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Nonrelativistic boson stars: Spherically symmetric configurations and their stability

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Resumen

En los últimos años, los campos escalares han adquirido un papel significativo en ciertos modelos de materia oscura. Las configuraciones autogravitantes de dichos campos pueden formar objetos compactos conocidos como estrellas de bosones: soluciones localizadas y estacionarias del sistema Einstein-Klein-Gordon propuestas como modelo para el núcleo de materia oscura en halos galácticos.

Esta tesis investiga la existencia y estabilidad de diversas configuraciones de estrellas de bosones en el límite no relativista. Mediante una combinación de métodos analíticos y numéricos se estudian las estrellas de ℓ-bosones (descritas mediante el sistema Schrödinger-Poisson multicampo) y las estrellas de bosones con autointeracción (modeladas por el sistema Gross-Pitaevskii-Poisson), incluyendo el efecto de una masa puntual central, la cual sirve como un modelo juguete para el agujero negro supermasivo en el centro de la galaxia.

Las estrellas de ℓ-bosones no relativistas consisten de una familia de campos escalares masivos, cada uno con momento angular y oscilaciones dependientes del tiempo, organizados de tal manera que la configuración total es estática y esféricamente simétrica. Utilizando el método directo del cálculo de variaciones, se presenta una prueba para la existencia de las estrellas de ℓ-bosones como estados de energía mínima. Además, se demuestra rigurosamente su estabilidad orbital bajo perturbaciones esféricas no lineales. Sin embargo, a través de un estudio de las ecuaciones linealizadas, se establece que las configuraciones con $\ell > 1$ son linealmente inestables ante perturbaciones no esféricas.

En el contexto de las estrellas de bosones con autointeracción, demostramos que, en el caso de interacciones repulsivas, el funcional de energía asociado alcanza un mínimo global, el cual corresponde a una solución fundamental del sistema Gross-Pitaevskii-Poisson que es orbitalmente estable ante perturbaciones esféricas. Adicionalmente, se realiza un estudio de estabilidad lineal bajo perturbaciones generales: interacciones repulsivas conducen a estados base estables, mientras que interacciones atractivas introducen un umbral crítico de masa que separa ramas estables e inestables. Los estados excitados son generalmente inestables; sin embargo, en teorías con autointeracción repulsiva, identificamos una banda de estabilidad para los primeros estados excitados, independiente de la masa del campo escalar o de los detalles específicos del potencial.

Palabras Clave: Estrellas de bosones, problema variacional, mínimo de energía, existencia de soluciones, campo escalar, estabilidad.

Abstract

In recent years, scalar fields have played a significant role in certain dark matter models. Self-gravitating configurations of such fields can form compact objects known as boson stars: localized, stationary solutions of the Einstein-Klein-Gordon system proposed as a model for the dark matter core in galactic halos.

This thesis investigates the existence and stability of various boson star configurations in the non-relativistic limit. Through a combination of analytical and numerical methods, we study ℓ-boson stars (described by the multi-field Schrödinger-Poisson system) and self-interacting boson stars (modeled by the Gross-Pitaevskii-Poisson system), including the effect of a central point mass, which serves as a toy model for the supermassive black hole at the center of the galaxy.

Nonrelativistic ℓ-boson stars consist of a family of massive scalar fields, each with angular momentum and time-dependent oscillations, arranged such that the total configuration is static and spherically symmetric. Using the direct method of calculus of variations, a proof for the existence of ℓ-boson stars as minimum energy states is presented. Furthermore, their orbital stability under non-linear spherical perturbations is rigorously demonstrated. However, through a study of the linearized equations, it is established that configurations with $\ell > 1$ are linearly unstable against non-spherical perturbations.

In the context of self-interacting boson stars, we demonstrate that for repulsive self-interactions the associated energy functional attains a global minimum, which corresponds to the ground state solution of the Gross-Pitaevskii-Poisson system and that is orbital stable against spherical perturbations. Additionally, a linear stability analysis under general perturbations is performed: repulsive interactions result in stable ground states, whereas attractive interactions introduce a critical mass threshold separating stable and unstable branches. Excited states are generally unstable; however, in theories with repulsive self-interaction, we identify a stability band for the first excited states, independent of the scalar field mass or the specific details of the potential.

Introduction

Boson stars are hypothetical astrophysical objects that, unlike conventional stars made of baryonic matter, are composed of a massive, self-gravitating scalar field. The idea that a field by itself could give rise to compact configurations dates back to the 1950's, when John Wheeler introduced the concept of geons [88]. These are exotic, particle-like objects constructed solely from an electromagnetic field interacting with gravity as described by general relativity. Although intriguing, Wheeler's geons were ultimately shown, even by Wheeler himself, to be unstable, which limited their physical relevance.

A few years later, in the 1960's, Kaup [49] proposed an alternative: replacing the electromagnetic field with a massive, complex scalar field, leading to the so-called Klein-Gordon geons. Almost simultaneously, Ruffini and Bonazzola [72] independently developed a similar model using a real quantum mechanical scalar field. Both approaches led to the same equations. The (numerical) solutions were found to be stable, giving rise to what are now known as boson stars. For a review on boson stars, their numerical construction and their properties, see for example [54].

At the theoretical level, boson stars are modeled through the Einstein-Klein-Gordon system. As the name suggest, this system combines the Einstein field equations, which allows one to obtain the spacetime curvature generated by the stress-energy-momentum of the scalar field itself, with the Klein-Gordon equation, which governs the dynamics of the field. From a physical point of view, boson stars can be understood as equilibrium configurations: on one hand, gravity tends to concentrate and collapse the scalar field, while on the other hand, the quantum pressure associated with the field (stemming from its dispersive nature) counter-reacts this collapse.

In recent years, boson stars have attracted growing attention within the community, not only because they have been proposed as black hole mimickers [84, 41], but also because they are considered viable candidates for modeling dark matter halo cores in galaxies. Dark matter itself is one of the greatest mysteries of modern physics. Whereas we have robust indirect evidence of its existence, stemming from observations such as galaxy rotation curves [71, 31], cosmological redshift [90, 77], or the cosmic microwave background [9], its direct detection and fundamental nature remains elusive.

Over the time, different theoretical models have been developed to explain its nature. Among them, the most widely accepted one is the Cold Dark Matter (CDM) model. This model posits dark matter consists of non-relativistic (hence the term cold), electrically neutral and non-baryonic particles, whose dominant interaction is gravitational. CDM has been highly successful in describing the universe at large scales [13]; however, it faces significant challenges at galactic scale, where it fails to reproduce certain observational features, such as the so-called "cusp-core" and "missing satellites" problems [14]. These shortcoming have motivated the search for alternative models of dark matter, among which ultralight dark matter has gained preeminence [45, 55, 60, 59, 58].

In the simplest model, the constituent particle is an ultralight boson (described by a scalar field) with mass around $m_0 \sim 10^{-22}\mathrm{eV}/c^2$. Assuming a typical dispersion velocity of $v \sim 250$ km/s in the galactic halo, a particle this light has a de Broglie wavelength of astronomical scale, meaning that a large collection of them can be treated as a single coherent (classical) wave [29]. This wavelike behavior is the reason why boson stars are proposed as a compelling candidate for the cores of galactic dark matter halos and thus an interesting framework for studying the properties of dark matter.

Several generalizations of this basic construction have been proposed, one of them consisting of multiple scalar fields [82, 38, 36]. Instead of a single field, this approach considers a fixed number, $K \geq 1$, of scalar fields. This raises a natural question: why introduce multiple fields? At first glance, such a requirement may seem artificial. However, this framework emerges naturally in the context of semiclassical gravity. As shown in [6], for the static case, the semiclassical Einstein-Klein-Gordon system for a single real quantum scalar field, whose state describes the excitation of K identical boson particles, can be effectively reduced to the Einstein-Klein-Gordon system for K complex classical scalar fields.

Recent studies have revealed that the Einstein-Klein-Gordon system with multiple fields admits a remarkably rich spectrum of static solutions, even within the spherically symmetric sector. This is because, when moving from a single field to $K \geq 3$ fields, the internal symmetry group $U(K)$ allows for non-trivial representations of the rotation group $SO(3)$. As a consequence, one can obtain configurations with nonzero orbital angular momentum but vanishing total angular momentum, thereby preserving spherical symmetry. In the particular case $K = 2\ell + 1$, choosing the irreducible representation of $SO(3)$ with integer spin ℓ leads to the so-called ℓ-boson stars [3]. In this way, ℓ-boson stars arise as an elegant extension of ordinary boson stars ($\ell = 0$).

Another interesting generalization of boson stars emerges when, in addition to the gravitational interaction, scalar fields interact with themselves (while the interaction with baryonic matter remains negligible), giving rise to self-interacting boson stars [23, 76, 30]. This extension is physically motivated because a repulsive force from self-interaction can support objects with a significantly larger maximum mass, making them more plausible astrophysical objects.

In this model, a repulsive self-interaction plays a role analogous to the Pauli exclusion principle in fermions, introducing an affective repulsive force between the constituent particles that helps counter-react the gravitational collapse. An attractive self-interaction, by contrast, will exacerbate the collapse and prevent the existence of a stable ground state.

As models for dark matter halo cores, boson stars are often studied in the Newtonian limit, since general relativistic effects are typically negligible at galactic scales. This approximation simplifies the underlying equations significantly: the full Einstein-Klein-Gordon system reduces to the more tractable Schrödinger-Poisson system. Specifically, the Klein-Gordon equation is replaced by the Schrödinger equation, and the Einstein equations by the Poisson equation. While a fully relativistic description is required to accurately describe the region near a galaxy's central black hole, the large-scale structure of boson stars solutions is known to be robust across both regimes [7]. This allows for the use of a toy model to approximate the black hole's gravitational effect, which consists of adding an external potential from a central point mass M to the system [24]. Furthermore, when self-interactions are considered, the scalar field is described by the Gross-Pitaevskii-Poisson system [39, 27].

The Schrödinger-Poisson system has been studied for several decades, both from a mathematical perspective and through numerical simulations. This system is also known as the Schrödinger-Newton equation or, in the context of plasma physics, as the Choquard equation [63]. The existence of static solutions was established in [53], using a variational approach combined with the technique of symmetric decreasing rearrangements of functions [42]. The resulting solution is spherically symmetric, monotonically decreasing, and corresponds to the ground state or minimum energy configuration. This technique enabled the treatment of various generalizations, such as the inclusion of self-interaction terms or the presence of a point mass in the system. Later, in [57], the more general concentration-compactness technique was introduces, allowing the existence of ground state solutions to be proven in a broader setting [32]. In this regard, variational methods are not only elegant but also particularly effective: showing that a solution minimizes the associated energy functional is already a promising indication for its stability.

Regarding excited states, in [62] the system was studied numerically, suggesting the existence of a family of stationary and spherically symmetric solutions labeled by an integer n, corresponding to the number of nodes in the radial profile of the wavefunction. Shortly thereafter, these numerical findings were rigorously confirmed in [83]. Furthermore, in [56] it was proven that the energy functional related with the Choquard equation (which is equivalent to the stationary Schrödinger-Poisson system when the wave function is stationary) possesses an infinitely number of critical points.

The existence of bosons stars in the relativistic regime was first rigorously proven in [10]. This work subsequently led to the establishment of the existence of Newtonian ℓ-boson stars in [22]. However, a limitation of these proofs is that the methodologies employed make it difficult to analyze physical properties of the solutions (e.g. stability) beyond confirming their existence. To address this issue, in this thesis we present an alternative proof that leverages the direct method of the calculus of variations [42] to establish the existence of Newtonian ℓ-boson stars.

One of the fundamental properties that any astrophysical object must possesses is stability, understood as the ability of the object to maintain equilibrium after being perturbed. Consequently, any viable dark matter configuration must be stable over cosmological timescales. There exist different concepts of stability, each characterizing a specific response to perturbations. The most elementary of these is linear stability, which assesses the local behavior near an equilibrium point by analyzing the system's linearization; if all eigenvalues of the linearized system have non-positive real parts, the equilibrium is considered linearly stable.

A more refined concept is orbital stability, which is important to this work. In this thesis, an equilibrium solution to a dynamical system (described by a Cauchy problem) is said to be orbital stable if, under sufficiently small perturbations to its initial data, its subsequent evolution remains close to the family of minimizers of the associated energy functional. A stronger condition is asymptotic stability, which further requires that the perturbed solution not only remains near to this family of minimizers but also converges to it over time.

The linear stability of boson stars in the relativistic regime has been widely studied [34, 35]. In particular, the stability of boson stars is controlled by the central amplitude of the scalar field, with a transition from stability to instability typically occurring at the maximum-mass configuration. Relativistic ℓ-boson stars have been shown to be stable under spherical perturbations, both linear and nonlinear [4, 5], as long as they are considered in their ground state. Moreover, based on 3D numerical simulations, the stability of relativistic ℓ-boson stars under non-spherical perturbations, without symmetries, has been studied in [47, 74] for the case $\ell = 1$, with no instabilities detected during the time span of the evolution. For a numerical investigation that includes the stability of non-relativistic ℓ-boson stars with $\ell = 1$, see [40].

In the Newtonian limit of ℓ-boson stars, linear stability against spherical perturbations was established in [70], while in the present thesis a rigorous result of orbital stability [17] under nonlinear spherical perturbations is presented (assuming global existence of solutions). Furthermore, following [65], a combined numerical and analytic analysis of linear stability under non-spherical perturbations is carried out, showing that Newtonian ℓ-boson stars are linearly unstable unless $\ell = 0, 1$. Moreover, building upon [21], we also present an analysis of linear stability under general perturbations in the case of self-interacting boson stars.

This thesis is organized as follows. Starting from the action corresponding to the Einstein-Klein-Gordon theory for a system of K minimally coupled and non self-interacting scalar fields, in Chapter 1 we derive the weak-field and slow-motion limits, which lead to the multi-field Schrödinger-Poisson system. A discussion about the symmetries of the system and the associated conserved quantities follows. We then extend this framework to include a generalization that incorporates the external gravitational potential generated by a point mass M, previously investigated in [24]. The chapter concludes with numerical examples for the case of a single field ($K = 1$) and zero point mass ($M = 0$).

To make this thesis self-contained, in Chapter 2 we provide a collection of definitions and important mathematical results used in the subsequent chapters. In Chapter 3, we reformulate the problem of finding static solutions of the multi-field Schrödinger-Poisson system as a minimization problem and prove the existence of a solution corresponding to the ground state.

In Chapter 4, we turn to the study of the Newtonian ℓ-boson stars. By modeling them as functions belonging to an appropriate function space which is invariant under the spin-ℓ representation of the rotation group $SO(3)$, we demonstrate their existence through a variational approach. Moreover, we prove that these configurations are orbital stable under spherical perturbations. Chapter 5, which is based on [65], extends this analysis by combining numerical and analytical techniques to investigate the linear stability of Newtonian ℓ-boson stars under non-spherical perturbations.

As we mentioned previously, the Gross-Pitaevskii-Poisson system can be used to model self-interacting Newtonian boson stars. In Chapter 6, we study this system in the presence of an external potential generated by a central point mass. In particular, by using variational methods, we prove the existence of stationary solutions and establish their orbital stability under spherical perturbations. Furthermore, building upon [21] and motivated by [75, 12], where it is suggested that self-interaction may stabilize excited states in the relativistic regime, we analyze the linear stability of self-interacting Newtonian boson stars. Finally, we present our conclusions in Chapter 7, while some mathematical results are detailed in Appendix A.

In this thesis, we work in natural units for which $c = \hbar = 1$. Within this convention, mass and energy have the same physical units, that is, $[M] = [E]$, while length and time both having units of inverse energy, $[L] = [T] = [E]^{-1}$. Furthermore, in Chapter 1 we use the $(-,+,+,+)$ signature convention for the metric.

Chapter 1

Newtonian boson stars

In this chapter, we derive the Newtonian limit of the multi-field Einstein-Klein-Gordon theory. We begin by considering the weak-field approximation of gravity and derive the corresponding action. Under the additional assumption of slow motion for the scalar fields, this further reduces the action to one leading to the Schrödinger-Poisson system. Next, we discuss the conserved quantities associated with the system and extend de framework to include the presence of a point mass M. Finally, we present numerical examples of solutions to the Schrödinger-Poisson system.

Our starting point is the Einstein-Klein-Gordon theory for fixed number K of complex scalar fields ϕ_k ($k = 1, 2, \ldots, K$) each with mass m_0. This theory is described by the action,

$$S[g, \phi_1, \ldots, \phi_K] = \int \left(\frac{R}{16\pi G_N} + \mathcal{L}_M \right) \sqrt{-g} d^4x, \tag{1.1}$$

which consists of the Einstein-Hilbert action with the matter term,

$$\mathcal{L}_M = -\sum_{k=1}^{K} \left[\nabla_\mu \overline{\phi}_k \nabla^\mu \phi_k + m_0^2 \overline{\phi}_k \phi_k \right]. \tag{1.2}$$

As usual, G_N is the gravitational Newton constant, $\sqrt{-g} d^4x$ denotes the invariant element on the spacetime manifold, R is the Ricci scalar, ∇_μ is the covariant derivative with respect to the metric and $\overline{\phi}_k$ represents the complex conjugate of ϕ_k. The variation with respect to the metric and the scalar fields yields the Einstein-Klein-Gordon system, which consist of the Einstein field equations

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \tag{1.3}$$

where the components of the stress-energy tensor $T_{\mu\nu}$ are given by

$$T_{\mu\nu} = \frac{1}{2} \sum_{k=1}^{K} \left[\nabla_\mu \overline{\phi}_k \nabla_\nu \phi_k + \nabla_\mu \phi_k \nabla_\nu \overline{\phi}_k - g_{\mu\nu} \left(\nabla_\alpha \overline{\phi}_k \nabla^\alpha \phi_k + m_0^2 \overline{\phi}_k \phi_k \right) \right], \tag{1.4}$$

and the Klein-Gordon equation for the scalar fields ϕ_k reads

$$\left(\nabla^\mu \nabla_\mu - m_0^2 \right) \phi_k = 0. \tag{1.5}$$

As explained in the introduction, in this thesis we are interested in the Newtonian limit of the Einstein-Klein-Gordon theory. To derive this limit, we follow the method explained to us in [26]: first, we compute the weak field limit of the action, and then we consider the slow-motion approximation of the complex scalar fields (see for example [33] for a different derivation of the Schrödinger-Poisson system from the Einstein-Klein-Gordon system).

1.1 Weak field limit of the action

We first investigate the regime in which the gravitational field is weak. To this end, we consider a smooth one-parameter family $\phi_k(\lambda)$ of scalar fields depending on a real parameter λ which characterizes the strength of the field, and satisfies $\phi_k(0) = 0$. Likewise, we introduce a smooth one-parameter family of metrics $g_{\mu\nu}(\lambda)$ such that $g_{\mu\nu}(0) = \eta_{\mu\nu}$ where $\eta_{\mu\nu}$ denotes the components of the Minkowski metric. Assuming that the scalar fields $\phi_k(\lambda)$ induce small deviations from the Minkowski metric, we write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{1.6a}$$
$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2), \tag{1.6b}$$

where $g^{\mu\nu}$ represents the components of the inverse metric and $h^{\mu\nu} := \eta^{\alpha\mu}\eta^{\beta\nu}h_{\alpha\beta}$. To obtain the weak field limit, expand the action $S[g, \phi_1, \ldots \phi_K]$ in powers of h and its derivatives. As usual, in Eq. (1.1) R denotes the Ricci scalar which is given by

$$R = g^{\mu\nu}R_{\mu\nu}, \tag{1.7}$$

where $R_{\mu\nu}$ refers to the components of the Ricci tensor. Hence, in order to obtain an explicit expression for the action (1.1), we need to calculate the components of the Ricci tensor, which are given by

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha{}_{\mu\nu} - \partial_\nu \Gamma^\alpha{}_{\alpha\mu} + \Gamma^\sigma{}_{\mu\nu}\Gamma^\alpha{}_{\sigma\alpha} - \Gamma^\sigma{}_{\mu\alpha}\Gamma^\alpha{}_{\sigma\nu}, \tag{1.8}$$

and the Christoffel symbols can be obtained from the following expression

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}\right). \tag{1.9}$$

By substituting the expressions for the components of the metric and its inverse given by Eq. (1.6), we obtain

$$\Gamma^\alpha{}_{\mu\nu} = \Gamma^{(1)\alpha}{}_{\mu\nu}[h_{\mu\nu}] + \Gamma^{(2)\alpha}{}_{\mu\nu}[h_{\mu\nu}] + \mathcal{O}(h^2\partial h), \tag{1.10}$$

where we defined

$$\Gamma^{(1)\alpha}{}_{\mu\nu}[h_{\mu\nu}] := \frac{1}{2}\eta^{\alpha\beta}\left(\partial_\mu h_{\beta\nu} + \partial_\nu h_{\beta\mu} - \partial_\beta h_{\mu\nu}\right), \tag{1.11a}$$
$$\Gamma^{(2)\alpha}{}_{\mu\nu}[h_{\mu\nu}] := -\frac{1}{2}h^{\alpha\beta}\left(\partial_\mu h_{\beta\nu} + \partial_\nu h_{\beta\mu} - \partial_\beta h_{\mu\nu}\right). \tag{1.11b}$$

Substituting Eq. (1.10) into Eq. (1.8) yields

$$R_{\mu\nu} = R^{(1)}_{\mu\nu}[h_{\mu\nu}] + R^{(2)}_{\mu\nu}[h_{\mu\nu}] + \mathcal{O}(h(\partial h)^2 + h^2\partial^2 h), \tag{1.12}$$

with

$$R^{(1)}_{\mu\nu}[h_{\mu\nu}] := \partial_\alpha \Gamma^{(1)\alpha}{}_{\mu\nu}[h_{\mu\nu}] - \partial_\nu \Gamma^{(1)\alpha}{}_{\alpha\mu}[h_{\mu\nu}], \tag{1.13a}$$

$$\begin{aligned} R^{(2)}_{\mu\nu}[h_{\mu\nu}] := {}& \partial_\alpha \Gamma^{(2)\alpha}{}_{\mu\nu}[h_{\mu\nu}] - \partial_\nu \Gamma^{(2)\alpha}{}_{\alpha\mu}[h_{\mu\nu}] \\ & + \Gamma^{(1)\sigma}{}_{\mu\nu}[h_{\mu\nu}]\Gamma^{(1)\alpha}{}_{\sigma\alpha}[h_{\mu\nu}] - \Gamma^{(1)\sigma}{}_{\mu\alpha}[h_{\mu\nu}]\Gamma^{(1)\alpha}{}_{\sigma\nu}[h_{\mu\nu}]. \end{aligned} \tag{1.13b}$$

Then, substituting Eqs. (1.6) and (1.12) into Eq. (1.7) we obtain the following expression for the Ricci scalar:

$$R = \eta^{\mu\nu} R^{(1)}_{\mu\nu}[h_{\mu\nu}] + \eta^{\mu\nu} R^{(2)}_{\mu\nu}[h_{\mu\nu}] - h^{\mu\nu} R^{(1)}_{\mu\nu}[h_{\mu\nu}] + \mathcal{O}(h(\partial h)^2 + h^2\partial^2 h). \tag{1.14}$$

Using the definitions (1.11) for the Christoffel symbols and after some algebraic manipulations, we obtain the Ricci scalar in terms of the components of the metric perturbation $h_{\mu\nu}$:

$$R = R^{(1)} + R^{(2)} + \mathcal{O}(h(\partial h)^2 + h^2\partial^2 h), \tag{1.15}$$

where

$$R^{(1)} = \Box h + \partial^\alpha \partial^\beta h_{\alpha\beta}, \tag{1.16}$$

$$\begin{aligned} R^{(2)} = {}& \partial_\mu V^\mu - \frac{1}{4}(\partial^\mu h^{\alpha\beta})(\partial_\mu h_{\alpha\beta}) - \frac{1}{2}(\partial^\mu h^{\alpha\beta})(\partial_\alpha h_{\mu\beta}) + (\partial_\alpha h^{\alpha\beta})(\partial^\mu h_{\mu\beta}) \\ & - \frac{1}{4}(\partial^\beta h)(\partial_\beta h). \end{aligned} \tag{1.17}$$

Here $h := \eta^{\mu\nu} h_{\mu\nu}$ represents the trace of the metric $h_{\mu\nu}$ and $V^\mu := h_{\alpha\beta}\partial^\mu h^{\alpha\beta} - 2h^{\mu\beta}\partial^\alpha h_{\alpha\beta} + h^{\mu\nu}\partial_\nu h$. Note that when the Ricci scalar (1.15) is introduced into the action (1.1), the terms $\Box h + \partial^\alpha\partial^\beta h_{\alpha\beta}$ and $\partial_\mu V^\mu$ contribute only boundary terms. Thus, considering that

$$\sqrt{-g} = 1 + \frac{1}{2}h + \mathcal{O}(h^2), \tag{1.18}$$

and discarding boundary terms, we get

$$\begin{aligned} \int R\sqrt{-g}\, d^4x = \int \Big[& \frac{1}{2}(\partial_\alpha h^{\alpha\beta})(\partial^\mu h_{\mu\beta}) - \frac{1}{4}(\partial^\mu h^{\alpha\beta})(\partial_\mu h_{\alpha\beta}) \\ & + \frac{h}{4}\left(\Box h + 2\partial^\alpha\partial^\beta h_{\alpha\beta}\right)\Big] d^4x + \mathcal{O}(h(\partial h)^2). \end{aligned} \tag{1.19}$$

Note that the terms of order h vanishes as a consequence of the fact that $\eta_{\mu\nu}$ is a solution of the Einstein field equations. On the other hand, substituting $g^{\mu\nu}$ from Eq. (1.6a) in Eq. (1.2), we obtain

$$\begin{aligned} \mathcal{L}_M\sqrt{-g} = -\sum_{k=1}^{K} \Big[& \eta^{\mu\nu}\partial_\mu\overline{\phi}_k\partial_\nu\phi_k + m_0^2|\phi_k|^2 \\ & - \frac{1}{2}\left(h\eta^{\mu\nu}\partial_\mu\overline{\phi}_k\partial_\nu\phi_k + m_0^2 h|\phi_k|^2 - 2h^{\mu\nu}\partial_\mu\overline{\phi}_k\partial_\nu\phi_k\right)\Big] + \mathcal{O}(h^2|\partial\phi_k|^2). \end{aligned} \tag{1.20}$$

Therefore, using the expression (1.18), we find

$$\int \mathcal{L}_M \sqrt{-g} d^4x = -\sum_{k=1}^{K} \Bigg\{ \int \left(\eta^{\mu\nu} \partial_\mu \overline{\phi}_k \partial_\nu \phi_k + m_0^2 |\phi_k|^2 \right) d^4x$$
$$+ \int h^{\mu\nu} \left[\partial_\mu \overline{\phi}_k \partial_\nu \phi_k - \frac{1}{2} \eta_{\mu\nu} \left(\eta^{\alpha\beta} \partial_\alpha \phi_k \partial_\beta \overline{\phi}_k + m_0^2 |\phi_k|^2 \right) \right] d^4x \Bigg\} + \mathcal{O}(h^2 |\partial \phi_k|^2). \quad (1.21)$$

Introducing the stress-energy tensor for the k-th scalar field $T^{(k)}$, whose components are defined by

$$T^{(k)}_{\mu\nu} := \frac{1}{2} \left[\partial_\mu \overline{\phi}_k \partial_\nu \phi_k + \partial_\mu \phi_k \partial_\nu \overline{\phi}_k - \eta_{\mu\nu} \left(\eta^{\alpha\beta} \partial_\alpha \phi_k \partial_\beta \overline{\phi}_k + m_0^2 |\phi_k|^2 \right) \right], \quad (1.22)$$

we can rewrite the expression (1.21) as follows:

$$\int \mathcal{L}_M \sqrt{-g} d^4x = -\sum_{k=1}^{K} \int \left(\partial^\mu \phi_k \partial_\mu \overline{\phi}_k + m_0^2 |\phi_k|^2 - h^{\mu\nu} T^{(k)}_{\mu\nu} \right) d^4x + \mathcal{O}(h^2 |\partial \phi_k|^2), \quad (1.23)$$

where we set $\partial^\mu \phi_k := \eta^{\mu\nu} \partial_\nu \phi_k$. Using Eqs. (1.19) and (1.23), we can express the action given by Eq. (1.1) in the following form:

$$S[g, \phi_1, \ldots, \phi_K] = S_H[h] + S_M[h, \phi_1, \ldots, \phi_K] + \mathcal{O}(h(\partial h)^2 + h^2 |\partial \phi_k|^2), \quad (1.24)$$

where the functionals S_H y S_M are define by

$$S_H[h_{\mu\nu}] := \frac{1}{16\pi G} \int \left[\frac{1}{2} (\partial_\alpha h^{\alpha\beta})(\partial^\mu h_{\mu\beta}) - \frac{1}{4} (\partial^\mu h^{\alpha\beta})(\partial_\mu h_{\alpha\beta}) + \frac{h}{4} \left(\Box h + 2 \partial^\alpha \partial^\beta h_{\alpha\beta} \right) \right] d^4x, \quad (1.25a)$$

$$S_M[h_{\mu\nu}, \phi_1, \ldots, \phi_K] := -\sum_{k=1}^{K} \int \left(\partial^\mu \phi_k \partial_\mu \overline{\phi}_k + m_0^2 |\phi_k|^2 - h^{\mu\nu} T^{(k)}_{\mu\nu} \right) d^4x. \quad (1.25b)$$

In a next step, we decompose the components of the metric perturbation $h_{\mu\nu}$ as follows [64, 87]

$$h_{00} = -2\Phi, \quad (1.26)$$
$$h_{0j} = \partial_j B - A_j, \quad (1.27)$$
$$h_{ij} = -2\delta_{ij} \Psi + 2 \left(\partial_i \partial_j E - \frac{1}{3} \delta_{ij} \Delta E \right) + 2 \partial_{(i} F_{j)} + W_{ij}, \quad (1.28)$$

where Φ, Ψ, B, E are scalar functions; A_j, F_j are the components of divergence-free vector fields; and W_{ij} are components of a tensor field satisfying the conditions $\delta^{ij} W_{ij} = 0$ and $\partial^i W_{ij} = 0$. Furthermore, recalling that in general relativity the action is invariant under diffeomorphisms, an infinitesimal transformation of coordinates modifies the components $h_{\mu\nu}$ according to

$$h_{\mu\nu} \mapsto h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \tag{1.29}$$

with ξ_μ denoting the components of an arbitrary vector field ξ. Using the Helmholtz theorem one can further decompose the spatial components ξ_j as follows

$$\xi_j = \partial_j X + Y_j, \tag{1.30}$$

where X is a scalar function and Y_j are the components of a divergence-free vector field. Therefore, using the coordinate transformation freedom (1.29) and expressions (1.26), we obtain

$$\widetilde{h}_{00} = -2(\Phi + \partial_0 \xi_0), \tag{1.31}$$

$$\widetilde{h}_{0j} = \partial_j(B + \xi_0) - A_j + \partial_0(Y_j + \partial_j X), \tag{1.32}$$

$$\widetilde{h}_{ij} = -2\delta_{ij}\Psi + 2\left(\partial_i\partial_j E - \frac{1}{3}\delta_{ij}\Delta E\right) + 2\partial_{(i}F_{j)} + W_{ij} \tag{1.33}$$

$$+ \frac{2}{3}\delta_{ij}\Delta X + 2\left(\partial_i\partial_j X - \frac{1}{3}\delta_{ij}\Delta X\right) + 2\partial_{(i}Y_{j)}. \tag{1.34}$$

Choosing $\xi_0 = -B$, $X = -E$, and $Y_j = -F_j$ yields

$$\widetilde{h}_{00} = -2\widetilde{\Phi}, \tag{1.35}$$

$$\widetilde{h}_{0j} = -\widetilde{A}_j, \tag{1.36}$$

$$\widetilde{h}_{ij} = -2\widetilde{\Psi}\delta_{ij} + W_{ij}, \tag{1.37}$$

where

$$\widetilde{\Phi} := \Phi + \partial_0 \xi_0, \tag{1.38}$$

$$\widetilde{A}_j := -A_j + \partial_0(Y_j + \partial_j X), \tag{1.39}$$

$$\widetilde{\Psi} := \Psi + \frac{1}{3}\Delta X. \tag{1.40}$$

We conclude that it is possible to set $B = E = 0$ and $F_j = 0$. This particular choice is known as the Newtonian gauge [87]. Thus, substituting the perturbations (1.35) in the functionals (1.25a) we obtain

$$S_H[h_{\mu\nu}] = \frac{1}{16\pi G}\left(S_H^{\text{scal}}[\Phi,\Psi] + S_H^{\text{vec}}[A_j] + S_H^{\text{ten}}[W_{ij}]\right), \tag{1.41}$$

where

$$S_H^{\text{scal}}[\Phi,\Psi] := 2\int\int\left[3\Psi\frac{\partial^2\Psi}{\partial t^2} + \Psi\Delta(2\Phi - \Psi)\right] dt d^3x, \tag{1.42a}$$

$$S_H^{\text{vec}}[A_j] := \frac{1}{2}\int\int A^j \Delta A_j dt d^3x \tag{1.42b}$$

$$S_H^{\text{ten}}[W_{ij}] := -\frac{1}{4}\int\int W^{ij}\Box W_{ij} dt d^3x. \tag{1.42c}$$

Similarly, introducing expressions (1.35) in Eq. (1.25b) we obtain

$$S_M[h_{\mu\nu}, \phi_1, \ldots, \phi_K] = -\sum_{k=1}^{K} \int\int \left(\partial^\mu \phi_k \partial_\mu \overline{\phi}_k + m_0^2 |\phi_k|^2\right) dt d^3x$$
$$-2\sum_{k=1}^{K} \int\int \left(\Phi T_{00}^{(k)} + \Psi \delta^{ij} T_{ij}^{(k)} - A^j T_{0j}^{(k)} - \frac{1}{2} W^{ij} T_{ij}^{(k)}\right) dt d^3x \tag{1.43}$$

with

$$T_{00}^{(k)} = \frac{1}{2}\left(\left|\frac{\partial \phi_k}{\partial t}\right|^2 + |\nabla \phi_k|^2 + m_0^2 |\phi_k|^2\right), \tag{1.44a}$$

$$T_{0j}^{(k)} = \mathrm{Re}\left(\frac{\partial \overline{\phi}_k}{\partial t} \partial_j \phi^{(k)}\right), \tag{1.44b}$$

$$T_{ij}^{(k)} = \mathrm{Re}\left(\partial_i \overline{\phi}_k \partial_j \phi_k\right) + \left(\left|\frac{\partial \phi_k}{\partial t}\right|^2 - |\nabla \phi_k|^2 - m_0^2 |\phi_k|^2\right) \delta_{ij}. \tag{1.44c}$$

The action (1.24) with S_H and S_M given by Eqs. (1.41) and (1.43), respectively, describes the Einstein-Klein-Gordon theory in the weak-field approximation. In the next section, we further reduce this action by applying the slow-motion approximation.

1.2 Slow-motion approximation

We begin by recalling that, in the non-relativistic limit, the energy point particle of mass m_0 and linear momentum $\vec{p}$, can be approximated as follows:

$$E = \sqrt{|\vec{p}|^2 + m_0^2} \approx m_0 + \frac{|\vec{p}|^2}{2m_0}, \tag{1.45}$$

This shows that the energy is dominated by the rest mass. Motivated by this and the de Broglie's principle, we propose the following ansatz for the scalar fields [37]

$$\phi_k(t, x) = \frac{1}{\sqrt{2m_0}} e^{-im_0 t} \psi_k(t, x), \tag{1.46}$$

where ψ_k are scalar functions of $t \in \mathbb{R}$ and $x \in \mathbb{R}^3$ that, as we will see later, represent the wave functions. Substituting this into Eq. (1.44), we obtain:

$$T_{00}^{(k)} = \frac{1}{2}\left[\frac{1}{2m_0}\left|\frac{\partial \psi_k}{\partial t}\right|^2 + i\mathrm{Im}\left(\overline{\psi}_k \frac{\partial \psi_k}{\partial t}\right) + \frac{1}{2m_0} |\nabla \psi_k|^2 + m_0 |\psi_k|^2\right], \tag{1.47a}$$

$$T_{0j}^{(k)} = \frac{1}{2}\mathrm{Re}\left(\frac{1}{m_0}\frac{\partial \overline{\psi}_k}{\partial t} \partial_j \psi_k + i\psi_k \partial_j \psi_k\right), \tag{1.47b}$$

$$T_{ij}^{(k)} = \frac{1}{2m_0}\mathrm{Re}\left(\partial_i \overline{\psi}_k \partial_j \psi_k\right)$$
$$+\left[\frac{1}{2m_0}\left|\frac{\partial \psi_k}{\partial t}\right|^2 + i\mathrm{Im}\left(\overline{\psi}_k \frac{\partial \psi_k}{\partial t}\right) - \frac{1}{2m_0} |\nabla \psi_k|^2\right] \delta_{ij}. \tag{1.47c}$$

Once again, motivated by the dispersion relation (1.45), we introduce a dimensionless small parameter ε, which leads to the following scaling relations:

$$\psi \sim \frac{m_0}{G}\varepsilon, \tag{1.48a}$$

$$\partial_t \psi \sim \varepsilon^{1/2}\partial_j \psi \sim \varepsilon m_0 \psi, \tag{1.48b}$$

$$h_{\mu\nu} \sim \varepsilon, \tag{1.48c}$$

$$\partial_t h_{\mu\nu} \sim \varepsilon^{1/2}\partial_j h_{\mu\nu} \sim \varepsilon m_0 h_{\mu\nu}. \tag{1.48d}$$

From Eq. (1.43), we observe that the term $A^j T_{0j}^{(k)}$ is of order $m_0^2\varepsilon^{7/2}/G$, whereas the terms $\Psi\delta^{ij}T_{ij}^{(k)}$ and $W^{ij}T_{ij}^{(k)}$ are of order $m_0^2\varepsilon^4/G$. Therefore, retaining only the leading-order terms in ε, namely those of order $m_0^2\varepsilon^3/G$, we obtain:

$$S_M[h_{\mu\nu}, \psi_1, \ldots, \psi_K] = \sum_{k=1}^{K} \int\int \left[\overline{\psi}_k \left(i\frac{\partial}{\partial t} + \frac{1}{2m_0}\Delta\right)\psi_k - m_0\Phi|\psi_k|^2\right] dtd^3x. \tag{1.49}$$

Similarly, from Eq. (1.41), the gravitational contribution reduces to

$$S_H[h_{\mu\nu}] = \frac{1}{16\pi G}\int\int \left[2\Psi\Delta(2\Phi - \Psi) + \frac{1}{2}A^j\Delta A_j + \frac{1}{4}W^{ij}\Delta W_{ij}\right] dtd^3x. \tag{1.50}$$

Finally, substituting expressions (1.49) and (1.50) in Eq. (1.24) we arrive at:

$$\begin{aligned} S[g, \phi_1, \ldots, \phi_K] = {}& \frac{1}{8\pi G}\int\int \left[\frac{1}{2}A^j\Delta A_j + \frac{1}{4}W^{ij}\Delta W_{ij} + \Psi\Delta\left(2\Phi - \Psi\right)\right] dtd^3x \\ &+ \sum_{k=1}^{K}\int\int \left[\frac{i}{2}\left(\overline{\psi}_k\frac{\partial \psi_k}{\partial t} - \psi_k\frac{\partial \overline{\psi}_k}{\partial t}\right) + \frac{1}{2m_0}\overline{\psi}_k\Delta\psi_k - m_0\Phi|\psi_k|^2\right] dtd^3x + \mathcal{O}(\varepsilon^{7/2}). \end{aligned} \tag{1.51}$$

The variation of this action with respect to A^j and W^{ij} leads to the equations $\Delta A_j = 0$ and $\Delta W_{ij} = 0$, respectively. Assuming that A_j and W_{ij} vanish at infinity, this implies that $A_j = 0$ and $W_{ij} = 0$. Thus ignoring the terms of order $\varepsilon^{7/2}$ or higher, we are left with the effective action

$$\begin{aligned} S_{eff}[\Phi, \Psi, \psi_k] = \int\int \Bigg\{ & \frac{1}{8\pi G}\Psi\Delta\left(2\Phi - \Psi\right) \\ &+ \sum_{k=1}^{n}\left[\frac{i}{2}\left(\overline{\psi}_k\frac{\partial \psi_k}{\partial t} - \psi_k\frac{\partial \overline{\psi}_k}{\partial t}\right) + \frac{1}{2m_0}\overline{\psi}_k\Delta\psi_k - m_0\Phi|\psi_k|^2\right]\Bigg\} dtd^3x. \end{aligned} \tag{1.52}$$

Next, the variation with respect to Ψ gives the equation $\Delta(\Phi - \Psi) = 0$ which, under the assumption that Φ and Ψ vanish at infinity, implies $\Psi = \Phi =: U$. Varying the action with respect to each $\overline{\psi}_k$ results in a Schrödinger equation for each ψ_k

$$i\frac{\partial \psi_k}{\partial t} = -\frac{1}{2m_0}\Delta\psi_k + m_0 U\psi_k. \tag{1.53a}$$

Finally, the variation with respect to U leads to the Poisson equation:

$$\Delta U = 4\pi G m_0 \sum_{k=1}^{K} |\psi_k|^2 . \tag{1.53b}$$

We refer to the system formed by Eqs. (1.53) as the multi-field Schrödinger-Poisson system.

It is important to remark that we have assumed the existence of weak-field solutions $g_{\mu\nu}(\lambda), \phi_k(\lambda)$ to the Einstein-Klein-Gordon system that satisfy the rescaling (1.48). An interesting and relevant question is whether such solutions actually exist.

1.3 The multi-field Schrödinger-Poisson system

In this section we study the multi-field Schrödinger-Poisson system (1.53). As shown in the previous section, this equations arises as the Newtonian limit of the Einstein-Klein-Gordon theory, and it can be used to model a system consisting of K indistinguishable, uncorrelated and spinless particles of mass m_0, whose only interaction is through the gravitational potential U generated by the particles themselves [70].

Introducing the following notation

$$\psi := \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_K \end{pmatrix}, \quad |\psi|^2 := \sum_{j=1}^{K} |\psi_j|^2. \tag{1.54}$$

we can rewrite the system (1.53) as follows

$$i\frac{\partial \psi(t,x)}{\partial t} = \left[-\frac{1}{2m_0}\Delta + m_0 U(t,x)\right]\psi(t,x), \tag{1.55a}$$

$$\Delta U(t,x) = 4\pi m_0 G |\psi(t,x)|^2. \tag{1.55b}$$

To uniquely determine the gravitational potential, we impose the condition $U(x) \to 0$ as $|x| \to \infty$. In what follows, we first discuss the symmetries of the system and their associated conserved quantities, and then proceed to search for stationary solutions using an elegant variational principle.

1.3.1 Symmetries of the multi-field Schrödinger-Poisson system

Using integration by parts, the action (1.52), with $\Phi = \Psi = U$, can be written as follows

$$\widetilde{S}_{eff}[\psi, \psi^*, U] = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \mathcal{L}_{SP}[\psi, \psi^*, U] d^3x dt, \tag{1.56}$$

where the Lagrangian $\mathcal{L}_{SP}$ is defined by

$$\mathcal{L}_{SP}[\psi, \psi^*, U] := \frac{i}{2}\left(\psi^* \frac{\partial \psi}{\partial t} - \frac{\partial \psi^*}{\partial t}\psi\right) - \frac{1}{2m_0}|\nabla\psi|^2 - m_0 U|\psi|^2 - \frac{1}{8\pi G}|\nabla U|^2. \tag{1.57}$$

Here and in the following $*$ denotes the conjugate transposed and we use the notation

$$|\nabla\psi|^2 := \sum_{j=1}^{K} |\nabla\psi_j|^2. \tag{1.58}$$

It is straightforward to verify that varying the action (1.56) with respect to ψ^* yields the Schrödinger-Poisson system (1.53).

Using Noether's theorem and the theory developed in [46] one can prove that the action (1.56) remains invariant under time and space translations, rotations, rigid unitary transformations and Galilean boosts, which leads to the following conserved quantities (see [69] for a systematicstudy of the symmetries of the Schrödinger-Poisson system):

(i) The invariance under time translations implies the conservation of the total energy:

$$\mathcal{E}[\psi] := \frac{1}{2m_0}\int_{\mathbb{R}^3} |\nabla\psi|^2 d^3x + \frac{m_0}{2}\int_{\mathbb{R}^3} U|\psi|^2 d^3x. \tag{1.59}$$

In the following, we call this the energy functional. It will be of great importance to prove the existence of solutions of the multi-field Schrödinger-Poisson system.

(ii) Invariance under space translations lead to the conservation of momentum:

$$P[\psi] := \mathrm{Im}\int_{\mathbb{R}^3} \psi^*\nabla\psi d^3x. \tag{1.60}$$

Here, the integrand is a column vector whose components are $(\psi^*\nabla\psi)_j = \psi^*\partial_j\psi$, with $j = 1, 2, 3$.

(iii) Invariance with respect to rotations leads to the conservation of angular momentum:

$$L[\psi] := \mathrm{Im}\int_{\mathbb{R}^3} \left[x \wedge (\psi^*\nabla\psi)\right] d^3x, \tag{1.61}$$

where the symbol $\wedge$ denotes the vector product in $\mathbb{R}^3$.

(iv) The invariance under rigid unitary transformations results in the conservation of the scalar product of the components of ψ in $L^2(\mathbb{R}^3)$:

$$(\psi_j, \psi_k)_{L^2} := \int_{\mathbb{R}^3} \overline{\psi}_j\psi_k d^3x. \tag{1.62}$$

with $j, k = 1, 2, \ldots, K$. Note that if $j = k$, the conservation of the L^2-scalar product implies the conservation of the L^2-norm of the components, i.e.

$$\|\psi_j\|_2^2 := \int_{\mathbb{R}^3} |\psi_j(x)|^2 d^3x, \quad \text{for all } j = 1, 2, 3, \ldots, K. \tag{1.63}$$

Furthermore, since the matrix Q, whose matrix elements are $Q_{jk} := (\psi_j, \psi_k)_{L^2}$, is hermitian, it can be diagonalized by means of a rigid unitary transformation. Therefore, it is possible and sufficient to impose the following initial condition at $t = 0$:

$$(\psi_j, \psi_k) = \sqrt{N_jN_k}\delta_{jk}, \tag{1.64}$$

where we define $N_j := \|\psi_j\|_2^2$ and $N_k = \|\psi_k\|_2^2$.

(v) Galilean boost. Consider the transformation

$$t \mapsto t, \quad x \mapsto x - vt, \quad v \in \mathbb{R}^3, \tag{1.65}$$

under which the Lagrangian $\mathcal{L}_{SP}$ transforms as

$$\mathcal{L}_{SP} \mapsto \mathcal{L}_{SP} + \frac{i}{2}[\psi^* D_v \psi - (D_v \psi^*)\psi], \tag{1.66}$$

where $D_v := v \cdot \nabla$ is the directional derivative along the vector v. In this case, the action is not invariant. However, if we supplement the Galilean boost with the following global phase transformation of the wave function:

$$t \mapsto t, \tag{1.67a}$$
$$x \mapsto x - vt, \tag{1.67b}$$
$$\psi \mapsto e^{im_0\left(v\cdot x - \frac{1}{2}|v|^2 t\right)}\psi, \tag{1.67c}$$

one can verify that the action (1.56) remains invariant. Thus, the invariance under this "Galilean boost" (1.67), gives the following conservation quantity:

$$C[\psi] := \int_{\mathbb{R}^3} x|\psi|^2 d^3x - tP[\psi], \tag{1.68}$$

which implies that the center of mass $\int_{\mathbb{R}^3} x|\psi|^2 d^3x$ follows a free trajectory.

As we said before, these conserved quantities will be useful to prove the existence of stationary solutions of the Schrödinger-Poisson system (1.53) and to analyze their stability.

1.3.2 Generalization to include a central point mass

As mentioned in the introduction, scalar fields have been proposed as dark matter candidates. In this context, the multi-field Schrödinger-Poisson system (1.55) can be used to model the dark matter halo of a galaxy. In this section, we generalize this system to include the gravitational influence of a central point mass M, representing a black hole located at the center of the galaxy. This setting serves as a simple toy model for describing the dynamics of a scalar field configurations in the presence of a black hole [25]. With this modification, the multi-field Schrödinger-Poisson system becomes

$$i\frac{\partial \psi(t,x)}{\partial t} = \left[-\frac{1}{2m_0}\Delta + m_0 U(t,x)\right]\psi(t,x), \tag{1.69a}$$
$$\Delta U(t,x) = 4\pi G\left[m_0|\psi(t,x)|^2 + M\delta(x)\right], \tag{1.69b}$$

where here $\delta(x)$ refers to the Dirac delta distribution, defined by

$$\int_{\mathbb{R}^3} \delta(x) f(x) d^3x = f(0), \tag{1.70}$$

for all test functions $f \in C_0^\infty(\mathbb{R}^3)$, that is, al functions $f : \mathbb{R}^3 \to \mathbb{R}$ which are infinitely differentiable and have compact support. The model is expected to be valid in the weak-field regime, that is, sufficiently far from the event horizon, where Newtonian gravity provides an accurate approximation; closer to the black hole, general relativity would be required.

For what follows, it is important to note that the presence of a fixed point mass at the origin breaks spatial translation and Galilean boost symmetries (see points (*ii*) and (*v*) of the previous section), since the gravitational potential possesses a distinguished point at the origin. Nevertheless, the system remains invariant under time translations and spatial rotations, which, by Noether's theorem, implies that the total energy, given by the energy functional (1.59), and the angular momentum, given by Eq. (1.61), are conserved quantities.

To incorporate the effect of the central mass into the energy of the system, we extend the energy functional to the form

$$\mathcal{E}[\psi] = \frac{1}{2m_0}\int_{\mathbb{R}^3} |\nabla\psi|^2 d^3x + \frac{m_0}{2}\int_{\mathbb{R}^3} U|\psi|^2 d^3x + \frac{M}{2}\int_{\mathbb{R}^3} U\delta(x) d^3x. \tag{1.71}$$

Special attention must be paid to the last term on the right-hand side of Eq. (1.71). By definition of the Dirac delta distribution, this contribution reduces to the evaluation of the gravitational potential at the origin, namely

$$\int_{\mathbb{R}^3} U\delta(x) d^3x = U(t,0). \tag{1.72}$$

The gravitational potential is obtained by solving the Poisson equation (1.69b), which leads to

$$U(t,x) = -Gm_0\int_{\mathbb{R}^3} \frac{|\psi(t,y)|^2}{|x-y|} d^3y - GM\int_{\mathbb{R}^3} \frac{\delta(y)}{|x-y|} d^3y. \tag{1.73}$$

However, from this expression it is clear that $U(t,0)$ is ill-defined: the second integral diverges at $x=0$. To address this, we approximate the delta distribution by means of a Dirac sequence

$$\rho_\lambda(x) = \frac{1}{\lambda^3}\rho\left(\frac{x}{\lambda}\right), \tag{1.74}$$

where ρ is a mollifier [11] and $\rho_\lambda \to \delta$ as $\lambda \to 0$. After the change of variables $z = y/\lambda$, the singular integral can be written as

$$\int_{\mathbb{R}^3} \frac{\delta(y)}{|x-y|} d^3y = \lim_{\lambda\to 0}\int_{\mathbb{R}^3} \frac{\rho(z)}{|x-\lambda z|} d^3z. \tag{1.75}$$

Substituting this equation into Eq. (1.73) and evaluating at $x=0$, we obtain

$$U(t,0) = -Gm_0\int_{\mathbb{R}^3} \frac{|\psi(t,y)|^2}{|y|} d^3y - GM\lim_{\lambda\to 0}\frac{1}{\lambda}\int_{\mathbb{R}^3} \frac{\rho(z)}{|z|} d^3z. \tag{1.76}$$

The last term diverges as $\lambda \to 0$, but this divergence can be given a physical interpretation: it originates from idealizing the central black hole as a point mass. The parameter λ can be understood as an effective radius, which is small compared to galactic scales but not exactly zero. Moreover, since this contribution does not depend on ψ, the equation of motion are unaffected and the term can be discarded. Therefore, substituting Eq. (1.76) (ignoring the divergence term of the right-hand side) and Eq. (1.73) into Eq. (1.71), the final expression for the energy functional, including the effects from the central mass, is given by

$$\mathcal{E}[\psi] = \frac{1}{2m_0}\int_{\mathbb{R}^3} |\nabla\psi|^2 d^3x - GMm_0\int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{|x|}d^3x - \frac{Gm_0^2}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|\psi(x)|^2|\psi(y)|^2}{|x-y|}d^3y d^3x. \tag{1.77}$$

1.4 Numerical example: spherical symmetric static solutions with one field

In order to present explicit numerical solutions of the system (1.69), it is convenient to eliminate the dependence on dimensional constants. For this purpose, we first rewrite the equations in dimensionless form

$$i\frac{\partial\widetilde{\psi}}{\partial\widetilde{t}}(\widetilde{t},\widetilde{x}) = [-\widetilde{\Delta} + \widetilde{U}(\widetilde{t},\widetilde{x})]\widetilde{\psi}(\widetilde{t},\widetilde{x}), \tag{1.78a}$$

$$\widetilde{\Delta}\widetilde{U}(\widetilde{t},\widetilde{x}) = |\widetilde{\psi}(\widetilde{t},\widetilde{x})|^2 + 4\pi\widetilde{M}\delta(\widetilde{x}), \tag{1.78b}$$

where we use the change of variables

$$\begin{aligned} &t = t_c\widetilde{t}/\Upsilon^2, \qquad x = d_c\widetilde{x}/\Upsilon, \\ &\psi = \Upsilon\widetilde{\psi}/\sqrt{d_c^3}, \quad U = 2\Upsilon v_c^2\widetilde{U}, \quad \widetilde{M} = M/4\pi\Upsilon m_0, \end{aligned} \tag{1.79}$$

with Υ an arbitrary positive dimensionless scale factor, $t \in \mathbb{R}$, $x \in \mathbb{R}^3$ and $v_c := d_c/t_c$ a characteristic velocity defined in terms of the characteristic distance and length

$$d_c := \frac{1}{8\pi G m_0^3} = 3.25812 \times 10^{-21}\left(\frac{M_{\text{Pl}}}{m_0}\right)^3 \text{GeV}^{-1} \tag{1.80}$$

$$t_c := \frac{1}{32\pi^2 G^2 m_0^5} = 2.59284 \times 10^{-22}\left(\frac{M_{\text{Pl}}}{m_0}\right)^5 \text{GeV}^{-1}, \tag{1.81}$$

where $M_{\text{Pl}} = 1/\sqrt{G} = 1.22089 \times 10^{19}$ GeV denotes the Planck mass. In order to simplify the notation, we shall omit the tilde and denote dimensionfull quantities with the superscript *phys* whenever necessary.

As a concrete example, in this section we consider the case $M = 0$ and a single scalar field, that is, we set $K = 1$. Furthermore, we focus on stationary states with spherical symmetry, which are described with the following ansatz

$$\psi(t,x) = e^{-i\omega t}f(r), \quad r := |x|, \tag{1.82}$$

where f is a real-valued scalar function and $\omega \in \mathbb{R}$ is a constant. Substituting this ansatz into Eqs. (1.78) (with $M = 0$) reduces the problem to the following system of

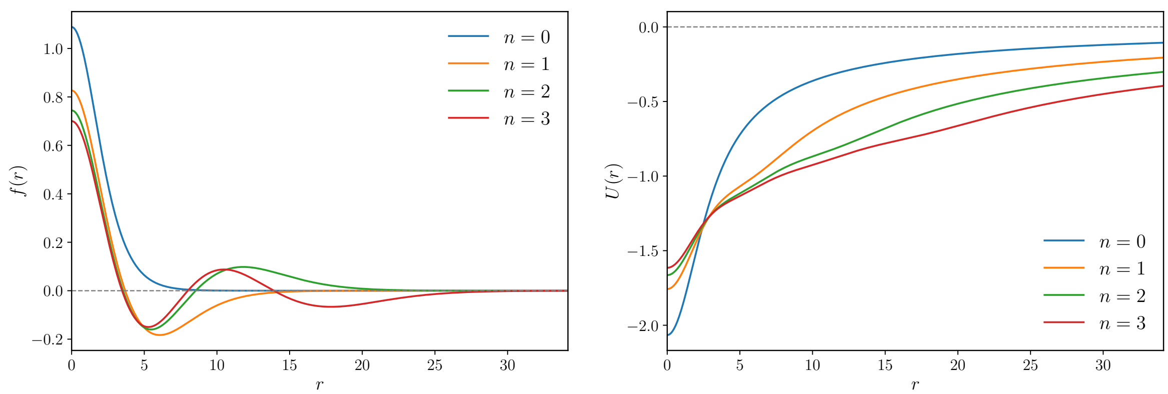

Figure 1.1: Profiles of the scalar field and gravitational potential. The left panel shows the scalar field for the ground state ($n = 0$) as well as the first ($n = 1$), second ($n = 2$) and third ($n = 3$) excited states. Similarly, the right panel shows the profile of the gravitational potential for the ground state ($n = 0$) and the first three excites states ($n = 1, 2, 3$).

radial equations:

$$-\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) + U(r)f = \omega f, \tag{1.83a}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) = f^2. \tag{1.83b}$$

Since these equations are singular at $r = 0$, regularity of the solutions require the boundary conditions

$$f(r = 0) = f_0, \quad \left.\frac{df}{dr}\right|_{r=0} = 0, \tag{1.84}$$

$$U(r = 0) = U_0, \quad \left.\frac{dU}{dr}\right|_{r=0} = 0, \tag{1.85}$$

and

$$f(r \to \infty) = 0, \tag{1.86}$$

$$U(r \to \infty) = 0, \tag{1.87}$$

where f_0 and U_0 are real parameters. Fig. 1.1 shows the numerical solution of system (1.83) for the ground state and the first three excited states. A more detailed discussion of the choice of the boundary conditions and the numerical method used for solving the system is given in Sec. 5.5 of Chapter 5.

It is worth to emphasize that in [53], Lieb proved the existence of the ground state solution to system (1.78) in the case $M = 0$ and $K = 1$, making use of variational methods. These methods will play an important role in this thesis, since in the following chapters we will extend them in order to establish the existence of solutions for arbitrary K and for central masses $M \geq 0$. In addition, in [62] the authors presented a numerical study of the radial system (1.83), and in [83] it was rigorously shown that there exist a family of solutions. This family can be indexed by a non-negative integer n, where the nth solution is characterized by the fact that the scalar field profile f possesses exactly n zeros.

Chapter 2

Mathematical tools

In preparation for what follows, we recall some definitions and useful results from functional analysis and the theory of group representation. This section provides a brief compilation of the main concepts and results that will be used in the subsequent chapters.

2.1 Basic definitions

Definition 2.1. A **metric space** (M, d) is a set M and a real valued function $d : M \times M \to \mathbb{R}$, which satisfies

(*i*) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$,

(*ii*) $d(x, y) = d(y, x)$,

(*iii*) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

The function d is called a metric on M.

Definition 2.2. A sequence (x_j) in a metric space (M, d) is said to converge to an element $x \in M$, if $d(x_j, x) \to 0$ as $j \to \infty$. It is usual to denote this by $x_j \to x$ or $\lim_{j\to\infty} x_j = x$.

Definition 2.3. A sequence (x_j) in a metric space (M, d) is called a **Cauchy sequence** if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $j, k \geq N$ implies $d(x_j, x_k) \leq \varepsilon$.

Remark 2.1. It is a well known result that every convergent sequence is a Cauchy sequence [67]. However, the converse does not hold in general.

Definition 2.4. A metric space (M, d) is **complete** if every Cauchy sequence in M converges to some element in M.

Definition 2.5. A **normed vector space** $(X, \|\cdot\|)$ is a vector space X, over $\mathbb{R}$ (or $\mathbb{C}$), and a function $\|\cdot\|$ from X to $\mathbb{R}$ which satisfies:

(*i*) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$, (*ii*) $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ and $\alpha \in \mathbb{R}$ (or $\mathbb{C}$),

(*iii*) $\|x + y\| \leq \|x\| + \|y\|$ for all x and y in X.

The function $\|\cdot\|$ is called a norm on X.

Theorem 2.1. *Every normed vector space $(X, \|\cdot\|)$ is a metric space, where the metric d is induced by the norm:*

$$d(x, y) := \|x - y\|_X, \quad \text{for all } x, y \in X. \tag{2.1}$$

Proof. It is straightforward to verify that the induced metric (2.1) satisfies the properties $(i) - (iii)$ of Definition 2.1. ■

Definition 2.6. A complete normed vector space is called a **Banach space**.

Definition 2.7. A complex vector space X is called an **inner product space** if there is a complex-valued function $(\cdot, \cdot)$ on $X \times X$ that satisfies the following conditions for all $x, y, z \in X$ and $\alpha \in \mathbb{C}$:

(*i*) $(x, x) \geq 0$ and $(x, x) = 0$ if and only if $x = 0$,

(*ii*) $(x, y + z) = (x, y) + (x, z)$,

(*iii*) $(x, \alpha y) = \alpha(x, y)$,

(*iv*) $(x, y) = \overline{(y, x)}$.

The function $(\cdot, \cdot)$ is called an inner product.

Theorem 2.2 (Cauchy-Schwarz inequality)**.** *Let X be a inner product space. Then,*

$$|(x, y)| \leq \|x\|_X \|y\|_X \quad \text{for all } x, y \in X. \tag{2.2}$$

Proof. See Chapter II in [68]. ■

Theorem 2.3. *Every inner product space X is a normed vector space with the induced norm*

$$\|x\| = \sqrt{(x, x)} \quad \text{for all } x \in X. \tag{2.3}$$

Proof. We only need to verify that the induced norm (2.3) satisfies the properties $(i) - (iii)$ of Definition 2.5. All of these properties, except the triangle inequality, follows intermediately from the properties $(i) - (iv)$ of Def. 2.7. Assume that $x, y \in X$. Then, by the Cauchy-Schwarz inequality

$$\|x + y\|^2 = \|x\|^2 + (x, y) + (y, x) + \|y\|^2 \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2, \tag{2.4}$$

and thus $\|x + y\| \leq \|x\| + \|y\|$. ■

Definition 2.8. An inner product space H is called a **Hilbert space** if the normed vector space $(H, \|\cdot\|)$, where $\|\cdot\|$ denotes the norm induced by the inner product, is a Banach space.

2.2 Weak convergence

One of the most useful concepts in the study of the existence of solutions to partial differential equations based on variational methods, is that of the *weak topology* of a Banach space, which, as its name suggests, is weaker then the topology induced by the norm. Proofs of the following results, as well as a more detailed study of this concept, can be found in [11].

Definition 2.9. Let $(X, \|\cdot\|)$ be a normed vector space. Then the dual space of X, denoted by X^*, is the space of all continuous linear functionals on X, i.e.

$$X^* := \{f : X \to \mathbb{R} : f \text{ linear and continuous}\}. \tag{2.5}$$

Remark 2.2. The dual space X^* is a normed vector space with (dual) norm defined by

$$\|f\|_{X^*} = \sup_{\substack{\|x\|\leq 1\\ x\in X}} |f(x)|. \tag{2.6}$$

Given $f \in X^*$ and $x \in X$, it is customary to write $\langle f, x\rangle$ instead of $f(x)$. Furthermore, it is well known [11] that X^* is a Banach space (even if X is not).

Definition 2.10. Let $(X, \|\cdot\|)$ be a normed vector space. The bidual X^{**} is the dual of X^* with norm

$$\|\xi\|_{X^{**}} = \sup_{\substack{\|f\|\leq 1\\ f\in X^*}} |\langle \xi, f\rangle|. \tag{2.7}$$

Moreover, there is a canonical injection $J : X \to X^{**}$ defined as follows: given $x \in X$, the map $f \mapsto \langle f, x\rangle$ is a continuous linear functional on X^*; thus it is an element of X^{**}, which we denote by Jx. We have

$$\langle Jx, f\rangle_{X^{**},\, X^*} = \langle f, x\rangle_{X^*,\, X} \quad \text{for all } x \in X \text{ and } f \in X^*. \tag{2.8}$$

Remark 2.3. It can be proven that J is linear and that J is an isometry, that is, $\|Jx\|_{X^{**}} = \|x\|_X$. It may be happen that J is not surjective from X onto X^{**}. However, it is convenient to identify X with a subspace of X^{**} using J.

Definition 2.11. Let X be a Banach space and let $J : X \to X^{**}$ be the canonical injection from X into X^{**}. The space X is said to be **reflexive** if J is surjective, that is, $J(X) = X^{**}$.

Remark 2.4. $L^p(\mathbb{R}^n)$ spaces with $1 < p < \infty$ and Hilbert spaces are reflexive [11] (see next section).

Definition 2.12. Let $(X, \|\cdot\|)$ be a Banach space. A sequence (x_j) in X is said to **converge weakly** to $x \in X$ if

$$\lim_{j\to\infty} f(x_j) = f(x), \tag{2.9}$$

for all f in the dual space X^* of X. In this case, we write

$$x_j \rightharpoonup x. \tag{2.10}$$

Remark 2.5. Strong convergence (norm convergence) $x_j \to x$ implies weak convergence $x_j \rightharpoonup x$. However, the converse is not necessary true, that is, weak convergence does not, in general, imply strong convergence.

The next two theorems are crucial for the remainder of this work and highlights the importance of weak convergence:

Theorem 2.4. *Assume that $x_j \rightharpoonup x$ weakly in a Banach space $(X, \|\cdot\|)$. Then*

(i) The sequence is bounded

$$\sup_{j\geq 1} \|x_j\| < \infty. \tag{2.11}$$

(ii) The norm $\|\cdot\| : X \to \mathbb{R}$ is weakly lower semi-continuous, that is

$$\|x\| \leq \liminf_{j\to\infty} \|x_j\|. \tag{2.12}$$

Theorem 2.5. *Let $(X, \|\cdot\|)$ be a reflexive Banach space and let (x_j) be a bounded sequence in X, that is,*

$$M := \sup_{j\geq 1} \|x_j\| < \infty. \tag{2.13}$$

Then there exist a subsequence (x_{j_k}) and an element $x \in X$ such that $x_{j_k} \rightharpoonup x$ weakly in X. Moreover, $\|x\| \leq M$.

Definition 2.13. A subset $A \subset X$ of a Banach space X is called **weakly closed** if, for every sequence (x_j) in A that converges weakly to some $x \in X$, it follows that $x \in A$.

Remark 2.6. If a subset $A \subset X$ is weakly closed, then it is also closed (in the norm topology). However, a subset A that is closed is not necessary weakly closed.

Lemma 2.1. *The following subsets of X are weakly closed:*

(i) The closed ball $B_R(y) := \{x \in X : \|x - y\| \leq N\}$ for any $y \in X$ and $R > 0$,

(ii) Any closed subspace of X,

(iii) Any closed and convex subset of X.

Proposition 2.1. *Assume that $(X, \|\cdot\|)$ is a uniformly convex Banach space. Let (x_j) be a sequence in X such that $x_j \rightharpoonup x$ weakly in X and*

$$\limsup_{j\to\infty} \|x_j\| \leq \|x\|. \tag{2.14}$$

Then $x_j \to x$ strongly.

Proof. See Proposition 3.32 in [11]. ■

Remark 2.7. The $L^p(\mathbb{R}^n)$ spaces are uniformly convex for $1 < p < \infty$, and every Hilbert space is uniformly convex as well [11], so that Proposition 2.1 applies to these cases.

Remark 2.8. By Theorem 2.4, it always holds that $\|x\| \leq \liminf_{j\to\infty} \|x_j\|$ if $x_j \rightharpoonup x$ weakly. Therefore, the condition (2.14) is equivalent to

$$\lim_{j\to\infty} \|x_j\| = \|x\|, \tag{2.15}$$

which, according to Proposition 2.1, yields strong convergence: $x_j \to x$ in X.

2.3 L^p-spaces

We present some important results of the L^p spaces. For $1 \leq p < \infty$ we define (see for example [11]):

$$L^p(\mathbb{R}^n) := \{u : \mathbb{R}^n \to \mathbb{R} \mid u \text{ is measurable and } |u|^p \text{ is Lebesgue integrable}\}, \tag{2.16}$$

with the norm

$$\|u\|_{L^p(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |u(x)|^p d^n x\right)^{1/p}. \tag{2.17}$$

The space $L^\infty(\mathbb{R}^n)$ is defined as

$$L^\infty(\mathbb{R}^n) := \left\{u : \mathbb{R}^n \to \mathbb{R} \;\middle|\; \begin{array}{l} f \text{ is measurable and there is a constant } C \\ \text{such that } |u(x)| \leq C \text{ a.e. on } \mathbb{R}^n \end{array}\right\}, \tag{2.18}$$

with

$$\|u\|_{L^\infty(\mathbb{R}^n)} = \inf\{C : |u(x)| \leq C \text{ a.e. on } \mathbb{R}^n\}. \tag{2.19}$$

Here and in the following, “a.e.” refers to almost everywhere, that is, everywhere except on a zero-measure set.

Definition 2.14. Let $m \in \mathbb{N}$. Then, for all $1 \leq p \leq \infty$, the space $L^p(\mathbb{R}^n, \mathbb{C}^m)$ is defined as the set of measurable functions $u : \mathbb{R}^n \to \mathbb{C}^m$, such that $|u| \in L^p(\mathbb{R}^n)$, where $|u|$ denotes de euclidean norm of u in $\mathbb{C}^m$. The norm on this space is defined by

$$\|u\|_{L^p(\mathbb{R}^n,\mathbb{C}^m)} := \||u|\|_{L^p(\mathbb{R}^n)}, \quad \text{for } 1 \leq p \leq \infty. \tag{2.20}$$

For the following, we will often abbreviate the norm as $\|u\|_p$ when there is no risk of confusion.

Remark 2.9. For $p = 2$ the space $L^2(\mathbb{R}^n, \mathbb{C}^m)$ is a Hilbert space with the inner product

$$(u, v)_{L^2} := \int_{\mathbb{R}^n} u^*(x) v(x) d^n x. \tag{2.21}$$

Proposition 2.2 (Hölder's inequality)**.** *Let $u_1, u_2, \ldots, u_k$ be functions such that*

$$u_i \in L^{p_i}(\mathbb{R}^n, \mathbb{C}), \quad 1 \leq i \leq k \tag{2.22}$$

with

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_k} \leq 1. \tag{2.23}$$

Then the product $u = u_1 u_2 \cdots u_k$ belongs to $L^p(\mathbb{R}^n, \mathbb{C})$ and

$$\|u\|_p \leq \|u_1\|_{p_1} \|u_2\|_{p_2} \cdots \|u_k\|_{p_k}. \tag{2.24}$$

Proof. See chapter 4 in [11]. ∎

Corollary 2.1 (Interpolation)**.** *Let $1 \leq p \leq q \leq \infty$ and assume that $u \in L^p(\mathbb{R}^n, \mathbb{C}^m) \cap L^q(\mathbb{R}^n, \mathbb{C}^m)$. Then $u \in L^r(\mathbb{R}^n, \mathbb{C}^m)$ for each $p \leq r \leq q$ and the following inequalities are satisfied*

$$\|u\|_r \leq \|u\|_p^{\alpha} \|u\|_q^{1-\alpha} \leq \alpha \|u\|_p + (1-\alpha)\|u\|_q, \tag{2.25}$$

where $\alpha \in [0,1]$ is such that

$$\frac{\alpha}{p} + \frac{1-\alpha}{q} = \frac{1}{r}. \tag{2.26}$$

Proof. If $u \in L^p(\mathbb{R}^n, \mathbb{C}^m) \cap L^q(\mathbb{R}^n, \mathbb{C}^m)$, then $|u| \in L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$. Since the corollary is valid for any function belonging to the space $L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ (see [11]) and $\|u\|_{L^p(\mathbb{R}^n, \mathbb{C}^m)} = \||u|\|_{L^p(\mathbb{R}^n)}$, the corollary is also valid for $u \in L^p(\mathbb{R}^n, \mathbb{C}^m) \cap L^q(\mathbb{R}^n, \mathbb{C}^m)$. ∎

Proposition 2.3 (Hardy-Littlewood-Sobolev inequality)**.** *Let $0 < \lambda < n$ and suppose that $u \in L^p(\mathbb{R}^n, \mathbb{C}^m)$, $v \in L^q(\mathbb{R}^n, \mathbb{C}^m)$ with $p^{-1} + q^{-1} + \lambda n^{-1} = 2$ and $1 < p, q < \infty$. Then*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)||v(y)|}{|x-y|^{\lambda}} d^n x d^n y \leq C \|u\|_p \|v\|_q. \tag{2.27}$$

Proof. The proof and a complete discussion about the sharpness of the constant C can be found in [42] for the case $m = 1$. Since, by Definition 2.14, $|u| \in L^p(\mathbb{R}^n)$ and $|v| \in L^q(\mathbb{R}^n)$, the result is also valid for arbitrary m. ∎

Proposition 2.4 (Young's convolution inequality)**.** *Let $1 \leq p, q, r \leq \infty$ and $p^{-1} + q^{-1} + r^{-1} = 2$. Let $u \in L^p(\mathbb{R}^n)$, $v \in L^q(\mathbb{R}^n)$, and $w \in L^r(\mathbb{R}^n)$. Then, there exists a constant $C_{p,q,r;n} > 0$ such that*

$$\left| \int_{\mathbb{R}^n} u(x) v(x-y) w(x) d^n x d^n y \right| \leq C_{p,q,r;n} \|u\|_p \|v\|_q \|w\|_r. \tag{2.28}$$

Proof. The proof can be found in [67]. ∎

2.4 Sobolev spaces

Definition 2.15. Let $s \geq 1$ be an integer. Then, the **Sobolev space** $W^{s,p}(\mathbb{R}^n, \mathbb{C}^m)$ is defined as

$$W^{s,p}(\mathbb{R}^n, \mathbb{C}^m) := \left\{ u : \mathbb{R}^n \to \mathbb{C}^m \;\middle|\; \begin{array}{l} u \in L^p(\mathbb{R}^3, \mathbb{C}^m) \text{ and } D^{\alpha} u \in L^p(\mathbb{R}^3, \mathbb{C}^m) \\ \forall \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leq s \end{array} \right\}, \tag{2.29}$$

where we use the *multi-index* notation $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$,

$$|\alpha| = \sum_{k=1}^{n} \alpha_k \quad \text{and} \quad D^{\alpha} u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}. \tag{2.30}$$

Here $\frac{\partial u}{\partial x_j}$ refers to the **weak partial derivative** of the function u (see for example [11] for a definition of the weak derivative of a funtion). Moreover, the space $W^{s,p}(\mathbb{R}^n, \mathbb{C}^m)$ is equipped with the norm

$$\|u\|_{W^{s,p}} := \left(\sum_{|\alpha| \leq s} \|D^{\alpha} u\|_p^p \right)^{1/p}. \tag{2.31}$$

Remark 2.10. Note that $u = (u_1, u_2, \dots, u_m)$ lies in $W^{s,p}(\mathbb{R}^n, \mathbb{C}^m)$ if and only if each component u_j lies in $W^{s,p}(\mathbb{R}^n) := W^{s,p}(\mathbb{R}^n, \mathbb{C})$.

Remark 2.11. It can be proved that the Sobolev spaces $W^{s,p}(\mathbb{R}^n, \mathbb{C}^m)$ are Banach spaces for $1 \leq p \leq \infty$ and reflexive for $1 < p < \infty$ (see [11] and the previous remark).

Remark 2.12. The most important case for this work is $p = 2$. In this case we use the notation $H^s(\mathbb{R}^n, \mathbb{C}^m) := W^{s,2}(\mathbb{R}^n, \mathbb{C}^m)$. This space is equipped with the scalar product

$$(u, v)_{H^s} := \sum_{|\alpha| \leq s} (D^\alpha u, D^\alpha v)_{L^2} = \sum_{|\alpha| \leq s} \int_{\mathbb{R}^n} D^\alpha u(x)^* D^\alpha v(x) d^n x, \tag{2.32}$$

which induces the norm $\|u\|_{H^s(\mathbb{R}^n, \mathbb{C}^m)}$

$$\|u\|_{H^s(\mathbb{R}^n, \mathbb{C}^m)} = \left(\sum_{|\alpha| \leq s} \|D^\alpha u\|_2^2 \right)^{1/2}, \tag{2.33}$$

and hence it is a Hilbert space. Here $*$ denotes the transpose conjugated, as before.

Of particular interest for this thesis is the space $H^1(\mathbb{R}^n, \mathbb{C}^m)$ with the scalar product

$$(u, v)_{H^1} = (u, v)_{L^2} + (\nabla u, \nabla v)_{L^2} = \int_{\mathbb{R}^n} \left[u(x)^* v(x) dx + \nabla u(x)^* \nabla v(x) \right] d^n x, \tag{2.34}$$

where we write

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n} \right), \tag{2.35}$$

and

$$\nabla u^* \nabla v = \sum_{j=1}^n \frac{\partial u^*}{\partial x_j} \frac{\partial v}{\partial x_j} = \sum_{j=1}^n \sum_{k=1}^m \frac{\partial u_k^*}{\partial x_j} \frac{\partial v_k}{\partial x_j}. \tag{2.36}$$

The following proposition gives a useful characterization of the Sobolev space $W^{1,p}(\mathbb{R}^n, \mathbb{C}^m)$.

Proposition 2.5. *Let $u \in L^p(\mathbb{R}^n, \mathbb{C}^m)$ with $1 < p < \infty$. The following properties are equivalent:*

(i) $u \in W^{1,p}(\mathbb{R}^n, \mathbb{C}^m)$.

(ii) There exist a constant $C > 0$ such that for all $h \in \mathbb{R}^n$ we have

$$\|\tau_h u - u\|_p \leq C|h|, \tag{2.37}$$

where $(\tau_h u)(x) := u(x + h)$.

Moreover, in the implication (ii) → (i), it follows that $\|\nabla u\|_p \leqslant C$.

Proof. The proof in the case of $W^{1,p}(\mathbb{R}^n)$ can be found in Proposition 9.3 [11] or Theorem 3 in section 5.8 in [15]. From this result, the extension to $W^{1,p}(\mathbb{R}^n, \mathbb{C}^m)$ can be obtained without additional difficulties. ■

Remark 2.13. From Proposition 2.4 it follows that if $u \in W^{1,p}(\mathbb{R}^n, \mathbb{C}^m)$, then $|u| \in W^{1,p}(\mathbb{R}^n)$ and $\||u|\|_{W^{1,p}(\mathbb{R}^n)} \leqslant \|u\|_{W^{1,p}(\mathbb{R}^n,\mathbb{C}^m)}$. Indeed, since for all $h \in \mathbb{R}^n$ the inequality

$$||u(x+h)| - |u(x)|| \leq |u(x+h) - u(x)| \tag{2.38}$$

holds for a.e. $x \in \mathbb{R}^n$, we obtain

$$\|\tau_h|u| - |u|\|_p \leq \|\tau_h u - u\|_p \leq |h|\|\nabla u\|_p. \tag{2.39}$$

Theorem 2.6 (Sobolev inequality)**.** *Let $1 \leqslant p < n$. Then*

$$W^{1,p}(\mathbb{R}^n, \mathbb{C}^m) \subset L^q(\mathbb{R}^3, \mathbb{C}^m), \text{ where } q \text{ is given by } \frac{1}{q} = \frac{1}{p} - \frac{1}{n}, \tag{2.40}$$

and there exist a constant $C = C(p, n)$ such that

$$\|u\|_q \leq C\|\nabla u\|_p \quad \text{for all } u \in W^{1,p}(\mathbb{R}^n, \mathbb{C}^m). \tag{2.41}$$

Proof. The case of $W^{1,p}(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n)$ is a well-known result and a proof can be found in [11]. For the case of $m \geq 0$ arbitrary, we recall the definition of the norm in $L^p(\mathbb{R}^n, \mathbb{C}^m)$

$$\|u\|_q = \||u|\|_{L^q(\mathbb{R}^n)}. \tag{2.42}$$

From Remark 2.13 we know that $|u| \in W^{1,p}(\mathbb{R}^n)$. Therefore,

$$\||u|\|_{L^q(\mathbb{R}^n)} \leq C\|\nabla|u|\|_{L^p(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^n,\mathbb{C}^m)}. \tag{2.43}$$

■

Proposition 2.6. *(Hardy's inequality) Let $u \in W^{1,p}(\mathbb{R}^n)$ and $K : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$ defined by $K(x) = 1/|x|$. Then, for $n \geq 2$ and $1 \leq p < n$, we have $Ku \in L^p(\mathbb{R}^n)$ and*

$$\|Ku\|_p \leq \frac{p}{n-p}\|\nabla u\|_p. \tag{2.44}$$

Proof. See, for example, Theorem 7 in section 5.8 in [15]. ■

The following two results extend the classical Sobolev embedding theorems for the case of vector-valued functions.

Theorem 2.7 (Embedding Theorems)**.** *Let $s \geq 1$ be an integer and let $p \in [1, \infty)$. We have*

(i) $W^{s,p}(\mathbb{R}^n, \mathbb{C}^m) \subset L^q(\mathbb{R}^n, \mathbb{C}^m)$, *where* $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{s}{n}$ *if* $\dfrac{1}{p} - \dfrac{s}{n} > 0$,

(ii) $W^{s,p}(\mathbb{R}^n, \mathbb{C}^m) \subset L^q(\mathbb{R}^n, \mathbb{C}^m)$, *for all* $q \in [p, \infty)$ *if* $\dfrac{1}{p} - \dfrac{s}{n} = 0$,

(iii) $W^{s,p}(\mathbb{R}^n, \mathbb{C}^m) \subset L^\infty(\mathbb{R}^n, \mathbb{C}^m)$ *if* $\dfrac{1}{p} - \dfrac{s}{n} < 0$,

and all these embeddings are continuous. Moreover, if $s-(n/p)>0$ is not an integer, set $k=[s-(n/p)]$ with [ ] denoting the integer part. Then

$$W^{s,p}(\mathbb{R}^n,\mathbb{C}^m)\subset C^k(\mathbb{R}^n,\mathbb{C}^m). \tag{2.45}$$

Proof. Let $u\in W^{s,p}(\mathbb{R}^n,\mathbb{C}^m)$, then

$$u=(u_1,u_2,\ldots,u_m) \tag{2.46}$$

where $u_i\in W^{s,p}(\mathbb{R}^n)$ denotes the i-th component of u, with $i=1,2,\ldots,m$. By the embedding theorems applied to the scalar case (see for example [11]), it follows that $u_i\in L^r(\mathbb{R}^n)$ with r depending on the cases (i), (ii) or (iii). From the definition of the norm in $L^p(\mathbb{R}^n,\mathbb{C}^m)$, we also conclude that $u\in L^r(\mathbb{R}^n,\mathbb{C}^m)$.

It remains to prove that embeddings are continuous. To this end, define the inclusion (linear) operator

$$\imath: W^{s,p}(\mathbb{R}^n,\mathbb{C}^m)\to L^r(\mathbb{R}^n,\mathbb{C}^m),\quad \imath(u):=u. \tag{2.47}$$

By the Closed Graph Theorem [11], it is sufficient to show that the graph of $\imath$, defined by

$$\Gamma(\imath):=\{(u,\imath(u)):u\in W^{s,p}(\mathbb{R}^n,\mathbb{C}^m)\}, \tag{2.48}$$

is closed in $W^{s,p}(\mathbb{R}^n,\mathbb{C}^m)\times L^r(\mathbb{R}^n,\mathbb{C}^m)$. Let (u_j) be a sequence in $W^{s,p}(\mathbb{R}^n,\mathbb{C}^m)$ such that $u_j\to u$ in $W^{s,p}(\mathbb{R}^n,\mathbb{C}^m)$ and $\imath(u_j)=u_j\to v$ in $L^r(\mathbb{R}^n,\mathbb{C}^m)$. By Theorem (4.9) in [11], there exists a subsequence (u_{j_k}) such that

$$u_{j_k}(x)\to u(x),\quad \text{a.e. on } \mathbb{R}^n. \tag{2.49}$$

On the other hand, $u_j\to v$ in $L^r(\mathbb{R}^n,\mathbb{C}^m)$ the same theorem guarantees

$$u_{j_k}(x)\to v(x),\quad \text{a.e. on } \mathbb{R}^n. \tag{2.50}$$

By uniqueness of the limit $u(x)=v(x)$ almost everywhere on $\mathbb{R}^n$. Hence, $\Gamma(\imath)$ is closed which, the Closed Graph Theorem implies that the linear operator $\imath$ is continuous. This completes the proof. ∎

Next, we discuss compact embeddings that will be important in the following chapters.

Definition 2.16. Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be Banach spaces such that $X\subset Y$. We say that X is **compactly embedded** in Y if every bounded sequence (u_j) in X contains a subsequence (u_{j_k}) that converges strongly in Y, that is $\|u_j\|_X$ bounded implies that there exist $u\in Y$ such that

$$\|u_{j_k}-u\|_Y\to 0,\quad \text{for some } u\in Y. \tag{2.51}$$

Theorem 2.8. *(Rellich-Kondrachov) Let $\Omega\subset\mathbb{R}^n$ be a bounded domain of class C^1, and let the spaces $L^p(\Omega,\mathbb{C}^m)$ and $W^{1,p}(\Omega,\mathbb{C}^m)$ be defined as in Definitions 2.14 and 2.15, replacing $\mathbb{R}^n$ by Ω. Then, we have the following compact embeddings:*

(i) $W^{1,p}(\Omega, \mathbb{C}^m) \subset L^q(\Omega, \mathbb{C}^m)$, *for all* $q \in [1, p^*)$ *where* $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$, *if* $p < n$,

(ii) $W^{1,p}(\Omega, \mathbb{C}^m) \subset L^q(\Omega, \mathbb{C}^m)$, *for all* $q \in [p, \infty)$ *if* $p = n$,

(iii) $W^{1,p}(\Omega, \mathbb{C}^m) \subset C(\overline{\Omega}, \mathbb{C}^m)$, *if* $p > n$.

In particular, $W^{1,p}(\Omega, \mathbb{C}^m) \subset L^p(\Omega, \mathbb{C}^m)$ *with compact injection for all* p *(and all* n*).*

Proof. Let $(u^{(j)})$ be a bounded sequence in $W^{1,p}(\Omega, \mathbb{C}^m)$, and write

$$u^{(j)} = (u_1^{(j)}, \dots, u_m^{(j)}), \tag{2.52}$$

where each $u_i^{(j)}$ denotes the i-th component of $u^{(j)}$ with $i = 1, 2, \dots, m$. Since $u^{(j)}$ is bounded, each component sequence $(u_i^{(j)})$ is bounded in $W^{1,p}(\Omega)$. By the Rellich–Kondrachov theorem applied to the scalar case (cf. [11]), for each fixed i there exists a subsequence $(u_i^{(j_{k(i)})})$ such that $u_i^{(j_{k(i)})} \to u_i$ strongly in $L^p(\Omega)$. Using a “diagonal trick” one obtains a single subsequence $(u_i^{(j_k)})$ such that, simultaneously for all $i = 1, 2, \dots, m$

$$u_i^{(j_k)} \to u_i, \quad \text{strongly in } L^q(\Omega). \tag{2.53}$$

Defining $u := (u_1, \dots, u_m)$ we have

$$\|u^{(j_k)} - u\|_q^q = \sum_{i=1}^{m} \|u_i^{(j_k)} - u_i\|_q^q, \tag{2.54}$$

and since each summand tends to zero, we conclude $u^{(j_k)} \to u$ strongly in $L^q(\Omega, \mathbb{C}^m)$. Therefore, every bounded sequence $W^{1,p}(\Omega, \mathbb{C}^m)$ admits a subsequence converging strongly in $L^q(\Omega, \mathbb{C}^m)$, which proves the theorem. ■

2.5 Symmetric decreasing rearrangement

We introduce the concept of the symmetric rearrangement of a measurable set, followed by the definition of the symmetric decreasing rearrangement of a measurable function. Loosely speaking, rearranging a subset of a space means transforming its geometry into a more convenient or symmetry form, without changing its size or measure. For example, in Euclidean space, a natural rearrangement of a subset is the ball centered at the origin whose volume is the same as the original one. This will turn out to be useful for proving the existence of stationary solutions of the multi-field Schrödinger-Poisson system. For a detailed discussion and proofs of the results presented below see, for example, [42]. See also [81] for a review on the theory of rearrangement of functions.

Definition 2.17. Let $A \subset \mathbb{R}^n$ be a measurable set with finite Lebesgue measure $|A| < \infty$. We define $A^\sharp$, the **symmetric rearrangement** of the set A, to be the open ball centered at the origin whose volume is that of A. Thus

$$A^\sharp := \{x \in \mathbb{R}^n : |x| < R\}, \tag{2.55}$$

where $|A^\sharp| = \Omega_n R^n = |A|$ with Ω_n denoting the volume of the n-sphere.

Definition 2.18. Let χ_A be the characteristic function of a measurable set A of finite Lebesgue measure, defined by

$$\chi_A(x) := \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases} \tag{2.56}$$

Then, we define

$$\chi_A^\sharp := \chi_{A^\sharp}, \tag{2.57}$$

that is,

$$\chi_A^\sharp(x) := \begin{cases} 1, & \text{if } \Omega_n |x|^n < |A|, \\ 0, & \text{otherwise.} \end{cases} \tag{2.58}$$

Definition 2.19. Let $f : \mathbb{R}^n \to [0, \infty]$ be a measurable function. Then, the **distribution function** $d_f : (0, \infty) \to [0, \infty]$ of f, is defined as

$$d_f(t) := |\{x \in \mathbb{R}^n : |f(x)| > t\}|. \tag{2.59}$$

Lemma 2.2. *Let $f \in L^p(\mathbb{R}^n)$ with $1 \le p < \infty$. Then, its distribution function $d_f(t)$ is finite for all $t > 0$.*

Proof. By definition of the distribution function, we have

$$d_f(t) = \int_{\mathbb{R}^n} \chi_{\{|f|>t\}}(x) d^n x, \tag{2.60}$$

where $\{|f| > t\} := \{x \in \mathbb{R}^n : |f(x)| > t\}$. Whenever $|f(x)| > t$, we have $|f(x)|^p / t^p > 1$ for a.e. $x \in \mathbb{R}^n$, hence

$$\chi_{\{|f|>t\}}(x) \le \left(\frac{|f(x)|}{t} \right)^p, \quad \text{a.e. on } x \in \mathbb{R}^n. \tag{2.61}$$

Inserting the estimate into Eq. (2.60) gives

$$d_f(t) \le \frac{1}{t^p} \|f\|_p^p, \tag{2.62}$$

which is finite for all $t > 0$. ■

Definition 2.20. Let $f : \mathbb{R}^n \to [0, \infty]$ be a measurable function whose distribution $d_f(t)$ is finite for all $t > 0$. Then, the **symmetric decreasing rearrangement** of f, denoted by $f^\sharp$, is defined by

$$f^\sharp(x) := \int_0^\infty \chi^\sharp_{\{f>t\}}(x) dt, \tag{2.63}$$

where $\{f > t\} := \{x \in \mathbb{R}^n : |f(x)| > t\}$. Moreover, if $g : \mathbb{R}^n \to \mathbb{C}$ is a measurable function with $d_g(t)$ finite for all $t \ge 0$, we define $g^\sharp := |g|^\sharp$.

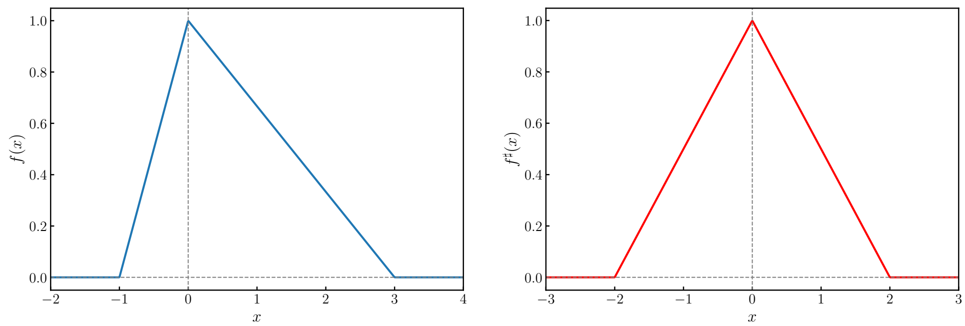

Figure 2.1: Left panel: plot of the function $f(x)$ for $a = 1$ and $b = 3$. Right panel: the symmetric decreasing rearrangement of the function $f(x)$.

Example. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 + a^{-1}x, & \text{if } -a \leq x \leq 0, \\ 1 - b^{-1}x, & \text{if } 0 \leq x \leq b, \\ 0, & \text{otherwise,} \end{cases}$$

where $a, b \in \mathbb{R}$ are positive constants (see Fig. 2.1). We compute its symmetric decreasing rearrangement. Since $0 \leq f(x) \leq 1$, it follows that $\{f > t\} = \emptyset$ for $t \geq 1$. Thus, we only consider $0 < t < 1$.

From the definition of f:

(1) For $x \in [-a, 0]$, the condition $f(x) > t$ is equivalent to $x > a(t-1)$.

(2) For $x \in [0, b]$, the condition $f(x) > t$ is equivalent to $x < b(1-t)$.

Hence, for $0 < t < 1$ we have

$$\{f > t\} = \{x \in \mathbb{R} : x \in [\, a(t-1),\, b(1-t)\,]\}. \tag{2.64}$$

Recalling Definition 2.17 of the symmetric rearrangement of a subset in $\mathbb{R}^n$, we obtain

$$\{f > t\}^\sharp = \left\{ x \in \mathbb{R} : |x| < \frac{(a+b)(1-t)}{2} \right\}. \tag{2.65}$$

Finally, by Definition 2.20,

$$f^\sharp(x) = \int_0^1 \chi_{\{f>t\}^\sharp}(x)dt = \int_0^{1-\frac{2|x|}{a+b}} dt = \max\left\{0, 1 - \frac{2|x|}{a+b}\right\}. \tag{2.66}$$

Remark 2.14. The rearrangement $f^\sharp$ has some important properties

(i) $f^\sharp : \mathbb{R}^n \to [0, \infty]$ is non-negative.

(ii) $f^\sharp$ is spherically symmetric and non-increasing. We say that $f^\sharp$ is *strictly symmetric decreasing* if $f^\sharp(x) > f^\sharp(y)$ whenever $|x| < |y|$, which in particular implies that $f^\sharp(x) > 0$ for all $x > 0$.

(iii) If $f \in L^p(\mathbb{R}^n, \mathbb{C})$, then $f^\sharp \in L^p(\mathbb{R}^n)$ and

$$\|f\|_p = \|f^\sharp\|_p, \tag{2.67}$$

for all $1 \leq p < \infty$.

Proposition 2.7. *Let f and g be nonnegative functions on $\mathbb{R}^n$, with distribution function $d_f(t)$ finite for all $t > 0$, and let $f^\sharp$ and $g^\sharp$ be their symmetric decreasing rearrangement. Then*

$$\int_{\mathbb{R}^n} f(x)g(x)d^n x \leq \int_{\mathbb{R}^n} f^\sharp(x)g^\sharp(x)d^n x, \tag{2.68}$$

with the understanding that when the left side is infinite so is the right side.

If f is strictly symmetric decreasing, then there is equality in (2.68) if and only if $g = g^\sharp$.

Proposition 2.8 (Riesz's rearrangement inequality)**.** *Let f, g and h be three nonnegative functions on $\mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(x-y)h(y)d^n x d^n y \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^\sharp(x)g^\sharp(x-y)h^\sharp(y)d^n x d^n y \tag{2.69}$$

with the understanding that when the left side is infinite so is the right side.

If g is strictly symmetric decreasing , then there is equality in (2.69) if and only if $f(x) = f^\sharp(x-y)$ and $h(x) = h^\sharp(x-y)$ for some $y \in \mathbb{R}^n$.

Theorem 2.9. *Let $u \in H^1(\mathbb{R}^n)$. Then $u^\sharp \in H^1(\mathbb{R}^n)$ and*

$$\|\nabla u^\sharp\|_2 \leq \|\nabla u\|_2. \tag{2.70}$$

2.6 Group representation

Finally, we recall some definitions of the theory of group representations [16], which will play a fundamental role in Chapter 4, in establishing the existence of ℓ-boson stars.

Definition 2.21. Let G be a group and let V a vector space over the field $K = \mathbb{R}$ or $\mathbb{C}$. A **representation** of G on V is a map

$$\rho : G \to GL(V), \tag{2.71}$$

such that ρ is a **group homomorphism**, that is, for all $g, h \in G$

$$\rho(gh) = \rho(g)\rho(h) \quad \text{and} \quad \rho(e) = \mathrm{Id}_V, \tag{2.72}$$

where $e \in G$ is the identity element of the group. Here $GL(V)$ denotes de group of invertible bounded linear operators on V. The vector space V is called the representation space and, and its dimension is called the dimension or the degree of the representation.

A representation of a group can be though as an action of the group on some vector space. Understanding the properties of representations is important, as they naturally appear in various areas of physics.

Definition 2.22. A representation of G on a Hilbert space H is a **unitary representation** if the operators $\rho(g) \in GL(H)$ are unitary for all $g \in G$, that is

$$\rho(g)\rho(g)^* = \rho(g)^*\rho(g) = \mathrm{Id}_H, \tag{2.73}$$

where $\rho(g)^*$ is the adjoint of $\rho(g)$.

Remark 2.15. If V is a finite-dimensional vector space with dimension n, then we can identify $GL(V)$ with the general linear group $GL(n; K)$, consisting of all invertible $n \times n$ matrices over the field K.

Definition 2.23. Let $\rho : G \to GL(V)$ a representation of the group G on a vector space V. A subspace $W \subset V$ is said to be G-**invariant** if

$$\rho(g)w \in W \quad \text{for all } g \in G \text{ and } w \in W. \tag{2.74}$$

Definition 2.24. A representation of a group G over the vector space V is called **irreducible** if the only G-invariant subspaces of V are $\{0\}$ and V itself.

To conclude this chapter, we present two examples of group representations. The first one illustrates an irreducible representation. The second will be useful in Chapter 4 to prove the existence of special stationary solutions, called ℓ-boson stars, of the multi-field Schrödinger-Poisson system.

Example 1. Consider the special orthogonal group $SO(3)$ and let

$$V_\ell := \operatorname{span}\{Y^{\ell m} : m = -\ell, \dots, \ell\}, \quad \ell \in \mathbb{N}_0, \tag{2.75}$$

where $Y^{\ell m}$ denotes the spherical harmonics. Next, define the map $\rho^{(\ell)} : SO(3) \to GL(V_\ell)$ by

$$\left(\rho^{(\ell)}(R)u\right)\left(\frac{x}{|x|}\right) = u\left(R^{-1}\frac{x}{|x|}\right), \tag{2.76}$$

where

$$u\left(R^{-1}\frac{x}{|x|}\right) = \sum_{m=-\ell}^{\ell} c_m Y^{\ell m}\left(R^{-1}\frac{x}{|x|}\right) = \sum_{m',m=-\ell}^{\ell} c_m D^{(\ell)}_{mm'}(R) Y^{\ell m'}\left(\frac{x}{|x|}\right), \tag{2.77}$$

with $D^{(\ell)}_{m'm}(R)$ the matrix elements of the Wigner D-matrix [50]. Then $\rho^{(\ell)}$ is an irreducible representation of $SO(3)$. Furthermore, the dimension of V_ℓ is $2\ell + 1$.

Example 2. Once again, we consider the group $SO(3)$ and define the map

$$\rho^{(\ell)} : SO(3) \to GL(L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1})), \tag{2.78}$$

by

$$[\rho^{(\ell)}(R)u](x) = D^{(\ell)}(R)u(R^{-1}x), \tag{2.79}$$

where $D^{(\ell)} : SO(3) \to GL(2\ell + 1; \mathbb{C})$ is a finite-dimensional irreducible unitary representation of $SO(3)$ on $\mathbb{C}^{2\ell+1}$. Then, $\rho^{(\ell)}$ is a unitary (but not irreducible) representation of $SO(3)$.

Chapter 3

Ground state solutions of the multi-field Schrödinger-Poisson system

In this chapter we carry out a rigorous study of the multi-field Schrödinger-Poisson system (1.78) with an arbitrary number K of scalar fields and a central mass $M \geq 0$. Specifically, in Section 3.1 we reformulate the problem of finding static solutions of this system as a variational problem. In Section 3.2 we state and proof the main theorem of the chapter, which demonstrates the existence of a minimizer of the energy functional (1.59). Finally, in Section 3.3 we discuss some important properties of this minimizer and, in particular, we show that it indeed provides a solution of the system (1.78).

For what follows, it is more convenient to write the system (1.78) as an (equivalent) integro-differential equation. Inverting the Laplace operator in Eq. (1.78b) and substituting into Eq. (1.78a) we obtain

$$i\frac{\partial \psi}{\partial t}(t,x) = \left[-\Delta - \frac{M}{|x|} - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{|\psi(t,y)|^2}{|x-y|} d^3y\right] \psi(t,x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3. \tag{3.1}$$

We are interested in stationary solutions of Eq (3.1), which are characterized by an harmonic dependency on time, such that

$$\psi(t,x) = e^{-i\omega t} u(x), \qquad t \in \mathbb{R},\ x \in \mathbb{R}^3, \tag{3.2}$$

with u a column vector where each component u_j ($j = 1, 2, \ldots, K$) is a complex-valued function. In principle, one could consider the case of a multi-frequency solution for which each component u_j oscillates with a different frequency. However, in the following, we will only consider stationary states for which ω is a real number. Substituting ansatz (3.2) into Eq. (3.1) we obtain the following non-linear equation

$$\left[-\Delta - \frac{M}{|x|} + \Delta^{-1}(|u|^2)\right] u = \omega u, \tag{3.3}$$

with Δ^{-1} denoting the inverse of the Laplace operator, defined formally by

$$\Delta^{-1}(f)(x) := -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{f(y)}{|x-y|}d^3y, \tag{3.4}$$

when acting on an arbitrary function f.

Definition 3.1. A **classical solution** of Eq. (3.3) is a function $u \in C^2(\mathbb{R}^3, \mathbb{C}^K)$ satisfying Eq. (3.3). A **strong solution** of Eq. (3.3) is a function $u \in H^2(\mathbb{R}^3, \mathbb{C}^K)$ satisfying Eq. (3.3). A **weak solution** of Eq. (3.3) is a function $u \in H^1(\mathbb{R}^3, \mathbb{C}^K)$ satisfying

$$\int_{\mathbb{R}^3}\left[\nabla v(x)^*\nabla u - \frac{M}{|x|}v(x)^*u(x) - v(x)^*\Delta^{-1}(|u|^2)(x)u(x) - \omega v(x)^*u(x)\right]d^3x = 0, \tag{3.5}$$

for all $v \in H^1(\mathbb{R}^3, \mathbb{C}^K)$.

The distinction between a classical, strong and weak solution lies in the required regularity of u and in the sense in which equation (3.3) is satisfied. A classical solution is twice continuously differentiable, so that all derivatives in the Eq. (3.3) are well defined and the equation holds pointwise for all $x \in \mathbb{R}^3$. A strong solution only requires $u \in H^2(\mathbb{R}^3, \mathbb{C}^K)$, so that the second derivatives exist in the weak sense and belong to $L^2(\mathbb{R}^3, \mathbb{C}^K)$; in this case Eq. (3.3) holds almost everywhere in $\mathbb{R}^3$. Finally, a weak solution only requires $u \in H^1(\mathbb{R}^3, \mathbb{C}^K)$, which is the natural space where the energy functional (3.6) is well defined (this will be proved in the next section); here the equation must hold after integration against all test functions $v \in H^1(\mathbb{R}^3, \mathbb{C}^K)$.

One can prove that the energy functional (1.77) remains a conserved quantity and takes the form $\mathcal{E}^{phys}[u] = 4m_0v_c^2\mathcal{E}[u]$, where

$$\mathcal{E}[u] = \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u(x)|^2d^3x - \frac{M}{2}\int_{\mathbb{R}^3}\frac{|u(x)|^2}{|x|}d^3x - \frac{1}{16\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|}d^3yd^3x. \tag{3.6}$$

The goal of this section is to prove the existence of solutions of Eq. (3.3) through a variational problem. More precisely, we look for a ground state solution, that is, a solution which minimize $\mathcal{E}[u]$ over a class of admissible functions.

Remark 3.1. Assume that $u \in H^1(\mathbb{R}^3, \mathbb{C}^K)$ is a weak solution of Eq. (3.3). Then u is a critical point of the energy functional (3.6) under the restriction $\|u\|_2^2 = N$. Indeed, for $v \in H^1(\mathbb{R}^3, \mathbb{C}^K)$ the first variation $\delta\mathcal{E}[u; v]$ (see Eq. (3.33) in the next section) of the energy functional is given by

$$\delta\mathcal{E}[u; v] = \int_{\mathbb{R}^3}\left[\nabla v(x)^*\nabla u - \frac{M}{|x|}v(x)^*u(x) - v(x)^*\Delta^{-1}(|u|^2)(x)u(x)\right]d^3x. \tag{3.7}$$

Since u is a weak solution of Eq. (3.3), it satisfies Eq. (3.5), which implies

$$\delta\mathcal{E}[u; v] = \omega\mathrm{Re}(v, u)_{L^2}. \tag{3.8}$$

Furthermore, the norm of u is fixed to N and this implies

$$2\mathrm{Re}(v, u)_{L^2} = 0. \tag{3.9}$$

Substituting in Eq. (3.8) it follows that

$$\delta\mathcal{E}[u; v] = 0. \tag{3.10}$$

3.1 Formulation as a variational problem

Looking at the first term of Eq. (3.6), it is natural to consider the Sobolev space $H^1(\mathbb{R}^3, \mathbb{C}^K)$ as the class of admissible functions. This space is defined by (see Definition 2.15 and Remark 2.12 for details)

$$H^1(\mathbb{R}^3, \mathbb{C}^K) := \left\{ u : \mathbb{R}^3 \to \mathbb{C}^K \mid u \in L^2(\mathbb{R}^3, \mathbb{C}^K) \text{ and } \frac{\partial u}{\partial x_j} \in L^2(\mathbb{R}^3, \mathbb{C}^K),\ j = 1,2,3 \right\}, \tag{3.11}$$

with the norm

$$\|u\|_{H^1} := \left(\|u\|_2^2 + \|\nabla u\|_2^2 \right)^{1/2} = \left\{ \int_{\mathbb{R}^3} \left[|u(x)|^2 + |\nabla u(x)|^2 \right] dx \right\}^{1/2}, \tag{3.12}$$

where

$$|\nabla u|^2 = \sum_{j=1}^{3} \left| \frac{\partial u}{\partial x_j} \right|^2. \tag{3.13}$$

Accordingly, we must verify that each term of the functional (3.6) is well defined and finite whenever $u \in H^1(\mathbb{R}^3, \mathbb{C}^K)$. To this end, we define $n := |u|^2$ and write the functional as

$$\mathcal{E}[u] = T[u] - L[n] - D[n, n], \tag{3.14}$$

with the functionals T, L, and D defined by

$$T[u] := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u(x)|^2 d^3x, \tag{3.15}$$

$$L[n] := \frac{M}{2} \int_{\mathbb{R}^3} \frac{n(x)}{|x|} d^3x, \tag{3.16}$$

$$D[n, n] := \frac{1}{16\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{n(x)n(y)}{|x - y|} d^3x d^3y. \tag{3.17}$$

The functional T is well defined and finite by definition of $H^1(\mathbb{R}^3, \mathbb{C}^K)$. It remains to show that the functional L and D are meaningful for $u \in H^1(\mathbb{R}^3, \mathbb{C}^K)$. First, defining the function $\mathcal{K} : \mathbb{R}^3 \backslash \{0\} \to (0, \infty)$ by

$$\mathcal{K}(x) := \frac{1}{|x|}, \tag{3.18}$$

one can split $\mathcal{K}(x) = \mathcal{K}_1(x) + \mathcal{K}_2(x)$, where

$$\mathcal{K}_1(x) := \begin{cases} \mathcal{K}(x), & \text{if } |x| \leq R, \\ 0, & \text{otherwise}, \end{cases} \tag{3.19}$$

and $\mathcal{K}_2(x) = \mathcal{K}(x) - \mathcal{K}_1(x)$. Then,

$$\mathcal{K}_1 \in L^p(\mathbb{R}^3),\ \mathcal{K}_2 \in L^q(\mathbb{R}^3), \quad \text{with } p \in [1, 3) \text{ and } q \in (3, \infty]. \tag{3.20}$$

In this case, we say that $\mathcal{K} \in L^p(\mathbb{R}^3) + L^q(\mathbb{R}^3)$. By Theorem 2.7, if $u \in H^1(\mathbb{R}^3, \mathbb{C}^K)$ then $u \in L^2(\mathbb{R}^3, \mathbb{C}^K) \cap L^6(\mathbb{R}^3, \mathbb{C}^K)$. By Corollary 2.1, this further implies that $u \in$

$L^p(\mathbb{R}^3, \mathbb{C}^K)$ for all $p \in [2,6]$, and hence $n \in L^p(\mathbb{R}^3, \mathbb{C}^K)$ for $p \in [1,3]$. Therefore, by Hölder's inequality 2.2 we have

$$L[n] \leq \|\mathcal{K}_1\|_{3/2}\|n\|_3 + \|\mathcal{K}_2\|_\infty \|n\|_1 < \infty. \tag{3.21}$$

Similarly, we use Young's convolution inequality (2.28) to obtain

$$D[n,n] \leq \|\mathcal{K}_1\|_{3/2}\|n\|_3\|n\|_1 + \|\mathcal{K}_2\|_\infty \|n\|_1^2 < \infty. \tag{3.22}$$

The conclusion is that $\mathcal{E} : H^1(\mathbb{R}^3, \mathbb{C}^K) \to \mathbb{R}$ is well defined and finite. Moreover, the functional $\mathcal{E}$ is continuous in $H^1(\mathbb{R}^3, \mathbb{C}^K)$ as the following proposition states.

Proposition 3.1. *The energy functional $\mathcal{E} : H^1(\mathbb{R}^3, \mathbb{C}^K) \to \mathbb{R}$ is continuous. More precisely, if (u_j) is a sequence such that $u_j \to u$ strongly in $H^1(\mathbb{R}^3, \mathbb{C}^K)$, then $\mathcal{E}[u_j] \to \mathcal{E}[u]$.*

Proof. Using the triangle inequality, we have

$$|\mathcal{E}[u_j] - \mathcal{E}[u]| \leq |T[u_j] - T[u]| + |L[n_j] - L[n]| + |D[n_j, n_j] - D[n,n]|, \tag{3.23}$$

where $n_j = |u_j|^2$ and $n = |u|^2$. We proceed to estimate term by term. First

$$\begin{aligned} |T[u_j] - T[u]| &\leq \frac{1}{2}\int_{\mathbb{R}^3} ||\nabla u|^2 - |\nabla u_j|^2| d^3x \\ &\leq \frac{1}{2}\int_{\mathbb{R}^3} \left[|\nabla u^*(\nabla u - \nabla u_j)| + (\nabla u - \nabla u_j)^* \nabla u_j\right] d^3x, \end{aligned} \tag{3.24}$$

and by Hölder's inequality we obtain

$$|T[u_j] - T[u]| \leq \frac{1}{2}(\|\nabla u_j\|_2 + \|\nabla u\|_2)\|u_j - u\|_{H^1} \to 0, \quad \text{as } u_j \to u. \tag{3.25}$$

For the second term we have

$$\begin{aligned} |L[n_j] - L[n]| &\leq \int_{\mathbb{R}^3} \frac{||u_j|^2 - |u|^2|}{|x|} d^3x \\ &\leq \int_{\mathbb{R}^3} \frac{|u_j + u||u_j - u|}{|x|} d^3x, \end{aligned} \tag{3.26}$$

and by Hölder's and Hardy's inequalities we obtain

$$|L[n_j] - L[n]| \leq C(\|\nabla u_j\|_2 + \|\nabla u\|_2)\|u_j - u\|_{H^1}, \tag{3.27}$$

with $C > 0$ a positive constant. Finally, for the third term

$$\begin{aligned} |D[n_j, n_j] - D[n,n]| &\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|n_j(x)n_j(y) - n(x)n(y)|}{|x-y|} d^3y d^3x \\ &\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|n_j(x)[n_j(y) - n(y)]| + |n(y)[n_j(x) - n(x)]|}{|x-y|} d^3y d^3x \\ &\leq C_1(\|n_j\|_{6/5} + \|n\|_{6/5})\|n_j - n\|_{6/5}, \end{aligned} \tag{3.28}$$

where $C_1 > 0$ is a constant and in the last estimate we use the Hardy-Littlewood-Sobolev inequality with $p = q = 6/5$. From Hölder's inequality and using that $||u_j(x)|^2 - |u(x)|^2| \leq |u_j(x) + u(x)||u_j(x) - u(x)|$ for all $x \in \mathbb{R}^3$, it follows that

$$\|n_j - n\|_{6/5} \leq (\|u_j\|_{12/5} + \|u\|_{12/5})\|u_j - u\|_{12/5}. \tag{3.29}$$

Hence, using this estimate in (3.28), and taking into account that $\|n_j\|_{6/5} = \|u_j\|^2_{12/5}$ and $\|n\|_{6/5} = \|u\|^2_{12/5}$, it follows that

$$|D[n_j, n_j] - D[n, n]| \leqslant C_1(\|u_j\|^2_{12/5} + \|u\|^2_{12/5})(\|u_j\|_{12/5} + \|u\|_{12/5})\|u_j - u\|_{12/5}. \tag{3.30}$$

By Sobolev's inequality (2.40) and Corollary 2.1 we have that if $u_j \to u$ strongly in $H^1(\mathbb{R}^3, \mathbb{C}^K)$, then $u_j \to u$ in $L^p(\mathbb{R}^3, \mathbb{C}^K)$ for $p \in [2, 6]$. Therefore, from expressions (3.25), (3.27) and (3.30) we deduce that

$$|\mathcal{E}[u_j] - \mathcal{E}[u]| \to 0, \quad \text{as } j \to \infty. \tag{3.31}$$

This proves the statement of the proposition. ■

The following lemma gives an important property of the function $\Delta^{-1}(|u|^2)u$ defined in Eq. (3.4).

Lemma 3.1. *For $s \geq 1$, the map $F : H^s(\mathbb{R}^3, \mathbb{C}^K) \to H^s(\mathbb{R}^3, \mathbb{C}^K)$ defined by*

$$F[u] := \left(\frac{1}{|x|} * |u|^2\right) u, \tag{3.32}$$

maps $H^s(\mathbb{R}^3, \mathbb{C}^K)$ into $H^s(\mathbb{R}^3, \mathbb{C}^K)$.

Proof. The proof for the case $K = 1$ is given in Lemma 3 in [52]. For $K > 1$, the result follows directly, since each component satisfies the lemma. ■

Having established that the energy functional is well defined and continuous, we proceed to examine its differentiability.

Definition 3.2. Let $F : U(u_0) \subset X \to \mathbb{R}$ be a functional defined on an open neighborhood $U(u_0)$ of the point u_0 in the normed vector space X. For fixed $h \in X$, the n**-th variation** $\delta^n F[u_0; h]$ of the functional F at the point u_0 in the direction of h is given by

$$\delta^n F[u_0; h] := \frac{d^n}{dt^n} F[u_0 + th]\bigg|_{t=0} \qquad n = 1, 2, 3, \ldots. \tag{3.33}$$

The functional F has a **Gâteaux derivative** $F'(u_0)$ at the point u_0 if and only if the first variation $\delta F[u_0; h]$ exists for each $h \in X$ and there exists a linear continuous functional $F'(u_0)$ on X such that

$$\delta F[u_0; h] = F'(u_0)[h] \quad \text{for all } h \in X. \tag{3.34}$$

Proposition 3.2. *The energy functional $\mathcal{E} : H^1(\mathbb{R}^3, \mathbb{C}^K) \to \mathbb{R}$ is Gâteaux differentiable. For every $u, v \in H^1(\mathbb{R}^3, \mathbb{C}^K)$ its Gâteaux derivative at u in the direction of v is*

$$\delta\mathcal{E}[u; v] := \operatorname{Re} \int_{\mathbb{R}^3} \left[\nabla v(x)^* \nabla u(x) - \frac{M}{|x|} v(x)^* u(x) + v(x)^* \Delta^{-1}(|u|^2)(x) u(x) \right] d^3x. \tag{3.35}$$

Proof. By definition of the Gâteaux derivative, we compute

$$\begin{aligned} \frac{d}{dt}\mathcal{E}[u + tv]\bigg|_{t=0} &= \frac{d}{dt}\left[\frac{1}{2} \int_{\mathbb{R}^3} |\nabla u(x) + t\nabla v(x)|^2 d^3x - \frac{M}{2} \int_{\mathbb{R}^3} \frac{|u(x) + tv(x)|^2}{|x|} d^3x \right. \\ &\quad \left. - \frac{1}{16\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x) + tv(x)|^2 |u(y) + tv(y)|^2}{|x - y|} d^3x d^3y \right] \Bigg|_{t=0} \end{aligned} \tag{3.36}$$

for fixed $u \in H^1(\mathbb{R}^3, \mathbb{C}^K)$ and each $v \in H^1(\mathbb{R}^3, \mathbb{C}^K)$. The terms of order t^2 or higher vanish when evaluated at $t = 0$, and the terms independent of t also vanish after differentiation with respect to t. Thus, we obtain

$$\delta\mathcal{E}[u, v] = \operatorname{Re} \int_{\mathbb{R}^3} \left[\nabla v(x)^* \nabla u - \frac{M}{|x|} v(x)^* u(x) - \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x - y|} dy\, v(x)^* u(x) \right] d^3x. \tag{3.37}$$

Using the definition of the inverse Laplace operator (3.4), this expression coincides with Eq. (3.35).

It is straightforward to verify that the functional $\mathcal{E}'(u) := \delta\mathcal{E}[u; \cdot]$ on $H^1(\mathbb{R}^3, \mathbb{C}^K)$ is linear, so it remains to prove that it is also continuous. Let (v_j) be a sequence in $H^1(\mathbb{R}^3, \mathbb{C}^K)$ such that $v_j \to v$ in $H^1(\mathbb{R}^3, \mathbb{C}^K)$. Then

$$\begin{aligned} |\delta\mathcal{E}[u; v_j] - \delta\mathcal{E}[u; v]| &\leq \int_{\mathbb{R}^3} |\nabla u(x)||\nabla v_j(x) - \nabla v(x)| d^3x + M \int_{\mathbb{R}^3} \frac{|u(x)|}{|x|} |v_j(x) - v(x)| d^3x \\ &\quad + \int_{\mathbb{R}^3} |\Delta^{-1}(|u|)(x) u(x)||v_j(x) - v(x)| d^3x. \end{aligned} \tag{3.38}$$

By Lemma 3.1, the function $\Delta^{-1}(|u|)u$ belongs to $H^1(\mathbb{R}^3, \mathbb{C}^K)$ and by Hölder's and Hardy's inequalities, it follows that

$$|\delta\mathcal{E}[u; v_j] - \delta\mathcal{E}[u; v]| \leq \left(\|u\|_{H^1} + 2M\|u\|_{H^1} + \|\Delta^{-1}(|u|)u\|_{H^1} \right) \|v_j - v\|_{H^1} \to 0. \tag{3.39}$$

We conclude that the functional $\mathcal{E}'(u)$ is linear and continuous. ■

As explained at the beginning of this chapter, our problem is to minimize $\mathcal{E}[u]$ under the condition $\|u\|_2^2 = N$ with $N > 0$ a fixed positive number. To this end, we define the subset of $H^1(\mathbb{R}^3, \mathbb{C}^K)$

$$\mathcal{A}_N := \left\{ u \in H^1(\mathbb{R}^3, \mathbb{C}^K) : \|u\|_2^2 = N \right\}, \tag{3.40}$$

and the constrained minimization problem

$$E(N) := \inf \{\mathcal{E}[u] : u \in \mathcal{A}_N\} . \tag{3.41}$$

It is convenient to define also the alternative minimization problem

$$E_{\leq}(N) := \inf \{\mathcal{E}[u] \ : \ u \in \mathcal{A}_{\leq N}\} , \tag{3.42}$$

where

$$\mathcal{A}_{\leq N} := \left\{ u \in H^1(\mathbb{R}^3, \mathbb{C}^K) \ : \ \|u\|_2^2 \leq N \right\} . \tag{3.43}$$

Note that unlike $\mathcal{A}_N$, $\mathcal{A}_{\leq N}$ is a convex set. Furthermore, since $\mathcal{A}_N \subset \mathcal{A}_{\leq N}$ we have $E_{\leq}(N) \leq E(N)$; however the following proposition shows that in fact the equality is fulfilled.

Proposition 3.3. *Let $N > 0$ and $M \geq 0$. Then the following statements hold*

(i) $E_{\leq}(N) \leq E(N) < 0$. Moreover, if u is a minimizer of the alternative problem (3.42), then u is also a minimizer of the constrained problem (3.41), that is, $\mathcal{E}[u] = E_{\leq}(N) = E(N)$.

(ii) If u is a minimizer of the alternative problem (3.42), then it has the form $u = f\mathbf{e}$ with $f := |u| \in H^1(\mathbb{R}^n)$ and $\mathbf{e} \in \mathbb{C}^K$ a unitary constant vector.

To prove this, we need the following lemma:

Lemma 3.2. *The functional $\mathcal{E}[u]$ satisfies the following rescaling properties:*

$$\mathcal{E}[\lambda u] = \lambda^2 T[u] - \lambda^2 L[n] - \lambda^4 D[n, n], \tag{3.44a}$$

$$\mathcal{E}[u_\lambda] = \lambda^2 T[u] - \lambda L[n] - \lambda D[n, n], \tag{3.44b}$$

for each $u \in H^1(\mathbb{R}^3, \mathbb{C}^K)$ and $\lambda > 0$, where $u_\lambda(x) := \lambda^{3/2} u(\lambda x)$ (note that $\|u_\lambda\|_2 = \|u\|_2$).

Proof. Direct verification. ■

Proof of Proposition 3.3. For part (i) we let $u \neq 0$ in $\mathcal{A}_N$ fixed. Using identity (3.44b) and the fact that $L[n], D[n, n] > 0$, we conclude that $\mathcal{E}[u] < 0$ for λ small enough, which shows that $E(N) < 0$. Next, let $u \in \mathcal{A}_{\leq N}$ such that $\mathcal{E}[u] = E_{\leq}(N)$. If we suppose by contradiction that $\widetilde{N} := \|u\|_2^2 < N$ with $\widetilde{N} > 0$ (note that $\widetilde{N}$ cannot be zero since in this case $E_{\leq}(N) = 0$, which would contradict $E(N) < 0$). We define $\alpha := \sqrt{N/\widetilde{N}} > 1$ and consider the function $v := \alpha u \in \mathcal{A}_N$. Using the rescaling (3.44a) we obtain that

$$\mathcal{E}[v] = \alpha^2 \left(T[u] - L[n] - \alpha^2 D[n, n] \right) \leq \alpha^2 \mathcal{E}[u], \quad n := |u|^2. \tag{3.45}$$

Since $\alpha^2 > 1$ and $\mathcal{E}[u] < 0$, we conclude $E(N) \leq \mathcal{E}[v] < \mathcal{E}[u] \leq E_{\leq}(N) \leq E(N)$, which is a contradiction.

For part (ii) we decompose u according to

$$u = f\mathbf{e}, \tag{3.46}$$

where $f := |u|$ refers to the euclidean norm of u in $\mathbb{C}^K$ and $\mathbf{e} : \mathbb{R}^3 \to \mathbb{C}^K$ is a unitary vector-valued function. Using expression (3.13) we obtain

$$|\nabla u|^2 = |\nabla f|^2 + f^2 |\nabla \mathbf{e}|^2. \tag{3.47}$$

From this, we deduce that the unitary vector-valued function $\mathbf{e} \in \mathbb{C}^K$ must be constant. Indeed, if $\mathbf{e}$ were not constant, the corresponding function $u = f\mathbf{e}$ would yield a larger value of the kinetic energy $T[u]$ whereas $L[n]$ and $D[n,n]$ would remain equal, thereby increasing the total energy $\mathcal{E}[u]$. Hence, such a function cannot be a minimizer of the energy functional. ■

Remark 3.2. Proposition 3.3 allow us to reduce the minimization problem (3.42) to

$$E^{\mathrm{red}}_{\leq}(N) = \inf\{\mathcal{E}[f] : f \in \mathcal{C}_{\leq N}\}, \tag{3.48}$$

with

$$\mathcal{C}_{\leq N} := \left\{ f \in H^1(\mathbb{R}^3) : \|f\|_2^2 \leq N \right\}, \tag{3.49}$$

where the energy functional $\mathcal{E} : \mathcal{C}_{\leq N} \to \mathbb{R}$ is defined as in Eq. (3.14) with $K = 1$. Therefore, if f_N is a minimizer of the reduced minimization problem (3.48), then $u_N = f_N\mathbf{e}$ is a minimizer of the problems (3.42) and (3.41).

Problem (3.48) was first solved by Lieb in [53] via symmetric decreasing rearrangements. Here, we adopt a slightly different approach, still employing this technique, to reduce first the problem to the subspace of spherically symmetric function, but using Strauss' lemma 4.1 to complete the proof. Following [53], we formulate the following lemma.

Lemma 3.3. *Let $N > 0$ and $M \geq 0$. Then, $\mathcal{E}[f^\sharp] \leq \mathcal{E}[f]$ for all $f \in H^1(\mathbb{R}^3)$. Moreover, $\mathcal{E}[f^\sharp] = \mathcal{E}[f]$ for $f \in C_{\leq N}$ holds if and only if $f(x) = \pm f^\sharp(x - y)$, where $y \in \mathbb{R}^3$ describes an arbitrary translation when $M = 0$ and $y = 0$ when $M > 0$.*

Proof. By Theorem 2.9, we have

$$T[f^\sharp] \leq T[f]. \tag{3.50}$$

Moreover, by Proposition 2.7 and by the Riesz's rearrangement inequality 2.8 we have

$$L[n] \leq L[n^\sharp] \quad \text{and} \quad D[n,n] \leq D[n^\sharp, n^\sharp], \tag{3.51}$$

where $n := |f|^2$ and $n^\sharp := |f^\sharp|^2$. Furthermore, the equality sign in the first equality only holds if $M = 0$ or $n = n^\sharp$, whereas the second inequality holds if and only if $n(x) = n^\sharp(x - y)$.

Therefore, inserting estimates (3.50) and (3.51) into the expression for the energy functional (3.14) we obtain the result. ■

Remark 3.3. The significance of the previous lemma lies in the fact that, in order to find the minimum of the energy functional $\mathcal{E}$ over the set $\mathcal{C}_{\leq N}$, it is sufficient to restrict the search to the subset consisting of spherically symmetric functions, that is,

$$E^r_{\leq}(N) := \inf\{\mathcal{E}[f] : f \in \mathcal{C}^r_{\leq N}\}, \tag{3.52}$$

where

$$\mathcal{C}^r_{\leq N} := \{f \in H^1_r(\mathbb{R}^3) : \|f\|_2^2 \leq N\} \subset \mathcal{C}_{\leq N}, \tag{3.53}$$

and

$$H^1_r(\mathbb{R}^3) := \{f \in H^1(\mathbb{R}^3) : f(Rx) = f(x) \text{ for all } R \in SO(3)\}, \tag{3.54}$$

is the subspace of $H^1(\mathbb{R}^3, \mathbb{R})$ consisting of spherically symmetric functions.

So far, we have reduced the problem of finding a minimizer of the variational problem (3.41) to the radial problem (3.52). This reduction is crucial, since in [80] (see also [66] for further properties of the space $H^1_r(\mathbb{R}^3)$), Strauss proved the following result

Proposition 3.4. *Then, the embedding*

$$H^1_r(\mathbb{R}^3) \subset L^p(\mathbb{R}^3), \tag{3.55}$$

is compact if $2 < p < 6$.

Proof. See Theorem A.1 in Appendix A. ∎

In the remainder of this chapter we focus on solving the radial problem (3.52). To this end, we first stablish the following key results.

Lemma 3.4. *The space $\mathcal{C}^r_{\leq N}$, defined for each $N > 0$, is weakly closed in the Sobolev space $H^1(\mathbb{R}^3)$. Moreover, if (f_j) is a sequence in $\mathcal{C}^r_{\leq N}$ such that $f_j \rightharpoonup f$ weakly in $H^1(\mathbb{R}^3)$, then (f_j) converges weakly to f in each space $L^p(\mathbb{R}^3)$ with $2 \leq p \leq 6$.*

Proof. Let (f_j) a sequence in $\mathcal{C}^r_{\leq N}$ such that $f_j \rightharpoonup f$ weakly in $H^1(\mathbb{R}^3)$. Since $H^1_r(\mathbb{R}^3)$ is weakly closed (see Proposition 4.1 in Appendix A), then $f \in H^1_r(\mathbb{R}^3)$. Moreover, from Theorem 2.4 it follows that the norm is weakly lower semi-continuous, so we have

$$\|f\|_2 \leq \liminf_{j\to\infty} \|f_j\|_2 = \sqrt{N}, \tag{3.56}$$

which implies that $f \in \mathcal{C}^r_{\leq N}$.

To prove the second statement, let $2 \leq p \leq 6$ and let $\varphi : L^p(\mathbb{R}^3) \to \mathbb{C}$ a bounded linear functional. By Theorem 2.7 the Sobolev embedding $H^1(\mathbb{R}^3) \subset L^p(\mathbb{R}^3)$ is continuous, the restriction

$$\tilde{\varphi} := \varphi\Big|_{H^1} : H^1(\mathbb{R}^3) \to \mathbb{C}, \tag{3.57}$$

is a bounded linear functional on $H^1(\mathbb{R}^3)$. Therefore, since $f_j \rightharpoonup f$ weakly in $H^1(\mathbb{R}^3)$, we conclude

$$\lim_{j\to\infty} \varphi(f_j) = \lim_{j\to\infty} \tilde{\varphi}(f_j) = \tilde{\varphi}(f) = \varphi(f), \tag{3.58}$$

which shows that $f_j \rightharpoonup f$ weakly in $L^p(\mathbb{R}^3)$. ∎

Remark 3.4. In fact, we have a stronger result: by Proposition 3.4, the space $H^1_r(\mathbb{R}^3)$ is compactly embedded in $L^p(\mathbb{R}^3)$ if $2 < p < 6$. This implies that every bounded sequence (f_j) in $H^1_r(\mathbb{R}^3)$ contains a subsequence (f_{j_k}) converging strongly to some function f in $L^p(\mathbb{R}^3)$. In particular, if $f_j \rightharpoonup f$ in $H^1_r(\mathbb{R}^3)$, then there exist a subsequence (f_{j_k}) such that $f_{j_k} \to f$ strongly in $L^p(\mathbb{R}^3)$, for all $2 < p < 6$.

Lemma 3.5. *Let $N > 0$ and $M \geq 0$. Then, the functional $\mathcal{E} : \mathcal{C}^r_{\leq N} \to \mathbb{R}$ defined by Eq.(3.14) is bounded below. Moreover, if (f_j) is a sequence in $\mathcal{C}^r_{\leq N}$ such that $\|f_j\|_{H^1} \to \infty$, then*

$$\mathcal{E}[f_j] \to \infty \quad \text{as } j \to \infty. \tag{3.59}$$

Proof. Theorem 2.6 implies

$$\|n\|_3 \leq 2CT[f] \quad \text{for } f \in H^1(\mathbb{R}^3), \tag{3.60}$$

with $n = |f|^2$. On the other hand, using inequality (3.21) and

$$\|\mathcal{K}_1\|_{3/2} = \left(\frac{8\pi}{3}\right)^{2/3} R, \quad \|\mathcal{K}_2\|_\infty = \frac{1}{R}, \tag{3.61}$$

we obtain the following estimate for the functional L defined in Eq. (3.16):

$$0 \leq L[n] \leq M\left[c_0 R\|n\|_3 + \frac{N}{R}\right], \tag{3.62}$$

with $c_0 := (8\pi/3)^{3/2}$ and for all $f \in \mathcal{C}^r_{\leq N}$. Similarly, using inequality (3.22) we obtain

$$0 \leq D[n,n] \leq \frac{N}{2}\left[c_0 R\|n\|_3 + \frac{N}{R}\right], \tag{3.63}$$

for all $f \in \mathcal{C}^r_{\leq N}$. Combining estimates (3.60), (3.62), and (3.63) yields

$$\mathcal{E}[f] \geq (1-\beta)T[f] - \frac{N}{2R}(2M+N), \tag{3.64}$$

where $\beta := (2M+N)c_0 CR$. Fixing R such that $\beta = 1$ we conclude that

$$E_{\leq}(N) > -\infty. \tag{3.65}$$

If we choose R such that $\beta = 1/2$ we obtain instead

$$\mathcal{E}[f] \geq \frac{1}{2}T[f] - \frac{N}{2R}(2M+N) = \frac{1}{4}\|f\|_{H^1} - \frac{N}{4} - \frac{N}{2R}(2M+N), \tag{3.66}$$

for all $f \in \mathcal{C}^r_{\leq N}$, which implies $\mathcal{E}[f] \to \infty$ if $\|f\|_{H^1} \to \infty$. ■

3.2 Existence theorem

We begin this section by stating the main theorem, which gives the existence of a minimizer of the variational problem (3.52):

Theorem 3.1. *Let $N > 0$ and $M \geq 0$. Then, there exist a minimizer $f \in \mathcal{C}^r_{\leq N}$ for the problem (3.52). Moreover, the minimizer satisfies $f = \pm f^\sharp$, i.e. (up to a sign) it is a spherically symmetric and decreasing function.*

Proof. Let (f_j) a minimizing sequence in $\mathcal{C}^r_{\leq N}$ such that $\mathcal{E}[f_j] \to E^r_{\leq}(N)$ as $j \to \infty$. According to Lemma 3.5, the sequence (f_j) is bounded in $H^1_r(\mathbb{R}^3)$; hence, there exist $f \in H^1_r(\mathbb{R}^3)$ and a subsequence (f_{j_k}) such that $f_{j_k} \rightharpoonup f$ weakly in $H^1_r(\mathbb{R}^3)$. Since $\mathcal{C}^r_{\leq N}$ is weakly closed, it follows that $f \in \mathcal{C}^r_{\leq N}$. Furthermore, by Lemma 3.4, $f_{j_k} \rightharpoonup f$ weakly in $L^p(\mathbb{R}^3)$ for $2 \leq p \leq 6$ and, from Remark 3.4, $f_{j_k} \to f$ strongly in $L^p(\mathbb{R}^3)$ for $2 < p < 6$. Therefore, it remains to prove that f is the minimum of the energy functional $\mathcal{E}$ over the set $\mathcal{C}^r_{\leq N}$ and in order to do this we analyze each term of the functional $\mathcal{E}$.

First, it is well known that the Dirichlet functional is weakly lower semi-continuous (see for example [42]), that is

$$T[f] \leq \liminf_{j\to\infty} T[f_{j_k}]. \tag{3.67}$$

Next, let $n := |f|^2$ and $n_k := f_{j_k}$. We now consider the convergence of $D[n_k, n_k]$ and $L[n_k]$ of $\mathcal{E}$ defined in Eqs. (3.16) and (3.17) respectively. For the functional D, we estimate

$$\begin{aligned}|D[n_k, n_k] - D[n, n]| &\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|n_k(x)n_k(y) - n(x)n(y)|}{|x-y|} d^3y d^3x \\ &\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|n_k(x)||n_k(y) - n(y)| + |n(y)||n_{j_k}(x) - n(x)|}{|x-y|} d^3y d^3x.\end{aligned} \tag{3.68}$$

By the Hardy-Littlewood-Sobolev inequality 2.3, we find

$$|D[n_k, n_k] - D[n, n]| \leq C(\|n_{j_k}\|_{6/5} + \|n\|_{6/5})\|n_{j_k} - n\|_{6/5}. \tag{3.69}$$

Since $\|n_k\|_{6/5} = \|f_{j_k}\|^2_{12/5}$, $\|n\|_{6/5} = \|f\|^2_{12/5}$ and $\|n_k - n\|_{6/5} \leq (\|f_{j_k}\|_{12/5} + \|f\|_{12/5})\|f_{j_k} - f\|_{12/5}$ by Hölder's inequality, we obtain

$$|D[n_k, n_k] - D[n, n]| \leq (\|f_{j_k}\|^2_{12/5} + \|f\|^2_{12/5})(\|f_{j_k}\|_{12/5} + \|f\|_{12/5})\|f_{j_k} - f\|_{12/5} \tag{3.70}$$

Therefore, Proposition 3.4 yields

$$\lim_{k\to\infty} D[n_k, n_k] = D[n, n]. \tag{3.71}$$

For the functional $L[n]$, we similarly estimate

$$\begin{aligned}|L[n_k] - L[n]| &\leq \int_{\mathbb{R}^3} \frac{||f_{j_k}(x)|^2 - |f(x)|^2|}{|x|} d^3x \\ &\leq \int_{\mathbb{R}^3} \frac{|f_{j_k}(x)||f_{j_k}(x) - f(x)| + |f(x)||f_{j_k}(x) - f(x)|}{|x|} d^3x.\end{aligned} \tag{3.72}$$

Recalling that the function $\mathcal{K}$ defined in Eq. (3.18) belongs to $L^p(\mathbb{R}^3) + L^q(\mathbb{R}^3)$ with $p \in [1, 3)$ and $q \in (3, \infty]$, we may again apply Hölder's inequality to deduce

$$|L[n_k] - L[n]| \leq \widetilde{C}\|u_{j_k} - u\|_{17/5}, \tag{3.73}$$

where $\tilde{C} > 0$ is given by

$$\widetilde{C} = \|\mathcal{K}_1\|_{17/6}(\|f\|_{17/6} + \|f_{j_k}\|_{17/6}) + \|\mathcal{K}_2\|_{17/5}(\|f\|_{17/7} + \|f_{j_k}\|_{17/7}). \tag{3.74}$$

Therefore, estimate (3.73) and Proposition 3.4 imply

$$\lim_{k\to\infty} L[n_k] = L[n]. \tag{3.75}$$

Combining (3.67), (3.71) and (3.73) yields

$$E^r_{\leq}(N) \leq \mathcal{E}[f] \leq \liminf_{k\to\infty} \mathcal{E}[f_{j_k}] = E^r_{\leq}(N). \tag{3.76}$$

We conclude that f is a minimizer of the variational problem (3.52). Moreover, by Lemma 3.3 it follows that $f = f^{\sharp}$. ■

Remark 3.5. Similarly to the proof of Proposition 3.3(i), one can show that the minimizer f_N of problem (3.52) satisfies $\|f_N\|_2^2 = N$. As we will see in the next chapters, one can prove that every minimizing sequence (f_j) of problem (3.52), with $\|f_j\|_2^2 = N$ for all $j \geq 0$, contains a subsequence (f_{j_k}) such that $f_{j_k} \to f_N$ strongly in $H^1_r(\mathbb{R}^3)$. In other words, the sequence (f_j) is **relatively compact** in $H^1_r(\mathbb{R}^3)$. This property is crucial for establishing the **orbital stability** [17] of the minimizer $f_N \in \mathcal{C}^r_{\leq N}$.

Corollary 3.1. *Assume that $N > 0$ and $M \geq 0$. Then, there exists a minimizer u of the problem (3.41). This minimizer must have the form $u(x) = f^{\sharp}(x-y)\mathbf{e}$, where $f^{\sharp}$ is a minimizer of the problem (3.52), $y \in \mathbb{R}^3$, and $\mathbf{e} \in \mathbb{C}^K$ a unitary complex vector. Furthermore, $y = 0$ if $M > 0$.*

Proof. By Proposition 3.3 and Lemma 3.3, the problem (3.41) reduces to find a minimizer of the problem (3.52) with $f = f^{\sharp}$. Therefore, the result follows directly from Theorem 3.1. ■

3.3 Properties of the minimizer

In this section, we present the main properties of the minimizer of the variational problem (3.41). In fact, the theorem below holds for any critical point of the energy function on A_N, not just the global minima.

Theorem 3.2. *Let $N > 0$, $M \geq 0$, and let $u \in H^1(\mathbb{R}^3, \mathbb{C}^K)$ be a critical point of the energy functional $\mathcal{E}$ defined in (3.14) over the set $\mathcal{A}_N$ defined in (3.40). Then*

(i) There exist a negative constant $\omega \in \mathbb{R}$ such that u satisfies (in the weak sense) the non-linear Schrödinger equation

$$-\Delta u - \frac{M}{|x|}u + \Delta^{-1}(|u|^2)u = \omega u, \tag{3.77}$$

where the potential $\Delta^{-1}(|u|^2) \in L^p(\mathbb{R}^3)$ for all $3 < p \leq \infty$.

(ii) The energy functional satisfies the virial relations:

$$\mathcal{E}[u] = -T[u] = \frac{1}{3}\left(\frac{\omega N}{2} - L[n]\right), \quad n := |u|^2. \tag{3.78}$$

Proof. (*i*) Let $v \in H^1(\mathbb{R}^3, \mathbb{C}^K)$ be fixed and consider the variation $u(t) := u + tv$ with $t \in \mathbb{R}$. Then,

$$\|u(t)\|_2^2 = \|u\|_2^2 + 2t\mathrm{Re}(u,v)_{L^2} + t^2\|v\|_2^2 = N + 2t\mathrm{Re}(u,v)_{L^2} + t^2\|v\|_2^2. \tag{3.79}$$

Hence, there exist a constant $\delta > 0$ such that $\|u(t)\|_2 > 0$ for all $|t| < \delta$ and

$$\frac{d}{dt}\|u(t)\|_2\bigg|_{t=0} = \frac{1}{\sqrt{N}}\mathrm{Re}(u,v)_{L^2}. \tag{3.80}$$

Next, define the rescaled variation

$$\hat{u}(t) := \sqrt{N}\frac{u(t)}{\|u(t)\|_2}, \quad |t| < \delta. \tag{3.81}$$

By construction $\hat{u}(0) = u$ and $\hat{u}(t) \in \mathcal{A}_N$ for all $|t| < \delta$. Since u is a critical point of $\mathcal{E}$ restricted to $\mathcal{A}_N$ and by Lemma 3.2, we have

$$\begin{aligned} 0 = \frac{d}{dt}\mathcal{E}[\hat{u}(t)]\bigg|_{t=0} &= \frac{d}{dt}\left(\frac{N}{\|u(t)\|_2^2}T[u(t)] - \frac{N}{\|u(t)\|_2^2}L[n(t)] - \frac{N^2}{\|u(t)\|_2^4}D[n(t),n(t)]\right)\bigg|_{t=0} \\ &= \frac{d}{dt}\mathcal{E}[u(t)]\bigg|_{t=0} - \frac{2}{\sqrt{N}}\frac{d}{dt}\|u(t)\|_2\bigg|_{t=0}\left(T[u] - L[n] - 2D[n,n]\right), \end{aligned} \tag{3.82}$$

where $n := |u|^2$ and $n(t) := |u(t)|^2$. On the other hand, $u(t) = u + tv$, $v \in H^1(\mathbb{R}^3, \mathbb{C}^K)$, and the energy functional $\mathcal{E}$ is Gâteaux differentiable on $H^1(\mathbb{R}^3, \mathbb{C}^K)$ with derivative given by

$$\frac{d}{dt}\mathcal{E}[u+tv]\bigg|_{t=0} = \mathrm{Re}\int_{\mathbb{R}^3}\left[\nabla v(x)^*\nabla u - \frac{M}{|x|}v(x)^*u(x) - v(x)^*\Delta^{-1}(|u|^2)(x)u(x)\right]d^3x, \tag{3.83}$$

as shown in Proposition 3.2. Using Eq. (3.80) and Eq. (3.83) we obtain

$$\mathrm{Re}\int_{\mathbb{R}^3}\left[\nabla v(x)^*\nabla u - \frac{M}{|x|}v(x)^*u(x) - v(x)^*\Delta^{-1}(|u|^2)(x)u(x) - \omega v(x)^*u(x)\right]d^3x = 0, \tag{3.84}$$

for all $v \in H^1(\mathbb{R}^3, \mathbb{C}^K)$, where

$$\omega := \frac{2}{N}\left(T[u] - L[n] - 2D[n,n]\right). \tag{3.85}$$

Similarly, replacing v with iv in Eqs. (3.80) and (3.83), we obtain

$$i\mathrm{Im}\int_{\mathbb{R}^3}\left[\nabla v(x)^*\nabla u - \frac{M}{|x|}v(x)^*u(x) - v(x)^*\Delta^{-1}(|u|^2)(x)u(x) - \omega v(x)^*u(x)\right]d^3x = 0. \tag{3.86}$$

Adding Eq. (3.86) to Eq. (3.84), yields

$$\int_{\mathbb{R}^3} \left[\nabla v(x)^* \nabla u - \frac{M}{|x|} v(x)^* u(x) - v(x)^* \Delta^{-1}(|u|^2)(x) u(x) - \omega v(x)^* u(x) \right] d^3x = 0, \tag{3.87}$$

which proves that u is a weak solution of Eq. (3.77). See Theorem 1 in chapter V in [78] for the proof that $\Delta^{-1}(|u|^2) \in L^p(\mathbb{R}^3)$ for $3 < p \leq \infty$.

(ii) Using the rescaling in Eq. (3.44b) (recall that $\|u_\lambda\|_2 = \|u\|_2$) implies

$$\frac{d}{d\lambda} \mathcal{E}[u_\lambda] \bigg|_{\lambda=1} = 2T[u] - L[n] - D[n,n] = 0. \tag{3.88}$$

Since $\mathcal{E}[u] = T[u] - L[n] - D[n,n]$, we deduce that

$$\mathcal{E}[u] = -T[u]. \tag{3.89}$$

Furthermore, from Eq. (3.88) we have $D[n,n] = 2T[u] - L[n]$. Substituting into Eq. (3.85) and taking into account Eq. (3.89) we obtain

$$\mathcal{E}[u] = \frac{1}{3} \left(\frac{N\omega}{2} - L[n] \right). \tag{3.90}$$

Finally, from Eq. (3.85) we obtain

$$\omega = \frac{2}{N} (\mathcal{E}[u] - D[n,n]) = -\frac{2}{N} (T[u] + D[n,n]). \tag{3.91}$$

From here and considering that $T[u], D[n,n] > 0$ it follows that $\omega < 0$. ■

The argument in the next chapter will show that in fact one has the strongest result:

Theorem 3.3. *Let $N > 0$, $M \geq 0$ and let $u \in H^1(\mathbb{R}^3, \mathbb{C}^K)$ be a critical point of the energy functional $\mathcal{E}$ over $\mathcal{A}_N$. Then, the critical point u is a strong solution of Eq. (3.77). Moreover, if $M = 0$, u is also a classical solution of Eq. (3.77).*

Proof. This result will be proven in a more general context in the following chapter, see Theorem 4.3. ■

Remark 3.6. Following [53], it is also possible to prove the uniqueness of the ground state solution for the case $M = 0$.

Chapter 4

Newtonian ℓ-boson stars

In this chapter, we focus on a special class of stationary solutions of the the multi-field Schrödinger-Poisson system, known as Newtonian ℓ-boson stars. From a physical perspective, these configurations can be interpreted as hypothetical self-gravitating objects composed of $K = 2\ell + 1$ massive complex scalar fields, where the integer $\ell \geq 0$ labels the orbital angular momentum of the field ψ.

Our aim in this chapter, is to give rigorous results for the existence and stability of ℓ-boson stars. The chapter unfolds as follows. In Section 4.1 we introduce the governing equations and, with the help of group representation theory, provide a precise definition of ℓ-boson stars. Section 4.2 develops a variational approach to prove the existence of ground-state solutions, while Section 4.3 presents some of their key properties. Section 4.4 is devoted to the stability problem, where we demonstrate orbital stability against spherically symmetric perturbations. Finally, in Section 4.5, we present the more familiar and explicit form of ℓ-boson stars, connecting the abstract framework with the standard formulation used in the literature, and provide some numerical examples.

4.1 Theoretical set up of ℓ-boson stars

Similarly to the previous chapters, we are interested in stationary solutions for the system (1.78) of the form

$$\psi(x) = e^{-i\omega t}u(x), \tag{4.1}$$

where u is a vector-valued function where each component u_j $(j = 1, 2, \ldots, K)$ is a complex-valued function and $\omega \in \mathbb{R}$. This ansatz leads the following system of equations

$$-\Delta u + Uu = \omega u, \tag{4.2}$$

$$\Delta U = |u|^2 + 4\pi M\delta, \quad \text{on } \mathbb{R}^3, \tag{4.3}$$

or, equivalently,

$$\left[-\Delta - \frac{M}{|x|} + \Delta^{-1}(|u|^2)\right] u = \omega u, \quad \text{on } \mathbb{R}^3, \tag{4.4}$$

where

$$\Delta^{-1}(|u|^2)(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{|u(y)|^2}{|x-y|}d^3y. \tag{4.5}$$

An ℓ-boson stars can be defined as a solution of Eq. (4.4), which are invariant under the (ℓ-spin) representation of $SO(3)$ defined at the end of chapter 2 in Eqs. (2.78) and (2.79). Specifically, we look for solutions satisfying

$$\rho^{(\ell)}(R)u = u, \quad \text{for all } R \in SO(3), \tag{4.6}$$

where $\ell \in \mathbb{N}_0$ and

$$\rho^{(\ell)} : SO(3) \to GL(L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1})), \tag{4.7}$$

is a unitary representation of the group $SO(3)$ on the Hilbert space $L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$, defined by

$$[\rho^{(\ell)}(R)u](x) := D^{(\ell)}(R)u(R^{-1}x). \tag{4.8}$$

Here $D^{(\ell)} : SO(3) \to GL(2\ell+1;\mathbb{C})$ is the finite-dimensional ℓ-spin irreducible unitary representation of the group $SO(3)$ on $\mathbb{C}^{2\ell+1}$.

It is natural to ask whether the variational methods employed in the previous chapters can be used to prove the existence of ℓ-boson stars. In other words, we are interested in determining whether the energy functional (3.6), when restricted to the class of functions in $H^1(\mathbb{R}^3, \mathbb{C}^K)$ satisfying Eq. (4.6), admits a minimizer. Note that for $\ell \geqslant 1$ this minimizer has a higher energy than the minimizer found in the previous section, since the condition (4.6) excludes the constant polarization state which describe the global minimum.

4.2 ℓ-boson stars as a minimum of a restricted energy functional

In order to formulate the variational problem, we must first specify the appropriate space of admissible functions. In contrast to Chapter 3, where the admissible space was $H^1(\mathbb{R}^3, \mathbb{C}^K)$, here it is necessary to restrict our attention to a proper subspace. Indeed, recall that ℓ-boson are required to satisfy Eq. (4.6). This condition naturally determines the subspace of $H^1(\mathbb{R}^3, \mathbb{C}^K)$ in which the variational problem should be posed. Therefore, we define the following minimization problem:

$$E^{(\ell)}(N) := \inf\{\mathcal{E}[u] : u \in H^1_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1}),\ \|u\|_2^2 = N\}, \tag{4.9}$$

where the energy functional $\mathcal{E}$ is defined as in Eq. (3.6) and

$$H^1_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1}) := \left\{u \in H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1}) \mid \rho^{(\ell)}(R)u = u \text{ for all } R \in SO(3)\right\}, \tag{4.10}$$

with the representation $\rho^{(\ell)} : SO(3) \to GL(L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1}))$ defined in Eq. (4.8). The space $H^1_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ is a Hilbert space with the scalar product inherited from $H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ (see Remark 2.12). When there is no risk of confusion, we write $H^1_{SO(3)}$ instead of $H^1_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ for brevity.

For what follows, it is convenient to define also the following Hilbert space

$$L^2_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1}) := \left\{u \in L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1}) \mid \rho^{(\ell)}(R)u = u \text{ for all } R \in SO(3)\right\}, \quad (4.11)$$

with the scalar product of $L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$.

Remark 4.1. Using that the matrices $D^{(\ell)}(R)$ are unitary and the rotation invariance of the Lebesgue measure in $\mathbb{R}^3$, one can verify that the Sobolev space $H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ is invariant under the action of the operators $\rho^{(\ell)}(R)$, that is,

$$\rho^{(\ell)}(R) : H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1}) \to H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1}). \quad (4.12)$$

The proofs of the following results can be found in Appendix A.

Proposition 4.1. *$H^1_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ is weakly closed in $H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$. Moreover, if (u_j) is a sequence in $H^1_{SO(3)}$ such that $u_j \rightharpoonup u$ weakly in $H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$, then (u_j) converge weakly to u in $L^p(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ for $2 \leq p \leq 6$.*

Lemma 4.1. *There is a constant $C > 0$, such that for every $u \in H^1_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$*

$$|u(x)| \leq \frac{C}{|x|}\|u\|_{H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})}, \quad \textit{for a.e. } x \in \mathbb{R}^3. \quad (4.13)$$

Theorem 4.1. *Let $2 < p < 6$. Then, the embedding*

$$H^1_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1}) \subset L^p(\mathbb{R}^3, \mathbb{C}^{2\ell+1}) \quad (4.14)$$

is compact.

Remark 4.2. Theorem 4.1 implies that if (u_j) is a bounded sequence in $H^1_{SO(3)}$, then there exist a subsequence (u_{j_k}) and a function $u \in L^p(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ such that $u_{j_k} \to u$ in $L^p(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ for $2 < p < 6$. Therefore, defining $n_k := |u_{j_k}|^2$ and $n := |u|^2$, and using Hölder's inequality, we obtain

$$\|n_k - n\|_q \leq (\|u_{j_k}\|_{2q} + \|u\|_{2q})\|u_{j_k} - u\|_{2q}, \quad (4.15)$$

which implies that $n_k \to n$ strongly in $L^q(\mathbb{R}^3, \mathbb{R})$ for $1 < q < 3$.

Similarly to the previous chapters, it turns out to be convenient to replace the variational problem (4.9) with the alternative minimization problem

$$E^{(\ell)}_{\leq}(N) := \inf\{\mathcal{E}[u] : u \in H^1_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1}),\ \|u\|_2^2 \leq N\}, \quad (4.16)$$

Proposition 4.2. *Let $N > 0$, $\ell \in \mathbb{N}_0$ and $M \geq 0$. Then, $E^{(\ell)}_{\leq}(N) \leq E^{(\ell)}(N) < 0$. Moreover, if u is a minimizer of the problem (4.16), then u is also a minimizer of the problem (4.9), that is, $\mathcal{E}[u] = E^{(\ell)}_{\leq}(N) = E^{(\ell)}(N)$.*

Proof. The proof is similar to the proof of part (i) of Proposition 3.3. ∎

The rescaling properties, seen in the previous chapters, still holds:

Lemma 4.2. *The functional $\mathcal{E}[u]$ satisfies the following rescaling properties:*

$$\mathcal{E}[\lambda u] = \lambda^2 T[u] - \lambda^2 L[n] - \lambda^4 D[n,n], \tag{4.17a}$$

$$\mathcal{E}[u_\lambda] = \lambda^2 T[u] - \lambda L[n] - \lambda D[n,n], \tag{4.17b}$$

for each $u \in H^1(\mathbb{R}^3, \mathbb{C}^K)$ and $\lambda > 0$, where $u_\lambda(x) := \lambda^{3/2} u(\lambda x)$ (note that $\|u_\lambda\|_2 = \|u\|_2$).

Proof. Direct verification. ∎

Lemma 4.3. *Let $N > 0$, $\ell \in \mathbb{N}_0$, and $M \geq 0$. Then, the functional $\mathcal{E} : \mathcal{A}^{SO(3)}_{\leq N} \to \mathbb{R}$ defined by Eq.(3.14) with*

$$\mathcal{A}^{SO(3)}_{\leq N} := \{u \in H^1_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1}) : \|u\|_2^2 \leq N\}. \tag{4.18}$$

is bounded below. Moreover, if (u_j) is a sequence in $\mathcal{A}^{SO(3)}_{\leq N}$ such that $\|u_j\|_{H^1} \to \infty$, then

$$\mathcal{E}[u_j] \to \infty, \quad \text{as } j \to \infty. \tag{4.19}$$

Proof. We can follow the same argument as the proof of Lemma 3.5, just changing the space $\mathcal{C}^r_{\leq N}$ by $\mathcal{A}^{SO(3)}_{\leq N}$. ∎

Theorem 4.2. *Suppose that $M \geq 0$. Then, for each $N > 0$ and $\ell \in \mathbb{N}_0$ we have:*

(i) The problem (4.9) admits a minimizer $u \in H^1_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$.

(ii) Every minimizing sequence (u_j) of (4.9) is relatively compact in $H^1_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$.

Proof. (i) It is sufficient to prove the existence of a minimizer of the alternative minimization problem (4.16), since by Proposition 4.2, such a minimizer satisfies $E^{(\ell)}_{\leq}(N) = E^{(\ell)}(N)$.

We proceed similarly to the proof of Theorem 3.1. Let (u_j) a minimizing sequence in $H^1_{SO(3)}$ such that $\|u_j\|_2^2 = N$ for all $j \geq 0$ and $\mathcal{E}[u_j] \to E^{(\ell)}_{\leq}(N)$. Lemma 4.3 implies that the sequence (u_j) is bounded in $H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ and by Theorem 2.5, there exist a subsequence (u_{j_k}) such that $u_{j_k} \rightharpoonup u$ weakly in $H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$. Furthermore, by Proposition 4.1, $u_{j_k} \rightharpoonup u$ weakly in $L^p(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ for $p \in [2, 6]$.

Since $H^1_{SO(3)}$ is weakly closed we have $u \in H^1_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ and by Theorem 2.4, $\|u\|_2^2 \leq N$. It remains to prove that u is a minimum of the energy functional $\mathcal{E}$.

First we recall that

$$T[u] \leq \liminf_{k\to\infty} T[u_{j_k}], \tag{4.20}$$

since the Dirichlet functional is weakly lower semi-continuous. Define $n := |u|^2$ and $n_k := |u_{j_k}|^2$. We study the convergence of the terms $L[n_k]$ and $D[n_k, n_k]$ defined in Eqs. (3.16) and (3.17) respectively. Similarly to the proof of Theorem 3.1 we obtain the following estimates for the functionals D and L

$$|D[n_k, n_k] - D[n, n]| \leq C_1 \|u_{j_k} - u\|_{12/5} \tag{4.21}$$

$$|L[n_k] - L[n]| \leq C_2 \|u_{j_k} - u\|_{17/5}, \tag{4.22}$$

where C_1 and C_2 are given by

$$C_1 = C(\|u_{j_k}\|_{12/5}^2 + \|u\|_{12/5}^2)(\|u_{j_k}\|_{12/5} + \|u\|_{12/5}), \tag{4.23}$$

$$C_2 = \|\mathcal{K}_1\|_{17/6}(\|u\|_{17/6} + \|u_{j_k}\|_{17/6}) + \|\mathcal{K}_2\|_{17/5}(\|u\|_{17/7} + \|u_{j_k}\|_{17/7}). \tag{4.24}$$

with $C > 0$ a constant. Theorem 4.1 implies that, up to a subsequence, $u_{j_k} \to u$ strongly in $L^p(\mathbb{R}^3)$ for $2 < p < 6$. Hence

$$\lim_{k\to\infty} D[n_k, n_k] = D[n, n] \quad \text{and} \quad \lim_{k\to\infty} L[n_k] = L[n]. \tag{4.25}$$

Combining (4.20) and (4.25) yields

$$E_{\leq}^{(\ell)}(N) \leq \mathcal{E}[u] \leq \liminf_{k\to\infty} \mathcal{E}[u_{j_k}] = E_{\leq}^{(\ell)}(N), \tag{4.26}$$

hence $E^{(\ell)}(N) = E_{\leq}^{(\ell)}(N) = \mathcal{E}[u]$ and u is a minimizer with $\|u\|_2^2 = N$.

(ii) Let (u_j) be a minimizing sequence of problem (4.16), such that $\|u_j\|_2^2 = N$ for all $j \geq 0$. Then, we can extract a subsequence (u_{j_k}) such that $u_{j_k} \rightharpoonup u$ weakly in $L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$. Furthermore, since all functions in the sequence have the same fixed L^2-norm, namely,

$$\|u_{j_k}\|_2^2 = \|u\|_2^2 = N, \quad \text{for all } k \geq 0, \tag{4.27}$$

it follows from Proposition 2.1 that $u_{j_k} \to u$ strongly in $L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$. In addition, from equality (4.26) we have

$$T[u] - L[n] - D[n, n] = \lim_{k\to\infty} \left(T[u_{j_k}] - L[n_k] - D[n_k, n_k]\right), \tag{4.28}$$

where, as before, $n := |u|^2$ and $n_k := |u_{j_k}|^2$. Since $D[n_k, n_k] \to D[n, n]$ and $L[n_k] \to L[n]$, it follows

$$\lim_{k\to\infty} T[u_{j_k}] = T[u]. \tag{4.29}$$

This implies that $\|\nabla u_{j_k}\|_2$ converges to $\|\nabla u\|_2$ and since we have already weak convergence in $H^1_{SO(3)}$, we apply again Proposition 2.1 to conclude that $\nabla u_{j_k} \to \nabla u$ strongly in $L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$. Therefore, $u_{j_k} \to u$ strongly in $H^1_{SO(3)}$, and the theorem is prove. ∎

Remark 4.3. The second statement of the Theorem 4.2 will be important for the study of orbital stability.

4.3 Properties of the minimum

This section is dedicated to present some key properties shared by any critical point of the variational problem (4.9). To this end, it is useful to recall the Hamiltonian operator

$$H := -\Delta - \frac{M}{|x|}, \tag{4.30}$$

describing the dynamics of the hydrogen atom in non-relativistic quantum mechanics.

First, using the fact that the Laplace operator $-\Delta$ with domain $H^2(\mathbb{R}^3)$ is self-adjoint in $L^2(\mathbb{R}^3)$, together with the Kato-Rellich theorem [67], one can show that H is also self-adjoint on the same domain. This guarantees that its spectrum is real.

The spectrum of H can be decomposed into the *point spectrum* $\sigma_{\mathrm{p}}(H)$ and the *essential spectrum* $\sigma_{\mathrm{ess}}(H)$, corresponding to the discrete and continuous parts, respectively. Explicitly, the point spectrum consist of the well-known energy levels

$$\sigma_{\mathrm{p}}(H) = \left\{ E_n = \frac{-M^2}{4n^2} : n = 1, 2, 3, \dots \right\}, \tag{4.31}$$

while the essential spectrum is given by

$$\sigma_{\mathrm{ess}}(H) = [0, \infty). \tag{4.32}$$

From a physical point of view, the values E_n in the point spectrum $\sigma_p(H)$ are the eigenvalues of the Hamiltonian operator H, and they correspond to the discrete energy levels of the hydrogen atom. Specifically, for each n, there exists a function $u_{n\ell m} \in H^2(\mathbb{R}^3)$ labeled by the quantum numbers n, ℓ, and m satisfying

$$H u_{n\ell m} = E_n u_{n\ell m}, \tag{4.33}$$

so that E_n represent the energy associated with the bound state $u_{n\ell m}$. In contrast, the essential spectrum $\sigma_{\mathrm{ess}}(H)$ corresponds to the continuum of energies associated with scattering states [73].

Theorem 4.3. *Assume that $N > 0$, $\ell \in \mathbb{N}_0$, and $M \geq 0$. Let u be a critical point of the energy functional $\mathcal{E}$ (defined in (3.6)) on the set $\{u \in H^1_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1}) : \|u\|_2^2 = N\}$. Then, u satisfies the following properties:*

(i) There exist a constant $\omega \in \mathbb{R}$ with $\omega < 0$, such that $u \in H^1_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ is a weak solution of the non-linear Schrödinger equation,

$$-\Delta u - \frac{M}{|x|} u + \Delta^{-1}(|u|^2) u = \omega u, \tag{4.34}$$

where the potential $\Delta^{-1}(|u|^2) \in L^p(\mathbb{R}^3)$ for all $3 < p \leq \infty$ is given by Eq. (3.4).

(ii) The critical point u belongs to $H^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ for $M > 0$ and to $H^s(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ for all $s \geq 1$ if $M = 0$. In particular, u is a strong solution of Eq. (4.34). Moreover, it is a classical solution if $M = 0$.

(iii) The energy functional satisfies the relations

$$\mathcal{E}[u] = -T[u] = \frac{1}{3}\left(\frac{\omega N}{2} - L[n] \right), \quad n := |u|^2. \tag{4.35}$$

Before to proving this theorem we give the following Lemmas (See Appendix A for the proofs):

Lemma 4.4. *The orthogonal projection*

$$P : L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1}) \to L^2_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1}) \tag{4.36}$$

satisfies the following properties:

(i) *P maps* $H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ *into* $H^1_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$.

(ii) *For all* $u, v \in H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$

$$(\nabla Pu, \nabla v)_{L^2} = (\nabla u, \nabla Pv)_{L^2}. \tag{4.37}$$

(iii) *Let* $f : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ *be a* $SO(3)$*-invariant measurable function and* $u \in L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ *such that the function* fu *belongs to* $L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$*. Then, P commutes with the multiplication by f, that is,* $P(fu) = fPu$.

Proof of Theorem 4.3.

(i) We proceed similarly to the proof of Theorem 3.2. Let $v \in H^1_{SO(3)}$ and define the variation

$$u(s) \ := \sqrt{N}\frac{u + sv}{\|u + sv\|_2}, \tag{4.38}$$

which is well define for $|s|$ small enough, since there exist $\delta > 0$ such that for $|s| < \delta$ we have $\|u + sv\|_2 > 0$. Then,

$$\frac{d}{ds}\mathcal{E}[u(s)]\bigg|_{s=0} = \delta\mathcal{E}[u; v] - \frac{2}{\sqrt{N}}\frac{d}{ds}\|u + sv\|_2\bigg|_{s=0} (T[u] - L[n] - 2D[n, n]), \tag{4.39}$$

where $n := |u|^2$. Since $\mathcal{E}$ is Gâteaux differentiable on $H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$, with derivative given in Eq. (3.35), and since u is a critical point of $\mathcal{E}$ on the space $H^1_{SO(3)}$ with $\|u\|_2^2 = N$, it follows that

$$\int_{\mathbb{R}^3}\left[\nabla v(x)^*\nabla u - \frac{M}{|x|}v(x)^*u(x) + v(x)^*\Delta^{-1}(|u|^2)(x)u(x) - \omega v(x)^*u(x)\right] d^3x = 0, \tag{4.40}$$

for all $v \in H^1_{SO(3)}$, where the Lagrange multiplier $\omega \in \mathbb{R}$ is given by

$$\omega := \frac{2}{N}(T[u] - L[n] - 2D[n, n]). \tag{4.41}$$

Equivalently, recalling the expression for the first variation of $\mathcal{E}$ is

$$\begin{aligned}\delta\mathcal{E}[u; v] &= \int_{\mathbb{R}^3}\left[\nabla v(x)^*\nabla u - \frac{M}{|x|}v(x)^*u(x) + v(x)^*\Delta^{-1}(|u|^2)(x)u(x)\right] d^3x \\ &= (\nabla v, \nabla u)_{L^2} - \left(\frac{M}{|x|}v, u\right)_{L^2} + (v, \Delta^{-1}(|u|^2)u)_{L^2},\end{aligned} \tag{4.42}$$

we can write Eq. (4.40) as follows

$$(\nabla v, \nabla u)_{L^2} + (v, Vu)_{L^2} - \omega(v, u)_{L^2} = 0, \quad \text{for all } v \in H^1_{SO(3)}, \tag{4.43}$$

where we have defined the function $V : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ by

$$V(x) := -\frac{M}{|x|} + \Delta^{-1}(|u|^2)(x). \tag{4.44}$$

Using the projector P defined in Lemma 4.4, we can further write

$$(\nabla Pv, \nabla u)_{L^2} + (Pv, Vu)_{L^2} - \omega(Pv, u)_{L^2} = 0, \quad \text{for all } v \in H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1}). \tag{4.45}$$

It is straightforward to prove that V is $SO(3)$-invariant and by Hardy's inequality and Lemma 3.1 we also have that $Vu \in L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$. Recalling that, by definition of an orthogonal projector, $(Pf, g)_{L^2} = (f, Pg)_{L^2}$ for all $f, g \in L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ and by Lemma 4.4, we obtain

$$(\nabla v, \nabla Pu)_{L^2} + (v, VPu)_{L^2} - \omega(v, Pu)_{L^2} = 0, \quad \text{for all } v \in H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1}). \tag{4.46}$$

Since $u \in H^1_{SO(3)}$, we have $Pu = u$. Therefore

$$\int_{\mathbb{R}^3} \left[\nabla v(x)^* \nabla u - \frac{M}{|x|} v(x)^* u(x) + v(x)^* \Delta^{-1}(|u|^2)(x) u(x) - \omega v(x)^* u(x) \right] d^3x = 0, \tag{4.47}$$

for all $v \in H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$. In other words, u is a weak solution of Eq. (4.34) in $H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$.

(ii) For any $\lambda \in \mathbb{C}$, Eq. (4.34) implies that u is a weak solution of

$$(H + \lambda)u = G, \quad G := -F[u] + (\omega + \lambda)u, \tag{4.48}$$

where the operator H is defined in expression (4.30) and $F : H^s(\mathbb{R}^3, \mathbb{C}^{2\ell+1}) \to H^s(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ for all $s \geq 1$ is defined by (see Lemma 3.1)

$$F[u] := \Delta^{-1}(|u|^2)u. \tag{4.49}$$

Thus, the function G on the right-hand side of Eq. (4.48) belongs to $H^1_{SO(3)}$. Since H is bounded from below, we can choose $\lambda \in \mathbb{R}$ large enough such that $H + \lambda$ is invertible [67]. Therefore,

$$u_1 := (H + \lambda)^{-1} G, \tag{4.50}$$

lies in $H^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ and is a weak solution of $(H + \lambda)u_1 = G$. By the uniqueness of the weak solution $u = u_1$, and hence u is a strong solution of Eq. (4.34). Furthermore, by the embedding theorem 2.7, it follows that $u \in C(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$.

If $M = 0$, we note that for $\lambda > 0$ and all $s \geq 0$, the operator $-\Delta + \lambda : H^{s+2}(\mathbb{R}^3, \mathbb{C}^{2\ell+1}) \to H^s(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ is invertible and the inverse satisfies $(-\Delta + \lambda)^{-1} : H^s(\mathbb{R}^3, \mathbb{C}^{2\ell+1}) \to H^{s+2}(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$. Therefore, $u \in H^4(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$. By induction we conclude that $u \in H^s(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ for all $s \geq 1$ and by the Sobolev embedding theorems 2.7, this implies that $u \in C^\infty(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$. Therefore, u is a classical solution of Eq. (4.34) for $M = 0$.

(iii) The proof is the same as point (ii) of Theorem 3.2.

■

Remark 4.4. For $M \geq 0$, Lemma 4.1 ensures that $u(x) \to 0$ as $|x| \to \infty$, with u decreasing asymptotically at least as $1/|x|$. In fact, the numerical solutions obtained from the shooting algorithm described in Chapter 5 indicates that the solutions decay exponentially for $|x| \to \infty$.

4.4 Orbital stability with respect to spherical perturbations

Assuming that there always exists a global solution of the Cauchy problem

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi - \frac{M}{|x|}\psi + \Delta^{-1}(|\psi|^2)\psi, \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^3 \tag{4.51a}$$

$$\psi(0,x) = \psi_0(x), \quad \text{with } \psi_0 \in H^1_{SO(3)}, \tag{4.51b}$$

in this section we prove the *orbital stability* of solutions of the form

$$\psi(t,x) = e^{-i\omega t}u(x), \tag{4.52}$$

where $u \in H^1_{SO(3)}$ is a minimum of the problem (4.9) and ω is given in Eq. (4.41).

Definition 4.1. Let X be a normed vector space and $A \subset X$ a subspace. Then, for $x \in X$ we define

$$\operatorname{dist}(x,A) := \inf_{y\in A} \|x-y\|_X. \tag{4.53}$$

Based in [17] and [32] we present the following definition of stability:

Definition 4.2 (Orbital stability)**.** Suppose that $N > 0$, $\ell \in \mathbb{N}_0$ and $M \geq 0$. Let $S_{N,\ell}$ denoting the set of minimizers of the variational problem (4.9), i. e.,

$$S_{N,\ell} := \{u \in H^1_{SO(3)} \ : \ \mathcal{E}[u] = E^{(\ell)}(N), \ \|u\|_2^2 = N\}. \tag{4.54}$$

Solutions of the form (4.52) with $u \in S_{N,\ell}$, are **orbital stable** if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\psi_0 \in H^1_{SO(3)}$ satisfies

$$\operatorname{dist}(\psi_0, S_{N,\ell}) \leq \delta, \tag{4.55}$$

then the corresponding solution $\psi(t)$ of the Cauchy problem (4.51) satisfies

$$\operatorname{dist}(\psi(t), S_{N,\ell}) \leq \varepsilon \quad \text{for all } t \geq 0. \tag{4.56}$$

Intuitively, a solution of the Cauchy problem (4.51) of the form (4.52) is said to be orbital stable if, under sufficiently small perturbations, its subsequent evolution remains close to the family of minimizers $S_{N,\ell}$.

Theorem 4.4. *Assuming that the Cauchy problem (4.51) has global solutions, the stationary solutions of the form (4.52), with $u \in S_{N,\ell}$ and ω given in Eq. (4.41) are orbital stable.*

Proof. Suppose by contradiction that the theorem is false. Then there exists $\varepsilon_0 > 0$, a sequence of initial data $(\psi_0^j) \in H^1_{SO(3)}$ and $t_j \geq 0$ such that

$$\inf_{u \in S_{N,\ell}} \|\psi_0^j - u\|_{H^1} \to 0 \quad \text{when } j \to \infty \tag{4.57}$$

and

$$\inf_{u \in S_{N,\ell}} \|\psi^j(t_j) - u\|_{H^1} \geq \varepsilon_0 \quad \text{for all } j \geq 0. \tag{4.58}$$

By continuity of the L^2-norm (indeed, every norm is continuous) and from (4.57) we deduce that $\|\psi_0^j\|_2^2 \to N$. Moreover, Proposition 3.1 ensures that the energy functional $\mathcal{E} : H^1_{SO(3)} \to \mathbb{R}$ is continuous. Since both, $\|\psi(t)\|_2$ and $\mathcal{E}[\psi(t)]$ are conserved quantities, we obtain

$$\|\psi^j(t_j)\|_2^2 = \|\psi_0^j\|_2^2 \to N \quad \text{and} \quad \mathcal{E}[\psi^j(t_j)] = \mathcal{E}[\psi_0^j] \to E^{(\ell)}(N) \tag{4.59}$$

If ψ_0^j satisfies $\|\psi_0^j\|_2^2 = N$, then it would follow from Eqs. (4.59) that $\psi^j(t_j)$ is a minimization sequence and according to Theorem 4.2 it would have a subsequence converging strongly to a minimizer u in $H^1_{SO(3)}$, which would lead to a contradiction with Eq. (4.58).

Otherwise, we define the rescaled sequence

$$\phi_j := \sqrt{\frac{N}{M_j}}\psi^j(t_j) \quad \text{with} \quad M_j := \|\psi^j(t_j)\|_2^2 = \|\psi_0^j\|_2^2, \tag{4.60}$$

which satisfies $\|\phi_j\|_2^2 = N$. From the boundedness of the sequence $(\psi^j(t_j))$ (see Lemma 3.5) and the convergence $M_j \to N$, we have

$$\|\phi_j - \psi^j(t_j)\|_{H^1} \leq C\left|1 - \frac{N}{M_j}\right| \to 0 \quad \text{as } j \to \infty, \tag{4.61}$$

for some constant $C > 0$. Once again, since the energy functional $\mathcal{E} : H^1_{SO(3)} \to \mathbb{R}$ is continuous, it follows

$$\mathcal{E}[\phi_j] - \mathcal{E}[\psi^j(t_j)] \to 0 \quad \text{as } j \to \infty. \tag{4.62}$$

Hence,

$$\lim_{j\to\infty} \mathcal{E}[\phi_j] = E^{(\ell)}(N) \quad \text{and} \quad \|\phi_j\|_2^2 = N \quad \text{for all } j \geq 0. \tag{4.63}$$

Thus, ϕ_j is a minimizing sequence for the problem (4.9). By Theorem 4.2, there exist a subsequence (ϕ_{j_k}) such that $\phi_{j_k} \to u$ strongly in $H^1_{SO(3)}$, for some $u \in S_{N,\ell}$, that is

$$\|\phi_{j_k} - u\|_{H^1} \to 0 \quad \text{as } j \to \infty. \tag{4.64}$$

Therefore, in view of expression (4.61) we have

$$\|\psi^{j_k}(t_{j_k}) - u\|_{H^1} \leq \|\phi_{j_k} - \psi^{j_k}(t_{j_k})\|_{H^1} + \|\phi_{j_k} - u\|_{H^1} \to 0 \quad \text{as } k \to \infty, \tag{4.65}$$

which is a contradiction with the inequality (4.58). ■

Remark 4.5. It is important to stress that (assuming global existence) Theorem 4.4 establishes the orbital stability of ℓ-boson stars with respect to spherically symmetric perturbations only (i.e. for perturbations within $H^1_{SO(3)}(\mathbb{R}^3,\mathbb{C}^{2\ell+1})$). Despite this limitation, our result generalizes the linearized study in [70] to the full nonlinear case and to arbitrary values of $\ell \in \mathbb{N}_0$, and it is also compatible with the results from previous stability studies for relativistic ℓ-boson stars [4, 5] with respect to spherical perturbations. A relevant open problem is whether orbital stability can also be established for arbitrary perturbations in $H^1(\mathbb{R}^3,\mathbb{C}^{2\ell+1})$. This is clearly the case for $\ell = 0$ (see previous chapter). Numerical results in both the nonrelativistic [40] and relativistic regimes [47, 74] indicate that $(\ell = 1)$-boson stars may also be orbital stable. However, the results presented in chapter 5 strongly suggest that ℓ-boson stars with $\ell \geqslant 2$ cannot be orbital stable.

Remark 4.6. For the case $M = 0$ and $\ell = 0$ global existence of solutions to the Cauchy problem (4.51) has been established in [48].

4.5 Explicit form of ℓ-boson stars

Proposition 4.3. *Assume $N > 0$, $\ell \in \mathbb{N}_0$ and $M = 0$. Let $u \in H^1_{SO(3)} \cap C^\infty(\mathbb{R}^3,\mathbb{C}^{2\ell+1})$ be a critical point of the energy functional $\mathcal{E}$ as in Theorem 4.3. Then:*

(i) There exists function f_ℓ belonging to space

$$X_r(\mathbb{R}^3) := \{f \in C^\infty(\mathbb{R}^3) : f(Rx) = f(x) \text{ for all } R \in SO(3)\} \tag{4.66}$$

such that

$$u(x) = f_\ell(x)\mathcal{Y}_\ell\left(\frac{x}{|x|}\right), \tag{4.67}$$

where

$$\mathcal{Y}_\ell\left(\frac{x}{|x|}\right) := \sqrt{\frac{4\pi}{2\ell+1}}\begin{pmatrix} Y^{\ell,-\ell}(x/|x|) \\ Y^{\ell,-\ell+1}(x/|x|) \\ \vdots \\ Y^{\ell,\ell}(x/|x|) \end{pmatrix} \quad \text{and} \quad f_\ell(x) = |x|^\ell g_\ell(x) \tag{4.68}$$

with $Y^{\ell,m}$ denoting the spherical harmonics and $g_\ell \in X_r(\mathbb{R}^3)$.

(ii) The function $f_\ell \in X_r(\mathbb{R}^3)$ satisfies the equation

$$-\Delta_r f_\ell + \frac{\ell(\ell+1)}{r^2} f_\ell + \Delta^{-1}(|f_\ell|^2) f_\ell = \omega f_\ell. \tag{4.69}$$

Proof. (i) Each representation of a Lie group G induces a corresponding representation of its Lie algebra $\mathfrak{g}$. In particular, the associated Lie algebra representation $\varrho^{(\ell)} : \mathfrak{so}(3) \to \mathfrak{gl}(L^2(\mathbb{R}^3,\mathbb{C}^{2\ell+1}))$ of the group representation defined in Eq. (4.8), is defined by

$$[\varrho^{(\ell)}(X)u](x) = \left.\frac{d}{ds}\right|_{s=0}\left[\rho^{(\ell)}\left(e^{isX}\right)u\right](x), \tag{4.70}$$

where $\mathfrak{so}(3)$ denotes the Lie algebra of $SO(3)$ and $\mathfrak{gl}(L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1}))$ is the Lie algebra of the general linear group $GL(L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1}))$, that is, the space of linear operators on $L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$.

A standard basis for $\mathfrak{so}(3)$ is given by the matrices

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

corresponding to the infinitesimal generators of rotations along the x, y and z axis, respectively. Then, from Eqs. (4.8) and (4.70), the induced Lie algebra acts as

$$[\varrho^{(\ell)}(X_k)u](x) = \left.\frac{d}{ds}\right|_{s=0} D^{(\ell)}\left(e^{isX_k}\right) u\left(e^{-isX_k}x\right) = J_k u(x). \tag{4.71}$$

where we define the operators

$$J_k := S_k + L_k, \quad k = 1, 2, 3, \tag{4.72}$$

with

$$S_k := \left.\frac{d}{ds}\right|_{s=0} D^{(\ell)}(e^{isX_k}) \quad \text{and} \quad L_k := -i(x \wedge \nabla)_k, \quad k = 1, 2, 3. \tag{4.73}$$

In what follows, we refer to $\hat{J} := (J_1, J_2, J_3)$ as the *total angular momentum* operator and to $\hat{S} = (S_1, S_2, S_3)$ and $\hat{L} = (L_1, L_2, L_3)$ as the *spin angular momentum* and *orbital angular momentum* operators, respectively. Hence, condition (4.6) yields

$$\hat{J}u = 0, \tag{4.74}$$

which in turn implies

$$\hat{J}^2 u = 0, \tag{4.75}$$

where the Casimir operator $\hat{J}^2$ is defined by

$$\hat{J}^2 = J_1^2 + J_2^2 + J_3^2. \tag{4.76}$$

On the other hand, the *tensor spherical harmonics* ${Y^{JM}}_{L\ell}$ describe the angular distribution and polarization of spin ℓ particles with total angular momentum J, total magnetic quantum number M, and orbital angular momentum L. They are constructed from the standard (scalar) spherical harmonics Y^{Lm} (which are eigenfunctions of the operators $\hat{L}^2$ and L_3) and basis spin functions $\xi^{\ell\sigma}$ (which are eigenfunctions of $\hat{S}^2$ and S_3) in accordance with the addition of angular momenta in quantum mechanics [50]:

$${Y^{JM}}_{L\ell}(\vartheta, \varphi) := \sum_{m,\sigma} {C^{JM}}_{Lm\ell\sigma} Y^{Lm}(\vartheta, \varphi)\xi^{\ell\sigma}. \tag{4.77}$$

Here, ${C^{JM}}_{Lm\ell\sigma}$ denote the Clebsch-Gordan coefficients and J, ℓ are nonnegative integer. Given a pair (J, ℓ), the admissible values for L and M are: $L = |J-\ell|, |J-\ell|+1, \ldots, J+\ell$ and $M = -J, \ldots, J$. The basis spin functions $\xi^{\ell\sigma}$ satisfy the conditions

$$\hat{S}^2\xi^{\ell\sigma} = \ell(\ell+1)\xi^{\ell\sigma}, \quad S_3\xi^{\ell\sigma} = \sigma\xi^{\ell\sigma}, \quad \sigma = -\ell, \ldots, \ell, \tag{4.78}$$

and since S_3 is self-adjoint, they form an orthonormal basis of $\mathbb{C}^{2\ell+1}$ after suitable normalization. Using these properties and those of the scalar spherical harmonics, it is not difficult to verify that

$$\hat{J}^2 Y^{JM}{}_{L\ell} = J(J+1)Y^{JM}{}_{L\ell}, \tag{4.79}$$

$$J_3 Y^{JM}{}_{L\ell} = MY^{JM}{}_{L\ell}, \tag{4.80}$$

$$\hat{L}^2 Y^{JM}{}_{L\ell} = L(L+1)Y^{JM}{}_{L\ell}, \tag{4.81}$$

$$\hat{S}^2 Y^{JM}{}_{L\ell} = \ell(\ell+1)Y^{JM}{}_{L\ell}. \tag{4.82}$$

Furthermore, it follows that the collection of tensor spherical harmonics with the same ℓ and all possible J, L, M constitutes a complete orthonormal set in $L^2(\mathbb{S}^2, \mathbb{C}^{2\ell+1})$, where $\mathbb{S}^2$ refers to the two-sphere. The orthonormality condition is

$$\int\limits_{\mathbb{S}^2} [Y^{JM}{}_{L\ell}(\vartheta,\varphi)]^* Y^{JM}{}_{L\ell}(\vartheta,\varphi) d\Omega = \delta^{JJ'}\delta_{MM'}\delta_{LL'}. \tag{4.83}$$

Hence, we can write any function $u \in C^\infty(\mathbb{R}^3, \mathbb{C}^{2\ell+1}) \cap L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ as follows

$$u(x) = \sum_{JML} v_{JM}{}^L(x) Y^{JM}{}_{L\ell}\left(\frac{x}{|x|}\right), \tag{4.84}$$

where $v_{JM}{}^L \in X_r(\mathbb{R}^3)$ for every J, M, L. Therefore, returning to Eq. (4.75) it follows

$$\hat{J}^2 u(x) = \sum_{JML} J(J+1) v_{JM}{}^L(x) Y^{JM}{}_{L\ell}\left(\frac{x}{|x|}\right) = 0, \quad \text{for every } x \in \mathbb{R}^3, \tag{4.85}$$

which in turn implies, by the orthogonality $Y^{JM}{}_{L\ell}$, that $v_{JM}{}^L = 0$ for all $J > 0$, such that

$$u(x) = v_{00}{}^\ell(x) Y^{00}_{\ell\ell}\left(\frac{x}{|x|}\right). \tag{4.86}$$

Using the identity

$$Y^{00}_{\ell\ell} = \frac{1}{\sqrt{4\pi}}\mathcal{Y}_\ell, \quad \mathcal{Y}_\ell := \sqrt{\frac{4\pi}{2\ell+1}}\begin{pmatrix} Y^{\ell,-\ell} \\ Y^{\ell,-\ell+1} \\ \vdots \\ Y^{\ell,\ell} \end{pmatrix}, \tag{4.87}$$

we finally obtain

$$u(x) = f_\ell(x)\mathcal{Y}_\ell\left(\frac{x}{|x|}\right), \tag{4.88}$$

where $f_\ell \in X_r(\mathbb{R}^3)$ is a radial function. By Theorem 4.3, u belongs to $H^1_{SO(3)} \cap C^\infty(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$, which implies that $f_\ell \mathcal{Y}_\ell \in H^1_{SO(3)} \cap C^\infty(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$. Every spherical harmonic $Y^{\ell m}$ can be written as [8]

$$Y^{\ell m}\left(\frac{x}{|x|}\right) = \frac{h_{\ell m}(x)}{|x|^\ell}, \quad \text{for every } x \in \mathbb{R}^3 \setminus \{0\}, \tag{4.89}$$

where $h_{\ell m}$ is an harmonic homogeneous polynomial of degree ℓ. Therefore, in order for $f_\ell \mathcal{Y}_\ell$ to belong $H^1_{SO(3)} \cap C^\infty(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$, the function f_ℓ must have the form

$$f_\ell(x) = |x|^\ell g_\ell(x), \tag{4.90}$$

with $g_\ell \in X_r(\mathbb{R}^3)$.

(ii) Substituting the explicit form (4.67) of u into Eq. (4.34) (with $M = 0$) and using the fact that

$$\Delta \mathcal{Y}_\ell = -\frac{\ell(\ell+1)}{r^2}\mathcal{Y}_\ell, \quad r := |x|, \tag{4.91}$$

we arrive at Eq. (4.69). ■

Finally we close this chapter presenting some results about excited states. Theorem 4.2 established the existence of a restricted minimizer of the energy functional $\mathcal{E}$ introduced in Eq. (3.6). Furthermore, Theorem 4.3 shows that, when $M = 0$, this minimizer is indeed a classical solution of Eq. (4.34) (or equivalently os system (4.2)). In other words, ground states of ℓ-boson stars exist for each fixed $\ell \geq 0$.

Having settled the question about ground states, it is natural to turn our attention to the excited states. To investigate their existence, we first reduce system (4.2) by substituting the explicit ansatz for ℓ-boson stars provided in Proposition 4.3 using spherical coordinates (r, ϑ, φ), namely

$$u(x) = r^\ell g_\ell(r)\mathcal{Y}_\ell(\vartheta, \varphi), \tag{4.92}$$

This leads to the following radial system of differential equations:

$$\frac{1}{r^{2(\ell+1)}}\frac{d}{dr}\left[r^{2(\ell+1)}\frac{dg_\ell}{dr}\right] = \mathcal{U} g_\ell, \tag{4.93a}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\mathcal{U}}{dr}\right) = r^{2\ell} g_\ell^2, \tag{4.93b}$$

with the boundary conditions

$$g_\ell(r=0) = a, \quad \left.\frac{dg_\ell}{dr}\right|_{r=0} = 0, \tag{4.94a}$$

$$\mathcal{U}(r=0) = \mathcal{U}_0, \quad \left.\frac{d\mathcal{U}}{dr}\right|_{r=0} = 0, \tag{4.94b}$$

where $\mathcal{U} := U - \omega$ and, a and $\mathcal{U}_0$ are real constants. A rigorously study of this system can be found in [22], and here we present the more relevant results.

Theorem 4.5 (Existence of a local solution)**.** *Let $a, \mathcal{U}_0 \in \mathbb{R}$ and $R > 0$ a positive real number. Then, there exist a twice continuously differentiable solution $(g_\ell, \mathcal{U}) : (0, R) \to$*

$\mathbb{R}^2$ *of the problem*

$$\frac{1}{r^{2(\ell+1)}}\frac{d}{dr}\left[r^{2(\ell+1)}\frac{dg_\ell}{dr}\right] = \mathcal{U}g_\ell, \tag{4.95a}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\mathcal{U}}{dr}\right) = r^{2\ell}g_\ell^2, \tag{4.95b}$$

$$\lim_{r\to 0}(g_\ell(r),\mathcal{U}(r)) = (a,\mathcal{U}_0), \tag{4.95c}$$

$$\lim_{r\to 0}\left(\frac{dg_\ell}{dr},\frac{d\mathcal{U}}{dr}\right) = (0,0). \tag{4.95d}$$

Proof. See Theorem 2 in [22]. ■

Definition 4.3. Let $r_*(a) > 0$ a real number defined by

$$r_*(a) := \sup\left\{ r_1 > 0 \;\middle|\; \begin{array}{l}\text{There exist solution } (g_\ell,\mathcal{U}) \in C^2((0,r_1),\mathbb{R}^2) \\ \text{of the problem (4.95)}\end{array}\right\}, \tag{4.96}$$

where $C^2((0,r_1),\mathbb{R}^2)$ denotes the space of functions $f : (0,r_1) \to \mathbb{R}^2$ which are twice continuously differentiable. Then, the maximal extension of the solution $(g_\ell,\mathcal{U}) : (0,r_*(a)) \to \mathbb{R}^2$ of the problem (4.95) is called the a-orbit.

Definition 4.4. The a-orbits which exist for all $r > 0$ are called globally defined.

Remark 4.7. It can be shown [22] that the a-orbits which are globally defined are automatically normalizable at infinity, that is, they satisfy

$$\int_0^\infty |g_\ell(r)|^2 r^2 dr < \infty, \tag{4.97}$$

such that $\|u\|_2^2 = N$, with $N > 0$ a positive constant.

Theorem 4.6. *For each $\mathcal{U}_0 < 0$, there exists a decreasing sequence of parameters a_n $(n = 0,1,2,\ldots)$ such that the corresponding a_n-orbits are globally defined, where the index n labels the number of nodes of the function g_ℓ. Moreover, the function $|g_\ell|$ decays exponentially.*

Proof. See Theorem 1 in [22]. ■

Examples for the radial profile of the ℓ-boson stars, for different values of ℓ, are shown in Fig. 4.1. Furthermore, Fig. 4.2 shows the Newtonian potential corresponding to the ground and first excited states and $\ell = 0,1,2,3$. For a detailed discussion about the numerical construction of solutions of system (4.93), see Section 5.5 of the next chapter.

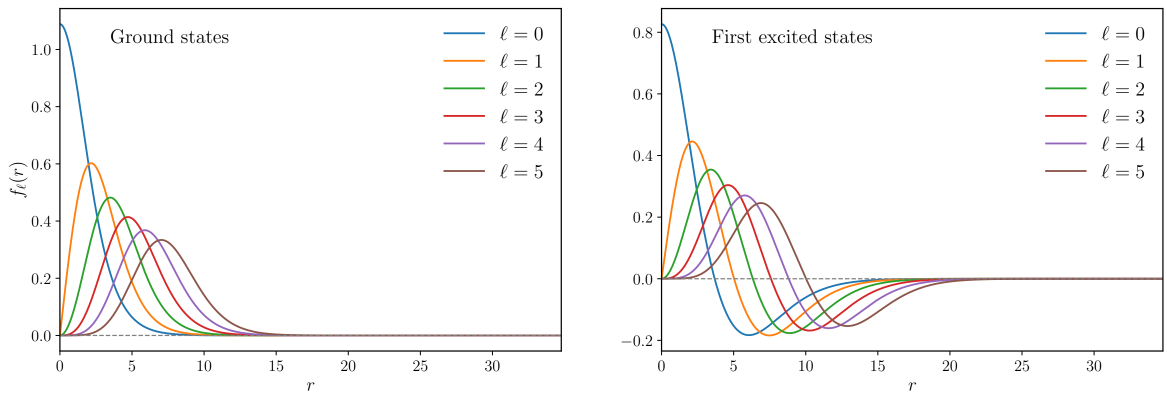

Figure 4.1: Radial profiles $f_\ell(r) = r^\ell g_\ell(r)$ for different values of ℓ. Left panel: configurations in the ground state with the ℓ-spin representation ($n = 0$). Right panel: configurations in the first excited state ($n = 1$).

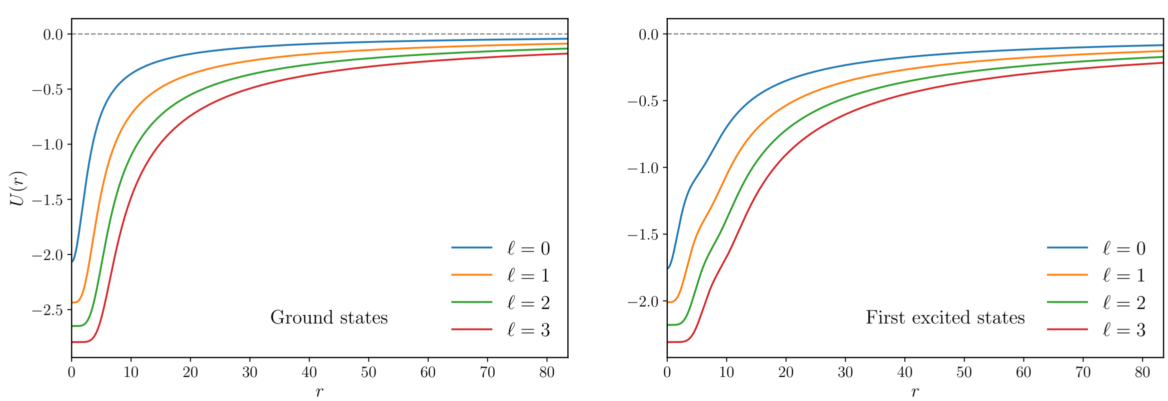

Figure 4.2: Profiles for the gravitational potential for the configurations shown in Fig. 4.1 corresponding to $\ell = 0, 1, 2, 3$.

Chapter 5

Linear stability of ℓ-boson stars under non-spherical perturbations

In the previous section we established the existence and orbital stability of solutions for the multi-field Schrödinger-Poisson equation (4.34) in the spherically symmetric sector, given a representation of $SO(3)$. These solutions, known as ℓ-boson stars, have the form (4.67). A distinctive feature of ℓ-boson stars is that both the orbital and spin angular momenta are nonzero, while the total angular momentum vanishes.

In this section, for simplicity, we take $M = 0$ in Eq. (3.16), which yields the following expression for the energy functional

$$\mathcal{E}[u] = T[u] - D[n, n], \tag{5.1}$$

and by Theorem 4.3, any critical point $u \in \mathcal{D}_{SO(3)}$ of the energy functional, such that $\|u\|_2^2 = N$ satisfies Eq. (4.34), that is,

$$-\Delta u + \Delta^{-1}(|u|^2)u = \omega u. \tag{5.2}$$

We want to study the stability of ℓ-boson stars under general (non-spherically symmetric) perturbations. To this end we begin by observing the following fact:

Proposition 5.1. *Assume $N \geq 0$ and $\ell \geq 1$. Let $u \in H^1_{SO(3)}$ be a critical point of the energy functional (5.1) such that $\|u\|_2^2 = N$. Then u is a saddle point of $\mathcal{E}$ on the space $\mathcal{A}_N$ defined in (3.40).*

Proof. Let $v \in H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ and consider the variation

$$u(s) := \sqrt{N}\frac{u + sv}{\|u + sv\|_2}, \tag{5.3}$$

which is well define for $|s|$ small enough. Then,

$$\delta_r^2\mathcal{E}[u; v] := \frac{d^2}{ds^2}\mathcal{E}[u(s)]\bigg|_{s=0} = \delta^2\mathcal{E}[u; v] - \omega\|v\|_2^2. \tag{5.4}$$

Note the difference between $\delta_r^2\mathcal{E}$ and $\delta^2\mathcal{E}$, which arises by constraining the variation to the subset $\mathcal{A}_N$ in the former case, whereas in the latter the second variation is computed with arbitrary $v \in H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ according to Eq. (3.33) and is given by

$$\delta^2\mathcal{E}[u;v] = \delta\mathcal{E}[v;v] - \frac{1}{2\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\mathrm{Re}[v^*(x)u(x)]\mathrm{Re}[v^*(y)u(y)]}{|x-y|}d^3yd^3x, \tag{5.5}$$

and consequently

$$\delta_r^2\mathcal{E}[u;v] = \delta\mathcal{E}[v;v] - \omega\|v\|_2^2 - \frac{1}{2\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\mathrm{Re}[v^*(x)u(x)]\mathrm{Re}[v^*(y)u(y)]}{|x-y|}d^3yd^3x. \tag{5.6}$$

Proposition 4.3 implies that the critical point has the form $u = f_\ell\mathcal{Y}_\ell$ and its components are given by

$$u_m = f_\ell Y^{\ell m}, \quad m \in \{-\ell, -\ell+1, \ldots, \ell-1, \ell\}. \tag{5.7}$$

Next, consider the variation $v := (v_1, v_2, \ldots, v_{2\ell+1})$ with components

$$v_m := i f_\ell Y^{LM}\delta_{mM}, \tag{5.8}$$

where $L \in \mathbb{N}_0$ is fixed, $M \in \{-L, -L+1, \ldots, L-1, L\}$ for $L < \ell$ and $M \in \{-\ell, -\ell+1, \ldots, \ell+1, \ell\}$ for $L > \ell$. Substituting into Eq. (5.6), taking into account that f_ℓ satisfies Eq. (4.69) and that $Y^{\ell m*}(x/|x|)Y^{Lm}(x/|x|) \in \mathbb{R}$ for every $x \in \mathbb{R}^3$ (which implies $\mathrm{Re}(v^*u) = 0$), we obtain

$$\delta_r^2\mathcal{E}[f_\ell, f_\ell] = [L(L+1) - \ell(\ell+1)]\left\|\frac{f_\ell}{|x|}\right\|_2^2. \tag{5.9}$$

By Hardy's inequality $\|f_\ell/|x|\|_2 < \infty$. Therefore, $\delta_r^2\mathcal{E}[u;v] < 0$ for $L < \ell$ and $\delta_r^2\mathcal{E}[u;v] > 0$ for $L > \ell$. ∎

Remark 5.1. It is important to note that even if a critical point u of the energy functional $\mathcal{E}$ is a saddle point, this does not necessary imply that u is unstable. To illustrate this, consider the following simple example: Let $H : \mathbb{R}^3 \to \mathbb{R}$ be a function defined by

$$H(x,y,z) = x^2 + y^2 - z^2, \tag{5.10}$$

which has a saddle point at $(x,y,z) = (0,0,0)$. Next, consider the dynamical system

$$\begin{aligned} &\dot{x}(t) = y(t), \quad \dot{y}(t) = x(t), \quad \dot{z}(t) = 0, \\ &x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z, \end{aligned} \tag{5.11}$$

where the dot denotes derivative with respect to time. Solving system (5.11) we obtain

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}, \tag{5.12}$$

and clearly $H(x(t), y(t), z(t)) = H(x_0, y_0, z_0)$ for all $t \in \mathbb{R}$. Since the dynamics in the (x,y)-plane describe circular motion the trajectories are confined to constant height z_0, tracing closed orbits in the (x, y)-plane. Consequently, the equilibrium point $(0, 0, 0)$ is orbital stable, despite the fact that it is a saddle point of the function H. This stability arises from the conservation of H and the structure of the dynamical system, which also conserves z under the evolution. Therefore, even though H decreases as z_0 increases and vice versa, the system cannot evolve in this direction since $z(t) = z_0$ is constant along trajectories.

It would be interesting to obtain a better stability criteria for saddle points. We hope to address this issue in future work. However, to gain more insight into the stability problem we now resort to linearized analysis.

The reminder of this chapter is based on the published article [65] and is devoted to studying the linear stability of ℓ-boson stars under non-spherical perturbations, using a combination of analytical and numerical techniques.

5.1 Linearization of the system

In this section we linearize Eq. (1.78) about a stationary solution u with the form

$$u(x) = f_\ell(r)\mathcal{Y}_\ell(\vartheta, \varphi). \tag{5.13}$$

where (r, ϑ, φ) denotes the spherical coordinates. In order to do this, we assume an expansion of ψ in terms of a small parameter $\varepsilon > 0$ of the form

$$\psi(t, x) = e^{-i\omega t}[u(x) + \varepsilon v(t, x) + \mathcal{O}(\varepsilon^2)], \quad t \in \mathbb{R}_+ \text{ and } x \in \mathbb{R}^3. \tag{5.14}$$

Here, v is a column vector in which each component is a complex-valued function and $(\omega,\, u)$ is a solution to Eq. (5.2). Substituting the expansion (5.14) into Eq. (1.78) and considering the first-order terms we arrive at the perturbed evolution equation

$$i\frac{\partial v}{\partial t} = \left(\hat{\mathcal{H}}_u - \omega\right)v + 2\Delta^{-1}\left(\mathrm{Re}\{u^* v\}\right)u, \tag{5.15}$$

where we have introduced the operator

$$\hat{\mathcal{H}}_u := -\Delta + \Delta^{-1}(|u|^2). \tag{5.16}$$

A natural question is wether the ℓ-boson stars are stable under nonspherical linear perturbations. To answer this we follow Refs. [44, 70] and separate the time and spatial parts of u using the ansatz

$$v(t, x) = e^{\lambda t}\left[\mathcal{A}(x) + \mathcal{B}(x)\right] + e^{\lambda^* t}\overline{\left[\mathcal{A}(x) - \mathcal{B}(x)\right]}, \tag{5.17}$$

where the bar denotes complex conjugation. Here $\mathcal{A}$ and $\mathcal{B}$ are complex vector-valued functions depending only on $x \in \mathbb{R}^3$ and λ is a complex number. Note that when $\lambda = \lambda^*$ is real, one can assume that $\mathcal{A}$ is real and $\mathcal{B}$ is purely imaginary.

Introducing Eq. (5.17) into Eq. (5.15) one obtains, after setting the coefficients in front of $e^{\lambda^* t}$ and $e^{\lambda t}$ to zero,

$$\begin{aligned} i\lambda\mathcal{A} &= \left(\hat{\mathcal{H}}_u - \omega\right)\mathcal{B} \qquad (5.18a)\\ &+ i\left\{\Delta^{-1}[u^*(\mathcal{A}+\mathcal{B}) + u^T(\mathcal{A}-\mathcal{B})]\right\}\mathrm{Im}(u),\\ i\lambda\mathcal{B} &= \left(\hat{\mathcal{H}}_u - \omega\right)\mathcal{A} \qquad (5.18b)\\ &+ \left\{\Delta^{-1}[u^*(\mathcal{A}+\mathcal{B}) + u^T(\mathcal{A}-\mathcal{B})]\right\}\mathrm{Re}(u). \end{aligned}$$

These two equations remain correct for the case in which λ is real, provided $\mathcal{A} = \mathcal{A}_R$ is assumed to be real and $\mathcal{B} = i\mathcal{B}_I$ is purely imaginary. In this case,

$$u^*(\mathcal{A}+\mathcal{B}) + u^T(\mathcal{A}-\mathcal{B}) = 2\mathrm{Re}(u)^T\mathcal{A}_R + 2\mathrm{Im}(u)^T\mathcal{B}_I, \qquad (5.19)$$

which is real. Note also that when u is real, Eqs. (5.18) simplify considerably.

5.2 Linear perturbation for arbitrary ℓ using tensor spherical harmonics

In this section we want to analyze the linear stability under non-spherical perturbations of ℓ-boson stars. To this end, we expand the fields in terms of tensor spherical harmonics ${Y^{JM}}_{L\ell}$ which are eigenfunctions of the operators $\hat{J}^2$, $\hat{L}^2$, $\hat{S}^2$ and $\hat{J}_z$ [50]. They are defined by

$${Y^{JM}}_{L\ell}(\vartheta,\varphi) := \sum_{m,\sigma} {C^{JM}}_{Lm\ell\sigma} Y^{Lm}(\vartheta,\varphi)\xi^{\ell\sigma}, \qquad (5.20)$$

with ${C^{JM}}_{Lm\ell\sigma}$ the Clebsch-Gordan coefficients and $\xi^{\ell\sigma}$ denoting an orthonormal basis of spin functions in $\mathbb{C}^{2\ell+1}$. Note that

$${Y^{00}}_{\ell\ell} = \frac{(-1)^\ell}{\sqrt{2\ell+1}}\sum_\sigma (Y^{\ell\sigma})^*\xi^{\ell\sigma}, \qquad (5.21)$$

and for a suitable choice of the basis functions $\xi^{\ell\sigma}$ and using that

$${C^{00}}_{\ell m\ell\sigma} = \frac{(-1)^{\ell-m}}{\sqrt{2\ell+1}}\delta_{m,-\sigma}. \qquad (5.22)$$

one obtains

$${Y^{00}}_{\ell\ell} = \frac{1}{\sqrt{4\pi}}\mathcal{Y}_\ell. \qquad (5.23)$$

However, for the following we shall assume that the basis spin functions satisfy the relation

$$\overline{\xi^{\ell\sigma}} = (-1)^\sigma\xi^{\ell-\sigma} \qquad (5.24)$$

for all $\sigma = -\ell,\dots,\ell$, which implies that

$$\overline{{Y^{JM}}_{L\ell}} = (-1)^{J+M+L+\ell}{Y^{J-M}}_{L\ell}, \qquad (5.25)$$

and, in particular, that ${Y^{00}}_{\ell\ell}$ is real-valued. For $\ell = 1$, for instance, this basis can be chosen as

$$\xi^{1-1} := \frac{1}{\sqrt{2}}(\hat{e}_x - i\hat{e}_y), \quad \xi^{11} := -\frac{1}{\sqrt{2}}(\hat{e}_x + i\hat{e}_y), \tag{5.26}$$

and $\xi^{10} := \hat{e}_z$, with $\hat{e}_x, \hat{e}_y, \hat{e}_z$ the usual Cartesian basis of $\mathbb{C}^3$, and this yields ${Y^{00}}_{11} = -\hat{\vec{x}}/\sqrt{4\pi}$.

Due to their completeness, the tensor spherical harmonics can be used to expand the fields $\mathcal{A}$ and $\mathcal{B}$ as follows:

$$\mathcal{A} = \sum_{JLM} {A_{JM}}^L(r) {Y^{JM}}_{L\ell}, \tag{5.27}$$

and similarly for $\mathcal{B}$. The fact that the background has zero total angular momentum implies that the different JM modes decouple in the linearized equations. To derive the mode equations and exhibit this decoupling, we use the identity

$$({Y^{00}}_{\ell\ell})^* {Y^{JM}}_{L\ell} = \frac{(-1)^\ell}{\sqrt{4\pi}} \sqrt{\frac{2L+1}{2J+1}} {C^{J0}}_{L0\ell 0} Y^{JM}, \tag{5.28}$$

which can be deduced from the product formula for the spherical harmonics, see for instance Eq. (10) in Sec. 5.6 in Ref. [50]. One obtains from this

$$\Delta^{-1}(u^*\mathcal{A})u = \sum_{JLM} {Q_{JM}}^L(r) {Y^{JM}}_{L\ell}, \tag{5.29}$$

with

$$\begin{aligned} {Q_{JM}}^L(r) &= f_\ell(r) \sum_{L'=|J-\ell|}^{J+\ell} \frac{\sqrt{(2L+1)(2L'+1)}}{2J+1} \\ &\times {C^{J0}}_{L0\ell 0} {C^{J0}}_{L'0\ell 0} \Delta_J^{-1} \left(f_\ell {A_{JM}}^{L'} \right)(r). \end{aligned} \tag{5.30}$$

The selection rules for the Clebsch-Gordan coefficients imply that ${C^{J0}}_{L0\ell 0}$ is different from zero only if $|J-\ell| \leqslant L \leqslant J+\ell$ and $J+L+\ell$ is even. Therefore, the only non-vanishing coefficients are ${Q_{JM}}^{|J-\ell|}, {Q_{JM}}^{|J-\ell|+2}, \ldots, {Q_{JM}}^{J+\ell}$. Likewise, only the amplitudes ${A_{JM}}^{|J-\ell|}, {A_{JM}}^{|J-\ell|+2}, \ldots, {A_{JM}}^{J+\ell}$ appear in the sum in the right-hand side of Eq. (5.30).

Using this observation, Eqs. (5.18) yields for each values of $J \in \{0,1,2,\ldots\}$ and $|M| \leqslant J$, the following decoupled system

$$i\lambda {A_{JM}}^L = \left(\hat{\mathcal{H}}^L_{f_\ell} - \omega \right) {B_{JM}}^L, \tag{5.31a}$$

$$i\lambda {B_{JM}}^L = \left(\hat{\mathcal{H}}^L_{f_\ell} - \omega \right) {A_{JM}}^L + 2{Q_{JM}}^L, \tag{5.31b}$$

for the coefficients $(\mathcal{A}_{JM}, \mathcal{B}_{JM}) := \{({A_{JM}}^L, {B_{JM}}^L)\}\big|_{L=|J-\ell|,\ldots,J+\ell}$ where $\mathcal{H}^L_{f_\ell}$ is defined as

$$\hat{\mathcal{H}}^L_{f_\ell} := -\Delta_L + \Delta^{-1}\left(|f_\ell|^2 \right), \tag{5.32}$$

$$\Delta_L = \Delta_r - \frac{L(L+1)}{r^2}. \tag{5.33}$$

Furthermore, the system decouples into two subsystems: the even-parity sector which contains $L = |J-\ell|, |J-\ell|+2, \dots J+\ell$ and has non-trivial coefficients ${Q_{JM}}^L$ given in Eq. (5.30) and the odd-parity sector with $L = |J-\ell|+1, |J-\ell|+3, \dots, J+\ell-1$ which has vanishing ${Q_{JM}}^L$.

5.3 Decoupling of the perturbed evolution equation

In this section we show that the evolution equation (5.15) for the linearized field v can be decoupled by expanding v in terms of tensor spherical harmonics. Writing

$$u := \sqrt{4\pi} f_\ell(r) {Y^{00}}_{\ell\ell}(\vartheta,\varphi), \tag{5.34a}$$
$$v := \sum_{JLM} {X_{JM}}^L(t,r) {Y^{JM}}_{L\ell}(\vartheta,\varphi), \tag{5.34b}$$

Eq. (5.15) yields

$$i\frac{\partial}{\partial t}{X_{JM}}^L(t,r) = \left(\hat{\mathcal{H}}^L_{f_\ell} - \omega\right){X_{JM}}^L(t,r) + {q_{JM}}^L(t,r), \tag{5.35}$$

where

$$\begin{aligned} {q_{JM}}^L(t,r) = f_\ell(r) &\sum_{L'=|J-\ell|}^{J+\ell} \frac{\sqrt{(2L+1)(2L'+1)}}{2J+1} \\ &\times {C^{J0}}_{L0\ell0}{C^{J0}}_{L'0\ell0}\Delta_J^{-1}\left(f_\ell(r){Z_{JM}}^{L'}(t,r)\right), \end{aligned} \tag{5.36}$$

and where we have defined ${Z_{JM}}^{L'}(t,r) := {X_{JM}}^{L'}(t,r) + (-1)^M \overline{{X_{J-M}}^{L'}(t,r)}$.

The properties of the Clebsch-Gordan coefficients discussed below Eq. (5.30) imply that this system further decouples into two subsystems:

1. The even-parity sector which contains the values $L = |J-\ell|, |J-\ell|+2, \dots J+\ell$ and has non-trivial coefficients ${q_{JM}}^L$.
2. The odd-parity sector which has $L = |J-\ell|+1, |J-\ell|+3, \dots, J+\ell-1$, for which ${q_{JM}}^L$ vanishes.

An important consequence of these observations is that in the odd-parity sector the right-hand side of Eq. (5.35) is characterized by the self-adjoint operators $\hat{\mathcal{H}}^L_{f_\ell} - \omega$ implying a unitary evolution. Consequently, one can have only oscillatory modes in the odd-parity sector, and unstable modes can only arise in the even-parity sector.

5.4 Properties of the solutions of the linearized system

Before numerically solving the linearized system (5.31), in this section we discuss some important general properties of its solutions. For the following, we assume that u is real-valued.

5.4.1 Quadruple symmetry

When u is real, it is simple to see that a solution $(\lambda, \mathcal{A}, \mathcal{B})$ of the system (5.18) gives rise to the three other solutions $(\overline{\lambda}, \overline{\mathcal{A}}, -\overline{\mathcal{B}})$, $(-\lambda, \mathcal{A}, -\mathcal{B})$, $(-\overline{\lambda}, \overline{\mathcal{A}}, \overline{\mathcal{B}})$. Likewise, any solution $(\lambda, A_{JM}{}^L, B_{JM}{}^L)$ of the system (5.31) yields the other three solutions $(-\lambda, A_{JM}{}^L, -B_{JM}{}^L)$, $(\overline{\lambda}, \overline{A}_{JM}{}^L, -\overline{B}_{JM}{}^L)$, and $(-\overline{\lambda}, \overline{A}_{JM}{}^L, \overline{B}_{JM}{}^L)$. This means that the eigenvalues come in pairs $(\lambda, -\lambda)$ if they are real or purely imaginary, and in quadruples $(\lambda, -\lambda, \overline{\lambda}, -\overline{\lambda})$ otherwise.

5.4.2 Stationary modes

Next, we analyze the presence of stationary modes, that is, solutions of the system (5.31) with $\lambda = 0$. In this case, Eq. (5.31a) implies that $B_{JM}{}^L$ must be an eigenfunction of $\hat{\mathcal{H}}^L_{f_\ell}$ with eigenvalue ω. When $L = \ell$, we know that $B_{JM}{}^\ell = \sigma_\ell^{(0)}$ satisfies this condition, because of the background equations (4.4). *A priori* it seems possible that ω also lies in the point spectrum of $\hat{\mathcal{H}}^L_{f_\ell}$ for values of L different from ℓ; however we do not pursue this issue further here. When $\lambda = 0$, Eq. (5.31b) leads to a homogeneous equation for $A_{JM}{}^L$. We only consider the trivial solution $A_{JM}{}^L = 0$, leaving open the problem of the existence of nontrivial solutions.

Summarizing, for given values of ℓ, $J \in \{0, 1, \ldots, 2\ell\}$ and $|M| \leqslant J$, there is a one-parameter family of zero modes of the form

$$(A_{JM}{}^L, B_{JM}{}^L) = \Gamma_{JM}(0, S_{JM}{}^L), \tag{5.37}$$

with Γ_{JM} an arbitrary complex constant and where the fields $S_{JM}{}^L$ are zero except when $L = \ell$ in which case it is equal to f_ℓ. This leads to a multivalue family of stationary solutions of the linearized equations (5.15) which is of the form

$$v(t, x) = f_\ell(r) \sum_{J=0}^{2\ell} \sum_{M=-J}^{J} \left[\Gamma_{JM} Y^{JM}{}_{\ell\ell}(\vartheta, \varphi) - c.c. \right], \tag{5.38}$$

where *c.c.* denotes complex conjugation. When $\ell = 0$ there is only one mode which describes a change in amplitude of the background field, as discussed in [70]. However, when $\ell > 0$, there are $(2\ell+1)^2$ of these modes and, except the one with $J = 0$, all these modes have an angular dependency which is different from the one of the background solution. As an example, consider ℓ-boson stars with $\ell = 1$. Then, we have stationary modes with angular dependency

$$\begin{aligned} Y^{10}{}_{11} &= \frac{1}{\sqrt{2}} \left[Y^{1-1} \xi^{11} + Y^{11} \xi^{1-1} \right] \\ &= -\frac{3}{8\pi} \sin\vartheta \left[\cos\varphi \hat{e}_x + \sin\varphi \hat{e}_y \right]. \end{aligned} \tag{5.39}$$

Note that the zero modes discussed here belong to the even-parity sector when J is even and to the odd-parity sector otherwise.

We conjecture that these modes lead to nonspherical stationary deformations of the ℓ-boson stars.

5.4.3 General properties and connection with the second variation of the energy functional

Multiplying both sides of Eq. (5.18a) from the left with $\mathcal{B}^*$ and integrating yields

$$i\lambda(\mathcal{B},\mathcal{A}) = (\mathcal{B},(\hat{\mathcal{H}}_u-\omega)\mathcal{B}), \tag{5.40}$$

where $(\cdot,\cdot)$ refers to the L^2-scalar product. Likewise, multiplying both sides of Eq. (5.18b) from the left with $\mathcal{A}^*$ and integrating gives

$$\begin{aligned} i\lambda(\mathcal{A},\mathcal{B}) &= (\mathcal{A},(\hat{\mathcal{H}}_u-\omega)\mathcal{A}) + 2(u^T\mathcal{A},\Delta^{-1}[u^T\mathcal{A}]) \\ &= \delta_r^2\mathcal{E}[\mathcal{A}_R] + \delta_r^2\mathcal{E}[\mathcal{A}_I], \end{aligned} \tag{5.41}$$

where $\mathcal{A}_R$ and $\mathcal{A}_I$ refer to the real and imaginary parts of $\mathcal{A}$, respectively and $\delta_r^2\mathcal{E}[\mathcal{A}_R]$ denotes to the second variation[1] (5.4) evaluated at $v=\mathcal{A}_R$ with fixed particle numbers N_j.

Several interesting features can be inferred from Eqs. (5.40, 5.41). For this, we first note that the right-hand sides of these equations are real, which implies that

$$-\lambda^2\left|(\mathcal{A},\mathcal{B})\right|^2 \in \mathbb{R}. \tag{5.42}$$

Hence, either λ^2 is real or $\mathcal{A}$ is orthogonal to $\mathcal{B}$. Taking into account the quadruple symmetry, we may consider the following cases:

(i) $\lambda=0$: These are the zero modes discussed previously.

(ii) $\lambda_R>0$ and $\lambda_I=0$: In this case we can assume that $\mathcal{A}=\mathcal{A}_R$ is real and $\mathcal{B}=i\mathcal{B}_I$ is purely imaginary. Eliminating $i\lambda\mathcal{A}$ on the left-hand side of Eq. (5.41) using Eq. (5.18a), one finds

$$-(\mathcal{B}_I,(\hat{\mathcal{H}}_u-E)\mathcal{B}_I) = \delta_r^2\mathcal{E}[\mathcal{A}_R]. \tag{5.43}$$

Below, we will use this identity to eliminate the possibility of having unstable modes with arbitrary high values of J.

(iii) $\lambda_R=0$ and $\lambda_I>0$: In this case one can choose both $\mathcal{A}$ and $\mathcal{B}$ to be real, and one obtains instead of Eq. (5.43),

$$(\mathcal{B}_R,(\hat{\mathcal{H}}_u-E)\mathcal{B}_R) = \delta_r^2\mathcal{E}[\mathcal{A}_R]. \tag{5.44}$$

(iv) $\lambda_R>0$ and $\lambda_I>0$: In this case $(\mathcal{A},\mathcal{B})=0$ and it follows from Eq. (5.41) that u is a saddle point of $\mathcal{E}$, provided that $\mathcal{E}[\mathcal{A}_R]\neq 0$.

In terms of the decomposition (5.27) into tensor spherical harmonics, the scalar product $(\mathcal{A},\mathcal{B})$ reads

$$(\mathcal{A},\mathcal{B}) = \sum_{JM}\left[(\mathcal{A}_{JM},\mathcal{B}_{JM})_{\text{even}} + (\mathcal{A}_{JM},\mathcal{B}_{JM})_{\text{odd}}\right], \tag{5.45}$$

[1] Note that the restricted second variation $\delta_r^2\mathcal{E}[u;v]$ defined in Eq. (5.4), corresponds to $\delta^2\mathcal{E}$ given in Eq. (19b) in [65]

with

$$
\begin{aligned}
&(\mathcal{A}_{JM}, \mathcal{B}_{JM})_{\text{even, odd}} \\
:= &\sum_{\substack{L=|J-\ell| \\ J+\ell-L \text{ even,odd}}}^{J+\ell} \int\limits_0^\infty \overline{A_{JM}{}^L(r)} B_{JM}{}^L(r) r^2 dr
\end{aligned}
\tag{5.46}
$$

denoting the corresponding products for the JM modes in the even and odd parity sectors. A similar decomposition can be performed for the previous equations in this subsection; for instance Eq. (5.41) yields

$$
i\lambda(\mathcal{A}_{JM}, \mathcal{B}_{JM})_{\text{even}} = \delta_r^2 \mathcal{E}_{JM,even}[\mathcal{A}_{JM}],
\tag{5.47}
$$

where $\delta_r^2 \mathcal{E}_{JM,even}[\mathcal{A}_{JM}]$ is given by

$$
\begin{aligned}
&\delta_r^2 \mathcal{E}_{JM,even}[\mathcal{A}_{JM}] \\
&= \sum_{\substack{L=|J-\ell| \\ J+\ell-L \text{ even}}}^{J+\ell} \int\limits_0^\infty (A_{JM}{}^L)^*(r)(\hat{\mathcal{H}}_{f_\ell}^L - \omega) A_{JM}{}^L(r) r^2 dr \\
&- \frac{1}{2J+1} \int_0^\infty \int_0^\infty \frac{r_<^J}{r_>^{J+1}} a_{JM}(r)^* a_{JM}(\tilde{r}) r^2 \tilde{r}^2 dr d\tilde{r},
\end{aligned}
\tag{5.48}
$$

where

$$
a_{JM}(r) := f_\ell(r) \sum_L \sqrt{\frac{2L+1}{2J+1}} C^{J0}{}_{L0\ell 0} A_{JM}{}^L(r).
\tag{5.49}
$$

In the next section, we shall use Eq. (5.47) to check numerically that $i\lambda(\mathcal{A}_{JM}, \mathcal{B}_{JM})_{\text{even}}$ is real.

5.4.4 Non-existence of unstable modes for sufficiently large values of J

Finally, in this section we prove that for modes with large enough values of the total angular momentum J, the second variation of the energy functional is positive definite. As we show below, this implies through Eq. (5.43) that there cannot exist unstable modes with large J. This reduces the stability problem to the analysis of a finite number of J.

Let $v \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$, then we have the following estimate for the second variation of the energy functional (see Appendix E of [65])

$$
\delta_r^2 \mathcal{E}[u; v] \geqslant \frac{1}{2}(\nabla v, \nabla v)_{L^2} + (v, [U_u - \omega] v)_{L^2} - C_1 \left\| \frac{v}{g} \right\|_2^2,
\tag{5.50}
$$

where g is a positive function of r which will be determined shortly, $C_1 > 0$ a positive constant depending on g and $U_u := \triangle^{-1}(|u|^2)$ is the gravitational potential of the background configuration. To show that $\delta_r^2\mathcal{E}$ is positive definite for large enough J we expand v in terms of the tensor spherical harmonics:

$$v(x) = \sum_{JLM} h_{JM}{}^L(r) Y_{L\ell}^{JM}(\vartheta,\varphi), \tag{5.51}$$

with coefficients $h_{JM}{}^L$ depending on r. Substituting this in the right-hand side of Eq. (5.50) and discarding the quadratic terms in the derivatives of $h_{JM}{}^L$ yields

$$\begin{aligned}\delta_r^2\mathcal{E} \geqslant \sum_{JLM} \Bigg\{ &\int_0^\infty |h_{JM}{}^L(r)|^2 \left[\frac{L(L+1)}{2} - \frac{C_1 r^2}{g(r)^2}\right] dr \\ &+ \int_0^\infty |h_{JM}{}^L(r)|^2 [U_u(r) - \omega] r^2 dr \Bigg\}. \end{aligned}\tag{5.52}$$

Consider first the integral on the second line, whose integrand contains the function $\alpha(r) := [U_u(r) - \omega]r^2$. Since U_u is regular at the center, one has $\alpha(0) = 0$, whereas $\alpha(r)$ is positive for large enough r since ω is negative. Together with the fact that U_u is continuous [53], this implies that $\alpha(r) \geqslant C_2$ for all $r \geqslant 0$, for some (negative) constant C_2. Next, choose $g(r) := \sqrt{1+r^2}$ which implies that $r^2/g(r)^2 \leqslant 1$ for all $r \geqslant 0$. Using these properties, the estimate (5.52) yields

$$\delta_r^2\mathcal{E} \geqslant \sum_{JLM} \int_0^\infty |h_{JM}{}^L(r)|^2 \left[\frac{L(L+1)}{2} - C_1 + C_2\right] dr. \tag{5.53}$$

Therefore, $\delta_r^2\mathcal{E}$ is positive definite if $h_{JM}{}^L$ vanishes identically for all L with $L(L+1)/2 - C_1 + C_2 \leqslant 0$. In particular, it follows that $\delta_r^2\mathcal{E}_{JM,even}$ is positive definite for J large enough, such that $L := |J-\ell|$ satisfies $L(L+1) > 2(C_1 - C_2)$.

Finally, we prove that this property implies the absence of unstable modes for large enough values of J. We do this by contradiction. Consider first case (iv) for which $\lambda_R, \lambda_I > 0$ and $(\mathcal{A},\mathcal{B}) = 0$. In this case, Eq. (5.41) and the positivity of $\delta_r^2\mathcal{E}$ would imply that $\mathcal{A}_R = \mathcal{A}_I = 0$ which also implies that $\mathcal{B} = 0$ according to Eqs. (5.18). The other case in which an instability could appear is case (ii). Here, a contradicion arises by observing that the right-hand side of Eq. (5.43) is positive definite, whereas the left-hand side is negative definite for large enough values of J, as can be shown using arguments similar to the ones following Eq. (5.52).

5.5 Numerical results

In this subsection we solve numerically the system (5.31). In the previous section, some general properties of the linearized system were established. In particular, it was proven that unstable modes cannot arise in the odd-parity sector nor in the even-parity sector with high values of J, thus reducing the problem to a finite number of decoupled systems. For this reason, we will focus on the even-parity sector for what follows.

We start in the next subsection with a short description of our numerical implementation and subsequently, we discuss our main results regarding the eigenvalues of (5.31).

5.5.1 Background solutions

First, it is necessary to construct the ℓ-boson stars numerically, that is, we must solve system (which is equivalent to Eq. (4.69))

$$-\Delta_r f_\ell + \frac{\ell(\ell+1)}{r^2} f_\ell + U f_\ell = \omega f_\ell, \tag{5.54}$$

$$\Delta_r U = |f_\ell|^2. \tag{5.55}$$

To perform the numerical integration, it is convenient to replace the gravitational background potential U with the shifted potential $\mathcal{U} := \omega_\ell - U$, which allows us to rewrite the problem (5.54) in the equivalent form

$$\Delta_r f_\ell = \left[\frac{\ell(\ell+1)}{r^2} - \mathcal{U}\right] f_\ell, \tag{5.56a}$$

$$\Delta_r \mathcal{U} = -|f_\ell|^2. \tag{5.56b}$$

In a next step, we identify the correct boundary conditions at $r = 0$ and at $r \to \infty$ that guarantee that the solution is normalizable and regular at the origin. As we explain in Proposition 4.3, we can write $f_\ell = r^\ell g$ for some function $g \in X_r$. Therefore, the boundary conditions for g are

$$g(r=0) = g_0, \quad \frac{dg}{dr}(r=0) = 0, \tag{5.57}$$

$$\mathcal{U}(r=0) = \mathcal{U}_0, \quad \frac{d\mathcal{U}}{dr}(r=0) = 0. \tag{5.58}$$

with constants g_0 and $\mathcal{U}_0$. Note that $\mathcal{U}_0$ must be positive for a global solution to exist [22]. Furthermore, one can prove that if $(f_\ell, \mathcal{U})$ is a solution of the system (5.56), then for any constant $\Lambda > 0$, the pair $(f_\ell^\Lambda, \mathcal{U}^\Lambda)$ defined by

$$f_\ell^\Lambda(r) := \Lambda^2 f_\ell(\Lambda r), \tag{5.59}$$

$$\mathcal{U}^\Lambda(r) := \Lambda^2 \mathcal{U}(\Lambda r), \tag{5.60}$$

is also a solution. Hence, one can assume without loss of generality that $\mathcal{U}_0 = 1$. In turn, the value of the constant g_0 is fine-tuned using a numerical shooting method which aims at the condition $\lim_{r\to\infty} g(r) = 0$.

The numerical integration of the system (5.56) with the boundary conditions (5.57) is performed using an adaptive explicit 5(4)-order Runge-Kutta routine[2] [86, 28, 51], where we rewrite the system as a first-order system for the fields $(g, \mathcal{U})$. For the fine-tuning, we use a methodology similar to the one described in [61], based on bisection. Additionally, we find it necessary to match the numerical solution obtained in this way to the asymptotic form of the fields $(g, \mathcal{U})$, given by

$$g(r) \approx \frac{C_1}{r^{1+\ell-M/(2\kappa)}} e^{-\kappa r}, \tag{5.61a}$$

$$\mathcal{U}(r) \approx \omega + \frac{M}{r}, \tag{5.61b}$$

[2] The integration is performed using the fifth-order accurate steps; the fourth-order steps are only performed in order to estimate the error.

with $\kappa := \sqrt{|\omega|}$ and constants C_1, ω and M. Here, ω and M represent, respectively, the (unrescaled) energy eigenvalue and total mass of the configuration. The form (5.61b) is obtained by recalling the fact that $\lim_{r\to\infty} U(r) = 0$ and the definition $\mathcal{U}(r) = \omega - U(r)$. The constants ω and M are determined using the methodology described in Appendix C in [70], whereas the constant C_1 is computed by fitting the profile of the right-hand side of Eq. (5.61a) to the last 10 points of the function g obtained from the shooting algorithm. This extension of the solution turns out to be necessary for the numerical analysis of the first-order equations discussed in the next section, which requires the knowledge of the background solution for values of r lying beyond the maximal radius obtained from the shooting algorithm. Typical examples for the radial profile of the ℓ-boson stars, for different values of ℓ, are shown in Fig. 4.1. Furthermore, Fig. 4.2 shows the Newtonian potential corresponding to the ground and first excited states and $\ell = 0, 1, 2, 3$.

5.5.2 Perturbations solutions

Next, we focus in the study of of the linearized system (5.31). Introducing the change of variables ${A_{JM}}^L = {a_{JM}}^L/r$, ${B_{JM}}^L = {b_{JM}}^L/r$ in (5.31) one obtains

$$b''{}_{JM}{}^L - {U_{\text{eff}}}^L {b_{JM}}^L = -i\lambda {a_{JM}}^L, \tag{5.62a}$$

$$a''{}_{JM}{}^L - {U_{\text{eff}}}^L {a_{JM}}^L - 2{q_{JM}}^L = -i\lambda {b_{JM}}^L, \tag{5.62b}$$

where a prime denotes differentiation with respect to r, ${U_{\text{eff}}}^L(r) := L(L+1)/r^2 - \mathcal{U}(r)$ is an effective potential, and the function ${q_{JM}}^L$ is defined by

$$\begin{aligned} {q_{JM}}^L(r) &:= f_\ell(r) \sum_{L'=|J-\ell|}^{J+\ell} \frac{\sqrt{(2L+1)(2L'+1)}}{2J+1} {C^{J0}}_{L0\ell0} \\ &\times {C^{J0}}_{L'0\ell0} \left(\frac{d^2}{dr^2} - \frac{J(J+1)}{r^2}\right)^{-1} \left[f_\ell {a_{JM}}^{L'}\right](r), \end{aligned} \tag{5.63}$$

with the operator $\left(\frac{d^2}{dr^2} - \frac{J(J+1)}{r^2}\right)^{-1} = r\triangle_J^{-1}(r^{-1})$ denoting the inverse of the operator $r\triangle_J(r^{-1})$ with homogeneous Dirichlet conditions at $r = 0$ and $r \to \infty$. Note that the system (5.62), like the system (5.31), is independent of the total magnetic quantum number M, and hence we do not need to specify it.

To solve the system (5.62) we need two boundary conditions for each equation. To determine these, one can study (heuristically) the dominant terms of the perturbed system near the origin and infinity. Using the fact that $J = 0, 1, 2, \ldots$ and $L = |J-\ell|, |J-\ell|+2, \ldots, J+\ell$, and that the background solution behaves as $f_\ell(r) \sim r^\ell$, one finds that the dominant terms at the center stem from the centrifugal terms $L(L+1)/r^2$ in the effective potential. Consequently, the regular solution at the center behaves as $({a_{JM}}^L, {b_{JM}}^L) \sim (r^{L+1}, r^{L+1})$ (see Appendix F in [65] for further details). This leads to the following boundary conditions for all $\ell \geqslant 0$ at the origin:

$${a_{JM}}^L(r=0) = 0, \qquad {b_{JM}}^L(r=0) = 0. \tag{5.64a}$$

In the asymptotic region f_ℓ decays exponentially and $\mathcal{U}(r) \to \omega$. Demanding that the fields $({a_{JM}}^L, {b_{JM}}^L)$ decay at infinity, one requires that

$$\lim_{r\to\infty} {a_{JM}}^L(r) = 0, \qquad \lim_{r\to\infty} {b_{JM}}^L(r) = 0. \tag{5.64b}$$

In order to solve numerically the system (5.62) using the previous Dirichlet boundary conditions we proceed as follows. First, we computed the background profiles f_ℓ, $\mathcal{U}$ and represent these, as well as the perturbed fields ${a_{JM}}^L, {b_{JM}}^L$, in terms of Chebyshev polynomials. The different operators e.g., derivative and its inverse are discretized using a standard spectral method (see, e.g., Ref. [85]), which leads to a finite-dimensional eigenvalue problem. For details of the numerical discretization procedure, we refer the reader to subsection IVA in Ref. [70].

The discrete version of the system (5.62) can be written as:

$$\begin{pmatrix} \mathsf{0} & \mathbb{D}^2 - \mathbb{U}_{\ell J} \\ \mathbb{D}^2 - \mathbb{U}_{\ell J} - 2\mathbb{F}_\ell \mathbb{Z}_{\ell J}(\mathbb{D}^2 - \mathbb{V}_J)^{-1}\mathbb{F}_\ell & \mathsf{0} \end{pmatrix} \begin{pmatrix} \mathsf{a}_{JM} \\ \mathsf{b}_{JM} \end{pmatrix} = -i\lambda \begin{pmatrix} \mathsf{a}_{JM} \\ \mathsf{b}_{JM} \end{pmatrix}, \tag{5.65}$$

where here $\mathsf{0}$ represents the $c_J(\mathsf{N}-1) \times c_J(\mathsf{N}-1)$ zero matrix, with N the number of Chebyshev points distributed as $x_j = \cos(j\pi/\mathsf{N}), j = 0, 1, \ldots, \mathsf{N}$. The constant c_J is defined as $c_J := J+1$ for $J < \ell$ and as $c_J := \ell + 1$ when $J \geqslant \ell$, and it corresponds to the number of possible values of L with non-trivial coefficients ${Q_{JM}}^L$ for a given tuple (ℓ, J). $\mathbb{D}^2, \mathbb{U}_{\ell J}, \mathbb{F}_\ell$ and $\mathbb{V}_J$ are $c_J(\mathsf{N}-1) \times c_J(\mathsf{N}-1)$ matrices whose diagonal contain c_J blocks of smaller $(\mathsf{N}-1) \times (\mathsf{N}-1)$ matrices,

$$\mathbb{D}^2 = \mathrm{diag}\left(\tilde{\mathbb{D}}^2_{\mathbb{N}}, \tilde{\mathbb{D}}^2_{\mathbb{N}}, \ldots, \tilde{\mathbb{D}}^2_{\mathbb{N}}\right), \tag{5.66a}$$

$$\mathbb{U}_{\ell J} = \mathrm{diag}\left({\mathcal{U}_{\text{eff}}}^{|J-\ell|}, {\mathcal{U}_{\text{eff}}}^{|J-\ell|+2}, \ldots, {\mathcal{U}_{\text{eff}}}^{J+\ell}\right), \tag{5.66b}$$

$$\mathbb{F}_\ell = \mathrm{diag}\left(F_\ell, F_\ell, \ldots, F_\ell\right), \tag{5.66c}$$

$$\mathbb{V}_J = \mathrm{diag}\left(V_J, V_J, \ldots, V_J\right). \tag{5.66d}$$

The matrix block $\tilde{\mathbb{D}}^2_{\mathbb{N}}$ corresponds to the discrete representation of the second derivative operator with implemented Dirichlet conditions. For details of its construction we refer the reader to Refs. [85, 70]. The blocks F_ℓ, V_A and ${\mathcal{U}_{\text{eff}}}^L$ are diagonal and are constructed as

$$\begin{aligned} F_\ell &= \mathrm{diag}\left(f_\ell(x_1), f_\ell(x_2), \ldots, f_\ell(x_{\mathsf{N}-1})\right), \\ V_A &= \mathrm{diag}\left(\frac{A(A+1)}{x_1^2}, \frac{A(A+1)}{x_2^2}, \ldots, \frac{A(A+1)}{x_{\mathsf{N}-1}^2}\right), \\ {\mathcal{U}_{\text{eff}}}^L &= V_L - \mathrm{diag}\left(\mathcal{U}(x_1), \mathcal{U}(x_2), \ldots, \mathcal{U}(x_{\mathsf{N}-1})\right), \end{aligned}$$

where the subscript A in V_A can take the labels J and L. The matrix $\mathbb{Z}_{\ell J}$ is non-diagonal, has dimension $c_J(\mathsf{N}-1) \times c_J(\mathsf{N}-1)$, and is obtained from

$$\mathbb{Z}_{\ell J} = \begin{pmatrix} Z^{L=|J-\ell|}_{JL'=|J-\ell|} & Z^{L=|J-\ell|}_{JL'=|J-\ell|+2} & \cdots & Z^{L=|J-\ell|}_{JL'=J+\ell} \\ Z^{L=|J-\ell|+2}_{JL'=|J-\ell|} & Z^{L=|J-\ell|+2}_{JL'=|J-\ell|+2} & \cdots & Z^{L=|J-\ell|+2}_{JL'=J+\ell} \\ \vdots & \cdots & \ddots & \vdots \\ Z^{L=J+\ell}_{JL'=|J-\ell|} & Z^{L=J+\ell}_{JL'=|J-\ell|+2} & \cdots & Z^{L=J+\ell}_{JL'=J+\ell} \end{pmatrix}, \tag{5.68}$$

where the blocks $Z^{L}_{JL'}$ are diagonals matrices of constant coefficients

$$Z^{L}_{JL'} = \frac{\sqrt{(2L+1)(2L'+1)}}{2J+1} C^{J0}{}_{L0\ell 0} C^{J0}{}_{L'0\ell 0} \times \mathbb{I}, \tag{5.69}$$

with $\mathbb{I}$ the identity matrix of dimension $(\mathsf{N}-1)\times(\mathsf{N}-1)$. The vector

$$\begin{pmatrix} \mathsf{a}_{JM} \\ \mathsf{b}_{JM} \end{pmatrix} = \Big(a^{|J-\ell|}(x_1), \dots, a^{|J-\ell|}(x_{\mathsf{N}-1}), a^{|J-\ell|+2}(x_1), \dots, a^{|J-\ell|+2}(x_{\mathsf{N}-1}), \dots\dots, a^{J+\ell}(x_{\mathsf{N}-1}), b^{|J-\ell|}(x_1), \dots, b^{|J-\ell|}(x_{\mathsf{N}-1}), b^{|J-\ell|+2}(x_1), \dots, b^{|J-\ell|+2}(x_{\mathsf{N}-1}), \dots\dots, b^{J+\ell}(x_{\mathsf{N}-1}) \Big)^T,$$

corresponds to the discrete representation of the eigenfields $r({A_{JM}}^L, {B_{JM}}^L)^T$.

We solve the discrete eigenvalue problem (6.65) using the SciPy library [86] for $\mathsf{N} := 3r_\star/4$ Chebyshev points where $r_\star := 200(n+1)$ for the n'th excited state of the background solution represents the physical radius of the outer boundary of our numerical domain. Our code is publicly available in [1].

5.5.3 Results: Ground state in non-relativistic ($\ell = 1$)-boson star

We first proceed to study the linear stability of the ground state corresponding to a nonrelativistic ($\ell = 1$)-boson star. This configuration is characterized by a radial scalar field profile f_1 without nodes ($n = 0$), whose gravitational potential $U_{f_1} = \triangle^{-1}(|f_1|^2)$ is monotonically increasing to zero as can be seen in Figs. 4.1 and 4.2. It follows from Eq. (5.17) that linear stability requires that the real part of each eigenvalue λ of the system (6.65) with $\ell = 1$ is zero.

By Theorem 4.4, ℓ-boson stars are orbital stable (assuming existence of global solutions) under radial perturbations ($J = 0$). The numerical results are in agreement with the theorem and indicate that they are also stable under linear nonspherical perturbations with $J = 1, 2, \dots, 10$. That is, we found only purely oscillatory modes with strictly imaginary eigenvalues that come in pairs $(\lambda, -\lambda)$ as was discussed in subsection 5.4.1.

Table 5.1 presents the three lowest positive frequencies λ for the first seven J values. Notice that for even $J = 0, 2$ values the first eigenvalue corresponds to the stationary mode with $\lambda_{st} := \lambda = 0$ discussed previously. Numerically we can *identify* these eigenvalues because although they are not zero to machine precision, they are several orders smaller in magnitude than the remaining eigenvalues. For example, for $J = 0$ and $\ell = 1$ the ratio with the first non-stationary eigenvalue is $|\lambda_{st}/\lambda| \sim 10^{-3}$. Their eigenfunctions fulfill the relation Eq. (5.37). In the case of odd values for J, the stationary modes belong to the odd-parity sector which we do not study numerically because it only contributes to oscillatory modes.

	λ [$1/t_c$] Components			λ [$1/t_c$] Components	
J	Real	Imaginary	J	Real	Imaginary
0	0	1.05201×10^{-5}	4	0	0.11368481
	0	0.06385090		0	0.16030883
	0	0.14328038		0	0.18097544
1	0	0.00086601	5	0	0.15468802
	0	0.05499307		0	0.18203923
	0	0.08194250		0	0.19756177
2	0	2.11418×10^{-6}	6	0	0.18105790
	0	0.09613376		0	0.19765391
	0	0.10590972		0	0.20832581
3	0	0.05557199	7	0	0.19756669
	0	0.13196275		0	0.20833027
	0	0.15373565		0	0.21570444

TABLE 5.1: (Table extracted from [65]). Table with the first three eigenvalue pairs $(\lambda, -\lambda)$ for perturbations with total angular momentum $J = 1, 2, \ldots, 7$ of a boson star with $\ell = 1$. All modes are purely oscillatory.

Finally, we observe from Table 5.1 that (when excluding the stationary modes) the slowest oscillating nonspherical modes with the largest period have total angular momentum $J = 1$.

5.5.4 Results: Ground state in other non-relativistic ℓ-boson star

Next, we generalize the above study to configurations with $\ell = 0, 2, 3, \ldots, 6$ and non-radial linear perturbations with $J = 1, \ldots, 10$. This extends Theorem 4.4, which proves that ℓ-boson stars are orbitally stable under radial perturbations $J = 0$. As a check of our results, we computed the respective eigenfunctions $\mathcal{A}_{JM}, \mathcal{B}_{JM}$ for every type of eigenvalues λ found: real, purely imaginary, complex with nonzero real and imaginary parts, and we validated that these satisfy the properties discussed in Sec. 5.4. In particular, we verified the quadruple symmetry and the fact that $i\lambda(\mathcal{A}_{JM}, \mathcal{B}_{JM})_{\text{even}}$ is real.

Similar to the $\ell = 1$ case, we show in Table 5.2 the three lowest positive eigenvalues for the configurations $\ell = 0, 2, 3$ with $J = 0, 1, 2, 3$. As can be appreciated, similar to configurations with $\ell = 1$, ($\ell = 0$)-boson stars only exhibit purely oscillation modes

J	$\ell=0,\ \lambda\ [1/t_c]$ Components Real	Imaginary	$\ell=2,\ \lambda\ [1/t_c]$ Components Real	Imaginary	$\ell=3,\ \lambda\ [1/t_c]$ Components Real	Imaginary
0	0	4.321×10^{-6}	0	8.766×10^{-6}	0	1.637×10^{-5}
	0	0.03412558	0	0.06408695	0	0.05903587
	0	0.06030198	0	0.16575905	0	0.16827468
1	0	0.00049999	0	0.00111802	0	0.00132286
	0	0.05555078	0	0.05842642	0	0.05925506
	0	0.06676208	0	0.07777395	0	0.06609698
2	0	0.05285868	4.32×10^{-7}	2.63×10^{-15}	7.99×10^{-7}	1.37×10^{-14}
	0	0.06564270	0.00644373	-5.11×10^{-13}	0.00907756	2.1×10^{-12}
	0	0.07136165	0	0.10237935	0	0.09954769
3	0	0.06571493	0	0.05550210	0.00363560	0.05313153
	0	0.07135829	0	0.06147704	0	0.05910000
	0	0.07442765	0	0.07000546	0.00085739	0.05953874

TABLE 5.2: (Table extracted from [65]). Table with the first three lowest eigenvalues of the perturbations with $J=0,1,\ldots,3$ for boson stars with $\ell=0,2,3$. For $\ell=0$ all modes are purely oscillatory, while the others have unstable modes.

and a stationary solution for $J=0$. In contrast, for $\ell=2$ and 3 we found a real eigenvalue in the sector with total angular momentum $J=2$. (Strictly speaking, this eigenvalue has a nonzero small imaginary part; however a convergence study reveals that by increasing the number N of Chebyshev points the imaginary part converges to zero.) Furthermore, Table 5.2, we observe that for $\ell=J=3$, complex eigenvalues with nonvanishing real and imaginary parts appear. In fact, we found that this type of eigenvalue is also present in configurations with $2\leqslant\ell\leqslant9$ (see the left panel in Fig. 5.1). The corresponding modes grow exponentially in time implying that the underlying background solution is linearly unstable. This leads us to conjecture that nonrelativistic $\ell-$boson stars with $\ell\geqslant3$ possess at least one exponentially in time growing mode characterized by a complex eigenvalue λ with $\lambda_I\neq0$.

Summarizing, for ground state configurations of the $\ell-$boson stars we verified that the eigenvalues of the linearized system (5.31) satisfy the properties discussed in Sec. 5.4. Furthermore, we found that they possess the following features:

(a) Configurations with $\ell=0,1$ only present oscillating modes whose largest periods correspond to the smallest J values.

(b) A family of stationary modes of the form Eq. (5.37) exist for a given set of values

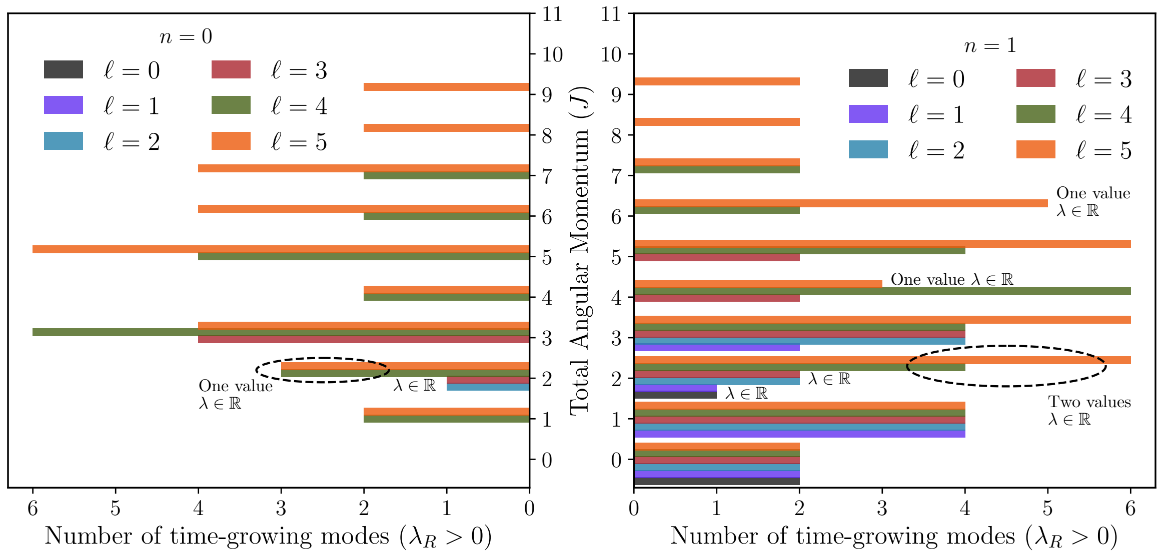

Figure 5.1: (Figure extracted from [65]). The relation between the total angular momentum J and the number of time-growing modes i.e., eigenvalues with positive real parts $\lambda_R > 0$. As a consequence of the symmetries discussed in subsection 5.4.1 we have one (two) growing mode (modes) for every real (complex) eigenvalue with $\lambda_R \neq 0$. The cases having real λ are pointed out in the figures, e.g., for $n = 0, \ell = 2, 3$ and $J = 2$ one has $\lambda \in \mathbb{R}$. Note that for large values of J there are only oscillating modes. The left/right panel corresponds to configurations in the ground/first-excited states.

ℓ, $J \in \{0, 1, \ldots, 2\ell\}$ and $|M| \leqslant J$. For $\ell = 2, 4, \ldots, 2n$ with $n \in \mathbb{N}$ and $J \neq 0$, they have an angular dependency that is different from the background solution; hence they are expected to give rise to stationary nonspherical deformation in the nonlinear case.

(c) Configurations with $\ell > 1$ have in the even-parity sector with $J = 2$ a real eigenvalue, for which all components of $\mathcal{B}_{JM}$ ($\mathcal{A}_{JM}$) are real (purely imaginary), and the component ${B_{2M}}^{\ell}$ is proportional to f_ℓ. These modes are exponentially growing in time.

(d) Configurations with $\ell \geqslant 3$ have at least one unstable mode that grows exponentially in time and is characterized by a complex eigenvalue with nonvanishing real and imaginary parts.

(e) Perturbations with large total angular momentum J have only purely oscillatory modes. As can be seen from the left panel of Fig. 5.1, the real parts of the eigenvalues λ vanish above a certain value of J, leaving only oscillatory modes. This result is compatible with the analytical results of subsection 5.4, where the absence of unstable modes for high enough values of J was proven. Therefore, the lowest J modes are the ones that determine the linear stability of the non-relativistic ℓ-boson stars.

5.5.5 Results: Excited states in non-relativistic ℓ-boson star

Finally, to close this section, we discuss briefly the mode stability of excited $\ell-$boson stars, i.e. background configurations with $n > 0$ nodes.

In [70] we conjectured that these configurations are linearly unstable – with exponentially in time growing modes – under radial perturbations $J = 0$. The findings in this section allow us to strengthen this conjecture: in addition to the unstable modes reported in our previous work, here we found unstable non-spherical modes characterized by purely real or complex eigenvalues. Furthermore, we found non-spherical stationary and purely oscillatory modes. Our results support the conclusion that excited states of nonrelativistic ℓ-boson stars are unstable.

Similar to the ground state configurations, under perturbations with large total angular momentum excited configurations only have oscillatory modes. In the right panel of Fig. 5.1 we show the number of eigenvalues with non-zero real parts as a function of J. Notice that in contrast to the ground state configurations, the real eigenvalues are not limited to the $J = 2$ sector; however they are constrained to even values of J.

Chapter 6

Self-interacting boson stars

Following the approach of previous chapters, this chapter applies the direct method of of the calculus of variations to the Gross-Pitaevskii-Poisson system. Our goal is to demonstrate the existence of a ground state solution in the presence of a central point mass and to establish its orbital stability. This analysis consider a single scalar field and differs from the preceding work by including a self-interaction potential.

Moreover, recent numerical studies by Sanchis-Gual *et al.* [75] and Brito *et al.* [12] suggest that excited boson stars may be stable if their self-interaction is repulsive and sufficiently strong. Their conclusion is based on the numerical evolution of perturbed boson stars in the spherically symmetric sector of the fully nonlinear Einstein-Klein-Gordon theory with quartic self-interactions. In particular, they find excited configurations in which no apparent instabilities are manifest during the time span of the evolution, indicating that these states are either stable or are unstable with a large time scale associated with the unstable modes. Of course, the restriction to spherical symmetry leaves open the possibility that such excited states, although stable with respect to radial perturbations, suffer from instabilities with respect to generic perturbations.

The purpose of the second part of this chapter is to shed new light on the stability problem of excited boson stars in the self-interacting case. For this, we concentrate on the Newtonian limit of the theory, where the complications of relativistic effects are absent. As we will show, this non-relativistic framework, which reduces the Einstein-Klein-Gordon system to the Gross-Pitaevskii-Poisson system, allows for a systematic, linearized study of the stability of excited states against generic perturbations.

Similarly to chapter 1, in this chapter we are interested in the Newtonian limit of the action (1.1) with the Lagrangian of matter given by

$$\mathcal{L}_M := -\nabla_\mu\overline{\phi}\nabla^\mu\phi - m_0^2|\phi|^2 - \lambda|\phi|^4. \tag{6.1}$$

Here $\phi : \mathbb{R}^4 \to \mathbb{C}$ is a complex-valued function and as before, the bar denotes complex conjugate. Following the procedure developed in Sections 1.1 and 1.2 we obtain the action

$$\begin{aligned} S[U,\psi] = \int dt \int d^3x \Bigg[&\frac{1}{8\pi G} U\Delta U \\ &+\psi^* \left(i\frac{\partial}{\partial t} + \frac{1}{2m_0}\Delta - \frac{\lambda}{4m_0^2}|\psi|^2 \right)\psi - m_0 U|\psi|^2 \Bigg], \end{aligned} \tag{6.2a}$$

where the functions ψ and U refers to the wave function and the gravitational potential, respectively. The variation with respect to these functions yields the Gross-Pitaevskii-Poisson system

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m_0}\Delta\psi + \frac{\lambda}{2m_0^2}|\psi|^2\psi + m_0 U\psi, \tag{6.3a}$$

$$\Delta U = 4\pi G m_0|\psi|^2, \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^3, \tag{6.3b}$$

where λ is a constant which is positive (negative) if the self-interaction is repulsive (attractive).

The action (6.2a) is invariant under time translations, which means that the evolution of the system is not affected by shifts in the time parameter t. Associated to this symmetry we can define the total energy of the configuration,

$$\mathcal{E}[\psi, U] = \int \left(\frac{1}{2m_0}|\nabla\psi|^2 + \frac{\lambda}{4m_0^2}|\psi|^4 + \frac{m_0}{2}U|\psi|^2\right) d^3x \tag{6.4}$$

which is conserved during the evolution. In addition, Eq. (6.2a) is also invariant under continuous shifts in the phase of the wave function, $\psi \mapsto e^{i\alpha}\psi$, with α a real constant, which results in the conservation of the particle number

$$N = \int |\psi|^2 d^3x. \tag{6.5}$$

One can prove using Noether's theorem that other conserved quantities associated with the Galilei group do exist (see Subsection 1.3.1 in Chapter 1).

It is convenient to formulate the Gross-Pitaevskii-Poisson system in terms of the following dimensionless quantities

$$\widetilde{t} := \frac{2m_0}{\Lambda}t, \quad \widetilde{x} := \frac{2m_0}{\Lambda^{1/2}}x, \tag{6.6a}$$

$$\widetilde{U} := \frac{\Lambda}{2}U, \quad \widetilde{\psi} := \left(\frac{\pi\Lambda^2}{2M_{\text{Pl}}^2 m_0}\right)^{1/2}\psi, \quad \widetilde{M} := GMm_0\Lambda^{1/2}, \tag{6.6b}$$

where we have defined

$$\Lambda := \frac{|\lambda| M_{\text{Pl}}^2}{2\pi m_0^2}, \quad \text{and} \quad M_{\text{Pl}} := \frac{1}{\sqrt{G}}. \tag{6.7}$$

Including a central potential generated by a central mass M (see Section 1.3.2) and in terms of these new variables the Gross-Pitaevskii-Poisson system (6.2) simplifies to

$$i\frac{\partial\psi}{\partial t} = \left(-\Delta + \kappa|\psi|^2 + U\right)\psi, \tag{6.8a}$$

$$\Delta U = |\psi|^2 + 4\pi M\delta, \tag{6.8b}$$

where $\kappa = +1$ if the self-interaction is repulsive and $\kappa = -1$ if it is attractive. We have omitted the tildes in order to simplify the presentation (as before we will denote dimensionfull quantities with the superscript *phys* whenever necessary).

Equivalently, Eqs. (6.8) can be expressed as an integro-differential nonlinear equation

$$i\frac{\partial\psi}{\partial t} = \left[-\Delta + \kappa|\psi|^2 - \frac{M}{|x|} + \Delta^{-1}(|\psi|^2)\right]\psi, \tag{6.9}$$

where $\Delta^{-1}(|\psi|^2)$ is defined as in Eq. (3.4).

As in previous chapters, we focus on scalar field configurations of the form:

$$\psi(t, x) = e^{-i\omega t}u(x), \tag{6.10}$$

where u is a real-valued function of $x \in \mathbb{R}^3$ and ω a real constant. Introducing this ansatz into system (6.2) we arrive at:

$$-\Delta u + \kappa|u|^2 u + Uu = \omega u, \tag{6.11a}$$

$$\Delta U = |u|^2 + 4\pi M\delta, \tag{6.11b}$$

or equivalently

$$\left[-\Delta + \kappa|u|^2 - \frac{M}{|x|} + \Delta^{-1}(|u|^2)\right]u = \omega u. \tag{6.12}$$

The system (6.12) determines the stationary solutions to the Gross-Pitaevskii-Poisson system.

The conserved energy functional (6.4) can be expressed in the form $\mathcal{E}^{phys}[\psi^{phys}] = [M_{\text{Pl}}^2/(m_0\Lambda^{3/2}\pi)]\mathcal{E}[\psi]$, with

$$\mathcal{E}[u] = T[u] - L[n] + \kappa F[n] - D[n, n], \quad n := |u|^2, \tag{6.13}$$

where the functionals T, F and D are defined by

$$T[u] := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u(x)|^2 dx, \tag{6.14a}$$

$$L[n] = \frac{M}{2}\int_{\mathbb{R}^3} \frac{|u|^2}{|x|} dx, \tag{6.14b}$$

$$F[n] := \frac{1}{4}\int_{\mathbb{R}^3} n(x)^2 dx, \tag{6.14c}$$

$$D[n, n] := \frac{1}{16\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{n(x)n(y)}{|x-y|} dydx. \tag{6.14d}$$

Lemma 6.1. *The functional $\mathcal{E}[u]$ satisfies the following rescaling properties:*

$$\mathcal{E}[\nu u] = \nu^2 T[u] - \nu^2 L[n] + \nu^4\kappa F[n] - \nu^4 D[n, n], \tag{6.15a}$$

$$\mathcal{E}[u_\nu] = \nu^2 T[u] - \nu L[n] + \nu^3\kappa F[n] - \nu D[n, n], \tag{6.15b}$$

for each $u \in H^1(\mathbb{R}^3, \mathbb{C}^K)$ and $\nu > 0$, where $u_\nu(x) := \nu^{3/2}u(\nu x)$.

As in previous chapters, we want to prove the existence of a minimizer of the energy functional (6.13) when it is defined over a admissible class of functions, and then to establish the existence of solutions to Eq. (6.9).

6.1 Variational problem

We define the variational problem

$$E(N) := \inf\{\mathcal{E}[u] : u \in \mathcal{A}_N\}, \tag{6.16}$$

where

$$\mathcal{A}_N := \{u \in H^1(\mathbb{R}^3) : \|u\|_2^2 = N\}. \tag{6.17}$$

We note that $F[u]$ can be written as

$$F[n] = \frac{1}{4}\|u\|_4^4. \tag{6.18}$$

Then, by the results of Chapter 1 (see estimate (3.22)) and by Theorem 2.7 and Corollary 2.1, the energy functional $\mathcal{E} : H^1(\mathbb{R}^3) \to \mathbb{R}$ is well defined, continuous and Gâteaux differentiable. However, the next lemma shows that $\mathcal{E}$ is not bounded from below in the attractive case.

Lemma 6.2. *Let $N > 0$, $M \geq 0$, and $\kappa = -1$. Then, the energy functional $\mathcal{E}$ is not bounded below.*

Proof. The statement follows from identity (6.15b) of Lemma 6.1, namely

$$\frac{\mathcal{E}[u_\nu]}{\nu^3} = -F[n] + \frac{1}{\nu}T[u] - \frac{1}{\nu^2}\left(L[n] + D[n,n]\right) \to -\infty \quad \text{when } \nu \to 0. \tag{6.19}$$

■

In view of this lemma in what follows we shall only consider the repulsive case $\kappa = +1$. Furthermore, let $u \in H^1(\mathbb{R}^3)$ and denote by $u^\sharp$ its symmetric decreasing rearrangement. Since $\|u\|_p = \|u^\sharp\|_p$ and by Theorem 3.3, we have

$$\mathcal{E}[u^\sharp] \leq \mathcal{E}[u], \tag{6.20}$$

which implies that we may restrict our attention to radial functions and consider instead the minimization problem

$$E^r(N) := \inf\{\mathcal{E}[u] : u \in \mathcal{A}_N^r\}, \tag{6.21}$$

where

$$\mathcal{A}_N^r := \{u \in H_r^1(\mathbb{R}^3) : \|u\|_2^2 = N\} \subset \mathcal{A}_N. \tag{6.22}$$

Theorem 6.1. *Let $N > 0$, $M \geq 0$, and $\kappa = +1$. Then the problem (6.21) admits a minimizer $u \in \mathcal{A}_N^r$ and $u = u^\sharp$.*

Proof. The proof follows the same strategy as that of Theorem 3.1. Therefore, using the fact that $H_r^1(\mathbb{R}^3)$ is compactly embedded in $L^p(\mathbb{R}^3)$ for $2 < p < 6$ (see Proposition 3.4), Theorem 2.4 and that the Dirichlet functional is weakly lower semi-continuous, we conclude that there exist a minimizer $u \in \mathcal{A}^r_{\leq N}$ of the energy functional $\mathcal{E}$, defined by (6.13), over the space

$$\mathcal{A}^r_{\leq N} := \{u \in H^1_r(\mathbb{R}^3) : \|u\|_2^2 \leq N\}. \tag{6.23}$$

It remains to prove that $\|u\|_2^2 = N$. To this end, define

$$E_{\leq}(N) := \inf\{\mathcal{E}[u] : u \in \mathcal{A}^r_{\leq N}\}. \tag{6.24}$$

Fix $u \in \mathcal{A}^r_N$ with $u \neq 0$. Since $L[n] > 0$ and $D[n,n] > 0$ and using the scaling identity (6.15b), we conclude that $\mathcal{E}[u_\nu] < 0$ for $\nu > 0$ small enough, which implies that $E(N) < 0$.

Let $u \in \mathcal{A}^r_{\leq N}$ be a minimizer of $\mathcal{E}$, so that

$$E_{\leq}(N) = \mathcal{E}[u] < 0. \tag{6.25}$$

Assume by contradiction that $\|u\|_2^2 < N$ and consider the function νu for $0 < \nu < 1$. Using the scaling identity (6.15a), we compute

$$\left.\frac{d}{d\alpha}\right|_{\alpha=1} \mathcal{E}[\alpha u] = 2T[u] - 2L[n] + 4F[n] - 4D[n,n] = 0, \quad n := |u|^2. \tag{6.26}$$

Similarly, defining $u_\alpha(x) = \alpha^{3/2}u(\alpha x)$ and using Eq. (6.15b) we have

$$\left.\frac{d}{d\alpha}\right|_{\alpha=1} \mathcal{E}[u_\alpha] = 2T[u] - L[n] + 3F[n] - D[n,n] = 0. \tag{6.27}$$

Combining Eqs.(6.26) and (6.27), we deduce

$$T[u] + F[n] + D[n,n] = 0, \tag{6.28}$$

which is a contradiction since each term is strictly positive. Hence $\|u\|_2^2 = N$, and therefore $u \in \mathcal{A}^r_N$ is a minimizer of the functional $\mathcal{E}$. ■

The following theorems can be proven by repeating the arguments of the proofs of Theorems 4.2(ii), 4.3(i) and 4.4 respectively.

Theorem 6.2. *Let $N > 0$, $\kappa = +1$, and $M \geq 0$, Then, every minimizing sequence (u_j) of (6.21) is relatively compact in $H^1_r(\mathbb{R}^3)$.*

Theorem 6.3. *Assume that $N > 0$, $\kappa = +1$, and $M \geq 0$. Let $u \in H^1_r(\mathbb{R}^3)$ be a critical point of the energy functional $\mathcal{E}$ (defined in (6.13)) on the set $\{H^1_r(\mathbb{R}^3) : \|u\|_2^2 = N\}$, such that $\|u\|_2^2 = N$. Then, there exist a constant $\omega \in \mathbb{R}$ with $\omega < 0$ given by*

$$\omega = \frac{2}{N}\left(T[u] + 2F[n] - L[n] - 2D[n,n]\right), \tag{6.29}$$

such that u is a weak solution in $H^1(\mathbb{R}^3)$ of the non-linear Schrödinger equation,

$$-\Delta u + |u|^2 u - \frac{M}{|x|}u + \Delta^{-1}(|u|^2)u = \omega u, \tag{6.30}$$

where the potential $\Delta^{-1}(|u|^2) \in L^p(\mathbb{R}^3)$ *for all* $4 \leq p \leq \infty$ *is given by Eq.* (3.4).

That is, for all $v \in H^1(\mathbb{R}^3)$

$$\int_{\mathbb{R}^3} \left[\nabla u \nabla v(x)^* + |u(x)|^2 u(x) v(x)^* - \frac{M}{|x|} u(x) v(x)^* \right. \tag{6.31}$$

$$\left. v(x)^* U_u(x) u(x) v(x)^* - \omega u(x) v(x)^* \right] d^3x = 0, \tag{6.32}$$

Moreover, the following virial relation is satisfied

$$\mathcal{E}[u] = -T[u] - 2F[n]. \tag{6.33}$$

Theorem 6.4. *Suppose that* $N > 0$, $\kappa = +1$, *and* $M \geq 0$. *Let* S_N *denote the set of minimizer of variational problem* (6.21)*, i. e.,*

$$S_N := \{u \in H^1_r(\mathbb{R}^3) \ : \ \mathcal{E}[u] = E(N), \ \|u\|_2^2 = N\}. \tag{6.34}$$

Solutions to the Cauchy problem (assuming these exist globally in time)

$$i\frac{\partial \psi}{\partial t} = -\Delta\psi + |\psi|^2\psi - \frac{M}{|x|}\psi + \Delta^{-1}(|\psi|^2)\psi, \quad \textit{in } \mathbb{R}_+ \times \mathbb{R}^3 \tag{6.35a}$$

$$\psi(0, x) = \psi_0(x), \quad \textit{with } \psi_0 \in H^1_r(\mathbb{R}^3), \tag{6.35b}$$

of the form

$$\psi(t, x) = e^{-it\omega} u(x), \tag{6.36}$$

are orbital stable (see Definition 4.2). Here $u \in S_N$ *and* $\omega < 0$ *is given by Eq.* (6.29).

6.2 Linear stability of self-interacting boson stars under non-spherical perturbations

The stability of stationary solutions is determined by the behavior of the small deviations about Eq. (6.10). To proceed, we take $M = 0$ and linearize the integro-differential equation (6.9) following the procedure presented in Chapter 5. With this is mind, we propose the following ansatz for the wave function

$$\psi(t, x) = e^{-i\omega t} \left[u(x) + \varepsilon v(t, x) + \mathcal{O}(\varepsilon^2) \right], \tag{6.37}$$

where ε is a small positive parameter. Here, (ω, u) is a solution of the nonlinear eigenvalue problem (6.12) and v is a complex-valued function depending on (t, x) that describes the linear perturbation.

Introducing the ansatz (6.37) into Eq. (6.9) one obtains, to linear order in ε,

$$i\frac{\partial v}{\partial t} = \left(\hat{\mathcal{H}} - \omega \right) v + 2u\hat{K} \left[u\mathrm{Re}(u) \right], \tag{6.38}$$

with the linear operators

$$\hat{\mathcal{H}} := -\Delta + \kappa |u|^2 + \Delta^{-1}(|u|^2), \tag{6.39}$$

$$\hat{K} := \kappa + \Delta^{-1}. \tag{6.40}$$

To separate the temporal from the spatial parts of σ we use the mode ansatz (see also Section. 5.2 in Chapter 5 for details):

$$v(t,x) = [A(x) + B(x)]\, e^{\lambda t} + [A(x) - B(x)]^*\, e^{\lambda^* t}, \tag{6.41}$$

where A and B are complex-valued functions depending only on $x \in \mathbb{R}^3$ and λ is a complex constant (not to be confused with the selfinteraction coupling constant λ of Eq. (6.1)). Substituting Eq. (6.41) into Eq. (6.38) and setting the coefficients in front of $e^{\lambda^* t}$ and $e^{\lambda t}$ to zero, we arrive at

$$i\lambda A = \left(\hat{\mathcal{H}} - \omega\right) B, \tag{6.42a}$$

$$i\lambda B = \left(\hat{\mathcal{H}} - \omega\right) A + 2u\hat{K}\,(uA)\,. \tag{6.42b}$$

This system constitutes a linear eigenvalue problem for the constant λ. Notice that linear instability is signaled by the existence of solutions with a positive real part λ_R of λ. The lifetime of the unstable configurations is expected to be of the order of $t_{\text{life}} \sim 1/\lambda_R^{\max}$, with $\lambda^{\max}$ the eigenvalue with the largest real part.

6.2.1 Basic properties of the stationary states

Next, we present some basic properties satisfied by the solutions of Eq. (6.12), based on the simple scaling argument given in Lemma 6.1. In Sect. 6.2.3 these properties will be used to shed light on the stability of stationary states in the attractive case.

To explain our scaling argument, recall that stationary states are critical points of the energy functional (6.13), assuming that the particle number $N = \|u\|_2^2$ is fixed. Now, let $\nu > 0$ be an arbitrary real and positive parameter and $u(x)$ a given scalar field. Consider the rescaled function

$$u_\nu(x) := \nu^{3/2} u(\nu x), \tag{6.43}$$

which leaves the particle number invariant: $\|u_\nu\|_2^2 = \|u\|_2^2$. Under this rescaling, the energy functional (6.13) transforms as

$$\mathcal{E}[u_\nu] = \nu^2 T[u] \pm \nu^3 F[n] - \nu D[n,n], \tag{6.44}$$

and the first and second variations of $\mathcal{E}[u_\nu]$ at $u_{\nu=1} = u$ are

$$\left.\frac{d}{d\nu}\mathcal{E}[u_\nu]\right|_{\nu=1} = 2T[u] \pm 3F[n] - D[n,n], \tag{6.45a}$$

$$\left.\frac{d^2}{d\nu^2}\mathcal{E}[u_\nu]\right|_{\nu=1} = 2T[u] \pm 6F[n]. \tag{6.45b}$$

In particular, if u is a stationary solution, the first variation is zero, which yields the relation

$$D[n,n] = 2T[u] \pm 3F[n]. \tag{6.46}$$

This expression is valid for any stationary solution and allows one to eliminate $D[n,n]$ in the energy functional and compute the energy for such states solely in terms of $T[u]$ and $F[n]$ according to

$$\mathcal{E}[u] = -T[u] - 2\kappa F[n]. \tag{6.47}$$

This expression implies that the energy of the stationary states is always negative in the repulsive case. In the attractive case, it follows from Eq. (6.44) that $\mathcal{E}[u_\nu]$ is not bounded from below, since it can be made arbitrarily negative by choosing ν large. According to Eq. (6.45b), the critical point at $\nu = 1$ corresponds to a local minimum of $\mathcal{E}[u_\nu]$ if $T - 3F > 0$ and to a local maximum if $T - 3F < 0$. It follows from these observations that in the attractive case a stationary state u cannot be a (local) minimum of the energy functional if $T < 3F$, that is, when the selfinteraction term dominates the kinetic term.

6.2.2 Properties of linearized equations

In this section we review some properties of the solutions to the system of Eqs. (6.42) that describe the behavior of linear perturbations. These properties and their derivation are completely analogous to the ones reported in subsection 5.4 of chapter 4; hence, we only mention the most relevant ones.

First, note that there always exists the zero-mode solution $(\lambda, A, B) = (0, 0, \beta u)$, with β an arbitrary complex constant. This corresponds to the trivial perturbation that rotates the real function $v(x)$ into the complex plane, resulting in a new configuration $\psi(t,x)$ that is indistinguishable from (6.10).

Second, the solutions to the Eqs. (6.42) appear always in quadruples, that is, if (λ, A, B) is a solution, then $(-\lambda, A, -B)$, $(\lambda^*, A^*, -B^*)$ and $(-\lambda^*, A^*, B^*)$ are also solutions. This implies that the eigenvalues λ appear always in the form $\{\lambda, -\lambda, \lambda^*, -\lambda^*\}$. Linear stability requires that all eigenvalues λ are purely imaginary. In the remaining of this work we shall count the number of unstable modes by the number of eigenvalues with distinct real part. Hence, when both the real and imaginary parts of λ are different from zero, we count the pair of eigenvalues λ and λ^* as one unstable mode, although they may belong to two linearly independent eigenfunctions.

Third, from Eq. (6.13) and analogously to expression (5.4), we obtain the following expression for the second variation of the energy functional,

$$\delta_r^2 \mathcal{E}[u; v] = (v, [\hat{\mathcal{H}} - \omega]v)_{L^2} + 2\kappa F[\delta n] - 2D[\delta n, \delta n], \tag{6.48}$$

where $\delta n := 2\mathrm{Re}(u^* v)$. The restricted second variation of the energy functional $\delta_r^2 \mathcal{E}$ is related to the linearized equations in the following way: multiplying Eq. (6.42b) by A^* and Eq. (6.42a) by B^* and integrating one obtains

$$i\lambda\,(A, B) = \delta_r^2 \mathcal{E}[A_R] + \delta_r^2 \mathcal{E}[A_I], \tag{6.49a}$$

$$i\lambda\,(B, A) = \left(B, [\hat{\mathcal{H}} - \omega]B\right), \tag{6.49b}$$

where A_R, A_I denotes the real and imaginary parts of A, respectively, and $\delta_r^2\mathcal{E}[A_R]$ is given by Eq. (6.48) replacing v with A_R. The right-hand sides of the previous equations are real; thus,

$$-\lambda^2 |\,(A,B)\,|^2 \in \mathbb{R}. \tag{6.50}$$

Finally, if λ is real one can choose A real and B purely imaginary. Eliminating $i\lambda A$ on the left-hand side of Eq. (6.49a) and using Eq. (6.42a) one gets

$$-\left(B,[\hat{\mathcal{H}}-\omega]B\right) = \delta_r^2\mathcal{E}[A]. \tag{6.51}$$

Equations (6.49), (6.50) and (6.51) are useful to rule out the existence of certain unstable modes. For example, consider a stationary state for which ω is the ground state energy of the Schrödinger operator $\hat{\mathcal{H}}$. Then, $\left(B,[\hat{\mathcal{H}}-\omega]B\right) \geqslant 0$ with equality if and only if B lies in the kernel of $\hat{\mathcal{H}}-\omega$. This immediately excludes the existence of modes with $\lambda_R \neq 0$ and $\lambda_I \neq 0$ since in this case Eq. (6.50) would imply that $(A,B)=0$ and then one could infer that $A=B=0$ using Eqs. (6.49b) and (6.42). If, in addition, this state is a local minimum of the energy functional, such that $\delta^2\mathcal{E}$ is positive or zero, then unstable modes with $\lambda_I = 0$ are also ruled out since in this case Eqs. (6.51) and (6.42) imply that $A=B=0$. These arguments show that stationary ground state configurations can have only real or purely imaginary λ and that real eigenvalues are excluded if these configurations are a local minimum of the energy functional $\mathcal{E}$.

6.2.3 Stationary and spherically symmetric configurations

For the remaining of the chapter we focus on stationary solutions which are spherically symmetric, i.e. $u(x)=u(r)$. In the next subsection we discuss the background (unperturbed) solutions, whereas some properties of the linearized perturbations are discussed in subsequent subsections.

Background configurations

Instead of solving directly the integro-differential equation (6.12), for a numerical analysis it is more convenient to work with the original Gross-Pitaevskii-Poisson system (6.8). Introducing the harmonic ansatz 6.10) into Eqs. (6.8) and defining the shifted potential $\mathcal{U}(r) := \omega - U(r)$, the dimensionless Gross-Pitaevskii-Poisson equations take the form

$$\Delta_r u = \left(\kappa u^2 - \mathcal{U}\right) u, \tag{6.52a}$$

$$\Delta_r \mathcal{U} = -u^2, \tag{6.52b}$$

where $\Delta_r := \frac{1}{r}\frac{d^2}{dr^2} r$ denotes the radial Laplace operator.

The system of equations (6.52) must be solved for some appropriate boundary conditions. By Theorem 6.3, u must be regular at the origin, thus $u(r=0)=u_0$, $u'(r=0)=0$, $\mathcal{U}(r=0)=\mathcal{U}_0$, and $\mathcal{U}'(r=0)=0$. (Here and in the following the prime refers to derivation with respect to r.) Then, given $\mathcal{U}_0$, the central amplitude of the field u is fine-tuned using a numerical shooting methodology based on the condition $\lim\limits_{r\to\infty} u(r) = 0$ at spatial infinity, which is required for the solution to be localized.

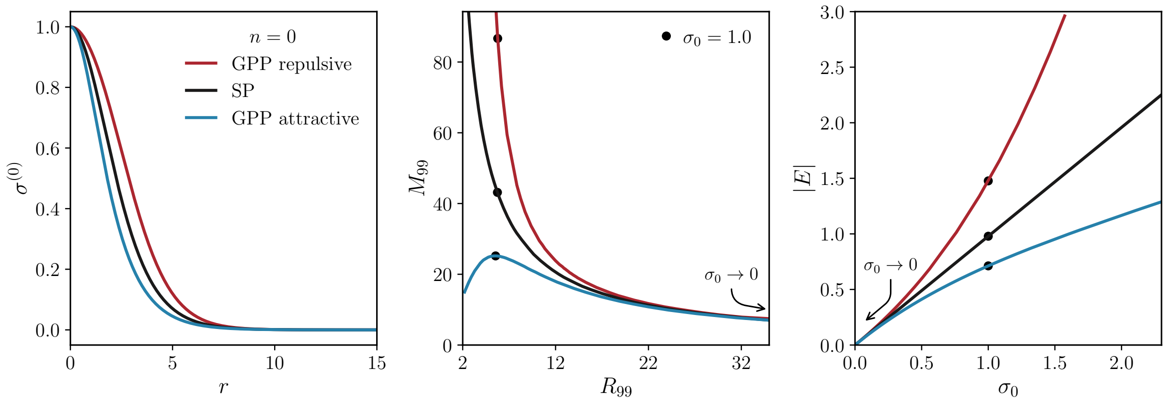

Figure 6.1: (Plot extracted from [21]). In the notation adopted in this thesis, $\sigma^{(0)}$ corresponds to u, σ_0 to u_0 and $|E|$ to $|\omega|$). Stationary and spherically symmetric solutions of the Gross-Pitaevskii-Poisson system with no nodes ($n = 0$). Red (blue) lines correspond to the repulsive (attractive) case, and we have included the solutions to the Schrödinger-Poisson system (black lines) for reference. *(Left panel)* The profile of the wave function $u(r)$ for a central value of unity, $u_0 = 1$. *(Center panel)* The effective mass of the configurations M_{99} as a function of the effective radius R_{99}. *(Right panel)* The magnitude of the energy eigenvalue $|\omega|$ as a function of the central amplitude u_0. The dots in the last two panels correspond to the configurations of unit amplitude. For $u_0 \to 0$ the effects of the selfinteractions become negligible.

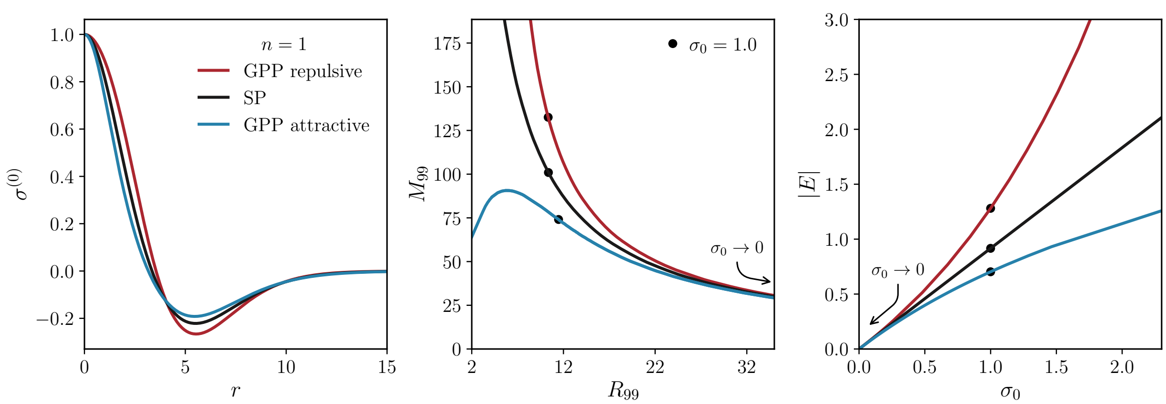

Figure 6.2: (Plot extracted from [21]). In the notation adopted in this thesis, $\sigma^{(0)}$ corresponds to u, σ_0 to u_0 and $|E|$ to $|\omega|$). Same as in Fig. 6.1 but for the stationary and spherically symmetric solutions of the Gross-Pitaevskii-Poisson system with one node ($n = 1$).

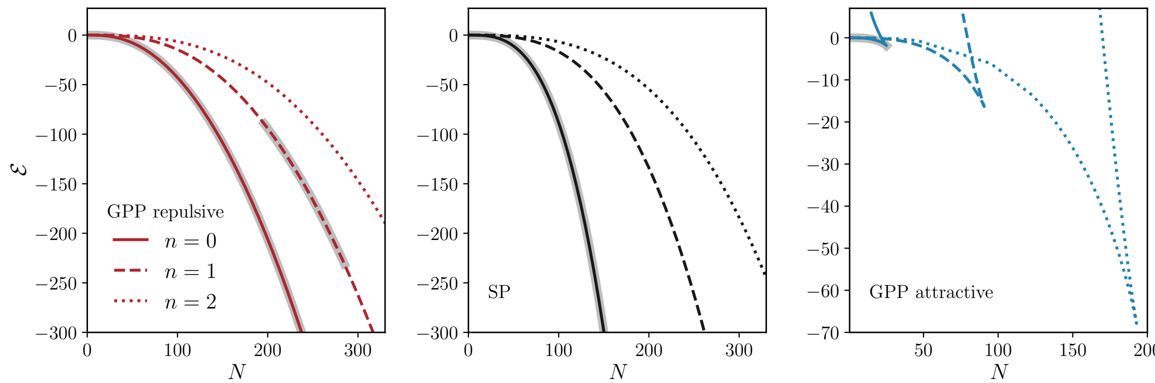

Figure 6.3: (Plot extracted from [21]). Relationship between the energy functional (6.47) and the particle number N for states with zero, one and two nodes in the repulsive, non-selfinteracting and attractive cases. Note that in the attractive case the number of nodes n does not necessarily label the number of the excited state. The shaded regions represent the stability bands with respect to generic linear perturbations discussed in Sec. 6.2.4.

The system (6.52) is solved numerically using an adaptive explicit 5(4)-order Runge-Kutta routine [86, 28, 51], where we rewrite the equations as a first-order system for the fields $(u, \mathcal{U})$. For the shooting, we use a methodology similar to the one described in previous Chapter in Sec. 5.5.

Some representative solutions of the stationary and spherically symmetric Gross-Pitaevskii-Poisson system are shown in Figs. 6.1 and 6.2 for the ground ($n = 0$) and the first excited ($n = 1$) states, respectively.

In these figures, the mass of the configurations M^{phys} has been calculated as their particle number Eq. (6.5) times the mass m_0 of the individual particles, which leads to $M^{phys} = [M_{\rm Pl}^2/(4\pi m_0 \Lambda^{1/2})]M$, where $M := 4\pi\int_0^\infty u(r)^2 r^2 dr$ is the dimensionless mass. Note that, formally, the radius of a boson star extends to infinity, and for that reason we have defined the effective radius R_{99} as the one encompassing 99% of the total mass of the configuration, M_{99}. On the other hand, the energy eigenvalue $\omega^{phys} = [2m_0/\Lambda]\omega$ associated with the state $u(r)$ can be computed from the formula

$$\omega = \mathcal{U}_0 - \int_0^\infty u(r)^2 r dr. \tag{6.53}$$

As expected, the profile of the configurations expands (shrinks) if the selfinteraction is repulsive (attractive), as can be appreciated in the left panels of Figs. 6.1 and 6.2. Furthermore, in the repulsive case the configurations become more compact as we increase the value of the central amplitude, as also occurs for the stationary and spherically symmetric solutions of the Schrödinger-Poisson system, wheres in the case in which the selfinteraction is attractive there exists a state of maximum mass corresponding to a central amplitude $u_0^{M_{99}^{\max}}$; see the central panel of Figs. 6.1 and 6.2 and Refs. [18, 20, 19]. Finally, it is also interesting to stress that in the limit $u_0 \to 0$ the attractive and the repulsive configurations approach the solutions to the Schrödinger-Poisson system, see the central and right panels of the same figures. This implies that

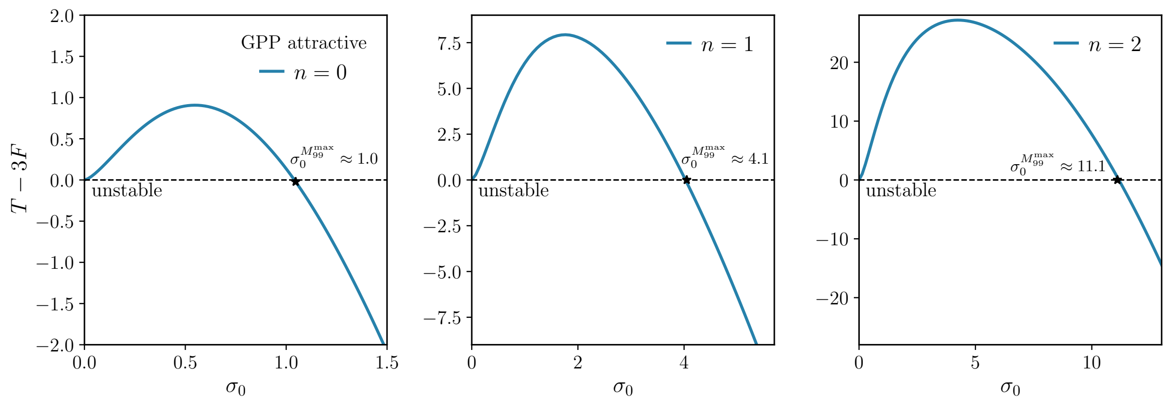

Figure 6.4: (Plot extracted from [21]. Here σ_0 corresponds to u_0 and $\sigma_0^{M_{99}^{\max}}$ to $u_0^{M_{99}^{\max}}$). Configurations with a negative value of $T - 3F$ correspond to stationary states that represent critical points of the energy functional which cannot be local minima, and hence they are expected to be unstable. The stars denote the maximum mass configurations at $u_0 = u_0^{M_{99}^{\max}}$; see also Fig. 6.5 which complements these plots.

the effects of the selfinteractions become negligible in the limit of low amplitudes, as is also apparent from the Gross-Pitaevskii-Poisson system (6.3).

In Fig. 6.3 we plot the total energy $\mathcal{E}$ of the configurations as a function of the total number of particles N for the ground and first two excited states. Note that the energy functional (6.4) consists of three terms: the first of them is a consequence of the "quantum pressure" and is positive definite. The second one is associated with the selfinteraction and can be positive or negative, depending on whether the selfinteraction is repulsive or attractive, respectively. Finally, the last term is associated with the gravitational interaction which is attractive and negative definite. It is interesting to note that in the repulsive case, as well as if there are no selfinteractions, the total energy is negative definite and its magnitude increases with the number of particles. Furthermore, for a given value of N, the energy also increases with the number of nodes n in the configuration. This allows us to label the number of the excited states with n, as is usually done. However, this behavior changes for the case of an attractive selfinteraction, where, for a given n, there is a state of minimum energy (which coincides with the state of maximum mass), and from this point onward the energy increases up to positive values.

Stability properties based on a scaling argument (attractive case)

Recall from Section 6.2.1 that in the attractive case, a stationary configuration cannot be a local minimum of the energy functional if $T - 3F < 0$. When the central value of the scalar field u_0 is small, the configurations resemble the ones of the Schrödinger-Poisson system, and hence the selfinteraction term F is negligible, such that $T > 3F$ is expected to hold. In contrast, as u_0 grows, the influence of the selfinteraction becomes more pronounced (see Figs. 6.1 and 6.2) and one expects $3F$ to surpass the value of T after some critical value of u_0.

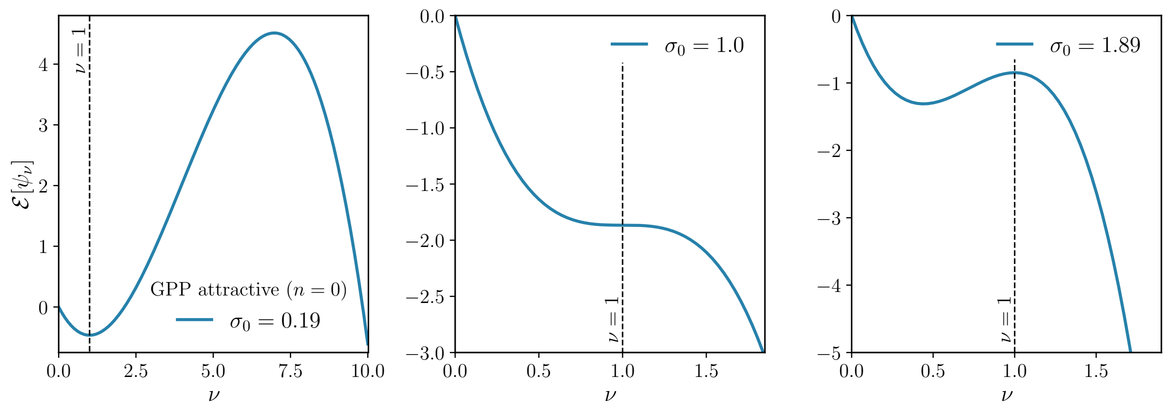

Figure 6.5: (Plot extracted from [21]. Here ψ_ν corresponds to u_ν and σ_0 to u_0). The energy functional $\mathcal{E}[u_\nu]$ as a function of ν for three ground state configurations. *(Left panel)* Stationary state with $u_0 = 0.19 < u_0^{M_{99}^{\max}}$ giving $T = 0.558$ and $F = 0.047$. This state could be a local (but not a global) minimum of the energy functional and hence could be stable. *(Center panel)* Stationary state with $u_0 = 1.0 = u_0^{M_{99}^{\max}}$ giving $T = 5.641$ and $F = 1.887$. This state resembles a saddle point of the energy functional. *(Right panel)* Stationary state with $u_0 = 1.89$ giving $T = 11.411$ and $F = 5.282$. This state cannot be a local minimum of the energy functional and is expected to be unstable.

The plots in Fig. 6.4 show that these expectations are indeed fulfilled. Interestingly, the critical value of u_0 for which $T - 3F = 0$ seems to coincide with the maximum of the mass in the center panels of Figs. 6.1 and 6.2. As will be discussed in the next section, when this critical value is surpassed, the number of unstable modes grows by one.

For completeness, Fig. 6.5 shows the behavior of the energy functional under the rescaling (6.43) and illustrates that stationary states with positive (negative) values of $T-3F$ represent a local minimum (maximum) with respect to this particular variation.

Decomposition of the linearized system into spherical harmonics

Since the background solution is spherically symmetric, the linearized equations can be decoupled into a family of purely radial systems by expanding the perturbations in terms of spherical harmonics Y^{LM}. In particular, the field $A(x)$ can be expanded as:

$$A(x) = \sum_{LM} A_{LM}(r) Y^{LM}(\vartheta, \varphi), \tag{6.54}$$

and similarly for $B(x)$. This reduces Eqs. (6.42) to the following system:

$$i\lambda A_{LM} = \left(\hat{\mathcal{H}}_L - \omega\right) B_{LM}, \tag{6.55a}$$

$$i\lambda B_{LM} = \left(\hat{\mathcal{H}}_L - \omega\right) A_{LM} + 2u\hat{K}_L\left[uA_{LM}\right], \tag{6.55b}$$

with the operators $\hat{\mathcal{H}}_L$ and $\hat{K}_L$ defined by

$$\hat{\mathcal{H}}_L := -\triangle_L + \kappa u^2 + \Delta_r^{-1}(u^2), \tag{6.56a}$$

$$\hat{K}_L := \kappa + \Delta_L^{-1}, \tag{6.56b}$$

where $\Delta_L := \Delta_r - L(L+1)/r^2$ and

$$\Delta_L^{-1}(f)(r) := -\frac{1}{2L+1}\int_0^\infty \frac{r_<^L}{r_>^{L+1}} f(\tilde{r})\tilde{r}^2 d\tilde{r}, \tag{6.57}$$

with $r_< := \min\{r, \tilde{r}\}$ and $r_> := \max\{r, \tilde{r}\}$. Note, in particular, that $\Delta_s = \Delta_{L=0}$.

In the next section this system will be solved numerically for $L = 0, 1, 2, \ldots$. Before doing that, however, we prove that no unstable modes are present if L is large enough, which, in principle, reduces the numerical analysis to a finite number of L values.

Non-existence of unstable modes for sufficiently large L

In this subsection we show that if the orbital angular momentum L of the perturbation is large enough, then there are no unstable modes. The arguments below generalize the ones in Section. 5.4.4 to include the self-interaction term.

Using the estimate (E8) in Appendix E in [65], one obtains from Eq. (6.48) the inequality[1]

$$\delta_r^2\mathcal{E} \geqslant \frac{1}{2}\|\nabla u\|_2^2 + (v, [\kappa u^2 + U - \omega]v) - C_1\|v/f\|_2^2 + 2\kappa F[\delta n], \tag{6.58}$$

where $U := \Delta^{-1}(|u|^2)$, $C_1 > 0$ is a positive constant and $f(x) := \sqrt{1+|x|^2}$. In the repulsive case, the term $+2F[\delta n]$ on the right-hand side of Eq. (6.58) is positive and hence can be discarded without altering the inequality. In the attractive case, we use the estimate $|\delta n|^2 \leqslant 4|u|^2|v|^2$, which implies

$$F[\delta n] \leqslant \max_{x\in\mathbb{R}^3}\left[(1+|x|^2)|u(x)|^2\right]\|v/f\|_2^2. \tag{6.59}$$

Since $|u|$ is smooth and decays exponentially to zero as $|x| \to \infty$, the maximum is finite. Therefore, the negative term $-2F[n]$ on the right-hand side of Eq. (6.58) can be removed after making the constant C_1 larger.

Based on these results, we can prove that $\delta_r^2\mathcal{E}$ is positive definite for sufficiently large values of the orbital angular momentum L. To this purpose we expand δu in spherical harmonics

$$v = \sum_{LM} h_{LM}(r)Y^{LM}(\vartheta, \varphi). \tag{6.60}$$

Substituting this in expression (6.58) and discarding the terms which involve $|h'_{LM}(r)|^2$ one gets

$$\begin{aligned}\delta_r^2\mathcal{E} \geqslant \sum_{LM}\Bigg\{&\int_0^\infty |h_{LM}(r)|^2\left[\frac{L(L+1)}{2} - C_1\right]dr \\ &+ \int_0^\infty |h_{LM}(r)|^2[\kappa u^2 + U(r) - \omega]r^2 dr\Bigg\}.\end{aligned} \tag{6.61}$$

[1] Note that the left-hand side of Eq. (E8) in [65] should read $D[\delta n, \delta n]$ instead of $D[\delta u, \delta u]$.

First we look at the integrand in the second line and observe that the function $g(r) := [\kappa u^2 + U(r) - \omega]r^2$ satisfies $g(0) = 0$ since u and U are regular at the center, whereas $g(r)$ is positive for large enough r. This implies (since U is continuous) that $g(r)$ has a minimum, that is, $g(r) \geqslant C_2$ for all $r \geqslant 0$ for some negative constant C_2. Using these properties, the estimate (6.61) yields

$$\delta_r^2 \mathcal{E} \geqslant \sum_{LM} \int_0^\infty |h_{LM}(r)|^2 \left[\frac{L(L+1)}{2} - C_1 + C_2 \right] dr. \tag{6.62}$$

Therefore, $\delta_r^2\mathcal{E}$ is positive definite if h_{LM} is identically zero for all L with $L(L+1)/2 - C_1 + C_2 \leqslant 0$. This proves that the second variation of the energy functional is positive definite on the subspace of perturbations with large enough values of L.

Now we can prove the non-existence of unstable modes for large L. We proceed by contradiction. If λ has real and imaginary parts different from zero, then Eq. (6.50) yields $(A, B) = 0$, and in this case Eq. (6.49) and the positivity of $\delta_r^2\mathcal{E}$ imply that $A = B = 0$. On the other hand, if λ is real, Eq. (6.51) leads to a contradiction since the right-hand side is positive whereas the left-hand side is negative for sufficiently large values of L.

6.2.4 Numerical study of the linearized system

To find the eigenvalues λ of the system (6.55) we use a similar methodology to the previous chapter, for which it is convenient to write the system of equations in a more appropriate form. For this, we define the new rescaled functions $a_L(r) := rA_{LM}(r)$ and $b_L(r) := rB_{LM}(r)$, in terms of which the system (6.55) reduces to

$$b_L'' \mp |u|^2 b_L + U_L^{\text{eff}} b_L = -i\lambda a_L, \tag{6.63a}$$

$$\begin{aligned} &a_L'' + U_L^{\text{eff}} a_L - 3\kappa |u|^2 a_L \\ &\qquad - 2u \left(\frac{d^2}{dr^2} - \frac{L(L+1)}{r^2} \right)^{-1} [u a_L] = -i\lambda b_L. \end{aligned} \tag{6.63b}$$

Here we have introduced the effective potential $U_L^{\text{eff}}(r) := \mathcal{U}(r) - L(L+1)/r^2$, and the operator $\left(d^2/dr^2 - L(L+1)/r^2\right)^{-1} = r\triangle_L^{-1} r^{-1}$ denotes the inverse of $r\triangle_L(r^{-1})$ with homogeneous Dirichlet boundary conditions at $r = 0$ and $r \to \infty$. Given that the system (6.55) is independent of the magnetic quantum number M, we have omitted this label in Eqs. (6.63).

These equations must be solved for some appropriate boundary conditions. To determine these conditions, one can study (heuristically) the dominant terms near the origin and at spatial infinity. Near $r = 0$ the equations are dominated by the centrifugal term $L(L+1)/r^2$ of the effective potential U_L^{eff}, which means that regular solutions at the center must behave as $\left(a_L, b_L\right) \sim \left(r^{L+1}, r^{L+1}\right)$. This leads to the following boundary conditions at the origin:

$$a_L(r = 0) = 0, \quad b_L(r = 0) = 0. \tag{6.64a}$$

On the other hand, in the asymptotic region, the radial profile of the background field $u(r)$ decays exponentially to zero and $\mathcal{U}(r)$ approaches ω. This implies that the fields (a_L, b_L) must vanish at infinity,

$$\lim_{r\to\infty} a_L(r) = 0, \qquad \lim_{r\to\infty} b_L(r) = 0, \tag{6.64b}$$

in order to have solutions with finite total energy.

To numerically solve the system (6.63) with the Dirichlet boundary conditions (6.64) we employ the following approach: first, the background profiles $u(r)$ and $\mathcal{U}(r)$ are computed following the method described in Section 6.2.3, and they are extended to large values of r using the procedure described in the previous chapter in Section 5.5. Second, these background profiles, as well as the perturbed fields a_L, b_L and the different operators, e.g. the derivative and its inverse, are discretized in terms of Chebyshev polynomials using a standard spectral method (see Sec. 5.5 or Ref. [85]), which leads to a finite-dimensional eigenvalue problem. For details of the numerical discretization procedure we refer the reader to Sec. IV of Ref. [70].

The discrete version of the system (6.63) can be writen as:

$$\begin{pmatrix} \mathsf{0} & \tilde{\mathbb{D}}_{\mathsf{N}}^2 \mp \mathbb{U}^2 + \mathsf{U}_L^{\text{eff}} \\ \tilde{\mathbb{D}}_{\mathsf{N}}^2 \mp 3\mathbb{U}^2 + \mathsf{U}_L^{\text{eff}} - 2\mathbb{U}\left(\tilde{\mathbb{D}}_{\mathsf{N}}^2 - \mathbb{L}\right)^{-1}\mathbb{U} & \mathsf{0} \end{pmatrix} \begin{pmatrix} \mathsf{a}_L \\ \mathsf{b}_L \end{pmatrix} = -i\lambda \begin{pmatrix} \mathsf{a}_L \\ \mathsf{b}_L \end{pmatrix}, \tag{6.65}$$

where here $\mathsf{0}$ represents the $(\mathsf{N}-1)\times(\mathsf{N}-1)$ zero matrix, with N the number of Chebyshev points distributed as $x_j = \cos(j\pi/\mathsf{N}), j = 0, 1, \dots, \mathsf{N}$. The quantities

$$\begin{aligned} \mathbb{U} &:= \mathbf{diag}\Big(u(x_1), u(x_2), \dots, u(x_{\mathsf{N}-1})\Big), \\ \mathsf{U}_L^{\text{eff}} &:= \mathbf{diag}\Big(U_L^{\text{eff}}(x_1), U_L^{\text{eff}}(x_2), \dots, U_L^{\text{eff}}(x_{\mathsf{N}-1})\Big), \\ \mathbb{L} &:= \mathbf{diag}\left(\frac{L(L+1)}{x_1^2}, \frac{L(L+1)}{x_2^2}, \dots, \frac{L(L+1)}{x_{\mathsf{N}-1}^2}\right), \end{aligned}$$

are the discrete representation of the background quantities and centrifugal term, respectively, and the discrete operator $\tilde{\mathbb{D}}_{\mathsf{N}}$ is defined in [70]. The vector

$$\begin{pmatrix} \mathsf{a}_L \\ \mathsf{b}_L \end{pmatrix} := \Big(a_L(x_1),\ \dots\ , a_L(x_{\mathsf{N}-1}), b_L(x_1), \dots, b_L(x_{\mathsf{N}-1})\Big)^T$$

represents the eigenfields $r(A_{LM}, B_{LM})^T$. We solve the discrete eigenvalue problem (6.65) using the SciPy library [86]. Our code is publicly available in [1].

In the following subsections we present the results of the eigenvalue problem, first for the ground state configurations and then for the excited states.

Ground state

For ground state configurations ($n = 0$) we have found that, for the explored range of the parameter space ($0 < u_0 < 10$, $0 \le L \le 12$ for the repulsive and $0 < u_0 < 5$,

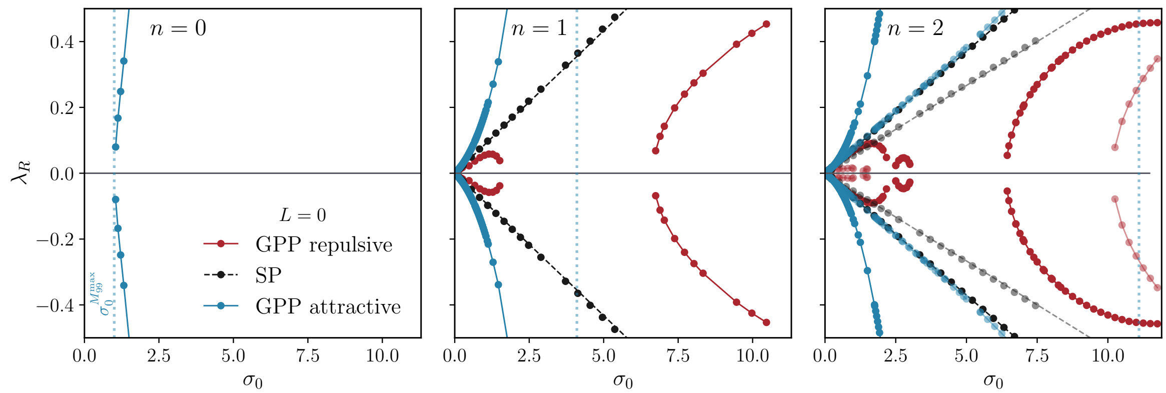

Figure 6.6: (Plot extracted from [21]. In the notation used in this thesis we identify σ_0 with u_0). The real part of the frequency λ as a function of the central amplitude u_0 for perturbations with $L = 0$ in configurations with $n = 0$, 1 and 2 nodes. The existence of modes with a real positive part of the frequency indicates the existence of an instability. The different color shades correspond to different unstable modes, and for reference we have included dotted vertical lines in blue to indicate the state of maximum mass $u_0^{M_{99}^{\max}}$ in the attractive case. The circles denote our numerical results, whereas the dashed lines represent the scaled results of the n unstable modes reported in Ref. [65] for the Schrödinger-Poisson system. For the repulsive and the attractive cases we use straight solid lines to connect the numerical results and improve the visualization. Ground state configurations ($n = 0$) are stable for any value of the amplitude if the selfinteraction is repulsive, or if there is no selfinteraction, and they are unstable for large enough amplitudes ($u_0 \gtrsim u_0^{M_{99}^{\max}} \approx 1.0$) if the selfinteraction is attractive. Excited states ($n \neq 0$) are in general unstable, although certain bands of stability in the amplitude u_0 appear if the selfinteraction is repulsive.

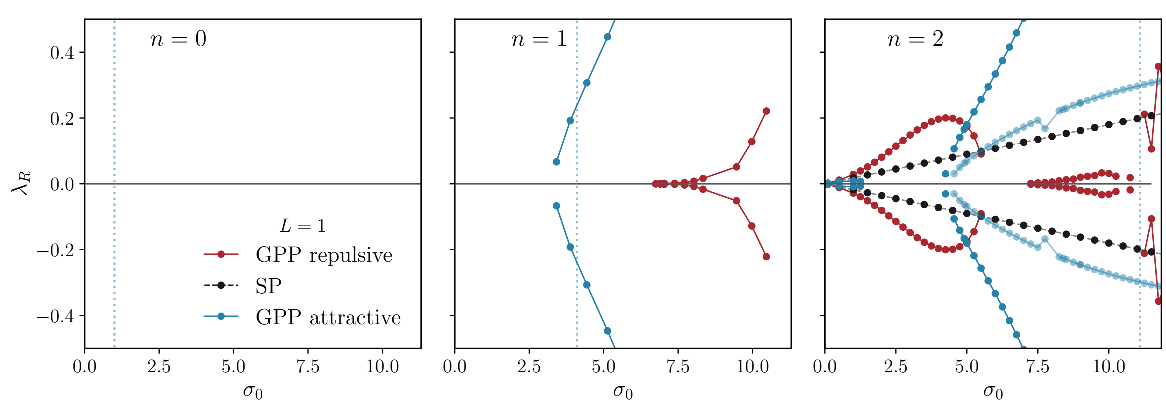

Figure 6.7: (Plot extracted from [21]. In the notation used in this thesis we identify σ_0 with u_0). Same as in Fig. 6.6 but for perturbations with $L = 1$. The ground state ($n = 0$) is always stable under such perturbations, no matter if the selfinteraction is attractive, repulsive or absent. Excited states ($n \neq 0$) present stability bands for the repulsive and attractive cases.

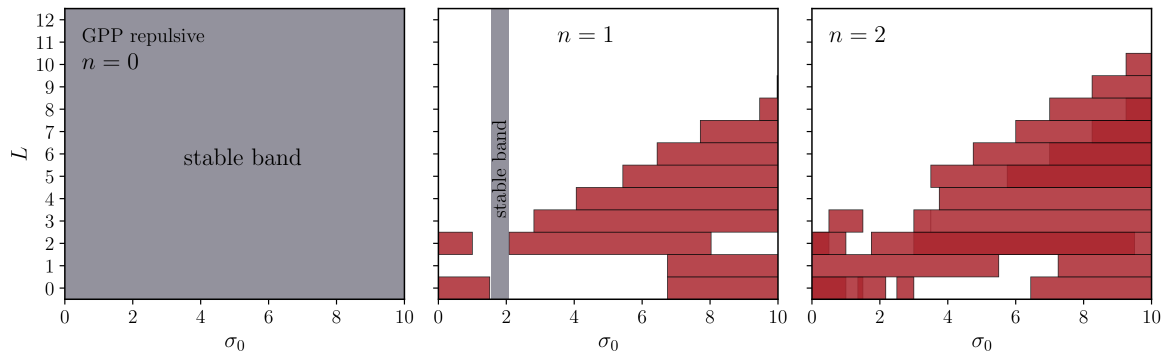

Figure 6.8: Plot extracted from [21]. In the notation used in this thesis we identify σ_0 with u_0. The shaded regions in red represent, for different values of L in the interval from 0 to 12, the amplitudes u_0 for which the system posses unstable modes. Lighter (darker) colors indicate the presence of one (two) unstable mode(s). Note that the ground state ($n = 0$) does not present unstable modes for any value of u_0 or L, so we can conclude that it is stable. In contrast, the first excited state ($n = 1$) presents a common stability band in the interval $1.55 \lesssim u_0 \lesssim 2.07$, whereas this band is empty for the second excited state ($n = 2$).

$0 \leq L \leq 6$ for the attractive case), the first two scenarios exhibit a similar behavior under linear perturbations. In particular, the configurations are always stable under spherically symmetric perturbations ($L = 0$), as can be appreciated from the left panel of Fig. 6.6. Furthermore, non-spherical perturbations ($L \neq 0$) do not either exhibit unstable modes that grow in time, at least for $L \leq 12$, as can be seen in the left panel of Fig. 6.7 for $L = 1$ and the left panel of Fig. 6.8 for other values of L in the repulsive case. The results corresponding to the scenario without selfinteractions are consistent with those reported in [65] and coincide with the repulsive and the attractive cases in the limit $u_0 \to 0$.

Contrary to the first two scenarios, the case with an attractive selfinteraction presents ground state configurations that are unstable under radial perturbations ($L = 0$). More precisely, we found unstable modes with $\lambda_R > 0$ when $u_0 \gtrsim 1$; see the left panel of Fig. 6.6. This gives rise to a family of solutions which is stable for $u_0 \lesssim 1$ and unstable for $u_0 \gtrsim 1$. Interestingly, the division between the stable and unstable states seems to coincide with the maximum mass configuration at $u_0 = u_0^{M_{99}^{\max}} \approx 1$. This is consistent with the turning-point principle; see Refs. [89, 43, 79], the Appendix C2 of the published version of [2], and in particular [35] for a discussion in the context of relativistic boson stars. Regarding non-spherical perturbations, no unstable modes appear when $L = 1$, as is appreciated from the left panel of Fig. 6.7. Based on a more extensive study within the aforementioned parameter range, we found no unstable modes for $L > 0$, as can be seen from the left panel of Fig. 6.9.

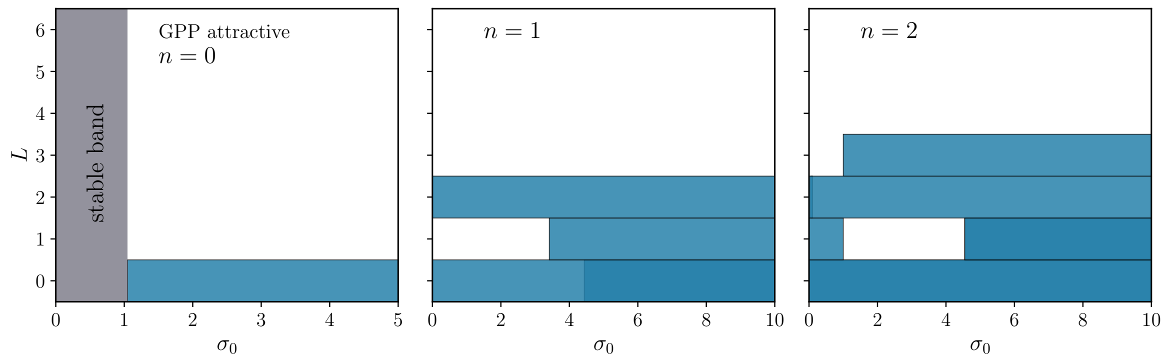

Figure 6.9: (Plot extracted from [21]. In the notation used in this thesis we identify σ_0 with u_0). Same as in Fig. 6.8 but for the attractive case. Note that the common stability band is empty for the first and second excited states and that no unstable modes are found for $L > 3$. Note also that for $n = 2$ and $L = 2$ there are two unstable modes in the interval $0 < u_0 < 0.1$, although they are barely visible.

Excited states (spherical perturbations)

The excited states of the Schrödinger-Poisson system are known to be unstable under radial perturbations [44]. However, in presence of selfinteractions, our analysis (carried out in the range $0 < u_0 < 14$ with $0 \leq L \leq 12$ and $0 \leq L \leq 6$ in the repulsive and attractive cases, respectively) reveals an interesting pattern of stability and instability bands, with a particular relevance for the case where the selfinteraction is repulsive. Our study focusses on the first two excited states ($n = 1$ and $n = 2$) of the Gross-Pitaevskii-Poisson equations, however, we expect that the behavior discussed below will continue to be valid for higher excited states.

The central and right panels of Fig. 6.6 present the results obtained for radial perturbations. As can be observed, there exist unstable modes with $\lambda_R > 0$ in the attractive, the repulsive as well as in the non-selfinteracting case. In the first and third cases, we have identified an unstable band that spans all values of the central amplitude u_0 that we have explored. For the n'th excited state, this band consists of n unstable modes that grow exponentially in time. In the attractive scenario, for one of these modes, λ_R grows rapidly as u_0 increases, signaling that the solutions have a lifetime that is much shorter than the state with the same central amplitude in the non-selfinteracting theory. Indeed, in absence of selfinteractions, one observes a linear relationship between the real part of the mode(s) frequency λ_R and the central amplitude u_0. This relationship arises from an inherent scaling freedom in the Schrödinger-Poisson system. For further details regarding this symmetry we refer the reader to Refs. [61, 70]. The dashed lines represent the scaled results of the n unstable modes reported in Ref. [65], while the dark circles denote our numerical results. As can be seen, both sets of results are in agreement.

A more interesting pattern appears for the repulsive scenario, in which the unstable radial modes cover only localized intervals of u_0, resulting in stability bands where

$\lambda_R = 0$;[2] see the central and right panels of Fig. 6.6. It is worth noting that, unlike ground state configurations, the boundary between stable and unstable solutions is not determined by a maximum mass state.

Excited states (non-spherical perturbations)

Next, we address the problem of what happens to the stability bands of the previous subsection when non-spherical perturbations are considered. Specifically, the relevant question is whether unstable modes with $L > 0$ appear inside these bands, which would imply that the configurations are unstable with respect to generic linear perturbations.

The central and right panels of Fig. 6.7 show unstable modes with $L = 1$ appearing in certain intervals of u_0, giving rise to new stability bands. Interestingly, such bands appear in the repulsive as well as in the attractive case, whereas for $L = 0$ they only occurred in the former case. Further stability bands arise as L increases. The central and right panels of Figs. 6.8 and 6.9 show the different stability bands (white regions) for different values of L. Notice that for high enough values of L, the stable band covers entirely the region of the parameter space we have explored, which is compatible with the result obtained in Subec. 6.2.3.

Furthermore, it is interesting to compare the colored regions of instability shown in Figs. 6.8 and 6.9 for small values of u_0. In this limit the selfinteraction term of the Gross-Pitaevskii equation (6.3) is subdominant, which would lead to all scenarios being equivalent (see the overlapping region in Figs. 6.1 and 6.2). As a consequence, one observes from Figs. 6.8 and 6.9 that in the limit $u_0 \to 0$ the instability bands (including the number of unstable modes) transit continuously to the corresponding instability bands belonging to the configurations of the Schrödinger-Poisson system. For the particular cases $n = 0$ and $n = 1$ we refer the reader to Fig. 4 of Ref. [65] for the case $\ell = 0$ where the respective regions of instability for the Schrödinger-Poisson system are reported.

We conclude this section by emphasizing that only configurations lying in the intersection of the stability bands for all L are stable with respect to generic linear perturbations. For $n = 1$ the resulting common stability band turns out to be very narrow in the repulsive case, as shown in Fig. 6.8, whereas it is empty in the attractive case, see Fig. 6.9. For $n = 2$ there are no stable configurations, neither in the repulsive nor in the attractive case. Similar results are expected for higher excited states.

[2] Note that the occurrence of such localized stability bands is excluded in the non-selfinteracting case due to the aforementioned rescaling freedom of the Schrödinger-Poisson system.

Chapter 7

Conclusions

In this thesis we have studied the multi-field Schrödinger-Poisson and the Gross-Pitaevskii-Poisson systems, both modified through the inclusion of an external gravitational potential generated by a point mass M. By minimizing the functional associated with the total energy keeping the total particle number fixed, we established the existence of static solutions of these systems. These solutions are strong (belonging to the Sobolev space H^2, controlling up to two derivatives in L^2) in the case $M \geq 0$ and classical solutions (controlling derivatives of arbitrary order) when $M = 0$. For the multi-field Schrödinger-Poisson system, the ground state obtained is characterized by the form $u = f\mathbf{e}$ where $\mathbf{e} \in \mathbb{C}^K$ is a constant unit vector and f is a real-valued, spherically symmetric and monotonically decreasing function.

Furthermore, by restricting the minimization of the energy functional to the space of functions invariant under the spin-ℓ unitary group representation $\rho^{(\ell)}$, we were able to rigorously demonstrate the existence of Newtonian ℓ-boson star. In addition, it was shown that these configurations are orbital stable within the class of perturbations which are invariant with respect to $\rho^{(\ell)}$ (under the assumption of global existence), and a linear stability analysis indicated that they are linearly unstable against non-spherically symmetric perturbations unless $\ell = 0$ or $\ell = 1$.

The advantage of demonstrating the existence of static solutions through variational methods, compared to the approach in [22], is that we are capable of proving that the solutions not only exist but also minimize the energy functional for each ℓ. In addition to existence, this allows us to establish other relevant properties of the solutions, such as their orbital stability.

Using similar methods, we also proved the existence and orbital stability of solutions to the Gross-Pitaevskii-Poisson system with a point mass M. In the particular case $M = 0$, we investigated the role of self-interaction on the linear stability of such solutions, with a special focus on excited states. We established that the ground state is stable under generic perturbations (in agreement with orbital stability), while excited states are, in general, linearly unstable.

This finding shows that in the Newtonian limit, self-interaction alone is insufficient to stabilize excited states of boson stars. This stands in contrast to recent results in the relativistic regime from Refs. [75, 12], where the authors have concluded that, for strong enough repulsive self-interaction, these configurations remain stable under spherical perturbations, at least for times of the order of $10^4 m_0^{-1}$.

It is often the case in physics that the rigorous question of the existence of solutions to partial differential equations is regarded as a technical subtlety. Nevertheless, we believe that such results provide a crucial mathematical foundation for physical models. In particular, they offer a solid framework for applying numerical methods to construct an approximate solution with confidence. Furthermore, they allow one to prove generic results beyond exploring isolated points in the solution space. A rigorous proof of existence also ensures that the ground state (the configuration of minimal energy) is a well defined entity. This mathematical certainty is also vital for stability analyses, as it confirms that the object under study corresponds to a genuine solution of the underlying equations and that its stability properties are intrinsic.

Despite the progress achieved in this work, several important questions remain open. On the one hand, it would be of great interest to extend the existence results to the case of multi-ℓ boson stars [6], where multiple values of ℓ are allowed simultaneously. This should be an immediate generalization of the methods presented here; it is sufficient to replace the spin-ℓ representation $\rho^{(\ell)}$ with a more general unitary representation of $SO(3)$ or $SU(2)$.

On the other hand, the problem of establishing the existence of ℓ-boson stars in the relativistic regime remains a challenging open problem. Even in the simplest case $\ell = 0$, were a rigorous proof of existence was provided in [10], it would still be desirable to obtain a proof based on variational methods, as such approach naturally yields additional and valuable properties of the solution, such as its stability.

Regarding stability, while both numerical simulations and linear stability analysis suggest that Newtonian ℓ-boson stars may be orbital stable for $\ell = 1$ with respect to arbitrary perturbations, it would be desirable to obtain a rigorous proof confirming this fact. Although we proved that $(\ell = 1)$-boson stars are saddle points of the energy functional if we consider non-spherical perturbations, this does not imply that they are unstable, as we explained in Remark 5.1.

In summary, the results presented in this thesis strengthen and extend the theoretical framework concerning the existence and stability of Newtonian boson stars. Furthermore, these findings reinforce the proposal of scalar fields as viable dark matter candidates. In particular, we highlighted multi-field boson stars–an extension of ordinary boson stars that arise naturally from semiclassical theory of gravity [6]–as an interesting candidate for modeling dark matter halo cores. We hope to address some of the open issues mentioned above in future work.

Appendix A

Strauss lemma for vector-valued functions

In this appendix we present the proofs of some results.

Proposition A.1. *$H^1_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ is weakly closed in $H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$. Moreover, if (u_j) is a sequence in $H^1_{SO(3)}$ such that $u_j \rightharpoonup u$ weakly in $H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$, then (u_j) converge weakly to f in $L^p(\mathbb{R}^n, \mathbb{C}^{2\ell+1})$ for $2 \leq p \leq 6$.*

Proof. Since $H^1_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ is a linear subspace of $H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ it suffices to show that it is strongly closed.

By definition, for each $R \in SO(3)$, the operator

$$\rho^{(\ell)}(R) : L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1}) \to L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1}), \tag{A.1}$$

is linear and bounded and extends naturally to $H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$. Therefore, the operator $\rho^{(\ell)}(R) - I$ (with I the identity operator) is also linear and bounded for each $R \in SO(3)$, and the kernel of a bounded linear operator is closed. Hence,

$$H^1_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1}) = \bigcap_{R \in SO(3)} \mathrm{Ker}(\rho^{(\ell)}(R) - I). \tag{A.2}$$

is an intersection of infinite closed subsets of $H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$. Since the intersection of closed sets is closed, we conclude that $H^1_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ is closed in $H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$. ■

Lemma A.1. *There exist a constant $C > 0$, such that for every $u \in H^1_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$*

$$|u(x)| \leq \frac{C}{|x|} \|u\|_{H^1(\mathbb{R}^3;\mathbb{C}^{2\ell+1})}, \quad \text{for a.e. } x \in \mathbb{R}^3. \tag{A.3}$$

Proof. Let $u \in H^1_{SO(3)}$, then for all $R \in SO(3)$ we have

$$|u(x)| = |u(R^{-1}x)|, \tag{A.4}$$

which implies that $|u(x)| = f(|x|)$ for some real-valued function $f : (0, \infty) \to \mathbb{R}$. Using the Strauss' radial Lemma [80] and Proposition 2.5, we obtain that there is a constant $C > 0$ such that

$$|u(x)| \leqslant \frac{C}{|x|} \||u|\|_{H^1(\mathbb{R}^3)} \leq \frac{C}{|x|} \|u\|_{H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})}, \quad \text{for a.e. } x \in \mathbb{R}^3. \tag{A.5}$$

where in the last inequality we use that $\||u|\|_{H^1(\mathbb{R}^3)} \leq \|u\|_{H^1(\mathbb{R}^3,\mathbb{C}^{2\ell+1})}$. ■

Theorem A.1. *Let $2 < p < 6$. Then, the embedding*

$$H^1_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1}) \subset L^p(\mathbb{R}^3, \mathbb{C}^{2\ell+1}) \tag{A.6}$$

is compact.

Proof. Let (u_j) a sequence in $H^1_{SO(3)}$ such that $u_j \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$. Then (u_j) is bounded in $H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ and, by the previous lemma, there exist a constant $C > 0$ such that

$$|u_j(x)| \leq \frac{C}{|x|}, \quad \text{for a. e. } x \in \mathbb{R}^3 \text{ and for all } j. \tag{A.7}$$

Fix $\varepsilon > 0$. Since $p > 2$, we can choose $R_\varepsilon > 0$ such that

$$|u_j(x)|^{p-2} \leq \varepsilon, \quad \text{for a. e. } |x| \geq R(\varepsilon) \text{ and for all } j. \tag{A.8}$$

Therefore,

$$\int_{\mathbb{R}^3 \setminus B_{R_\varepsilon}(0)} |u_j(x)|^p d^3x \leq \varepsilon \int_{\mathbb{R}^3 \setminus B_{R_\varepsilon}(0)} |u_j(x)|^2 d^3x \leq C\varepsilon. \tag{A.9}$$

On the other hand, since $u_j \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ and the embedding

$$H^1(B_{R_\varepsilon}(0), \mathbb{C}^{2\ell+1}) \hookrightarrow L^p(B_{R_\varepsilon}(0), \mathbb{C}^{2\ell+1}) \tag{A.10}$$

is compact for $p < 6$ (see Theorem 2.8). We conclude that there exist a subsequence (u_{j_k}) in $H^1_{SO(3)}(B_{R_\varepsilon}(0), \mathbb{C}^{2\ell+1})$ such that

$$\int_{B_{R_\varepsilon}(0)} |u_{j_k}(x)|^p d^3x \to 0. \tag{A.11}$$

Putting both expressions (A.9) and (A.11) together:

$$\limsup_{k\to\infty} \int_{\mathbb{R}^3} |u_{j_k}(x)|^p d^3x \leq C\varepsilon, \tag{A.12}$$

and since ε is arbitrary, we deduce $\|u_{j_k}\|_p \to 0$, that is, $u_{j_k} \to 0$ strongly. Now, suppose $u_j \rightharpoonup u$ weakly in $H^1_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$. Since $H^1_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ is weakly closed, it follows that $u \in H^1_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$. Define $v_j := u_j - u$, then $v_j \rightharpoonup 0$ weakly in $H^1_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$, and by the argument above there exist a subsequence (v_{j_k}) such that $v_{j_k} \to 0$ strongly in $L^p(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$. Therefore,

$$\|u_{j_k} - u\|_p = \|v_{j_k}\|_p \to 0, \tag{A.13}$$

which proves that $u_{j_k} \to u$ strongly in $L^p(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$. Hence, the embedding is compact. ■

Lemma A.2. *The Fourier transform $\mathcal{F} : L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1}) \to L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ maps the subspace $L^2_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ into itself and also maps the orthogonal complement, denoted by $[L^2_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1})]^\perp$, into itself.*

Proof. Let $u \in L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$. Then, for all $\xi \in \mathbb{R}^3$

$$[\rho^{(\ell)}(R)\mathcal{F}u](\xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} D^{(\ell)}(R)u(x)e^{-ix\cdot R^{-1}\xi} d^3x. \tag{A.14}$$

Performing the change of variables $y = Rx$, yields ($dx = dy$ since $R \in SO(3)$ preserves the measure)

$$[\rho^{(\ell)}(R)\mathcal{F}u](\xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} D^{(\ell)}(R)u(R^{-1}y)e^{-iR^{-1}y\cdot R^{-1}\xi} d^3y. \tag{A.15}$$

Since $R^{-1}y \cdot R^{-1}\xi = y \cdot \xi$, we obtain

$$[\rho(R)^{(\ell)}\mathcal{F}u](\xi) = \mathcal{F}[\rho^{(\ell)}(R)u](\xi), \tag{A.16}$$

which proves that the Fourier transform commutes with the operator $\rho^{(\ell)}(R)$. Therefore, assuming that u is $SO(3)$-invariant, that is, $\rho^{(\ell)}(R)u = u$ it follows

$$[\rho^{(\ell)}(R)\mathcal{F}u](\xi) = \mathcal{F}[\rho^{(\ell)}(R)u](\xi) = (\mathcal{F}u)(\xi). \tag{A.17}$$

We conclude that if $u \in L^2_{SO(3)}$ then $\mathcal{F}u \in L^2_{SO(3)}$.

For the second part, let $u^\perp \in [L^2_{SO(3)}]^\perp$, that is, $(u, u^\perp)_{L^2} = 0$ for all $u \in L^2_{SO(3)}$. Hence by the Plancharel theorem [67] it follows that

$$(\mathcal{F}u, \mathcal{F}u^\perp)_{L^2} = 0, \tag{A.18}$$

for all $\mathcal{F}u \in L^2_{SO(3)}$. Since $\mathcal{F}$ is invertible and $\mathcal{F}^{-1}$ also leaves $L^2_{SO(3)}$ invariant, it follows that $\mathcal{F}u^\perp \in [L^2_{SO(3)}]^\perp$.

■

Lemma A.3. *The orthogonal projection*

$$P : L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1}) \to L^2_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1}) \tag{A.19}$$

satisfies the following properties:

(i) P maps $H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ into $H^1_{SO(3)}(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$.

(ii) For all $u, v \in H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$

$$(\nabla Pu, \nabla v)_{L^2} = (\nabla u, \nabla Pv)_{L^2}. \tag{A.20}$$

(iii) Let $f : \mathbb{R}^3 \backslash \{0\} \to \mathbb{R}$ a $SO(3)$-invariant measurable function and $u \in L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ such that the function fu belongs to $L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$. Then, P commutes with the multiplication by f, that is, $P(fu) = fPu$.

Proof. (iii) For any $u \in L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$, we decompose $u = Pu + u^\perp$ with $u^\perp$ belonging to the orthogonal compliment of $L^2_{SO(3)}$. Hence,

$$P(fu) = P[f(Pu + u^\perp)] = fPu + P(fu^\perp). \tag{A.21}$$

Now, let $v \in L^2_{SO(3)}$. Since f is $SO(3)$-invariant, we have that $fv \in L^2_{SO(3)}$. Therefore,

$$(fu^\perp, v)_{L^2} = (u^\perp, fv)_{L^2} = 0, \tag{A.22}$$

This shows that $fu^\perp$ lies in the orthogonal compliment of $L^2_{SO(3)}$, and thus $P(fu^\perp) = 0$. Consequently,

$$P(fu) = fPu, \tag{A.23}$$

which proves the claim.

(i) Recall that the Sobolev space H^1 admits a Fourier characterization: a function $u \in L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ belongs to $H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$ if and only if [67]

$$\int_{\mathbb{R}^3} (1 + |\xi|^2)|\mathcal{F}u(\xi)|^2 d^3\xi < \infty. \tag{A.24}$$

Let $u \in H^1$, then by definition of the projector P, the function Pu is $SO(3)$-invariant, that is, $\rho(R)Pu = Pu$ for all $R \in SO(3)$. It remains to prove that $Pu \in H^1$, in other words we must prove that

$$\int_{\mathbb{R}^3} (1 + |\xi|^2)|\mathcal{F}[Pu(\xi)]|^2 d^3\xi < \infty. \tag{A.25}$$

We can decompose $u = Pu + u^\perp$ with $Pu \in L^2_{SO(3)}$ and $u^\perp \in [L^2_{SO(3)}]^\perp$. Hence, by the linearity of P and the Fourier transform

$$P[\mathcal{F}u] = P[\mathcal{F}(Pu)] + P[\mathcal{F}u^\perp]. \tag{A.26}$$

By Lemma **??**, $\mathcal{F}[Pu] \in L^2_{SO(3)}$ and $\mathcal{F}u^\perp \in [L^2_{SO(3)}]^\perp$, which implies $P[\mathcal{F}(Pu)] = \mathcal{F}(Pu)$ and $P[\mathcal{F}u^\perp] = 0$, Therefore

$$P[\mathcal{F}u] = \mathcal{F}[Pu], \tag{A.27}$$

that is, the Fourier transform $\mathcal{F}$ commutes with the projector P. Thus

$$\int_{\mathbb{R}^3} (1 + |\xi|^2)|(\mathcal{F}[Pu])(\xi)|^2 d^3\xi = \int_{\mathbb{R}^3} (1 + |\xi|^2)|(P[\mathcal{F}u])(\xi)|^2 d^3\xi \tag{A.28}$$

$$= \int_{\mathbb{R}^3} |(P[\mathcal{F}u])(\xi)|^2 d^3\xi + \int_{\mathbb{R}^3} ||\xi|(P[\mathcal{F}u])(\xi)|^2 d^3\xi. \tag{A.29}$$

Using that $\|Pu\|_2 \le \|u\|_2$ for all $u \in L^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$, we obtain

$$\int_{\mathbb{R}^3} (1 + |\xi|^2)|(\mathcal{F}[Pu])(\xi)|^2 d^3\xi \le \int_{\mathbb{R}^3} |\mathcal{F}u(\xi)|^2 d^3\xi + \int_{\mathbb{R}^3} ||\xi|(P[\mathcal{F}u])(\xi)|^2 d^3\xi. \tag{A.30}$$

On the other hand, for any $v \in H^1$ we have

$$(P(|\xi|\mathcal{F}u), \mathcal{F}v)_{L^2} = (\mathcal{F}u, |\xi|P(\mathcal{F}v))_{L^2}. \tag{A.31}$$

Since $|R^{-1}\xi| = |\xi|$ for all $R \in SO(3)$, we have that $|\xi|P(\mathcal{F}u) \in L^2_{SO(3)}$, that is, $P[|\xi|P(\mathcal{F}u)] = |\xi|P(\mathcal{F}u)$. This implies

$$(\mathcal{F}u, |\xi|P(\mathcal{F}v))_{L^2} = (|\xi|P(\mathcal{F}u), \mathcal{F}v)_{L^2}, \tag{A.32}$$

and then

$$(P(|\xi|\mathcal{F}u), \mathcal{F}v)_{L^2} = (|\xi|P(\mathcal{F}u), \mathcal{F}v)_{L^2}, \tag{A.33}$$

for all $v \in H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$. Thus,

$$P(|\xi|\mathcal{F}u) = |\xi|P(\mathcal{F}u), \tag{A.34}$$

for $u \in H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$. Therefore,

$$\int_{\mathbb{R}^3} ||\xi|(P[\mathcal{F}u])(\xi)|^2 d^3\xi = \int_{\mathbb{R}^3} |P[|\xi|\mathcal{F}u](\xi)|^2 d^3\xi \leq \int_{\mathbb{R}^3} |\xi|^2 |\mathcal{F}u(\xi)|^2 d^3\xi. \tag{A.35}$$

Finally, returning to estimate (A.30) it follows

$$\int_{\mathbb{R}^3} (1 + |\xi|^2)|(\mathcal{F}[Pu])(\xi)|^2 d^3\xi \leq \int_{\mathbb{R}^3} (1 + |\xi|^2)|\mathcal{F}u(\xi)|^2 d^3\xi < \infty. \tag{A.36}$$

(ii) Let $u, v \in C^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1}) \cap H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$. Using integration by parts we have

$$(\nabla Pu, \nabla v)_{L^2} = (Pu, -\Delta v)_{L^2}. \tag{A.37}$$

By the Plancharel theorem and using that the Fourier transform commutes with the projector P and $|\xi|P(\mathcal{F}u) = P(|\xi|\mathcal{F}u)$, it follows

$$(Pu, -\Delta v)_{L^2} = (\mathcal{F}[Pu], \mathcal{F}[-\Delta v])_{L^2} = (\mathcal{F}u, |\xi|^2\mathcal{F}[Pv])_{L^2}, \tag{A.38}$$

Substituting in Eq (A.37) we obtain

$$(\nabla Pu, \nabla v)_{L^2} = (\mathcal{F}u, |\xi|^2\mathcal{F}[Pv])_{L^2}. \tag{A.39}$$

Using again integration by parts and the Plancharel theorem yields the result

$$(\nabla Pu, \nabla v)_{L^2} = (\nabla u, \nabla Pv)_{L^2}, \tag{A.40}$$

for all $u, v \in C^2(\mathbb{R}^3, \mathbb{C}^{2\ell+1}) \cap H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$. By a density argument, the same result is valid for $u, v \in H^1(\mathbb{R}^3, \mathbb{C}^{2\ell+1})$. ■

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Emmanuel Chávez Nambo

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