13}s\pm 4J_4)}\ln\big(m^2-(u\Lambda_u+\lambda_u\pm J_{41})\big)\right] \tag{C.14}$$

In our generalization, due to the introduction of J's, which were initially accompanied with the first derivative of the Green's functions, it becomes necessary to consider all elements of (C.13). This ensures the application of translation invariance and allows fixing one $u_i = 0$. In this work, we set $u_3 = 0$. At this stage, the two possible orders for solving the integrals are equivalent. Thus, we choose the order $1 > u_1 > u_2 > 0$ and include a factor of 2. Additionally, by relabeling the variables as $u_1 = x$ and

$u_2 = y$, we obtain the following :

$$
\begin{aligned}
I_4(i_1, j_1; \ldots; i_4, j_4) \quad &= \prod_{n=1}^{4} \frac{\partial}{\partial J_{i_n j_n}} \\
&\Bigg(\int_0^1 dx \int_0^x dy \frac{(c_1 + c_2 x + ux^2 + y(c_3 - ux))^3 \ln\left(c_1 + c_2 x + ux^2 + y(c_3 - ux)\right)}{c_4 + c_5 x + c_6 y} \\
&+ \int_0^1 dx \int_0^x dy \frac{(c_7 + c_2 x + ux^2 + y(c_3 - ux))^3 \ln\left(c_7 + c_2 x + ux^2 + y(c_3 - ux)\right)}{c_8 - c_5 x - c_6 y} \\
&+ \int_0^1 dx \int_0^x dy \frac{(c_9 + c_{10} x + y(c_{11} - tx) + ty^2)^3 \ln\left(c_9 + c_{10} x - txy + c_{11} y + ty^2\right)}{c_{12} + c_5 x + c_6 y} \\
&+ \int_0^1 dx \int_0^x dy \frac{(c_1 + c_{10} x + y(c_{11} - tx) + ty^2)^3 \ln\left(c_1 + c_{10} x + y(c_{11} - tx) + ty^2\right)}{-c_4 - c_5 x - c_6 y} \\
&+ \int_0^1 dx \int_0^x dy \frac{(c_{13} + c_{10} x + y(c_{14} + sx))^3 \ln\left((c_{13} + c_{10} x + y(c_{14} + sx)\right)}{-c_8 + c_5 x + c_6 y} \\
&+ \int_0^1 dx \int_0^x dy \frac{(c_9 + c_{10} x + y(c_{14} + sx))^3 \ln\left(c_9 + c_{10} x + y(c_{14} + sx)\right)}{-c_{12} - c_5 x - c_6 y} \Bigg) \\
&= \prod_{n=1}^{4} \frac{\partial}{\partial J_{i_n j_n}} \left(\int_0^1 dx (I_{y1} + I_{y2} + I_{y3} + I_{y4} + I_{y5} + I_{y6}) \right)
\end{aligned}
\tag{C.15}
$$

Where:

$$
\begin{aligned}
c_1 &= m^2 - J_{23} + J_3 + J_2 - J_{14} \\
c_2 &= 2(J_1 + J_4) - u \\
c_3 &= 2J_2 + u \\
c_4 &= 6u - 12J_4 \\
c_5 &= -6u \\
c_6 &= -6t \\
c_7 &= m^2 - J_{23} + J_3 + J_2 + J_{14} \\
c_8 &= 12J_4 \\
c_9 &= m^2 + J_{21} - J_1 - J_2 + J_{34} \\
c_{10} &= 2J_1 \\
c_{11} &= -2(J_1 + J_3) \\
c_{12} &= -12J_4 - 6s \\
c_{13} &= m^2 + J_{21} - J_1 - J_2 - J_{34} \\
c_{14} &= (2J_2 - s)
\end{aligned}
\tag{C.16}
$$

We obtain the following type of integral.

$$
\mathrm{b}i_1 = \int_0^1 dx \int_0^x dy \frac{(a(x)y + b(x))^3 \ln\left(a(x)y + b(x)\right)}{d(x) + cy} \tag{C.17}
$$

Our functions bi_1 still contain two integrals, but the number of terms arising from solving this type of integral is very large. However, to reduce the number of terms in the integration over the variable y, we apply a trick: introducing a derivative with respect to a new parameter Υ. After differentiating, Υ is set to zero, allowing us to express the integral over y of bi_1 as:

$$
\begin{aligned}
\int_0^x dy \frac{(a(x)y + b(x))^3 \ln\left(a(x)y + b(x)\right)}{d(x) + cy} &= \int_0^x dy \frac{\partial}{\partial \Upsilon} \frac{(a(x)y + b(x))^{3+\Upsilon}}{d(x) + cy} \Bigg|_{\Upsilon=0} \\
&= \frac{\partial}{\partial \Upsilon} \int_0^x dy \frac{(a(x)y + b(x))^{3+\Upsilon}}{d(x) + cy} \Bigg|_{\Upsilon=0}
\end{aligned}
\tag{C.18}
$$

We start by integrating:

$$\int_0^x dy \frac{(a(x)y+b(x))^{3+\Upsilon}}{d(x)+cy} = \frac{1}{bc-ad}\left(\frac{b^{4+\Upsilon}}{4+\Upsilon}\,{}_2F_1\left(1,4+\Upsilon,5+\Upsilon,\frac{bc}{bc-ad}\right)\right.$$
$$\left. -\frac{(b+ax)^{4+\Upsilon}}{4+\Upsilon}\,{}_2F_1\left(1,4+\Upsilon,5+\Upsilon,\frac{c(b+ax)}{bc-ad}\right)\right) \tag{C.19}$$

In the previous result equation, the Gaussian hypergeometric function ${}_2F_1$ is introduced. This function is defined as follows:

$${}_2F_1(\alpha,\beta,\gamma,z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k z^k}{(\gamma)_k k!} \tag{C.20}$$

Where $(\alpha)_k$ is the rising factorial function, also known as the Pochhammer symbol and it is defined as:

$$(q)_n = q(q+1)(q+2)\cdots(q+n-1), \tag{C.21}$$

where n is a non-negative integer. For $n=0$, the convention is:

$$(q)_0 = 1. \tag{C.22}$$

Alternatively, it can be expressed using the Gamma function:

$$(q)_n = \frac{\Gamma(q+n)}{\Gamma(q)}. \tag{C.23}$$

Additionally, for simplicity, we ignore the dependence on the variable x and it is also useful to denote the arguments of ${}_2F_1$as follows:

$$\begin{aligned} z_1 &= \frac{bc}{bc-ad} \\ z_2 &= \frac{c(b+ax)}{bc-ad} \end{aligned} \tag{C.24}$$

The derivative with respect to Υ is defined as $d_\Upsilon(b,z) := \frac{\partial}{\partial \Upsilon}\frac{b^{4+\Upsilon}}{4+\Upsilon}\,{}_2F_1\left(1, 4+\Upsilon, 5+\Upsilon, z\right)|_{\Upsilon=0}$ such that:

$$\begin{aligned}
d_\Upsilon(b,z) &= \frac{\partial}{\partial \Upsilon}\frac{b^{4+\Upsilon}}{4+\Upsilon}\sum_{k=0}^{\infty}\frac{(1)_k(4+\Upsilon)_k z^k}{(5+\Upsilon)_k k!}\Bigg|_{\Upsilon=0} = \frac{\partial}{\partial \Upsilon} b^{4+\Upsilon}\sum_{k=0}^{\infty}\frac{z^k}{4+\Upsilon+k}\Bigg|_{\Upsilon=0} \\
&= \frac{\partial}{\partial \Upsilon} b^{4+\Upsilon}\Phi(z,1,4+\gamma)\Bigg|_{\Upsilon=0} \\
&= b^{4+\gamma}(\ln(b)\Phi(z,1,4+\gamma) - \Phi(z,2,4+\gamma))\Bigg|_{\Upsilon=0} \\
&= \frac{b^4}{z^3}\left(\ln(b)\left(-1-\frac{z}{2}-\frac{z^2}{3}-\frac{\ln(1-z)}{z}\right)+\left(1+\frac{z}{4}+\frac{z^2}{9}-\frac{\mathrm{Li}_2(z)}{z}\right)\right)
\end{aligned} \tag{C.25}$$

So,

$$\frac{\partial}{\partial \Upsilon}\int_0^x dy \frac{(a(x)y+b(x))^{3+\Upsilon}}{d(x)+cy}\Bigg|_{\Upsilon=0} = \frac{d_\Upsilon(b,z_1)+d_\Upsilon(b+ax,z_2)}{bc-ad} \tag{C.26}$$

With the previous analysis, we found that:

$$
\begin{aligned}
I_{y1} &= \int_0^x dy \frac{(c_1 + c_2 x + u x^2 + y(c_3 - u x))^3 \ln(c_1 + c_2 x + u x^2 + y(c_3 - u x))}{c_4 + c_5 x + c_6 y} \\
&= \frac{1}{36 c_6^4} \quad c_6(-c_3 + ux)\Big[(c_3 - ux)^2(4c_6^2 x^2 - 9c_6 x(c_4 + c_5 x) + 36(c_4 + c_5 x)^2) \\
&\quad + 3c_6(c_3 - ux)(7c_6 x - 30(c_4 + c_5 x))(c_1 + x(c_2 + ux)) + 66 c_6^2 (c_1 + x(c_2 + ux))^2\Big] \\
&\quad + 6\ln[c_1 + (c_2 + c_3)x]\Big[c_6(c_1 + (c_2 + c_3)x)\Big((c_3 - ux)^2(2c_6^2 x^2 - 3c_6 x(c_4 + c_5 x) + 6(c_4 + c_5 x)^2) \\
&\quad + c_6(c_3 - ux)(7c_6 x - 15(c_4 + c_5 x))(c_1 + x(c_2 + ux)) + 11 c_6^2 (c_1 + x(c_2 + ux))^2\Big) \\
&\quad + 6\Big((c_4 + c_5 x)(-c_3 + ux) + c_6(c_1 + x(c_2 + ux))\Big)^3 \\
&\quad \times \ln\left(\frac{(c_4 + (c_5 + c_6)x)(-c_3 + ux)}{c_1 c_6 + c_2 c_6 x - c_3(c_4 + c_5 x) + ux(c_4 + (c_5 + c_6)x)}\right)\Big] \\
&\quad + 6\ln[c_1 + x(c_2 + ux)]\Big[c_6(c_1 + x(c_2 + ux))\Big(- 6(c_4 + c_5 x)^2 (c_3 - ux)^2 \\
&\quad + 15 c_6 (c_4 + c_5 x)(c_3 - ux)(c_1 + x(c_2 + ux)) - 11 c_6^2 (c_1 + x(c_2 + ux))^2\Big) \\
&\quad - 6\Big((c_4 + c_5 x)(-c_3 + ux) + c_6(c_1 + x(c_2 + ux))\Big)^3 \\
&\quad \times \ln\left(\frac{(c_4 + c_5 x)(-c_3 + ux)}{c_1 c_6 + c_2 c_6 x - c_3(c_4 + c_5 x) + ux(c_4 + (c_5 + c_6)x)}\right)\Big] \\
&\quad + 36\Big((c_4 + c_5 x)(-c_3 + ux) + c_6(c_1 + x(c_2 + ux))\Big)^3 \\
&\quad \times \Big[\mathrm{Li}_2\left(2, \frac{c_6(c_1 + (c_2 + c_3)x)}{c_1 c_6 + c_2 c_6 x - c_3(c_4 + c_5 x) + ux(c_4 + (c_5 + c_6)x)}\right) \\
&\quad - \mathrm{Li}_2\left(2, \frac{c_6(c_1 + x(c_2 + ux))}{c_1 c_6 + c_2 c_6 x - c_3(c_4 + c_5 x) + ux(c_4 + (c5 + c_6)x)}\right)\Big]\Big).
\end{aligned}
\tag{C.27}
$$

$$
\begin{aligned}
I_{y2} &= \int_0^x dy \frac{(c_7+c_2x+ux^2+y(c_3-ux))^3 \ln(c_7+c_2x+ux^2+y(c_3-ux))}{c_8-c_5x-c_6y} \\
&= \frac{1}{36c_6^4}\Bigg(-c_6(-c_3+ux)\Big[(c_3-ux)^2(4c_6^2x^2+9c_6x(c_8-c_5x)+36(c_8-c_5x)^2) \\
&\quad +3c_6(30c_8-30c_5x+7c_6x)(c_3-ux)(c_7+x(c_2+ux))+66c_6^2(c_7+x(c_2+ux))^2\Big] \\
&\quad +6\ln[c_7+x(c_2+ux)]\Big[-c_6(c_7+x(c_2+ux))\Big(-6(c_8-c_5x)^2(c_3-ux)^2 \\
&\quad +15c_6(c_8-c_5x)(-c_3+ux)(c_7+x(c_2+ux))-11c_6^2(c_7+x(c_2+ux))^2\Big) \\
&\quad -6\Big((c_8-c_5x)(-c_3+ux)-c_6(c_7+x(c_2+ux))\Big)^3 \\
&\quad \times \ln\left(\frac{(c_8-c_5x)(-c_3+ux)}{(c_8-c_5x)(-c_3+ux)-c_6(c_7+x(c_2+ux))}\right)\Big] \\
&\quad +6\ln[c_7+(c_2+c_3)x]\Big[-c_6(c_7+(c_2+c_3)x) \\
&\quad \times\Big((c_3-ux)^2(2c_6^2x^2+3c_6x(c_8-c_5x)+6(c_8-c_5x)^2) \\
&\quad +c_6(15c_8-15c_5x+7c_6x)(c_3-ux)(c_7+x(c_2+ux))+11c_6^2(c_7+x(c_2+ux))^2\Big) \\
&\quad +6\Big((c_8-c_5x)(-c_3+ux)-c_6(c_7+x(c_2+ux))\Big)^3 \\
&\quad \times \ln\left(\frac{(c_8-(c_5+c_6)x)(-c_3+ux)}{(c_8-c_5x)(-c_3+ux)-c_6(c_7+x(c_2+ux))}\right)\Big] \\
&\quad +36\Big((c_8-c_5x)(-c_3+ux)-c_6(c_7+x(c_2+ux))\Big)^3 \\
&\quad \times\Big[\mathrm{Li}_2\left(\frac{c_6(c_7+(c_2+c_3)x)}{(c_8-c_5x)(c_3-ux)+c_6(c_7+x(c_2+ux))}\right) \\
&\quad -\mathrm{Li}_2\left(\frac{c_6(c_7+x(c_2+ux))}{(c_8-c_5x)(c_3-ux)+c_6(c_7+x(c_2+ux))}\right)\Big]\Bigg). \qquad (C.28)
\end{aligned}
$$

$$
\begin{aligned}
I_{y5} &= \int_0^x dy \frac{(c_{13}+c_{10}x+y(c_{14}+sx))^3 \ln((c_{13}+c_{10}x+y(c_{14}+sx))}{-c_8+c_5x+c_6y} \\
&= \frac{1}{36c_6^4}\Bigg(c_6(-c_{14}-sx)\Big[66c_6^2(c_{13}+c_{10}x)^2+3c_6(c_{13}+c_{10}x)(30c_8-30c_5x+7c_6x)(c_{14}+sx) \\
&\quad +(c_{14}+sx)^2(4c_6^2x^2+36(c_8-c_5x)^2-9c_6x(-c_8+c_5x))\Big] \\
&\quad +6\ln[c_{13}+c_{10}x]\Big[c_6(c_{13}+c_{10}x)\Big(-11c_6^2(c_{13}+c_{10}x)^2+15c_6(c_{13}+c_{10}x)(-c_8+c_5x)(c_{14}+sx) \\
&\quad -6(c_8-c_5x)^2(c_{14}+sx)^2\Big)-6\Big(c_6(c_{13}+c_{10}x)+(c_8-c_5x)(c_{14}+sx)\Big)^3 \\
&\quad \times\ln\left(\frac{(-c_8+c_5x)(c_{14}+sx)}{-c_6(c_{13}+c_{10}x)+(-c_8+c_5x)(c_{14}+sx)}\right)\Big] \\
&\quad +6\ln[c_{13}+x(c_{10}+c_{14}+sx)]\Big[c_6(c_{13}+x(c_{10}+c_{14}+sx))\Big(11c_6^2(c_{13}+c_{10}x)^2 \\
&\quad +c_6(c_{13}+c_{10}x)(15c_8-15c_5x+7c_6x)(c_{14}+sx) \\
&\quad +(c_{14}+sx)^2(2c_6^2x^2+6(c_8-c_5x)^2-3c_6x(-c_8+c_5x))\Big) \\
&\quad +6\Big(c_6(c_{13}+c_{10}x)+(c_8-c_5x)(c_{14}+sx)\Big)^3 \\
&\quad \times\ln\left(\frac{(c_8-(c_5+c_6)x)(c_{14}+sx)}{c_{13}c_6+c_{14}(c_8-c_5x)+x(c_{10}c_6+c_8s-c_5sx)}\right)\Big] \\
&\quad +36\Big(c_6(c_{13}+c_{10}x)+(c_8-c_5x)(c_{14}+sx)\Big)^3 \\
&\quad \times\Big[-\mathrm{Li}_2\left(\frac{c_6(c_{13}+c_{10}x)}{c_6(c_{13}+c_{10}x)+(c_8-c_5x)(c_{14}+sx)}\right) \\
&\quad +\mathrm{Li}_2\left(\frac{c_6(c_{13}+x(c_{10}+c_{14}+sx))}{c_6(c_{13}+c_{10}x)+(c_8-c_5x)(c_{14}+sx)}\right)\Big]\Bigg).
\end{aligned}
\tag{C.29}
$$

$$\begin{aligned}
I_{6y} &= \int_0^x dy \frac{(c_9 + c_{10}x + y(c_{14} + sx))^3 \ln\left(c_9 + c_{10}x + y(c_{14} + sx)\right)}{-c_{12} - c_5 x - c_6 y}\Bigg) \\
&= \frac{1}{36c_6^4}\Bigg(- c_6 x(-c_{14} - sx)\Big[66c_6^2(c_9 + c_{10}x)^2 + 3c_6(c_9 + c_{10}x)(-30c_{12} - 30c_5 x + 7c_6 x)(c_{14} + sx) \\
&\quad +(c_{14} + sx)^2(4c_6^2x^2 - 9c_6 x(c_{12} + c_5 x) + 36(c_{12} + c_5 x)^2)\Big] \\
&\quad +6\ln[c_9 + c_{10}x]\Big[- c_6(c_9 + c_{10}x)\Big(- 11c_6^2(c_9 + c_{10}x)^2 + 15c_6(c_9 + c_{10}x)(c_{12} + c_5 x)(c_{14} + sx) \\
&\quad -6(c_{12} + c_5 x)^2(c_{14} + sx)^2\Big) - 6(-c_6(c_9 + c_{10}x) + (c_{12} + c_5 x)(c_{14} + sx))^3 \\
&\quad \times \ln\left(\frac{(c_{12} + c_5 x)(c_{14} + sx)}{-c_6(c_9 + c_{10}x) + c_{12}(c_{14} + sx) + c_5 x(c_{14} + sx)}\right)\Big] \\
&\quad +6\ln[c_9 + x(c_{10} + c_{14} + sx)]\Big[- c_6(c_9 + x(c_{10} + c_{14} + sx))\Big(11c_6^2(c_9 + c_{10}x)^2 \\
&\quad +c_6(c_9 + c_{10}x)(-15c_{12} - 15c_5 x + 7c_6 x)(c_{14} + sx) \\
&\quad +(c_{14} + sx)^2(2c_6^2x^2 - 3c_6 x(c_{12} + c_5 x) + 6(c_{12} + c_5 x)^2)\Big) \\
&\quad +6(-c_6(c_9 + c_{10}x) + (c_{12} + c_5 x)(c_{14} + sx))^3 \\
&\quad \times \ln\left(\frac{(c_{12} + (c_5 + c_6)x)(c_{14} + sx)}{-c_6(c_9 + c_{10}x) + c_{12}(c_{14} + sx) + c_5 x(c_{14} + sx)}\right)\Big] \\
&\quad +36(-c_6(c_9 + c_{10}x) + (c_{12} + c_5 x)(c_{14} + sx))^3 \\
&\quad \times\Big[- \mathrm{Li}_2\left(\frac{c_6(c_9 + c_{10}x)}{c_6(c_9 + c_{10}x) - c_{12}(c_{14} + sx) - c_5 x(c_{14} + sx)}\right) \\
&\quad +\mathrm{Li}_2\left(\frac{c_6(c_9 + x(c_{10} + c_{14} + sx))}{c_6(c_9 + c_{10}x) - c_{12}(c_{14} + sx) - c_5 x(c_{14} + sx)}\right)\Big]\Bigg).
\end{aligned} \tag{C.30}$$

The other type of integral that we have to solve is:

$$\mathrm{b}i_2 = \int_0^1 dx \int_0^x dy \frac{(\alpha y^2 + a(x)y + b(x))^3 \ln\left(\alpha y^2 + a(x)y + b(x)\right)}{d(x) + cy} \tag{C.31}$$

In contrast to the preceding calculation, the quadratic term in y makes things more complicated. However, using brute force, we find that:

$$
\begin{aligned}
I_{y3} &= \int_0^x dy \frac{(c_9 + c_{10}x + y(c_{11} - tx) + ty^2)^3 \ln(c_9 + c_{10}x - txy + c_{11}y + ty^2)}{c_{12} + c_5 x + c_6 y} \\
&= \frac{1}{60c_6^7}\Bigg[\ln(t)\Bigg(c_6 x\Big(t^3(10c_6^5x^5 - 12c_6^4x^4(c_{12}+c_5x) + 15c_6^3x^3(c_{12}+c_5x)^2 \\
&\quad - 20c_6^2x^2(c_{12}+c_5x)^3 + 30c_6x(c_{12}+c_5x)^4 - 60(c_{12}+c_5x)^5) \\
&\quad + 10c_6^3(c_{11}-tx)\big(18c_6^2(c_9+c_{10}x)^2 \\
&\qquad + 9c_6(c_9+c_{10}x)(c_{11}-tx)(c_6x - 2(c_{12}+c_5x)) \\
&\qquad + (c_{11}-tx)^2(2c_6^2x^2 - 3c_6x(c_{12}+c_5x) + 6(c_{12}+c_5x)^2)\big) \\
&\quad + 3c_6t^2\big(12c_6^4x^4(c_{11}-tx) \\
&\qquad + 20c_6^2x^2(c_{12}+c_5x)(-c_6(c_9+c_{10}x) + (c_{12}+c_5x)(c_{11}-tx)) \\
&\qquad + 60(c_{12}+c_5x)^3(-c_6(c_9+c_{10}x) + (c_{12}+c_5x)(c_{11}-tx)) \\
&\qquad + 15c_6^3x^3(c_6(c_9+c_{10}x) + (c_{12}+c_5x)(-c_{11}+tx)) \\
&\qquad + 30c_6x(c_{12}+c_5x)^2(c_6(c_9+c_{10}x) + (c_{12}+c_5x)(-c_{11}+tx))\big) \\
&\quad + 15c_6^2t\big(3c_6^3x^3(c_{11}-tx)^2 \\
&\qquad - 4c_6^2x^2(c_{11}-tx)(-2c_6(c_9+c_{10}x) + (c_{12}+c_5x)(c_{11}-tx)) \\
&\qquad + 6c_6x(c_6(c_9+c_{10}x) + (c_{12}+c_5x)(-c_{11}+tx))^2 \\
&\qquad - 12(c_{12}+c_5x)(c_6(c_9+c_{10}x) + (c_{12}+c_5x)(-c_{11}+tx))^2\big)\Bigg) \\
&\quad + 60\big(c_6^2(c_9+c_{10}x) + (c_{12}+c_5x)(-c_{11}c_6 + t(c_{12}+(c_5+c_6)x))\big)^3 \\
&\quad \times \big(-\ln(c_{12}+c_5x) + \ln(c_{12}+(c_5+c_6)x)\big)\Bigg] \\
&\quad + \frac{1}{3600c_6^7t^3}\Bigg[\\
&\quad 30c_6^2\Bigg(-c_6^4(c_{11}-tx)^6 \\
&\qquad + c_6^3(c_{11}-tx)^5\left(-3t(c_{12}+c_5x) + c_6\sqrt{-4t(c_9+c_{10}x) + (c_{11}-tx)^2}\right) \\
&\qquad - 2t^3(c_9+c_{10}x)\Big(55c_6^4(c_9+c_{10}x)^2 \\
&\qquad\quad + 3c_6^2(c_9+c_{10}x)(c_{12}+c_5x)\big(25t(c_{12}+c_5x) - 8c_6\sqrt{-4t(c_9+c_{10}x) + (c_{11}-tx)^2}\big) \\
&\qquad\quad + 10t(c_{12}+c_5x)^3\big(3t(c_{12}+c_5x) - c_6\sqrt{-4t(c_9+c_{10}x) + (c_{11}-tx)^2}\big)\Big) \\
&\qquad + 3c_6^2t(c_{11}-tx)^4\Big(5c_6^2(c_9+c_{10}x) \\
&\qquad\quad + (c_{12}+c_5x)\big(15t(c_{12}+c_5x) + c_6\sqrt{-4t(c_9+c_{10}x) + (c_{11}-tx)^2}\big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&- c_6 t(c_{11}-tx)^3\Big(5t(c_{12}+c_5x)^2\big(14t(c_{12}+c_5x)+9c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad + c_6^2(c_9+c_{10}x)\big(120t(c_{12}+c_5x)+13c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\\
&- 6t^2(c_{11}-tx)\Big(5t^2(c_{12}+c_5x)^4\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\\
&\qquad + 20c_6t(c_9+c_{10}x)(c_{12}+c_5x)^2\big(-c_{12}t-c_5tx+c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad + c_6^3(c_9+c_{10}x)^2\big(-25t(c_{12}+c_5x)+19c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\\
&+ 2t^2(c_{11}-tx)^2\Big(45c_6^4(c_9+c_{10}x)^2\\
&\qquad + 5t(c_{12}+c_5x)^3\big(3t(c_{12}+c_5x)+7c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad + 3c_6^2(c_9+c_{10}x)(c_{12}+c_5x)\big(5t(c_{12}+c_5x)+21c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\Bigg)\\
&\times \ln\Big(c_{11}-tx-\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\Big)\\
&- 30c_6^2\Bigg(-c_6^4(c_{11}-tx)^6\\
&+ c_6^3(c_{11}-tx)^5\Big(-3t(c_{12}+c_5x)+c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\Big)\\
&- 2t^3(c_9+c_{10}x)\Big(55c_6^4(c_9+c_{10}x)^2\\
&\qquad + 3c_6^2(c_9+c_{10}x)(c_{12}+c_5x)\big(25t(c_{12}+c_5x)-8c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad + 10t(c_{12}+c_5x)^3\big(3t(c_{12}+c_5x)-c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\\
&+ 3c_6^2t(c_{11}-tx)^4\Big(5c_6^2(c_9+c_{10}x)\\
&\qquad + (c_{12}+c_5x)\big(15t(c_{12}+c_5x)+c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\\
&- c_6 t(c_{11}-tx)^3\Big(5t(c_{12}+c_5x)^2\big(14t(c_{12}+c_5x)+9c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad + c_6^2(c_9+c_{10}x)\big(120t(c_{12}+c_5x)+13c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\\
&- 6t^2(c_{11}-tx)\Big(5t^2(c_{12}+c_5x)^4\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\\
&\qquad + 20c_6t(c_9+c_{10}x)(c_{12}+c_5x)^2\big(-c_{12}t-c_5tx+c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad + c_6^3(c_9+c_{10}x)^2\big(-25t(c_{12}+c_5x)+19c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\\
&+ 2t^2(c_{11}-tx)^2\Big(45c_6^4(c_9+c_{10}x)^2\\
&\qquad + 5t(c_{12}+c_5x)^3\big(3t(c_{12}+c_5x)+7c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad + 3c_6^2(c_9+c_{10}x)(c_{12}+c_5x)\big(5t(c_{12}+c_5x)+21c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\Bigg)\\
&\times \ln\Big(c_{11}+tx-\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\Big)
\end{aligned}
$$

$$
\begin{aligned}
& - 60t^2 \ln \left(-\frac{-c_{11} + tx + \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}}{2t} \right) \\
& \quad \times \quad 30c_6(c_{11}c_6 - t(c_{12} + (c_5 + c_6)x))\Big(3c_6^4(c_9 + c_{10}x)^2 \\
& \qquad + (c_{12} + c_5x)^2(c_{11}c_6 - t(c_{12} + (c_5 + c_6)x))^2 \\
& \qquad + 3c_6^2(c_9 + c_{10}x)(c_{12} + c_5x)(-c_{11}c_6 + t(c_{12} + (c_5 + c_6)x))\Big) \\
& \qquad \times (c_{11} - tx - \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2})\Bigg) \\
& + 60t\left(c_6^2(c_9 + c_{10}x) + (c_{12} + c_5x)(-c_{11}c_6 + t(c_{12} + (c_5 + c_6)x))\right)^3 \\
& \quad \times \ln\left(\frac{2t(c_{12} + c_5x)}{-c_{11}c_6 + 2c_{12}t + 2c_5tx + c_6tx + c_6\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}}\right) \qquad +t \\
& \quad - 15c_6^4(c_{11} - tx)^4\left(c_6tx - 6t(c_{12} + c_5x) + 2c_6\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}\right) \\
& \quad + 15c_6^3t(c_{11} - tx)^3\Big(-28c_6^2(c_9 + c_{10}x) + 150t(c_{12} + c_5x)^2 \\
& + c_6^2x\big(14tx + \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}\big) \\
& - 3c_6(c_{12} + c_5x)\big(11tx + 2\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}\big)\Big) \\
& + t^2\Big(330c_6^4(c_9 + c_{10}x)^2\big(-24t(c_{12} + c_5x) + 5c_6(tx + \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2})\big) \\
& + 5c_6^2t(c_9 + c_{10}x)\big(-1920t(c_{12} + c_5x)^3 + 450c_6(c_{12} + c_5x)^2(tx + \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}) \\
& + 35c_6^3x^2(3tx + 2\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}) \\
& - 48c_6^2x(c_{12} + c_5x)(4tx + 3\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2})\big) \\
& + t^2\big(-3600t(c_{12} + c_5x)^5 + 900c_6(c_{12} + c_5x)^4(tx + \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}) \\
& + 75c_6^3x^2(c_{12} + c_5x)^2(3tx + 2\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}) \\
& - 100c_6^2x(c_{12} + c_5x)^3(4tx + 3\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}) \\
& + 20c_6^5x^4(5tx + 3\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}) \\
& - 18c_6^4x^3(c_{12} + c_5x)(8tx + 5\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2})\big)\Big) \\
& + 5c_6^2t(c_{11} - tx)^2\Big(3c_6^2(c_9 + c_{10}x)\big(83c_6tx - 414t(c_{12} + c_5x) + 24c_6\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}\big) \\
& + t\big(-1740t(c_{12} + c_5x)^3 + 135c_6(c_{12} + c_5x)^2(3tx + 2\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}) \\
& + 2c_6^3x^2(48tx + 19\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}) \\
& - 3c_6^2x(c_{12} + c_5x)(58tx + 27\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2})\big)\Big) \\
& + 3c_6t^2(c_{11} - tx)\Big(1870c_6^4(c_9 + c_{10}x)^2
\end{aligned}
$$

$$+ 5c_6^2(c_9 + c_{10}x)\big(1110t(c_{12} + c_5x)^2 - 6c_6(c_{12} + c_5x)(41tx + 33\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2})$$
$$+ c_6^2x(102tx + 59\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2})\big)$$
$$+ t\big(3300t(c_{12} + c_5x)^4 - 100c_6(c_{12} + c_5x)^3(8tx + 7\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2})$$
$$+ 25c_6^2x(c_{12} + c_5x)^2(14tx + 9\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2})$$
$$- 5c_6^3x^2(c_{12} + c_5x)(39tx + 22\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2})$$
$$+ c_6^4x^3(124tx + 65\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2})\big)\Big)\Bigg)$$
$$+ 60t\ln\left(\frac{c_{11} + tx - \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}}{2t}\right)$$
$$\times \Bigg(c_6\Big(30c_6^3(c_{12} + c_5x)^2(c_{11} - tx)^4$$
$$- 10c_6^2(c_{11} - tx)^3\big(-2c_6^3tx^3 + 3c_6^2tx^2(c_{12} + c_5x) + 9c_6^2(c_9 + c_{10}x)(c_{12} + c_5x)$$
$$+ 9t(c_{12} + c_5x)^3 + 3c_6(c_{12} + c_5x)^2(-2tx + \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2})\big)$$
$$+ 15c_6(c_{11} - tx)^2\big(6c_6^4(c_9 + c_{10}x)^2 + 6c_6^2(c_9 + c_{10}x)(c_6^2tx^2 + 2t(c_{12} + c_5x)^2$$
$$+ c_6(c_{12} + c_5x)(-2tx + \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2})) + t(3c_6^4tx^4 - 4c_6^3tx^3(c_{12} + c_5x)$$
$$+ 6c_6^2tx^2(c_{12} + c_5x)^2 + 6t(c_{12} + c_5x)^4 + 6c_6(c_{12} + c_5x)^3(-2tx + \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}))\big)$$
$$- 3(c_{11} - tx)\big(30c_6^4(c_9 + c_{10}x)^2(c_{12}t + c_5tx - 2c_6tx + c_6\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2})$$
$$+ 10c_6^2t(c_9 + c_{10}x)(-4c_6^3tx^3 + 6c_6^2tx^2(c_{12} + c_5x) + 3t(c_{12} + c_5x)^3$$
$$+ 6c_6(c_{12} + c_5x)^2(-2tx + \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}))$$
$$+ t^2(-12c_6^5tx^5 + 15c_6^4tx^4(c_{12} + c_5x) - 20c_6^3tx^3(c_{12} + c_5x)^2$$
$$+ 30c_6^2tx^2(c_{12} + c_5x)^3 + 10t(c_{12} + c_5x)^5$$
$$+ 30c_6(c_{12} + c_5x)^4(-2tx + \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}))\big)$$
$$+ t\big(90c_6^4(c_9 + c_{10}x)^2(c_6tx^2 + (c_{12} + c_5x)(-2tx + \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}))$$
$$+ 15c_6^2t(c_9 + c_{10}x)(3c_6^3tx^4 - 4c_6^2tx^3(c_{12} + c_5x) + 6c_6tx^2(c_{12} + c_5x)^2$$
$$+ 6(c_{12} + c_5x)^3(-2tx + \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}))$$
$$+ t^2(10c_6^5tx^6 - 12c_6^4tx^5(c_{12} + c_5x) + 15c_6^3tx^4(c_{12} + c_5x)^2$$
$$- 20c_6^2tx^3(c_{12} + c_5x)^3 + 30c_6tx^2(c_{12} + c_5x)^4$$
$$+ 30(c_{12} + c_5x)^5(-2tx + \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}))\big)\Big)\Bigg)$$

$$+ 60t\left(c_6^2(c_9 + c_{10}x) + (c_{12} + c_5x)(-c_{11}c_6 + t(c_{12} + (c_5 + c_6)x))\right)^3$$
$$\times \ln\left(\frac{2t(c_{12} + (c_5 + c_6)x)}{-c_{11}c_6 + 2c_{12}t + 2c_5tx + c_6tx + c_6\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}}\right)$$
$$+ 3600t^3\left(c_6^2(c_9 + c_{10}x) + (c_{12} + c_5x)(-c_{11}c_6 + t(c_{12} + (c_5 + c_6)x))\right)^3$$
$$\times \mathrm{Li}_2 \quad \frac{c_6(-c_{11} - tx + \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2})}{-c_{11}c_6 + 2c_{12}t + 2c_5tx + c_6tx + c_6\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}}\Bigg)$$
$$- 3600t^3\left(c_6^2(c_9 + c_{10}x) + (c_{12} + c_5x)(-c_{11}c_6 + t(c_{12} + (c_5 + c_6)x))\right)^3$$
$$\times \mathrm{Li}_2 \quad \frac{c_6(-c_{11} + tx + \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2})}{-c_{11}c_6 + 2c_{12}t + 2c_5tx + c_6tx + c_6\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}}\Bigg)\Bigg]$$
$$+ \frac{1}{3600c_6^7t^3}\Bigg[$$
$$- 30c_6^2 \quad c_6^4(c_{11} - tx)^6 + c_6^3(c_{11} - tx)^5\left(3t(c_{12} + c_5x) + c_6\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}\right)$$
$$+ c_6t(c_{11} - tx)^3\Big(c_6^2(c_9 + c_{10}x)\big(120t(c_{12} + c_5x) - 13c_6\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}\big)$$
$$+ 5t(c_{12} + c_5x)^2\big(14t(c_{12} + c_5x) - 9c_6\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}\big)\Big)$$
$$- 2t^2(c_{11} - tx)^2\Big(45c_6^4(c_9 + c_{10}x)^2$$
$$+ 3c_6^2(c_9 + c_{10}x)(c_{12} + c_5x)\big(5t(c_{12} + c_5x) - 21c_6\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}\big)$$
$$+ 5t(c_{12} + c_5x)^3\big(3t(c_{12} + c_5x) - 7c_6\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}\big)\Big)$$
$$- 3c_6^2t(c_{11} - tx)^4\Big(5c_6^2(c_9 + c_{10}x)$$
$$+ (c_{12} + c_5x)\big(15t(c_{12} + c_5x) - c_6\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}\big)\Big)$$
$$+ 2t^3(c_9 + c_{10}x)\Big(55c_6^4(c_9 + c_{10}x)^2$$
$$+ 10t(c_{12} + c_5x)^3\big(3t(c_{12} + c_5x) + c_6\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}\big)$$
$$+ 3c_6^2(c_9 + c_{10}x)(c_{12} + c_5x)\big(25t(c_{12} + c_5x) + 8c_6\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}\big)\Big)$$
$$- 6t^2(c_{11} - tx)\Big(5t^2(c_{12} + c_5x)^4\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}$$
$$+ 20c_6t(c_9 + c_{10}x)(c_{12} + c_5x)^2\big(c_{12}t + c_5tx + c_6\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}\big)$$
$$+ c_6^3(c_9 + c_{10}x)^2\big(25t(c_{12} + c_5x) + 19c_6\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}\big)\Big)$$
$$\times \ln\left(c_{11} - tx + \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}\right)$$

$$
\begin{aligned}
&+30c_6^2\Bigg(c_6^4(c_{11}-tx)^6\\
&\quad+c_6^3(c_{11}-tx)^5\left(3t(c_{12}+c_5x)+c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\right)\\
&\quad+c_6t(c_{11}-tx)^3\Big(c_6^2(c_9+c_{10}x)\big(120t(c_{12}+c_5x)-13c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\quad+5t(c_{12}+c_5x)^2\big(14t(c_{12}+c_5x)-9c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\\
&\quad-2t^2(c_{11}-tx)^2\Big(45c_6^4(c_9+c_{10}x)^2\\
&\qquad+3c_6^2(c_9+c_{10}x)(c_{12}+c_5x)\big(5t(c_{12}+c_5x)-21c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad+5t(c_{12}+c_5x)^3\big(3t(c_{12}+c_5x)-7c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\\
&\quad-3c_6^2t(c_{11}-tx)^4\Big(5c_6^2(c_9+c_{10}x)\\
&\qquad+(c_{12}+c_5x)\big(15t(c_{12}+c_5x)-c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\\
&\quad+2t^3(c_9+c_{10}x)\Big(55c_6^4(c_9+c_{10}x)^2\\
&\qquad+10t(c_{12}+c_5x)^3\big(3t(c_{12}+c_5x)+c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad+3c_6^2(c_9+c_{10}x)(c_{12}+c_5x)\big(25t(c_{12}+c_5x)+8c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\\
&\quad-6t^2(c_{11}-tx)\Big(5t^2(c_{12}+c_5x)^4\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\\
&\qquad+20c_6t(c_9+c_{10}x)(c_{12}+c_5x)^2\big(c_{12}t+c_5tx+c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad+c_6^3(c_9+c_{10}x)^2\big(25t(c_{12}+c_5x)+19c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\Bigg)\\
&\times\ln\left(c_{11}+tx+\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\right)\\
&-60t^2\ln\left(\frac{c_{11}-tx+\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}}{2t}\right)\\
&\quad\times\Bigg(30c_6(c_{11}c_6-t(c_{12}+(c_5+c_6)x))\Big(3c_6^4(c_9+c_{10}x)^2\\
&\qquad+(c_{12}+c_5x)^2(c_{11}c_6-t(c_{12}+(c_5+c_6)x))^2\\
&\qquad+3c_6^2(c_9+c_{10}x)(c_{12}+c_5x)(-c_{11}c_6+t(c_{12}+(c_5+c_6)x))\Big)\\
&\qquad\times(c_{11}-tx+\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2})\Bigg)\\
&+60t\left(c_6^2(c_9+c_{10}x)+(c_{12}+c_5x)(-c_{11}c_6+t(c_{12}+(c_5+c_6)x))\right)^3\\
&\quad\times\ln\left(-\frac{2t(c_{12}+c_5x)}{-2t(c_{12}+c_5x)+c_6(c_{11}-tx)+c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+t\Bigg(c_6x\Bigg(-30c_6^5(c_{11}-tx)^5\\
&\quad-15c_6^4(c_{11}-tx)^4\left(-c_6tx+6t(c_{12}+c_5x)+2c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\right)\\
&\quad+15c_6^3t(c_{11}-tx)^3\Big(28c_6^2(c_9+c_{10}x)-150t(c_{12}+c_5x)^2\\
&\qquad+c_6(c_{12}+c_5x)\big(33tx-6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad+c_6^2x\big(-14tx+\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\\
&\quad+t^2\Big(330c_6^4(c_9+c_{10}x)^2\big(24t(c_{12}+c_5x)+5c_6(-tx+\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2})\big)\\
&\qquad+5c_6^2t(c_9+c_{10}x)\big(1920t(c_{12}+c_5x)^3+48c_6^2x(c_{12}+c_5x)(4tx-3\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2})\\
&\qquad\quad+450c_6(c_{12}+c_5x)^2(-tx+\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2})\\
&\qquad\quad+35c_6^3x^2(-3tx+2\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2})\big)\\
&\qquad+t^2\big(3600t(c_{12}+c_5x)^5+18c_6^4x^3(c_{12}+c_5x)(8tx-5\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2})\\
&\qquad\quad+100c_6^2x(c_{12}+c_5x)^3(4tx-3\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2})\\
&\qquad\quad+900c_6(c_{12}+c_5x)^4(-tx+\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2})\\
&\qquad\quad+75c_6^3x^2(c_{12}+c_5x)^2(-3tx+2\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2})\\
&\qquad\quad+20c_6^5x^4(-5tx+3\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2})\big)\Big)\\
&\quad+5c_6^2t(c_{11}-tx)^2\Big(3c_6^2(c_9+c_{10}x)\big(-83c_6tx+414t(c_{12}+c_5x)+24c_6\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad+t\big(1740t(c_{12}+c_5x)^3+3c_6^2x(c_{12}+c_5x)(58tx-27\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2})\\
&\qquad\quad+135c_6(c_{12}+c_5x)^2(-3tx+2\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2})\\
&\qquad\quad+2c_6^3x^2(-48tx+19\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2})\big)\Big)\\
&\quad-3c_6t^2(c_{11}-tx)\Big(1870c_6^4(c_9+c_{10}x)^2\\
&\qquad+t\big(3300t(c_{12}+c_5x)^4+c_6^4x^3(124tx-65\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2})\\
&\qquad\quad+25c_6^2x(c_{12}+c_5x)^2(14tx-9\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2})\\
&\qquad\quad+100c_6(c_{12}+c_5x)^3(-8tx+7\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2})\\
&\qquad\quad+5c_6^3x^2(c_{12}+c_5x)(-39tx+22\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2})\big)\\
&\qquad+5c_6^2(c_9+c_{10}x)\big(1110t(c_{12}+c_5x)^2+c_6^2x(102tx-59\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2})\\
&\qquad\quad+6c_6(c_{12}+c_5x)(-41tx+33\sqrt{-4t(c_9+c_{10}x)+(c_{11}-tx)^2})\big)\Big)\Bigg)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+ 60t \ln\left(\frac{c_{11} + tx + \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}}{2t}\right) \\
&\quad \times \Bigg(c_6 \Big(30c_6^3(c_{12} + c_5x)^2(c_{11} - tx)^4 \\
&\qquad + 10c_6^2(c_{11} - tx)^3\big(2c_6^3tx^3 - 3c_6^2tx^2(c_{12} + c_5x) - 9c_6^2(c_9 + c_{10}x)(c_{12} + c_5x) \\
&\qquad\quad - 9t(c_{12} + c_5x)^3 + 3c_6(c_{12} + c_5x)^2(2tx + \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2})\big) \\
&\qquad + 15c_6(c_{11} - tx)^2\big(6c_6^4(c_9 + c_{10}x)^2 + 6c_6^2(c_9 + c_{10}x)(c_6^2tx^2 + 2t(c_{12} + c_5x)^2 \\
&\qquad\quad - c_6(c_{12} + c_5x)(2tx + \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2})) + t(3c_6^4tx^4 - 4c_6^3tx^3(c_{12} + c_5x) \\
&\qquad\quad + 6c_6^2tx^2(c_{12} + c_5x)^2 + 6t(c_{12} + c_5x)^4 \\
&- 6c_6(c_{12} + c_5x)^3(2tx + \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}))\big) \\
&\qquad + 3(c_{11} - tx)\big(30c_6^4(c_9 + c_{10}x)^2(-c_{12}t - c_5tx + 2c_6tx + c_6\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}) \\
&\qquad\quad + 10c_6^2t(c_9 + c_{10}x)(4c_6^3tx^3 - 6c_6^2tx^2(c_{12} + c_5x) - 3t(c_{12} + c_5x)^3 \\
&\qquad\qquad + 6c_6(c_{12} + c_5x)^2(2tx + \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2})) \\
&\qquad\quad + t^2(12c_6^5tx^5 - 15c_6^4tx^4(c_{12} + c_5x) + 20c_6^3tx^3(c_{12} + c_5x)^2 \\
&\qquad\qquad - 30c_6^2tx^2(c_{12} + c_5x)^3 - 10t(c_{12} + c_5x)^5 + 30c_6(c_{12} + c_5x)^4 \\
&\times (2tx + \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}))\big) \\
&\qquad + t\big(90c_6^4(c_9 + c_{10}x)^2(c_6tx^2 - (c_{12} + c_5x)(2tx + \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2})) \\
&\qquad\quad + 15c_6^2t(c_9 + c_{10}x)(3c_6^3tx^4 - 4c_6^2tx^3(c_{12} + c_5x) + 6c_6tx^2(c_{12} + c_5x)^2 \\
&\qquad\qquad - 6(c_{12} + c_5x)^3(2tx + \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2})) \\
&\qquad\quad + t^2(10c_6^5tx^6 - 12c_6^4tx^5(c_{12} + c_5x) + 15c_6^3tx^4(c_{12} + c_5x)^2 \\
&\qquad\qquad - 20c_6^2tx^3(c_{12} + c_5x)^3 + 30c_6tx^2(c_{12} + c_5x)^4 \\
&\qquad\qquad - 30(c_{12} + c_5x)^5(2tx + \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2})))\big)\Big)\Bigg) \\
&- 3600t^3\left(c_6^2(c_9 + c_{10}x) + (c_{12} + c_5x)(-c_{11}c_6 + t(c_{12} + (c_5 + c_6)x))\right)^3 \\
&\quad \times \mathrm{Li}_2\left(\frac{c_6(c_{11} - tx + \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2})}{-2t(c_{12} + c_5x) + c_6(c_{11} - tx) + c_6\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}}\right) \\
&+ 3600t^3\left(c_6^2(c_9 + c_{10}x) + (c_{12} + c_5x)(-c_{11}c_6 + t(c_{12} + (c_5 + c_6)x))\right)^3 \\
&\quad \times \mathrm{Li}_2\left(\frac{c_6(c_{11} + tx + \sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2})}{-2t(c_{12} + c_5x) + c_6(c_{11} - tx) + c_6\sqrt{-4t(c_9 + c_{10}x) + (c_{11} - tx)^2}}\right)\Bigg].
\end{aligned}
$$

And:

$$I_{y4} = \int_0^x dy \frac{(c_1 + c_{10}x + y(c_{11} - tx) + ty^2)^3 \ln\left(c_1 + c_{10}x + y(c_{11} - tx) + ty^2\right)}{-c_4 - c_5 x - c_6 y}$$

$$\begin{aligned}
&= -\frac{1}{60c_6^7}\Bigg[\ln(t)\Bigg(-c_6 x\Big(t^3(-10c_6^5x^5 + 12c_6^4x^4(c_4 + c_5x) \\
&- 15c_6^3x^3(c_4 + c_5x)^2 + 20c_6^2x^2(c_4 + c_5x)^3 - 30c_6x(c_4 + c_5x)^4 + 60(c_4 + c_5x)^5)\Big) \\
&- 10c_6^3(c_{11} - tx)\Big(18c_6^2(c_1 + c_{10}x)^2 - 9c_6(c_1 + c_{10}x)(-2c_4 + (-2c_5 + c_6)x)(-c_{11} + tx) \\
&+ (c_{11} - tx)^2(2c_6^2x^2 - 3c_6x(c_4 + c_5x) + 6(c_4 + c_5x)^2)\Big) \\
&- 3c_6t^2\Big(12c_6^4x^4(c_{11} - tx) + 15c_6^3x^3(c_1c_6 - c_{11}(c_4 + c_5x) + x(c_{10}c_6 + c_4t + c_5tx)) \\
&\qquad - 20c_6^2x^2(c_4 + c_5x)(c_1c_6 - c_{11}(c_4 + c_5x) + x(c_{10}c_6 + c_4t + c_5tx)) \\
&\qquad + 30c_6x(c_4 + c_5x)^2(c_1c_6 - c_{11}(c_4 + c_5x) + x(c_{10}c_6 + c_4t + c_5tx)) \\
&\qquad - 60(c_4 + c_5x)^3(c_1c_6 - c_{11}(c_4 + c_5x) + x(c_{10}c_6 + c_4t + c_5tx))\Big) \\
&+ 15c_6^2t\Big(-3c_6^3x^3(c_{11} - tx)^2 \\
&\quad - 4c_6^2x^2(c_{11} - tx)(2c_6(c_1 + c_{10}x) + (c_4 + c_5x)(-c_{11} + tx)) \\
&\quad - 6c_6x(c_1c_6 - c_{11}(c_4 + c_5x) + x(c_{10}c_6 + c_4t + c_5tx))^2 \\
&\quad + 12(c_4 + c_5x)(c_1c_6 - c_{11}(c_4 + c_5x) + x(c_{10}c_6 + c_4t + c_5tx))^2\Big)\Bigg) \\
&+ 60\left(c_6^2(c_1 + c_{10}x) + (c_4 + c_5x)(-c_{11}c_6 + t(c_4 + (c_5 + c_6)x))\right)^3 \\
&\quad \times \left(-\ln(-c_4 - c_5x) + \ln(-c_4 - (c_5 + c_6)x)\right)\Bigg]
\end{aligned}$$

$$\begin{aligned}
&-\frac{1}{3600c_6^7t^3}\Bigg[\\
&\quad 30c_6^2\Bigg(-c_6^4(c_{11} - tx)^6 \\
&\quad + c_6^3(c_{11} - tx)^5\left(-3c_4t - 3c_5tx + c_6\sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2}\right)
\end{aligned}$$

$$
\begin{aligned}
&+3c_6^2 t(c_{11}-tx)^4\Big(5c_6^2(c_1+c_{10}x)\\
&\qquad +(c_4+c_5x)\big(15c_4t+15c_5tx+c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\\
&-2t^3(c_1+c_{10}x)\Big(55c_6^4(c_1+c_{10}x)^2\\
&\qquad +10t(c_4+c_5x)^3\big(3c_4t+3c_5tx-c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad -3c_6^2(c_1+c_{10}x)(c_4+c_5x)\big(-25c_4t-25c_5tx+8c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\\
&+c_6t(c_{11}-tx)^3\Big(-5t(c_4+c_5x)^2\big(14c_4t+14c_5tx+9c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad -c_6^2(c_1+c_{10}x)\big(120c_4t+120c_5tx+13c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\\
&-6t^2(c_{11}-tx)\Big(5t^2(c_4+c_5x)^4\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\\
&\qquad +20c_6t(c_1+c_{10}x)(c_4+c_5x)^2\big(-c_4t-c_5tx+c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad +c_6^3(c_1+c_{10}x)^2\big(-25c_4t-25c_5tx+19c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\\
&+2t^2(c_{11}-tx)^2\Big(45c_6^4(c_1+c_{10}x)^2\\
&\qquad +5t(c_4+c_5x)^3\big(3c_4t+3c_5tx+7c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad +3c_6^2(c_1+c_{10}x)(c_4+c_5x)\big(5c_4t+5c_5tx+21c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\Bigg)\\
&\times\ln\left(c_{11}-tx-\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\right)\\
&-30c_6^2\Bigg(-c_6^4(c_{11}-tx)^6\\
&+c_6^3(c_{11}-tx)^5\left(-3c_4t-3c_5tx+c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\right)\\
&+3c_6^2 t(c_{11}-tx)^4\Big(5c_6^2(c_1+c_{10}x)\\
&\qquad +(c_4+c_5x)\big(15c_4t+15c_5tx+c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\\
&-2t^3(c_1+c_{10}x)\Big(55c_6^4(c_1+c_{10}x)^2\\
&\qquad +10t(c_4+c_5x)^3\big(3c_4t+3c_5tx-c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad -3c_6^2(c_1+c_{10}x)(c_4+c_5x)\big(-25c_4t-25c_5tx+8c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ c_6 t(c_{11}-tx)^3\Big(-5t(c_4+c_5x)^2\big(14c_4t+14c_5tx+9c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad - c_6^2(c_1+c_{10}x)\big(120c_4t+120c_5tx+13c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\\
&- 6t^2(c_{11}-tx)\Big(5t^2(c_4+c_5x)^4\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\\
&\qquad + 20c_6t(c_1+c_{10}x)(c_4+c_5x)^2\big(-c_4t-c_5tx+c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad + c_6^3(c_1+c_{10}x)^2\big(-25c_4t-25c_5tx+19c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\\
&+ 2t^2(c_{11}-tx)^2\Big(45c_6^4(c_1+c_{10}x)^2\\
&\qquad + 5t(c_4+c_5x)^3\big(3c_4t+3c_5tx+7c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad + 3c_6^2(c_1+c_{10}x)(c_4+c_5x)\big(5c_4t+5c_5tx+21c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\Bigg)\\
&\times \ln\left(c_{11}+tx-\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\right)\\
&- 60t^2\ln\left(-\frac{-c_{11}+tx+\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}}{2t}\right)\\
&\times\Bigg(-30c_6(-c_{11}c_6+t(c_4+(c_5+c_6)x))\Big(3c_6^4(c_1+c_{10}x)^2\\
&\qquad + (c_4+c_5x)^2(c_{11}c_6-t(c_4+(c_5+c_6)x))^2\\
&\qquad + 3c_6^2(c_1+c_{10}x)(c_4+c_5x)(-c_{11}c_6+t(c_4+(c_5+c_6)x))\Big)\\
&\qquad \times\left(c_{11}-tx-\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\right)\Bigg)\\
&+ 60t\left(c_6^2(c_1+c_{10}x)+(c_4+c_5x)(-c_{11}c_6+t(c_4+(c_5+c_6)x))\right)^3\\
&\quad\times\ln\left(\frac{2t(c_4+c_5x)}{-c_{11}c_6+2c_4t+2c_5tx+c_6tx+c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}}\right)\\
&+ t\Bigg(c_6x\Bigg(-30c_6^5(c_{11}-tx)^5\\
&\quad + 15c_6^4(c_{11}-tx)^4\left(-6c_4t-6c_5tx+c_6tx+2c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\right)\\
&\quad - 15c_6^3t(c_{11}-tx)^3\Big(-28c_6^2(c_1+c_{10}x)+150t(c_4+c_5x)^2\\
&\qquad + c_6^2x\big(14tx+\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad - 3c_6(c_4+c_5x)\big(11tx+2\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\\
&\quad + t^2\Big(-330c_6^4(c_1+c_{10}x)^2\big(-24c_4t-24c_5tx+5c_6tx+5c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad + 5c_6^2t(c_1+c_{10}x)\big(1920t(c_4+c_5x)^3-450c_6(c_4+c_5x)^2(tx+\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2})
\end{aligned}
$$

$$
\begin{aligned}
&- 35c_6^3x^2(3tx + 2\sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2})\\
&+ 48c_6^2x(c_4 + c_5x)(4tx + 3\sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2})\big)\\
&+ t^2\big(3600t(c_4 + c_5x)^5 - 900c_6(c_4 + c_5x)^4(tx + \sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2})\\
&- 75c_6^3x^2(c_4 + c_5x)^2(3tx + 2\sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2})\\
&+ 100c_6^2x(c_4 + c_5x)^3(4tx + 3\sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2})\\
&- 20c_6^5x^4(5tx + 3\sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2})\\
&+ 18c_6^4x^3(c_4 + c_5x)(8tx + 5\sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2})\big)\Big)\\
&+ 5c_6^2t(c_{11} - tx)^2\Big(-3c_6^2(c_1 + c_{10}x)\big(-414c_4t - 414c_5tx + 83c_6tx\\
&+ 24c_6\sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2}\big)\\
&+ t\big(1740t(c_4 + c_5x)^3 - 135c_6(c_4 + c_5x)^2(3tx + 2\sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2})\\
&- 2c_6^3x^2(48tx + 19\sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2})\\
&+ 3c_6^2x(c_4 + c_5x)(58tx + 27\sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2})\big)\Big)\\
&- 3c_6t^2(c_{11} - tx)\Big(1870c_6^4(c_1 + c_{10}x)^2\\
&+ 5c_6^2(c_1 + c_{10}x)\big(1110t(c_4 + c_5x)^2 - 6c_6(c_4 + c_5x)(41tx + 33\sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2})\\
&+ c_6^2x(102tx + 59\sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2})\big)\\
&+ t\big(3300t(c_4 + c_5x)^4 - 100c_6(c_4 + c_5x)^3(8tx + 7\sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2})\\
&+ 25c_6^2x(c_4 + c_5x)^2(14tx + 9\sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2})\\
&- 5c_6^3x^2(c_4 + c_5x)(39tx + 22\sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2})\\
&+ c_6^4x^3(124tx + 65\sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2})\big)\Big)\Bigg)
\end{aligned}
$$

$$+ 60t\ln\left(\frac{c_{11}+tx-\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}}{2t}\right)\bigg)$$

$$\times\ \bigg(30c_6^4(c_4+c_5x)^2(c_{11}-tx)^4$$

$$+10c_6^3(c_{11}-tx)^3\big(2c_6^3tx^3-3c_6^2tx^2(c_4+c_5x)-9c_6^2(c_1+c_{10}x)(c_4+c_5x)$$
$$-9t(c_4+c_5x)^3+3c_6(c_4+c_5x)^2(2tx-\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2})\big)$$
$$+15c_6^2(c_{11}-tx)^2\big(6c_6^4(c_1+c_{10}x)^2+6c_6^2(c_1+c_{10}x)(c_6^2tx^2+2t(c_4+c_5x)^2$$
$$-c_6(c_4+c_5x)(2tx-\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}))+t(3c_6^4tx^4-4c_6^3tx^3(c_4+c_5x)$$
$$+6c_6^2tx^2(c_4+c_5x)^2+6t(c_4+c_5x)^4$$
$$-6c_6(c_4+c_5x)^3(2tx-\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}))\big)$$
$$+3c_6(c_{11}-tx)\big(30c_6^4(c_1+c_{10}x)^2(-c_4t-c_5tx+2c_6tx-c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2})$$
$$+10c_6^2t(c_1+c_{10}x)(4c_6^3tx^3-6c_6^2tx^2(c_4+c_5x)-3t(c_4+c_5x)^3$$
$$+6c_6(c_4+c_5x)^2(2tx-\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}))$$
$$+t^2(12c_6^5tx^5-15c_6^4tx^4(c_4+c_5x)+20c_6^3tx^3(c_4+c_5x)^2$$
$$-30c_6^2tx^2(c_4+c_5x)^3-10t(c_4+c_5x)^5$$
$$+30c_6(c_4+c_5x)^4(2tx-\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}))\big)$$
$$-c_6t\big(90c_6^4(c_1+c_{10}x)^2(-c_6tx^2+(c_4+c_5x)(2tx-\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}))$$
$$+15c_6^2t(c_1+c_{10}x)(-3c_6^3tx^4+4c_6^2tx^3(c_4+c_5x)-6c_6tx^2(c_4+c_5x)^2$$
$$+6(c_4+c_5x)^3(2tx-\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}))$$
$$+t^2(-10c_6^5tx^6+12c_6^4tx^5(c_4+c_5x)-15c_6^3tx^4(c_4+c_5x)^2$$
$$+20c_6^2tx^3(c_4+c_5x)^3-30c_6tx^2(c_4+c_5x)^4$$
$$+30(c_4+c_5x)^5(2tx-\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}))\big)\bigg)$$

$$\times\ \mathrm{Li}_2\left(\frac{c_6(-c_{11}-tx+\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2})}{-c_{11}c_6+2c_4t+2c_5tx+c_6tx+c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}}\right)$$

$$-3600t^3\left(c_6^2(c_1+c_{10}x)+(c_4+c_5x)(-c_{11}c_6+t(c_4+(c_5+c_6)x))\right)^3$$

$$\times\ \mathrm{Li}_2\left(\frac{c_6(-c_{11}+tx+\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2})}{-c_{11}c_6+2c_4t+2c_5tx+c_6tx+c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}}\right)\Bigg]$$

$$
\begin{aligned}
&-\frac{1}{3600c_6^7t^3}\Bigg[-30c_6^2\Bigg(c_6^4(c_{11}-tx)^6\\
&\quad+c_6^3(c_{11}-tx)^5\left(3c_4t+3c_5tx+c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\right)\\
&\quad-3c_6^2t(c_{11}-tx)^4\Big(5c_6^2(c_1+c_{10}x)\\
&\quad+(c_4+c_5x)\big(15c_4t+15c_5tx-c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\\
&\quad+2t^3(c_1+c_{10}x)\Big(55c_6^4(c_1+c_{10}x)^2\\
&\qquad+10t(c_4+c_5x)^3\big(3c_4t+3c_5tx+c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad+3c_6^2(c_1+c_{10}x)(c_4+c_5x)\big(25c_4t+25c_5tx+8c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\\
&\quad-c_6t(c_{11}-tx)^3\Big(-5t(c_4+c_5x)^2\big(14c_4t+14c_5tx-9c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad+c_6^2(c_1+c_{10}x)\big(-120c_4t-120c_5tx+13c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\\
&\quad-6t^2(c_{11}-tx)\Big(5t^2(c_4+c_5x)^4\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\\
&\qquad+20c_6t(c_1+c_{10}x)(c_4+c_5x)^2\big(c_4t+c_5tx+c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad+c_6^3(c_1+c_{10}x)^2\big(25c_4t+25c_5tx+19c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\\
&\quad-2t^2(c_{11}-tx)^2\Big(45c_6^4(c_1+c_{10}x)^2\\
&\qquad+5t(c_4+c_5x)^3\big(3c_4t+3c_5tx-7c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad-3c_6^2(c_1+c_{10}x)(c_4+c_5x)\big(-5c_4t-5c_5tx+21c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\Bigg)\\
&\times\ln\left(c_{11}-tx+\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ 30c_6^2\Bigg(c_6^4(c_{11}-tx)^6 + c_6^3(c_{11}-tx)^5\left(3c_4t+3c_5tx+c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\right)\\
&\quad - 3c_6^2t(c_{11}-tx)^4\Big(5c_6^2(c_1+c_{10}x)\\
&\qquad + (c_4+c_5x)\big(15c_4t+15c_5tx-c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\\
&\quad + 2t^3(c_1+c_{10}x)\Big(55c_6^4(c_1+c_{10}x)^2\\
&\qquad + 10t(c_4+c_5x)^3\big(3c_4t+3c_5tx+c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad + 3c_6^2(c_1+c_{10}x)(c_4+c_5x)\big(25c_4t+25c_5tx+8c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\\
&\quad - c_6t(c_{11}-tx)^3\Big(-5t(c_4+c_5x)^2\big(14c_4t+14c_5tx-9c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad + c_6^2(c_1+c_{10}x)\big(-120c_4t-120c_5tx+13c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\\
&\quad - 6t^2(c_{11}-tx)\Big(5t^2(c_4+c_5x)^4\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\\
&\qquad + 20c_6t(c_1+c_{10}x)(c_4+c_5x)^2\big(c_4t+c_5tx+c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad + c_6^3(c_1+c_{10}x)^2\big(25c_4t+25c_5tx+19c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\\
&\quad - 2t^2(c_{11}-tx)^2\Big(45c_6^4(c_1+c_{10}x)^2\\
&\qquad + 5t(c_4+c_5x)^3\big(3c_4t+3c_5tx-7c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad - 3c_6^2(c_1+c_{10}x)(c_4+c_5x)\big(-5c_4t-5c_5tx+21c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\Bigg)\\
&\times \ln\left(c_{11}+tx+\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\right)\\
&- 60t^2\ln\left(\frac{c_{11}-tx+\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}}{2t}\right)\\
&\quad \times\Bigg(-30c_6(-c_{11}c_6+t(c_4+(c_5+c_6)x))\Big(3c_6^4(c_1+c_{10}x)^2\\
&\qquad + (c_4+c_5x)^2(c_{11}c_6-t(c_4+(c_5+c_6)x))^2\\
&\qquad + 3c_6^2(c_1+c_{10}x)(c_4+c_5x)(-c_{11}c_6+t(c_4+(c_5+c_6)x))\Big)\\
&\qquad \times (c_{11}-tx+\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2})\Bigg)\\
&+ 60t\left(c_6^2(c_1+c_{10}x)+(c_4+c_5x)(-c_{11}c_6+t(c_4+(c_5+c_6)x))\right)^3\\
&\quad \times\ln\left(\frac{2t(c_4+c_5x)}{-c_{11}c_6+2c_4t+2c_5tx+c_6tx-c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ t\Bigg(-c_6x\Bigg(30c_6^5(c_{11}-tx)^5\\
&\quad - 15c_6^4(c_{11}-tx)^4\left(-6c_4t - 6c_5tx + c_6tx - 2c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\right)\\
&\quad - 15c_6^3t(c_{11}-tx)^3\Big(28c_6^2(c_1+c_{10}x) - 150t(c_4+c_5x)^2\\
&\qquad + 3c_6(c_4+c_5x)\big(11tx - 2\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad + c_6^2x\big(-14tx + \sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\Big)\\
&\quad + 3c_6t^2(c_{11}-tx)\Big(1870c_6^4(c_1+c_{10}x)^2\\
&\qquad + 5c_6^2(c_1+c_{10}x)\big(1110t(c_4+c_5x)^2 + c_6^2x(102tx - 59\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2})\\
&\qquad\quad - 6c_6(c_4+c_5x)(41tx - 33\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2})\big)\\
&\qquad + t\big(3300t(c_4+c_5x)^4 + c_6^4x^3(124tx - 65\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2})\\
&\qquad\quad - 5c_6^3x^2(c_4+c_5x)(39tx - 22\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2})\\
&\qquad\quad + 25c_6^2x(c_4+c_5x)^2(14tx - 9\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2})\\
&\qquad\quad - 100c_6(c_4+c_5x)^3(8tx - 7\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2})\big)\Big)\\
&\quad + 5c_6^2t(c_{11}-tx)^2\Big(3c_6^2(c_1+c_{10}x)\big(-414c_4t - 414c_5tx + 83c_6tx\\
&- 24c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\qquad + t\big(-1740t(c_4+c_5x)^3 - 3c_6^2x(c_4+c_5x)(58tx - 27\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2})\\
&\qquad\quad + 2c_6^3x^2(48tx - 19\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2})\\
&\qquad\quad + 135c_6(c_4+c_5x)^2(3tx - 2\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2})\big)\Big)\\
&\quad + t^2\Big(330c_6^4(c_1+c_{10}x)^2\big(-24c_4t - 24c_5tx + 5c_6tx - 5c_6\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2}\big)\\
&\quad + 5c_6^2t(c_1+c_{10}x)\big(-1920t(c_4+c_5x)^3 - 48c_6^2x(c_4+c_5x)(4tx - 3\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2})\\
&\qquad\quad + 35c_6^3x^2(3tx - 2\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2})\\
&\qquad\quad + 450c_6(c_4+c_5x)^2(tx - \sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2})\big)\\
&\qquad + t^2\big(-3600t(c_4+c_5x)^5 - 18c_6^4x^3(c_4+c_5x)(8tx - 5\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2})\\
&\qquad\quad - 100c_6^2x(c_4+c_5x)^3(4tx - 3\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2})\\
&\qquad\quad + 20c_6^5x^4(5tx - 3\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2})\\
&\qquad\quad + 75c_6^3x^2(c_4+c_5x)^2(3tx - 2\sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2})\\
&\qquad\quad + 900c_6(c_4+c_5x)^4(tx - \sqrt{-4t(c_1+c_{10}x)+(c_{11}-tx)^2})\big)\Big)\Bigg)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+ 60t \ln\left(\frac{c_{11} + tx + \sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2}}{2t}\right) \\
&\times \Bigg(-c_6\Big(-30c_6^3(c_4 + c_5x)^2(c_{11} - tx)^4 \\
&\quad + 10c_6^2(c_{11} - tx)^3\big(-2c_6^3tx^3 + 3c_6^2tx^2(c_4 + c_5x) + 9c_6^2(c_1 + c_{10}x)(c_4 + c_5x) \\
&\qquad + 9t(c_4 + c_5x)^3 - 3c_6(c_4 + c_5x)^2(2tx + \sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2})\big) \\
&\quad - 15c_6(c_{11} - tx)^2\big(6c_6^4(c_1 + c_{10}x)^2 + 6c_6^2(c_1 + c_{10}x)(c_6^2tx^2 + 2t(c_4 + c_5x)^2 \\
&\qquad - c_6(c_4 + c_5x)(2tx + \sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2})) + t(3c_6^4tx^4 - 4c_6^3tx^3(c_4 + c_5x) \\
&\quad + 6c_6^2tx^2(c_4 + c_5x)^2 + 6t(c_4 + c_5x)^4 - 6c_6(c_4 + c_5x)^3(2tx + \sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2}))\big) \\
&\quad + 3(c_{11} - tx)\big(30c_6^4(c_1 + c_{10}x)^2(t(c_4 + c_5x) - c_6(2tx + \sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2})) \\
&\qquad + 10c_6^2t(c_1 + c_{10}x)(-4c_6^3tx^3 + 6c_6^2tx^2(c_4 + c_5x) + 3t(c_4 + c_5x)^3 \\
&\qquad\quad - 6c_6(c_4 + c_5x)^2(2tx + \sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2})) \\
&\qquad + t^2(-12c_6^5tx^5 + 15c_6^4tx^4(c_4 + c_5x) - 20c_6^3tx^3(c_4 + c_5x)^2 \\
&\quad + 30c_6^2tx^2(c_4 + c_5x)^3 + 10t(c_4 + c_5x)^5 - 30c_6(c_4 + c_5x)^4(2tx + \sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2}))\big) \\
&\quad + t\big(90c_6^4(c_1 + c_{10}x)^2(-c_6tx^2 + (c_4 + c_5x)(2tx + \sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2})) \\
&\qquad + 15c_6^2t(c_1 + c_{10}x)(-3c_6^3tx^4 + 4c_6^2tx^3(c_4 + c_5x) - 6c_6tx^2(c_4 + c_5x)^2 \\
&\qquad\quad + 6(c_4 + c_5x)^3(2tx + \sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2})) \\
&\qquad + t^2(-10c_6^5tx^6 + 12c_6^4tx^5(c_4 + c_5x) - 15c_6^3tx^4(c_4 + c_5x)^2 \\
&\qquad\quad + 20c_6^2tx^3(c_4 + c_5x)^3 - 30c_6tx^2(c_4 + c_5x)^4 \\
&\qquad\quad + 30(c_4 + c_5x)^5(2tx + \sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2}))\big)\Big)\Bigg) \\
&- 3600t^3\left(c_6^2(c_1 + c_{10}x) + (c_4 + c_5x)(-c_{11}c_6 + t(c_4 + (c_5 + c_6)x))\right)^3 \\
&\times \mathrm{Li}_2\left(\frac{c_6(c_{11} - tx + \sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2})}{c_{11}c_6 - 2c_4t - 2c_5tx - c_6tx + c_6\sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2}}\right) \\
&+ 3600t^3\left(c_6^2(c_1 + c_{10}x) + (c_4 + c_5x)(-c_{11}c_6 + t(c_4 + (c_5 + c_6)x))\right)^3 \\
&\times \mathrm{Li}_2\left(\frac{c_6(c_{11} + tx + \sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2})}{c_{11}c_6 - 2c_4t - 2c_5tx - c_6tx + c_6\sqrt{-4t(c_1 + c_{10}x) + (c_{11} - tx)^2}}\right)\Bigg].
\end{aligned}
$$

The analytical expressions derived using this approach are considerably complex. Furthermore, subsequent integration over x and differentiation with respect to the J variables are still required. Nevertheless, a viable alternative strategy is to perform the differentiation with respect to the J variables first, and then carry out the

x-integration numerically. We have confirmed that this sequence produces results identical to those detailed in Chapter 4.
Having established the previous results, we now turn to the treatment of the spurious divergence. We begin with Equation (C.12) from Chapter 2. To address it, dimensional regularization is applied.* The integral of interest can be expressed as:

$$\int_0^\infty dT T^{\varepsilon-4} \mathrm{e}^{-T(m^2-b)} = (m^2-b)^{3-\varepsilon}\Gamma(\varepsilon-3) \tag{C.32}$$

$$\approx \frac{(m^2-b)^3}{6}\left(\gamma - \frac{1}{\varepsilon} - \frac{11}{6}\right) \tag{C.33}$$

$$+\frac{(m^2-b)^3}{6}\ln(m^2-b) + \mathcal{O}(\varepsilon) \tag{C.34}$$

In the previous equation, γ denotes the Euler-Mascheroni constant. Due to the expansion of the Gamma function, we obtain terms proportional to ε^{-1} and ε^0. Higher-order terms are discarded, as we are interested in the limit $\varepsilon \to 0$. Only the logarithmic contribution remains, as shown in Equation (C.12).
At this point, the integrals over du_4 and dT have already been resolved. For the remaining integrals over u_1, u_2, and u_3, we fix $u_1 = 0$ and focus on the sector where $1 \geq u_2 \geq u_3 \geq 0$. However, resolving the remaining integrals from (C.2) results in a very lengthy expression. Nonetheless, summing all contributions proportional to $\frac{1}{6}\left(\gamma - \frac{1}{\varepsilon} - \frac{11}{6}\right)$ leads to a more compact result, which is significantly shorter than the one obtained from a single element.

$$\begin{aligned}
\frac{1}{6}\left(\gamma - \frac{1}{\varepsilon} - \frac{11}{6}\right)\Bigg[\int_0^1 du_2 \int_0^{u_2} du_3 &\frac{\left(m^2-(u\Lambda_u+\lambda_u+J_{41})\right)^3}{u+\dot{G}_{12}t+\dot{G}_{13}s+4J_4} + \frac{\left(m^2-(u\Lambda_u+\lambda_u-J_{41})\right)^3}{u-\dot{G}_{12}t-\dot{G}_{13}s-4J_4} \\
&+ \frac{\left(m^2-(t\Lambda_t+\lambda_t+J_{42})\right)^3}{t+\dot{G}_{23}s+\dot{G}_{21}u+4J_4} + \frac{\left(m^2-(t\Lambda_t+\lambda_t-J_{42})\right)^3}{t-\dot{G}_{23}s-\dot{G}_{21}u-4J_4} \\
&+ \frac{\left(m^2-(s\Lambda_s+\lambda_s+J_{43})\right)^3}{s+\dot{G}_{31}u+\dot{G}_{32}t+4J_4} + \frac{\left(m^2-(s\Lambda_s+\lambda_s-J_{43})\right)^3}{s-\dot{G}_{31}u-\dot{G}_{32}t-4J_4}\Bigg]\Bigg|_{u_1=0} \\
= \frac{1}{720}\Big(&-60J_{12}^2-60J_{13}^2-60J_{14}^2-60J_{23}^2-60J_{24}^2-60J_{34}^2 \\
&+120J_{23}m^2-180m^4+6J_{12}(20m^2-s)+2J_{14}s+6J_{23}s \\
&-4J_{24}s+2J_{34}s-s^2-2J_{14}t+6J_{23}t2J_{24}t+4J_{34}t \\
&-st-t^2+6J_{13}(-20m^2+t)\Big).
\end{aligned} \tag{C.35}$$

*For readers unfamiliar with the concept of dimensional regularization, refer to Chapter 6 of [57] or Chapter 14 of [17].

As mentioned throughout this work, one of the disadvantages of solving integrals in the worldline formalism by dividing into sectors is the need to calculate the results separately for each sector and then sum them. In this particular case, for sector $1 \geq u_3 \geq u_2 \geq 0$, the result is:

$$\frac{1}{6}\left(\gamma - \frac{1}{\varepsilon} - \frac{11}{6}\right)\left[\int_0^1 du_3 \int_0^{u_3} du_2 \frac{\left(m^2-(u\Lambda_u+\lambda_u+J_{41})\right)^3}{u+\dot{G}_{12}t+\dot{G}_{13}s+4J_4} + \frac{\left(m^2-(u\Lambda_u+\lambda_u-J_{41})\right)^3}{u-\dot{G}_{12}t-\dot{G}_{13}s-4J_4} \right.$$
$$+ \frac{\left(m^2-(t\Lambda_t+\lambda_t+J_{42})\right)^3}{t+\dot{G}_{23}s+\dot{G}_{21}u+4J_4} + \frac{\left(m^2-(t\Lambda_t+\lambda_t-J_{42})\right)^3}{t-\dot{G}_{23}s-\dot{G}_{21}u-4J_4}$$
$$\left.\left. + \frac{\left(m^2-(s\Lambda_s+\lambda_s+J_{43})\right)^3}{s+\dot{G}_{31}u+\dot{G}_{32}t+4J_4} + \frac{\left(m^2-(s\Lambda_s+\lambda_s-J_{43})\right)^3}{s-\dot{G}_{31}u-\dot{G}_{32}t-4J_4}\right]\right|_{u_1=0}$$
$$= \frac{1}{720}\Big(-60J_{12}^2 - 60J_{13}^2 - 60J_{14}^2 - 60J_{23}^2 - 60J_{24}^2 - 60J_{34}^2$$
$$+ 120J_{13}m^2 - 120J_{23}m^2 - 180m^4 - 2J_{14}s - 6J_{23}s + 4J_{24}s$$
$$- 2J_{34}s - s^2 + 6J_{12}(-20m^2+s) - 6J_{13}t + 2J_{14}t$$
$$- 6J_{23}t + 2J_{24}t - 4J_{34}t - st - t^2\Big). \quad \text{(C.36)}$$

Let us recall that the introduction of $J's$ is accompanied by derivatives, and it is easy to see that differentiating both results more than once will yield zero. Therefore, we can disregard the contributions proportional to $\frac{1}{6}\left(\gamma - \frac{1}{\varepsilon} - \frac{11}{6}\right)$.

In this appendix, we've presented an alternative method, the Generating Function approach, to tackle the complexities of four-photon calculations within the worldline formalism. While this approach has led to analytically cumbersome and lengthy expressions, it's crucial to note that their extended nature doesn't equate to computational difficulty for modern computers. These expressions, though long, are straightforward for computational implementation.
We intend to continue investigating the potential for term elimination and simplification. Our future work will focus on exploring techniques to manage these expressions more effectively, with the aim of determining whether this alternative approach can offer further insights or computational advantages in the study of multi-photon interactions.

Appendix D

Relevant Formulas for the Worldline Formalism

In this research, we focused on solving circular integrals within the worldline formalism. This approach led us to explore various alternatives and techniques to handle such integrals, some of which are not commonly found in the literature. In this appendix, we compile alternative representations that could be useful for further studies in this area.
Solving circular integrals within this formalism using the bracket technology presented in Chapter 2 led us to explore the following general integral:Studying the cases of N=3 and N=4, we found the following representations:

$$\int_0^1 du_1 du_2 \dots du_N \prod_{i<j=1}^{N} G_{ij}^{n_{ij}} \tag{D.1}$$

Studying the cases of N=3 and N=4, we found the following representations:

$N = 3$

$$\int\limits_{1,2,3} G_{12}^a G_{13}^b G_{23}^c \quad = \quad a!b!c! \sum_{\substack{i=1\\j=1\\k=1}}^{a,b,c} h_i^a h_j^b h_k^c \left(\hat{B}_{2i} \hat{B}_{2j} \hat{B}_{2k} - \hat{B}_{2(i+j+k)} \right) \tag{D.2}$$

Where:

$$
\begin{aligned}
\hat{B}_{2i} &= \frac{B_{2i}}{(2i)!} && \text{(D.3)}\\
h_n^i &= (-1)^{i+1}\frac{2(2n-1)!}{(2n-i-1)!(2i-2n+1)!} && \text{(D.4)}
\end{aligned}
$$

$N = 4$

$$
\begin{aligned}
\int\limits_{1,2,3,4} G_{12}^a G_{13}^b G_{14}^c G_{23}^d G_{24}^f G_{34}^g &= a!b!c!d!f!g! \sum_{\substack{i=1\\j=1\\k=1}}^{a,b,c} \sum_{\substack{l=1\\p=1\\q=1}}^{d,f,g} h_i^a h_j^b h_k^c h_l^d h_p^f h_q^g \Big\{ \hat{B}_{2i}\hat{B}_{2j}\hat{B}_{2k}\hat{B}_{2l}\hat{B}_{2p}\hat{B}_{2q}\\
&\quad -\hat{B}_{2(j+k+q)}\hat{B}_{2i}\hat{B}_{2l}\hat{B}_{2p} - \text{Perm.}\\
&\quad -\hat{B}_{2(i+j+p+q)}\hat{B}_{2k}\hat{B}_{2l} - \text{Perm.}\\
&\quad +I_3^{2(j+k),2(l+p),2q}\hat{B}_{2i} + \text{Perm.} - I_6 \Big\}
\end{aligned}
\tag{D.5}
$$

Where:

$$
\begin{aligned}
I_3^{2i,2j,2k} &= \int_0^1 dudv \left\langle u\middle|\partial^{-2i}\middle|v\right\rangle \left\langle u\middle|\partial^{-2j}\middle|v\right\rangle \left\langle u\middle|\partial^{-2k}\middle|v\right\rangle\\
&= \sum_{a=2i}^{2i+2j-1} C_i^a \left(\hat{B}_a\hat{B}_{2i+2j+2k-a} + (-1)^a\hat{B}_{2i+2j+2k}\right) + \{i \leftrightarrow j\} && \text{(D.6)}
\end{aligned}
$$

with

$$
C_i^a := (-1)^a \binom{a-1}{2i-1} \tag{D.7}
$$

And

$$
\begin{aligned}
I_6 &= \int_0^1 dudv \left\langle u\middle|\partial^{-2i}\middle|v\right\rangle \left\langle u\middle|\partial^{-2j}\middle|v\right\rangle \left\langle u\middle|\partial^{-2k}\middle|v\right\rangle \left\langle u\middle|\partial^{-2l}\middle|v\right\rangle \left\langle u\middle|\partial^{-2p}\middle|v\right\rangle \left\langle u\middle|\partial^{-2q}\middle|v\right\rangle \\
&= -\sum_{a_1=2i}^{2i+2j-1} C_i^{a_1} \Bigg[\sum_{a_2=2l}^{2l+2p-1} \sum_{a_3=0}^{2i+2j-a_1-1} C_l^{a_2,a_3} \\
&\quad \times\Big(\hat{B}_{2i+2j-a_1-a_3} I_3^{2l+2p-a_2,2k+a_1+a_2+a_3,2q} + I_3^{2l+2p-a_2,2i+2j+2q-a_1-a_3,2k+a_1+a_2+a_3} \Big) \\
&\quad + \sum_{a_2=2p}^{2l+2p-1} \sum_{a_3=0}^{2i+2j-a_1-1} C_p^{a_2,a_3} \\
&\quad \times\Big(\hat{B}_{2k+a_1+a_2+a_3} I_3^{2l+2p-a_2,2i+2j-a_1-a_3,2q} + I_3^{2l+2p-a_2,2i+2j-a_1-a_3,2k+2q+a_1+a_2+a_3} \Big) \\
&\quad + \sum_{a_2=2i+2j-a_1}^{2i+2j+2p-a_1-1} \sum_{a_3=0}^{2l-1} C_{i+j-a_1}^{a_2,a_3} \\
&\quad \times\Big(\hat{B}_{2l-a_3} I_3^{2i+2j+2p-a_1-a_2,2k+a_1+a_2+a_3,2q} + I_3^{2l+2q-a_3,2i+2j+2p-a_1-a_2,2k+a_1+a_2+a_3} \Big) \\
&\quad + (-1)^{a_1} I_3^{2i+2j+2k,2l+2p,2q} \Bigg] + \{i \leftrightarrow j, p \leftrightarrow q\}
\end{aligned}
\tag{D.8}
$$

where now:

$$
C_i^{a_1,a_2} \quad := \quad (-1)^{a_1+a_2} \binom{a_1+a_2-1}{i-1} \binom{a_1+a_2-i}{a_1-i}
\tag{D.9}
$$

We conclude this section by using formula (B.10), presented in [**?**]:

$$
2\sum_{n=0}^{\infty}(2iF)^n \langle u_i|\partial^{-(n+2)}|u_j\rangle = \frac{1}{2F^2}\left(\frac{F}{\sin(F)}\mathrm{e}^{-iF\dot{G}_{ij}} + iF\dot{G}_{ij} - 1\right)
\tag{D.10}
$$

To explore an alternative representation for the 3-point case in ϕ^3 theory. After applying equation 3.3 to the following integral:

$$
I_3 = \int_{1,2,3} \mathrm{e}^{T(\lambda_{12}G_{12}+\lambda_{23}G_{23}+\lambda_{31}G_{31})},
\tag{D.11}
$$

we obtain:

$$
I_3 = \mathrm{e}^{\frac{T}{4}(\lambda_{12}+\lambda_{23}+\lambda_{31})} \int_{1,2,3} \mathrm{e}^{-\frac{T}{4}\left(\lambda_{12}\dot{G}_{12}^2+\lambda_{23}\dot{G}_{23}^2+\lambda_{31}\dot{G}_{31}^2\right)}
\tag{D.12}
$$

Let's proceed to solve:

$$\dot{I}_3 = \int_{1,2,3} \mathrm{e}^{\bar{x}_{12}\dot{G}_{12}+\bar{x}_{23}\dot{G}_{23}+\bar{x}_{31}\dot{G}_{31}} \tag{D.13}$$

$$\begin{aligned} \mathrm{e}^{-iF\dot{G}_{ij}} &= \frac{\sin(F)}{F}\left(4F^2\sum_{n=0}^{\infty}(2iF)^n\langle u_i|\partial^{-(n+2)}|u_j\rangle - iF\dot{G}_{ij}+1\right) \\ &= F\sin(F)\left(4\sum_{n=0}^{\infty}(2iF)^n\langle u_i|\partial^{-(n+2)}|u_j\rangle - \frac{2i}{F}\langle u_i|\partial^{-1}|u_j\rangle + \frac{1}{F^2}\right) \end{aligned} \tag{D.14}$$

If $F = ix$:

$$\begin{aligned} \mathrm{e}^{x_{ij}\dot{G}_{ij}} &= -x_{ij}\sinh(x_{ij})\left(4\sum_{n=0}^{\infty}(-2x_{ij})^n\langle u_i|\partial^{-(n+2)}|u_j\rangle - \frac{2}{x_{ij}}\langle u_i|\partial^{-1}|u_j\rangle - \frac{1}{x_{ij}^2}\right) \\ &= x_{ij}\sinh(x_{ij})\left(4\sum_{n=0}^{\infty}(-)^{n+1}(2x_{ij})^n\langle u_i|\partial^{-(n+2)}|u_j\rangle + \frac{2}{x_{ij}}\langle u_i|\partial^{-1}|u_j\rangle + \frac{1}{x_{ij}^2}\right) \end{aligned} \tag{D.15}$$

$$\begin{aligned} \dot{I}_3 &= x_{12}\sinh(x_{12})x_{23}\sinh(x_{23})x_{31}\sinh(x_{31})\left[\frac{1}{x_{12}^2x_{23}^2x_{31}^2} + 2^2\left(\frac{s_1(x_{12})}{x_{23}x_{31}} + \frac{s_1(x_{23})}{x_{12}x_{31}} + \frac{s_1(x_{31})}{x_{12}x_{23}}\right)\right. \\ &\left. -2\left(\frac{s_2(x_{12},x_{23})}{x_{31}} + \frac{s_2(x_{12},x_{31})}{x_{23}} + \frac{s_2(x_{23},x_{31})}{x_{12}}\right) + s_3(x_{12},x_{23},x_{31})\right] \end{aligned} \tag{D.16}$$

Where:

$$
\begin{aligned}
s_1(x) &= \sum_{n=0}^{\infty} 2^{n+2}(-x)^n \hat{B}_{n+4} \\
&= \frac{3x\coth(x) - x^2 - 3}{12x^4} \\
s_2(x_1, x_2) &= \sum_{n_1,n_2=0}^{\infty} 2^{n_1+n_2+4}(-x_1)^{n_1}(-x_2)^{n_2}\hat{B}_{n_1+n_2+5} \\
&= \frac{x_1 + x_2}{12x_1^2x_2^2} - \frac{i\zeta(3)}{2x_1x_2\pi^3} + \frac{i}{2\pi(x_1 - x_2)}\left(\frac{H_{\frac{ix_2}{\pi}}}{x_2^3} - \frac{H_{\frac{ix_1}{\pi}}}{x_1^3}\right) \\
s_3(x_1, x_2, x_3) &= \sum_{n_1,n_2,n_3=0}^{\infty} 2^{n_1+n_2+n_3+6}(-x_1)^{n_1}(-x_2)^{n_2}(-x_3)^{n_3}\hat{B}_{n_1+n_2+n_3+6} \\
&= -\frac{x_1x_2 + x_1x_3 + x_2x_3}{6x_1^2x_2^2x_3^3} + \frac{i\zeta(3)}{x_1x_2x_3\pi^3} \\
&\quad + \frac{i}{\pi(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)}\left((x_1 - x_2)\frac{H_{\frac{ix_3}{\pi}}}{x_3^3} + (x_2 - x_3)\frac{H_{\frac{ix_1}{\pi}}}{x_1^3} + (x_3 - x_1)\frac{H_{\frac{ix_2}{\pi}}}{x_2^3}\right)
\end{aligned}
\tag{D.17}
$$

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