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Dos Estudios Sobre No Linealidad en Teorías de Campo

Tesis para Obtener el Grado de Doctor en Ciencias en el Área de Física

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Abstract

In the first part, we present a framework for discrete spacetime classical field theory inwhich solutions to the field equations over elementary spacetime cells may be amalgamated if they satisfy simple gluing conditions matching the composition rules of cellular amplitudes in spin foam models. Furthermore, the formalism is endowed with a multisymplectic structure responsible for local conservation laws. Some models within our framework are effective theories modeling a system at a given scale.

In the second part, we present a new and highly efficient formalism for the calculation of helicity-decomposed amplitudes in Born-Infeld theory based on (off-shell) self-duality and (on-shell) helicity.

Born-Infeld theory is a highly nonlinear quantum field theory which is of exceptional interest for field theory, since it displays a number of symmetries, both hidden and visible ones, and also for string and D-brane theory. We confirm the 6-point tree-level computation of Boels et al and perform the first 8-point calculation in this theory.

To my dear husband, Hamzeh
and
my sweet daughters, Ala and Raha.

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Part I

New Spin Foam Models For Quantum Gravity

Chapter 1

Introduction to part I

There are two main approaches to Quantum Field theory. The operator approach based on the canonical quantisation of physical observables and the Feynman path integral.

It is not always easy to recover the classical limit by the canonical formulation, however it is clearly visible in path integral approach. Also path integrals allow for a formulation of non-perturbative approaches to the solution of quantum mechanical problems. In fact this formulation has many advantages.

The path integral formulation is based on fundamental connections between quantum and classical mechanics, in other words, the Lagrangian is used in this formulation since it is based on variational principles. Also it preserves all symmetries of the theory. Finally the path integral approach reveals a closed analogy between quantum field theory and statistical mechanics.

In path integral approach, renormalization group for low momentum indicates the hierarchy problem. One of the solution to this problem is replacing the space-time continuum with a discrete mesh of lattice points and then writing the kinematic term for the gauge fields which was done by Wilson.

The lattice is a beautiful regularization because it is local and it respects local gauge symmetries however it violates some space-time symmetries. We

try to recover these symmetries in the continuum limit.

In quantum mechanics the Hamilton operator induces the infinitesimal translation in time. The analogy of this operator in Lattice gauge theory is the transfer matrix which induces translation by the smallest possible distance; in spin foam models we compose the elementary cellular amplitudes/propagators.

Spin foams can be seen as higher-dimensional analogs of Feynman diagrams in quantum gravity and other gauge theories in the continuum or in lattice gauge theory and they are generated by evolution of spin networks.

There are several formulation of spin foam models for computing amplitudes of spin foams and the transition amplitude of spin networks from spin foams vertices, edges and faces. These models are possible candidates for quantizing gravity.

One of these possible models is lattice-discretizations of the path integral of gravity and generally covariant gauge theories which was proposed by Reisenberger. In this model, Reisenberger replaced the space-time manifold by a complex of four dimensional cells.

In this work, we present a framework for classical field theory in discrete spacetime. Our framework is tailored to spin foam models in the case of gauge theories. This formalism leads to a multisymplectic framework and respects local conservation laws.

This part of the thesis is organized as follows: We start in chapter 2 with a review of necessary background material that we use in the following chapter. The differential geometry and jet bundles are explained in this chapter since these are the main materials in our work.

In chapter 3, we present a geometric-variational framework of discrete mechanics and field theory. The advantage of our formalism is that the quantisation of our framework is natural in spin foam models. Some models are effective theories at a given scale, in our framework. As an example we apply our framework to 2d gravity treated as a constrained BF theory with gauge group $U(1)$.

Our formalism provides a multisymplectic structure and respects local conservation laws, therefore we see Noether's theorem within this formalism. Our framework allows us to study coarse graining and the continuum limit.[1]

Chapter 2

Mathematical Background

2.1 Differential Geometry

In this chapter first we review some preliminary mathematical concepts which all the physics student should be familiar with. Then we explain jet bundles and dual jet bundles which are more advanced but useful material in our work and finally we see the definition and some properties of symplectic and multisymplectic manifolds. [2],[3], [4]

2.1.1 Vector Fields

A **vector field** v on M is a function from $C^\infty(M)$ to $C^\infty(M)$ which satisfies

$$\begin{aligned} v(f+g) &= v(f) + v(g)\ , \\ v(\alpha f) &= \alpha v(f)\ , \\ v(fg) &= v(f)g + fv(g)\ . \end{aligned}$$

The **tangent space** is a vector space with all tangent vectors at $P \in M$. It is not needed to say that a tangent vector at $P \in M$ is a function from $C^\infty(M)$ to $\mathbb{R}$ satisfying above properties at $P \in M$.

If we have a tangent vector $v \in T_pM$ and a smooth function $\phi : M \to N$, we

can define a tangent vector $\phi_* v \in T_{\phi(p)}N$ which called **pushforward** of v by ϕ

$$\phi_* : T_pM \to T_{\phi(p)}N$$

$$(\phi_* v)(f) = v(f \circ \phi) \, .$$

2.1.2 Differential Forms

For any vector space V, the corresponding **dual** vector space, V^*, is defined to be the space of all linear functionals $\omega : V \to \mathbb{R}$. For example the **cotangent space** T_pM^* is the dual of the tangent space T_pM.

1-form is an object which acts on a vector field at point p and gives a number or we can say 1-forms are dual to vector fields.

By definition, a 1-form on any manifold M is a map from $\mathrm{Vect(M)}$ to $C^\infty(M)$ that is linear over $C^\infty(M)$. By $\mathrm{Vect(M)}$ we mean the set of all vector fields on M.

If we have a pushforward map $\phi(p) = q$ and $\phi_* : T_pM \to T_{\phi(p)}N$ we can write a dual map ϕ^* as

$$\phi^* : T_q^*N \to T_p^*M \, .$$

We call $\phi^*\omega$ the **pullback** of ω by ϕ.

We call 1-form the **differential** of f,df, as the **exterior derivative** of f [6], [3].

2.1.3 Chains

If $\{s_r, i\}$ is a labeled set of r-simplexes in M, we define a **r-chain** in M by a sum of $\{s_r, i\}$ with $\mathbb{Z}$-coefficients

$$c = \sum_i m_i \{s_r, i\} \qquad\qquad m_i \in \mathbb{Z} \, .$$

r-chains in M form the **chain group** which we call $C_r(M)$.

Let σ_r be an r-simplex in $\mathbb{R}^r$ and let $f : \sigma_r \to M$ be a smooth map. If s_r is the image of σ_r in M then $\partial s_r = f(\partial \sigma_r)$ is a set of $(r-1)$-simplexes in M and is called the **boundary** of s. Note that boundary of r-simplex, $\partial\sigma_r$, is a $r-1$ chain in M.

We could define an integration of an r-form ω on an r-simplex s_r by

$$\int_{s_r} \omega = \int_{\bar{\sigma_r}} f^* \omega \, ,$$

where $f : \bar{\sigma_r} \to M$ is a smooth map such that $s_r = f(\bar{\sigma_r})$. Note that $f^*\omega$ is an r-form in $\mathbb{R}^r$.

By this definition we can now define an integration of r-form over an r-chain :

$$\int_c \omega = \sum m_i \int_{s_r,i} \omega \, .$$

It is good to recall a simple but useful theorem[5].

Theorem (Stokes theorem)**.** *Let* $\omega \in \Omega^{r-1} M$ *and* $c \in C_r(M)$*. Then*

$$\int_c d\omega = \int_{\partial c} \omega \, .$$

2.1.4 Jet Bundles

A **bundle** is a structure consisting of a manifold E, a manifold M, and an onto map $\pi : E \to M$. M is called **base space**, E is the **total space** and the map π is called the **projection** map. For each point $p \in M$ the space $E_p = \{q \in E : \pi(q) = p\}$ is called **fiber** over p [6], [8].

Vector bundles are always of interest to physicists. The **tangent bundle** of a manifold is a good example. In a tangent bundle, the total space is $TM = \bigcup_{p \in M} T_p M$, so the fibers are the tangent spaces over any $p \in M$ and the projection is $\pi : TM \to M$.

To go from the classical mechanics to the field theory, we should generalize tangent bundles. The analogue of tangent bundle in this case is called **first jet bundle**, $J^1 Y$.

$J^1 Y$ is the set of all local sections ,ϕ_i, of the bundle Y considered only up to the first order of their Taylor expansion.

In this way, given a point of the base, $x \in X$, two local sections ϕ_i and ϕ_j are identified if $\phi_i(x)$=$\phi_j(x)$ and their derivatives at x also agree; this equivalent class, $[\Phi]_x$, becomes an element of $J^1_x Y$ which is the fiber over x in $J^1 Y$.

In other word, if $\phi : X \to Y$ is a section of π_{XY} , its tangent map $T_x\phi$ at $x \in X$ is an element of $J^1_{\phi(x)}Y$. Thus, the map $x \mapsto T_x\phi$ is a section of J^1Y regarded as a bundle over X. This section is denoted $j^1\phi$ and is called the **first jet prolongation** of ϕ. In coordinates, $j^1\phi$ is given by

$$x^\mu \mapsto (x^\mu, \phi^A(x^\mu), \partial_\mu\phi^A(x^\mu)) \, .$$

2.2 Lagrangian Mechanics and Differential Geometry

Mechanics has two main points of view, Lagrangian mechanics and Hamiltonian mechanics. The Lagrangian formulation of mechanics is based on the variational principles. Since the action is independent of the choice of coordinate thus the Euler-Lagrange equations are also. Today, using of differential geometry language in mechanics is a standard way of expressing Euler-Lagrange equations. In addition, important properties of solutions have simple expressions in geometric language.[4], [7]

2.2.1 Symplectic and Multisymplectic Manifolds

A **symplectic manifold** is a pair (P, Ω), where P is a manifold and Ω is a closed nondegenerate 2-form on P.

Ω is required to be closed,$d\Omega = 0$, so that the flows of Hamiltonian vector fields will consist of canonical transformations.

In many mechanics problems, the phase space is the cotangent bundle T^*Q of a configuration space Q. There is an intrinsic symplectic structure on T^*Q.

Assume that Q is n-dimensional, with local coordinates $(q^1, ..., q^n)$ on Q. Since$(dq^1, ..., dq^n)$ is a basis of T^*_qQ, we can write any $\alpha \in T^*_qQ$ as $\alpha = p_i dq^i$. Which means that $(q^1, ..., q^n; p^1, ..., p^n)$ are induced local coordinates on T^*Q.

We can define the canonical **symplectic 2-form** on T^*Q by

$$\Omega = dq^i \wedge dp_i \, .$$

This Ω is a closed 2-form that is independent of the choice of coordinates $(q^1, ..., q^n)$.

So far we talked about the cotangent space, T^*Q. We know that always we can switch from T^*Q to TQ with a Legendre transformation. Since we want to work with lagrangian, we continue this section working with TQ

In the variational approach of mechanics, we vary the action and obtain

$$\begin{aligned} dS(q)\cdot\delta q &= \frac{d}{d\epsilon}|_{\epsilon=0}\int_a^b L\left(q_\epsilon^i(t), \frac{dq_\epsilon^i}{dt}(t)\right)dt \\ &= \int_a^b \delta q^i(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i})dt + \frac{\partial L}{\partial \dot{q}^i}\delta q^i|_a^b\,, \end{aligned}$$

That last term is a linear pairing of the function $\frac{\partial L}{\partial \dot{q}^i}$, a function of q^i and $\dot{q}^i$, with the tangent vector δq^i. Thus, it is a 1-form on TQ; namely the 1-form $\frac{\partial L}{\partial \dot{q}^i}dq^i$. We call this 1-form **Lagrange 1-form**.

The Lagrange 1-form, θ_L, is the *boundary part* of the the functional derivative of the action when the boundary is varied, then presymplectic form is the (negative of) the exterior derivative of θ_L.

$$\Omega_L = -d\theta_L.$$

A **multisymplectic manifold** is a manifold with a closed k-form which is nondegenerate. It is obvious that symplectic manifolds is a special case of multisymplectic manifold when $k = 2$.

Here the analogue of Lagrange 1-form is $(n+1)$-form Θ_L which is defined on J^1Y and called **Cartan form**.[8] Also we define the $(n+2)$-form, Ω_L, as

$$\Omega_L = -d\Theta_L\,.$$

Chapter 3

Multisymplectic Effective General Boundary Field Theory

3.1 Variational Approach to Discrete Lagrangian Mechanics

In Lagrangian mechanics when we use Hamilton's principle; we consider an n-dimensional configuration space, Q, then the Lagrangian is a map $L : TQ \to \mathbb{R}$ and the action

$$S(q) \equiv \int_a^b L\left(q^i(t), \frac{dq^i}{dt}(t)\right) dt, \tag{3.1}$$

where the history is denoted by $q : [a,b] \to Q$.

Hamilton's principle determines the curve $q(t)$ which the action is stationary under the variations of $q(t)$ with fixed end points.

$$\begin{aligned} dS(q)\cdot\vec{\delta}q &= \frac{d}{d\epsilon}|_{\epsilon=0}\int_a^b L\left(q_\epsilon^i(t), \frac{dq_\epsilon^i}{dt}(t)\right) dt \\ &= \int_a^b \delta q^i(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i})dt + \frac{\partial L}{\partial \dot{q}^i}\delta q^i|_a^b \ . \end{aligned} \tag{3.2}$$

Here we assume q_ϵ as a smooth family of histories with $q_0 = q$ and $(d \setminus d\epsilon)\ |_{\epsilon=0}\ q_\epsilon = \delta q$.

As we saw in the previous chapter, the lagrange 1-form is the *boundary*

part of the functioanl derivative of the action . Therefore, we obtain the Lagrange 1-form θ_L when we remove the boundary condition. There is no needed to mention that the bulk part of the eq.(3.2) is the Euler-Lagrange equation. Then the symplectic form is

$$\omega_L = -d\theta_L. \tag{3.3}$$

3.1.1 A Discretization of Lagrangian Mechanics

In Lagrangian mechanics if we divide the time to small cells (t_i, t_{i+1}), then we see that the history ,q, could be written as $q = q_0^-, q_0, q_0^+ = q_1^-, ..., q_{n-1}^+ = q_n^-, q_n, q_n^+$. Note that in this model in the boundary of two cells $i-1$ and i, we have $q_{i-1}^+ = q_i^-$. The points are denoted by q^- and q^+ in the left and the right side of q.

In the Veselov's model the analogue of the tangent space TQ is the space $Q \times Q$. For more details see [appendix A] [9] [10].

We subdivide each cells (q_i, q_{i+1}) to two cells, therefore in our framework the analogue of the tangent space breaks in to two spaces $Q \times Q^-$ and $Q^+ \times Q$. We define two discrete Lagrangian densities $L^- : Q \times Q^- \longrightarrow \mathbb{R}$ and $L^+ : Q^+ \times Q \longrightarrow \mathbb{R}$ which are determined by the data shown in the fig.(3.1.1). It is clear that Q^+ and Q^- are two copies of Q.

Figure 3.1: *There is a point* $q \in Y$ *over each point* $t \in X$.

The action is written as

$$S(q) = \sum_{k=0}^{n} (L^-(q_k, q_k^-) + L^+(q_k^+, q_k)). \tag{3.4}$$

We calculate the variation of this action with nonzero variation on the endpoints

$$dS(q) \cdot \vec{\delta} q = dS(q_0^-, q_0, q_0^+, ..., q_n, q_n^+) \cdot (\vec{\delta} q_0^-, ..., \vec{\delta} q_n^+) =$$

$$\begin{aligned} &\frac{\partial L^-}{\partial q^{-A}}(q_0, q_0^-)\delta q_0^{-A} + \sum_{k=0}^{n-1} \big(\frac{\partial L^+}{\partial q^{+A}}(q_k^+, q_k) + \frac{\partial L^-}{\partial q^{-A}}(q_{k+1}, q_{k+1}^-)\big)\delta q_k^{+A} \\ &+ \sum_{k=1}^{n-1} \big(\frac{\partial L^+}{\partial q^A}(q_k, q_k^-) + \frac{\partial L^-}{\partial q^A}(q_k^+, q_k)\big)\delta q_k^A + \frac{\partial L^+}{\partial q^{+A}}(q_n^+, q_n)\delta q_n^{+A}, \end{aligned} \tag{3.5}$$

where a variation $(\delta q_1^-, ..., \delta q_n^+)$ is denoted by δq and by $\vec{\delta} q_i^-$ we mean the vector made by these numbers:

$$\vec{\delta} q_i = \delta q_i^A \frac{\partial}{\partial q^A}$$

From eq.(3.5), we obtain the Euler-Lagrange equations

$$\frac{\partial L^+}{\partial q_k^+}(q_k^+, q_k) + \frac{\partial L^-}{\partial q_k^+}(q_{k+1}, q_{k+1}^-) = 0 \qquad (q_k^+ = q_{k+1}^-), \tag{3.6}$$

and

$$\frac{\partial L^+}{\partial q_k}(q_k, q_k^-) + \frac{\partial L^-}{\partial q_k}(q_k^+, q_k) = 0. \tag{3.7}$$

We see that there are two sets of equation of motions for each $k = 0, ..., n$.

From the boundary terms of eq. (3.5) we obtain the discrete analogue of Lagrange 1-form. In fact in this situation the pair (θ^-, θ^+) is the discrete analogue of the Lagrange 1-form.

$$\theta_L^-(q_k, q_k^-) \cdot (\vec{\delta} q_k, \vec{\delta} q_k^-) \equiv \frac{\partial L^-}{\partial q^{-A}}(q_k, q_k^-) dq^A \vec{\delta} q_k^-,$$

$$\theta_L^+(q_k^+, q_k) \cdot (\delta q_k^+, \delta q_k) \equiv \frac{\partial L^+}{\partial q^+}(q_k^+, q_k) dq^A \vec{\delta} q_k^+. \tag{3.8}$$

Here we mention that since the potentials are functions of the main points q's (not q^+ or q^-), the potential doesn't appear in θ_L^- and θ_L^+, in contrast to Veselov's original framework [see appendix A].

We use the solutions of the equations of motion to define the discrete flow. Note that in this model, we have one discrete flow for each interval (t^+, t) and (t^-, t)

$$F^- : Q \times Q^- \longrightarrow Q^+ \times Q$$

$$(q_k, q_k^-) \mapsto (q_k^+, q_k), \tag{3.9}$$

and

$$F^+ : Q^+ \times Q \longrightarrow Q \times Q^-$$

$$(q_k^+, q_k) \mapsto (q_{k+1}, q_{k+1}^- = q_k^+). \tag{3.10}$$

If we define the discrete flow F to be

$$F \equiv F^+ \circ F^-, \tag{3.11}$$

we can show the symplecticity of the flow. To have more sense of this definition, we mention that $FF^-(q_k, q_k^-) = (q_{k+1}^+, q_{k+1})$ or $F^2F^-(q_k, q_k^-) = (q_{k+2}^+, q_{k+2})$.

With the solutions which are parameterized by initial data, the eq.(3.5) becomes

$$dS = \theta_L^- + (F^- F^i)^* \theta_L^+, \tag{3.12}$$

where $i = 0, ..., n$ depends on our arbitrary interval (q_k^-, q_{k+i}^+) which we concern about.

By using the fact that $ddS = 0$ on eq.(3.12) for each individual interval, we

obtain

$$(F^- F^i)^* d\theta_L^+ = -d\theta_L^-. \tag{3.13}$$

Now we can study the conservation of symplectic structure by the discrete flow F^i.

We define symplectic form as

$$\omega_L \equiv d\theta_L^-, \tag{3.14}$$

Eq.(3.12) and definition (3.14), imply

$$(F^i)^* \omega_L = \omega_L, \tag{3.15}$$

which shows the symplecticity condition of the flow F.

Let us assume that the Lagrangian is invariant under the action of a Lie group$G$, with Lie algebra $\mathfrak{g}$; therefore, the action S is G-invariant and this group sends the critical points of the action to other critical points; with this assumption if we write the eq.(3.12) in the space of solutions which is parametrized by initial conditions, we have

$$0 = \xi_{Q\times Q} \lrcorner dS = \xi_{Q\times Q} \lrcorner \theta_L^- + \xi_{Q\times Q} \lrcorner \big(F^- F^i)^* \theta_L^+\big), \tag{3.16}$$

for $\xi \in \mathfrak{g}$. It is easy to see that with G-invariant Lagrangian then the discrete flow, F, is G-equivariant; hence

$$\xi_{Q\times Q} \lrcorner \theta_L^- = -(F^- F^i)^* \big(\xi_{Q\times Q} \lrcorner \theta_L^+\big). \tag{3.17}$$

From the equation $\theta_L^- + (F^-)^* \theta_L^+ = dL$ which we have in the space of solutions, we can see

$$\xi_{Q\times Q} \lrcorner \theta_L^- = -(F^-)^* \big(\xi_{Q\times Q} \lrcorner \theta_L^+\big),$$

thus if we define the discrete momentum to be $J \equiv \xi_{Q\times Q} \lrcorner \theta_L^-$, we have conservation equation of the momentum

$$J = \xi_{Q\times Q} \lrcorner \theta_L^- = (F^i)^* J. \tag{3.18}$$

This is a discrete version of Noether's theorem which we obtained by restricting the action to space of solutions of the variational principle. The momentum J is conserved under infinitesimal transformation associated to a lie algebra elementξ

3.1.2 Comparison of Two Frameworks

We would like to investigate the relation between our framework and Marsden-Veselov's one [9]to get a better understanding of our model. Also it is useful to get a clear picture of the connection between two models and see the advantage of our model for the spinfoams.

To write the Lagrangian density in Veselov's framework, Marsden et al write the potential of a cell as a function of the average of the vertices of that cell. For instance in one dimensional interval (q_{k+1}, q_k), they write the Lagrangian density as

$$L(q_{k+1}, q_k) = \frac{m}{2}(\frac{q_{k+1} - q_k}{\Delta t})^2 - V(\frac{q_{k+1} + q_k}{2}).$$

So there is potential in all points of the discrete spacetime. Unlike Marsden et al implementation of Veselov's work, in our intermediate points(q_k^+ or q_k^-) there is no potential so all we have, is the main points with potential in the bulk and some points without any potential in the boundary. In this way, we can see that the Lagrange 1-from in our framework does not depend on potential and this is the important difference between our model and Marsden et al's. However, our framework and Marsden et al's are equivalent in the sense described below.

First we talk in 1-dimensional spacetime, however for higher dimensions it will be similar. We do the comparison in higher dimension in appendix B. The same as Veselov, we write the Lagrangian density for each interval but the difference is about our intervals and Veselov's. The Lagrangian density in our model is $L(q_k^+, q_k, q_k^-)$ but the Veselov's one is $L(q_{k+1}, q_k)$, in fact Veselov's interval is half of a step shift of ours.

In order to compare our model with their model, we write the same potential in our action.

We start with a general example which is a particle in Euclidean space with

arbitrary potential. In our framework the action of such particle is

$$S(q)=\sum_{k=1}^{n-1}L(q_{k+1},q_k^+,q_k)\Delta t=\sum_{k=1}^{n-1}\big(\frac{m}{4}(\frac{q_{k+1}-q_{k+1}^-}{\frac{1}{2}\Delta t})^2+\frac{m}{4}(\frac{q_k^+-q_k}{\frac{1}{2}\Delta t})^2-2V(\frac{q_k+q_{k+1}}{2})\big)\Delta t. \tag{3.19}$$

We write the time interval in our action $\Delta t'$, as $\frac{1}{2}\Delta t$. Note that we write the action in our framework for an arbitrary interval (q_1,q_n) which is the Veselov's interval.

To write the equations of motion we need to write the variation of the action

$$\begin{aligned}dS(q_1,q_1^+,...,q_{n-1}^+,q_n)\cdot(\vec{\delta}q_n,\vec{\delta}q_n^+,...,\vec{\delta}q_{n-1}^+,\vec{\delta}q_n)=\\ \frac{\partial L}{\partial q_1^A}(q_2,q_1^+,q_1)\delta q_1^A+\sum_{k=1}^{n-1}\big(\frac{\partial K^+}{\partial q_k^{+A}}(q_k^+,q_k)+\frac{\partial K^-}{\partial q_k^{+A}}(q_{k+1},q_{k+1}^-)\big)\delta q_k^{+A}\\ +\sum_{k=2}^{n-1}\big(\frac{\partial L}{\partial q_k^A}(q_k,q_k^-,q_{k-1})+\frac{\partial L}{\partial q_k^A}(q_{k+1},q_k^+,q_k)\big)\delta q_k^A+\frac{\partial L}{\partial q_n^A}(q_n,q_n^-,q_{n-1})\delta q_n^A,\end{aligned} \tag{3.20}$$

where we define the kinetic energy of each mini interval as $K^{\pm}$, so $K^+(q_k^+,q_k)=\frac{m}{2}(q_k^+-q_k)^2$ and $K^-(q_k,q_k^-)=\frac{m}{2}(q_k-q_k^-)^2$. Note that $q_k^+=q_{k+1}^-$.

The discrete Euler-Lagrange equations are

$$\frac{\partial K^+}{\partial q^+}(q_k^+,q_k)+\frac{\partial K^-}{\partial q^-}(q_{k+1},q_{k+1}^-)=0, \tag{3.21}$$

and

$$\frac{\partial L}{\partial q}(q_k,q_k^-,q_{k-1})+\frac{\partial L}{\partial q}(q_{k+1},q_k^+,q_k)=0. \tag{3.22}$$

With the solutions of the eqs.(3.21), we can see that

$$q_{k+1}-q_k=q_{k+1}-q_k^++q_k^+-q_k=2(q_{k+1}-q_k^+)=2(q_k^+-q_k). \tag{3.23}$$

If we substitute the equality eq.(3.23) in the action eq.(3.19), we obtain the

action

$$S=\sum_{k=1}^{n}\Big(2\frac{m}{4}(\frac{q_{k+1}-q_k}{\Delta t})^2-V(\frac{q_k+q_{k+1}}{2})\Big)\Delta t. \tag{3.24}$$

We see that our action eq.(3.24) becomes exactly the action of Veselov's model. We also notice that the potential term can be written as $V(\frac{q_k+q_{k+1}}{2})=V(q_i^+)$ using the solutions, so compare this potential to our potential, $V(q_i)$, the Veselov's one is ours which is shifted half a step.

Using the solutions of the eqs.(3.21,3.22) for two consecutive intervals, we obtain the equality $q_{k+1}-q_k=q_{k+2}-q_{k+1}$ which is the solution of the equations of motion in Veselov's framework. Hence, the solutions of two frameworks completely agree.

It is good to mention that the equivalence between the actions is not in the space of solutions of Veselov's. In fact, if we restrict our framework to the space of the solutions of the eqs.(3.21), we reproduce the action of Veselov's framework in the whole space of histories.

We can read the Lagrange 1-form in our framework from eqs.(3.20)and (3.19),

$$\begin{aligned}&\theta_L\cdot\vec{\delta}q=\frac{\partial L}{\partial q_1^A}(q_2,q_1^+,q_1)\delta q_1^A+\frac{\partial L}{\partial q_n^A}(q_n,q_n^-,q_{n-1})\delta q_n^A=\\&m(\frac{q_1-q_1^+}{\Delta t'})\delta q_1^A+m(\frac{q_n-q_n^-}{\Delta t'})\delta q_n^A-2\Delta t'\frac{\partial V}{\partial q_1^A}(\frac{q_1+q_2}{2})\delta q_1^A-2\Delta t'\frac{\partial V}{\partial q_n^A}(\frac{q_n+q_{n-1}}{2})\delta q_n^A\end{aligned} \tag{3.25}$$

It is easy to see that using the solutions of eq.(3.21) for $k=1$ and $k=n-1$, eq.(3.25) becomes

$$\theta_L\cdot\vec{\delta}q=m(\frac{q_1-q_2}{\Delta t})\delta q_1^A+m(\frac{q_n-q_{n-1}}{\Delta t})\delta q_n^A-\Delta t\frac{\partial V}{\partial q_1^A}(\frac{q_1+q_2}{2})\delta q_1^A-\Delta t\frac{\partial V}{\partial q_n^A}(\frac{q_n+q_{n-1}}{2})\delta q_n^A. \tag{3.26}$$

Therefore, the Lagrange 1-form in our framework is exactly the Veselov's one in the space of histories, if written for the Veselov's interval.

3.2 Discretization of Multisymplectic Field Theory

In this section we want to develop our model of discretization of Lagrangian mechanics to the discrete multisymplectic field theory. This will be a modification of what Marsden et al did[8].

We denote the discrete spacetime by X^{disc} and the simplices in X^{disc} by Greek letters ν, τ and σ. We want the spacetime, X^{disc}, to be oriented simplicial complex with $n+1$-simplices ν's. We have a look to Reisenberger's model in 4d so we work with the points in the center of the simplices. We call the center of each ν as C_ν, also there are C_τ's in the center of the n-simplices τ's and C_σ's in the center of the n-1-simplices, σ.

We ask to have $C_\nu < U^\circ$ for all ν's, where the interior of U is presented by U°. If $U \subseteq X^{disc}$ such that $U = \sum_{\nu \subset U} \nu$ then the boundary of U is $\partial U = \sum_{\nu \subset U} \partial\nu$.

The discrete analogue of J^1Y in our framework is defined as

$$J^1Y^{disc} \equiv \{(\nu, y^A; \{y^A_\tau\}_{\tau<\nu}) \mid y^A, y^A_\tau \in F\}, \tag{3.27}$$

where y^A is the the element of F over the base point C_ν and the elements over C_τ's are denoted by y^A_τ's.

And the **first jet extension** of section ϕ is

$$j^1\phi(\nu) \equiv \big(\nu, \phi(C_\nu); \phi(C_\tau)_{\tau<\nu}\big). \tag{3.28}$$

We define the **discrete Lagrangian density** $L : J^1_\nu Y^{disc} \to \mathbb{R}$ to be a function $L(\nu, y^A; \{y^A_\tau\}_{\tau<\nu}\})$,

Marsden et al. write the potential term as $V(\frac{\sum_{vertices<\nu} \phi(y^A)}{v})$ for each cell ν, here v is the number of vertices of ν which depends on the discretization and dimension of X. Although we could do the same for the potential and show the equality of two models, [see App. B]; the structure of our framework in enhanced if we write the potential as a function of the fields $\phi(C_\nu)$'s .
With all these definition we define the action

$$S(\phi) \equiv \sum_{\nu \subset U} L \circ j^1\phi(\nu). \tag{3.29}$$

As we had for the classical mechanics we define the Discrete Euler-Lagrange field equations,

$$\frac{\partial L}{\partial y^A}(\nu, y^A; \{y^A_{\tau'}\}_{\tau'<\nu}) = 0, \tag{3.30}$$

for all $\nu \subset U$.
And

$$\sum_{\tau<\nu} \frac{\partial L}{\partial y^A_\tau}(\nu, y^A; \{y^A_{\tau'}\}_{\tau'<\nu}) = 0, \tag{3.31}$$

for all $\tau \subset U^\circ$.
It is good to mention that in eq.(3.30), $L(\nu, y^A; \{y^A_{\tau'}\}_{\tau'<\nu})$ may involve a summation over all $\tau < \nu$.
There exists only one $\nu \subset U$ related to each $\tau \subset \partial U$, then we define

$$\Theta^\tau_L(\nu, y^A; \{y^A_{\tau'}\}_{\tau'<\nu}) \equiv \frac{\partial L}{\partial y^A_\tau}(\nu, y^A; \{y^A_{\tau'}\}_{\tau'<\nu}) dy^A_\tau \tag{3.32}$$

If $\pi_\nu : C_U \to J^1_\nu Y^{disc}$ is a map from the set of all sections of Y over U to $J^1_\nu Y^{disc}$
Then the **Lagrange 1-form** is defined as

$$\theta_L \equiv \sum_{\tau<\partial U} \pi^*_\nu \Theta^\tau_L. \tag{3.33}$$

In the space of solutions which is parameterized by the initial conditions the variation of S is just the boundary contribution

$$dS(\phi) = \sum_{\nu;\nu\cap\partial U\neq\emptyset} \sum_{\tau<\nu;\tau<\partial U} \theta^\tau_L\big(j^1\phi(C_\nu), \{j^1\phi(C_\tau)\}_{\tau<\nu}\big). \tag{3.34}$$

By the fact $d^2S = 0$, we obtain the **multisymplectic formula** which is given by

$$0 = d\theta_L[V, W] = \sum_{\nu;\nu\cap\partial U\neq\emptyset} \sum_{\tau<\nu;\tau<\partial U} j^1V \lrcorner j^1W \lrcorner \Omega^\tau_L, \tag{3.35}$$

where V and W are first variations and $\Omega^\tau_L = -d\Theta^\tau_L$ and j^1V is defined the first variation of $j^1\phi(V)$

To write the analogue of the Noether's theorem, we assume that Lagrangian and so the action are G-invariant, where G is Lie group with associated Lie

algebra $\mathfrak{g}$. For each 1-form Θ^{τ}_{L} we define a momentum map to be a $\mathfrak{g}^{*}$-valued function on $\nu \subset U$ as

$$J^{\tau}_{\xi} \equiv \xi_{j^{1}Y^{disc}} \lrcorner \Theta^{\tau}_{L},$$

for any$\xi \in \mathfrak{g}$.
We obtain the **discrete analogue of the Noether's theorem**

$$\sum_{\nu;\nu\cap\partial U\neq\emptyset} \sum_{\tau<\nu;\tau<\partial U} \pi^{*} J^{\tau}_{\xi} = 0, \tag{3.36}$$

by using the fact that $\xi_{Y^{disc}} \lrcorner \theta_{L} = 0$ as a result of the invariance of action. It is clear that $\pi_{*}\xi_{Y^{disc}} = \xi_{j^{1}Y^{disc}}$.
As a simple example, we consider the action for scalar field which is written in the discrete spacetime U with unit hypercubic simplices ν. It is clear that $C_{\tau} = C_{\nu} \mp 1$, $\tau < \nu$, therefore, the discrete action for the scalar fields is

$$S(\phi) = \sum_{\nu<U} \Bigg(-\frac{1}{2}\big(\phi(C_{\nu}+a_{0}) - \phi(C_{\nu})\big)^{2} - \frac{1}{2}\big(\phi(C_{\nu}-a_{0}) - \phi(C_{\nu})\big)^{2}$$
$$+ \sum_{i=1}^{n} \frac{1}{2}\big(\phi(C_{\nu}+a_{i}) - \phi(C_{\nu})\big)^{2} + \frac{1}{2}\big(\phi(C_{\nu}-a_{i}) - \phi(C_{\nu})\big)^{2} + m^{2}\phi^{2}(C_{\nu}) \Bigg), \tag{3.37}$$

where in $n+1$ dimensions, $a_{0} = (1, 0, ..., 0)$ with the first of $n+1$ components as one and $a_{i} = (0, ..., 1, 0, ..., 0)$ with one in $i+1^{th}$ place.
It is easy to write the variation of this action and solve the DELF's. Also we can write the Lagrange 1-form

$$\theta.\delta\phi_{\tau} = \sum_{\nu;\nu\cap\partial U\neq\emptyset} \sum_{\tau<\nu;\tau<\partial U} -\big(\phi(C_{\tau_{0}}) - \phi(C_{\nu})\big)\delta\phi_{\tau_{0}} + \big(\phi(C_{\tau_{i}}) - \phi(C_{\nu})\big)\delta\phi_{\tau_{i}},$$

which by $C_{\tau_{0}}$, we mean $C_{\nu}+a_{0}$ or $C_{\nu}-a_{0}$ and $C_{\tau_{i}} = C_{\nu}+a_{i}$ or $C_{\nu}-a_{i}$.

3.2.1 Gauge Theory

In the case of gauge theory, we need to define some new objects in the discrete spacetime, X^{disc}. We define a wedge $s(\nu,\sigma)$ with corners on four points C_ν, $C_{\sigma_{<\nu}}$ and the center of the two τ's which share σ. This wedge $s(\nu,\sigma)$ was first introduced by Reisenberger. He called each edge from C_ν to C_τ where $\tau<\nu$ as $l(\nu,\tau)$ and the edge from C_τ to C_σ as $r(\tau,\sigma)$. Note that each wedge $s(\nu,\sigma)$ has an induced orientation from ν and σ.

We ask to have the $l(\nu,\tau) < U^\circ$ as the link with ends on ν and τ, also we define $r(\tau,\sigma) < \partial\nu$ as the boundary links for each $n+1$-simplex ν (see figure 2).

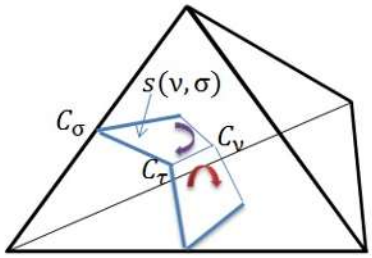


Figure 3.2:

The definition of the boundary of U is $\partial U = \sum_{\nu\subset U}\partial\nu$, which is the same as we had before.

We define the first jet bundle as

$$J^1Y^{disc} \equiv \{(\nu,\{h_{l(\nu,\tau)}\}_{\tau<\nu};\{k_{r(\tau,\sigma)}\}_{\sigma<\tau,\tau<\nu}) \mid h_l,k_r\in G\}, \tag{3.38}$$

where the element of the fiber over the link $l(\nu,\tau)$ is denoted by $h_{l(\nu,\tau)}$ and the element of the fiber over the link $r(\tau,\sigma)$ is $k_{r(\tau,\sigma)}$. Here we have to mention

that the information relates to the curvature comes from the closed holonomy around the boundary of $s(\nu,\sigma)$, $g_{\partial s}$, which is convenient to the definition eq.(3.38).

Our discrete analogue of $j^1\phi(x)$ in gauge theory is

$$j^1\phi(\nu) \equiv \big(\nu, \{\phi(l(\nu,\tau))\}_{\tau<\nu}; \{\phi(r(\tau,\sigma))\}_{\sigma<\tau,\tau<\nu}\big). \tag{3.39}$$

The action is defined as

$$S(\phi) \equiv \sum_{\nu\subset U} L \circ j^1\phi^{disc}(\nu), \tag{3.40}$$

where, here, the **discrete Lagrangian density** $L : J^1_\nu Y^{disc} \to \mathbb{R}$ is a function $L(\nu, \{h_{l(\nu,\tau)}\}_{\tau<\nu}; \{k_{r(\tau,\sigma)}\}_{\sigma<\tau,\tau<\nu})$.

The Discrete Euler-Lagrange field equations, are defined as

$$\frac{\partial L}{\partial h_{l(\nu,\tau)}}(\nu', \{h_{l'(\nu',\tau')}\}_{\tau'<\nu'}; \{k_{r'(\tau',\sigma')}\}_{\sigma'<\tau',\tau'<\nu'}) = 0, \tag{3.41}$$

for all $\nu \subset U$ and

$$\sum_{\nu'>\tau} \frac{\partial L}{\partial k_{r(\tau,\sigma)}}(\nu', \{h_{l'(\nu',\tau')}\}_{\tau'<\nu'}; \{k_{r'(\tau',\sigma')}\}_{\sigma'<\tau',\tau'<\nu'}) = 0, \tag{3.42}$$

for all $\tau \subset U^\circ$.

We define

$$\Theta^\tau_L(\nu', \{h_{l'(\nu',\tau')}\}_{\tau'<\nu'}; \{k_{r'(\tau',\sigma')}\}_{\sigma'<\tau',\tau'<\nu'}), \equiv \sum_{r<\tau} \frac{\partial L}{\partial k_r}(\nu', \{h_{l'(\nu',\tau')}\}_{\tau'<\nu'}; \{k_{r'(\tau',\sigma')}\}_{\sigma'<\tau',\tau'<\nu'})dk_r \tag{3.43}$$

for each $r(\tau,\sigma)$.

In the space of solutions which is parameterized by the initial conditions the variation of S is just the boundary contribution

$$dS(\phi) = \theta_L. \tag{3.44}$$

Where θ_L is Lagrange 1-form which is defined, using eq.(3.43).

By the fact $d^2S = 0$, we obtain the **multisymplectic formula** as

$$0 = d\theta_L[V,W] = \sum_{\nu;\nu\cap\partial U\neq\emptyset} \sum_{\tau<\nu;\tau<\partial U} V \lrcorner W \lrcorner \pi_\nu^* \Omega_L^\tau, \tag{3.45}$$

where V and W are first variations and $\Omega_L^\tau = -d\Theta_L^\tau$.

In the space of solutions, when all fields are the solutions, to write the analogue of the Noether's theorem we assume that Lagrangian and so the action are G-invariant, where G is Lie group with associated Lie algebra $\mathfrak{g}$. For each 1-form Θ_L^τ, we define a momentum map to be a $\mathfrak{g}^*$-valued function on $\nu \subset U$ as

$$J_\xi^r \equiv \xi_{j^1 Y^{disc}} \lrcorner \Theta_L^\tau,$$

for any$\xi \in G$. Notice that $\pi_* \xi_{Y^{disc}} = \xi_{j^1 Y^{disc}}$.
We obtain the **discrete analogue of the Noether's theorem**

$$\sum_{\nu;\nu\cap\partial U\neq\emptyset} \sum_{\tau<\nu;\tau<\partial U} \pi^* J_\xi^\tau = 0, \tag{3.46}$$

by using the fact that $\xi_{Y^{disc}} \lrcorner \theta_L = 0$ as a result of the invariance of action.

3.3 "Macroscopic Description" for a Particle in a Circle

In this example the bundle over each point of X^{disc} is 1-dimensional configuration space which is S^1, so the discrete analogue of the tangent bundle is $S^1 \times S^1$.
For particle in a circle, since the configuration space is S^1, the points determine the angles in the circle. To write the action, we have to consider that the velocity of the particle is not only determined by the difference of two angles which the particle starts from and ends up to, but an integer number to determine that how many times the particle has turned around the circle.
If the angles determine by $\varphi(z)$'s and the integer numbers by z's, the action for

particle in a circle is

$$S = \sum_{k=1}^{n} L^-(\varphi_k(z), \varphi_k^-(z^-)) + L^+(\varphi_k^+(z^+), \varphi_k(z)),$$

$$L^-(\varphi_k(z), \varphi_k^-(z^-)) = \frac{m}{2}(\varphi_k(z) - \varphi_k^-(z^-) + 2\pi z_k^-)^2 - V(\varphi_k(z)),$$

$$L^+(\varphi_k^+(z^+), \varphi_k(z)) = \frac{m}{2}(\varphi_k^+(z^+) - \varphi_k(z) + 2\pi z_k^+)^2 - V(\varphi_k(z)). \tag{3.47}$$

For simplicity we write $\varphi^\pm(z^\pm)$ as $\varphi^\pm$.

We can vary this action to drive the discrete Euler Lagrange equations.

$$dS(\varphi_1, \varphi_1^+, ..., \varphi_n, \varphi_n^+) \cdot (\vec{\delta}\varphi_1, ..., \vec{\delta}\varphi_n^+) =$$

$$\frac{\partial L^-}{\partial \varphi^-}(\varphi_1, \varphi_1^-)\delta\varphi_1^- + \sum_{k=1}^{n-1}\Big(\frac{\partial L^+}{\partial \varphi^+}(\varphi_k^+, \varphi_k) + \frac{\partial L^-}{\partial \varphi^+}(\varphi_{k+1}, \varphi_{k+1}^-)\Big)\delta\varphi_k^+$$
$$+ \sum_{k=2}^{n-1}\Big(\frac{\partial L^+}{\partial \varphi}(\varphi_k, \varphi_k^-) + \frac{\partial L^-}{\partial \varphi}(\varphi_k^+, \varphi_k)\Big)\delta\varphi_k + \frac{\partial L^+}{\partial \varphi^+}(\varphi_n^+, \varphi_n)\delta\varphi_n^+. \tag{3.48}$$

With initial condition $(\varphi_1^+, \varphi_1) = (\phi^+, \phi)$, we can solve the equations of motion. The first set of equations is

$$\frac{\partial L^+}{\partial \varphi^+}(\varphi_k^+, \varphi_k) + \frac{\partial L^-}{\partial \varphi_k^+}(\varphi_{k+1}, \varphi_{k+1}^-) = 0,$$

Which gives

$$m(\varphi_k^+ - \varphi_k + 2\pi z_k^-) + m(\varphi_k^+ - \varphi_{k+1} - 2\pi z_{k+1}^-) = 0. \tag{3.49}$$

From the second set of equations

$$\frac{\partial L^+}{\partial \varphi}(\varphi_k, \varphi_k^-) + \frac{\partial L^-}{\partial \varphi_k}(\varphi_k^+, \varphi_k) = 0,$$

we can see that

$$m(\varphi_k - \varphi_k^- + 2\pi z_k^-) + \frac{\partial V(\varphi_k)}{\partial \varphi_k} + m(\varphi_k - \varphi_k^+ - 2\pi z_k^+) + \frac{\partial V(\varphi_k)}{\partial \varphi_k} = 0, \quad (3.50)$$

therefore in the space of solutions, the discrete analogue of Lagrange 1-form is

$$\theta_L(\varphi_1, \varphi_1^-, \varphi_n^+, \varphi_n) \cdot (\vec{\delta}\varphi_1, \vec{\delta}\varphi_1^-, \vec{\delta}\varphi_n^+, \vec{\delta}\varphi_n) = \frac{\partial L^-}{\partial \varphi^-}(\varphi_1, \varphi_1^-)\delta\varphi_1^- + \frac{\partial L^+}{\partial \varphi^+}(\varphi_n^+, \varphi_n)\delta\varphi_n^+ =$$

$$m(\varphi_1^- - \varphi_1 + 2\pi z_1^-)\delta\varphi_1^- + m(\varphi_n^+ - \varphi_n + 2\pi z_n^+)\delta\varphi_n^+. \quad (3.51)$$

By using the initial condition that we have, the Lagrange 2-form is

$$\omega_L(\varphi_1^-, \varphi_1).(\vec{\delta}\varphi_1, \vec{\delta}\varphi_1^-) = m\,\delta\varphi_1\delta\varphi_1^- \quad (3.52)$$

and

$$\omega_L(\varphi_n^-, \varphi_n).(\vec{\delta}\varphi_n, \vec{\delta}\varphi_n^-) = m\,\delta\varphi_n\delta\varphi_n^- \quad (3.53)$$

eq.(3.52)and (3.53)implies the conservation of the symplectic structure by the flow $F : (\varphi_k, \varphi_k^-) \mapsto (\varphi_{k+1}, \varphi_{k+1}^-)$.

Hence we can see the symplecticity of the flow

$$(F^n)^*\omega_L = \omega_L.$$

3.3.1 Free Particle in Circle and the Noether's Theorem

In last example, we study a particle in a circle. Now, let us assume that the particle is free, so there is no potential in the action eq.(3.47) and in the second set of the eqs.(3.50). Note that since in our model the potential never appears in Lagrange 1-form, the Lagrange 1-from and its derivative remain the same as before. It is easy to see that Lagrangian density is invariant under $SO(2)$ transformation and so the flow is $SO(2)$-equivariant, therefore the

discrete momentum map is defined by

$$J_\xi \equiv \xi_{Q\times Q} \lrcorner \theta_L,$$

for any $\xi \in \mathbb{R}$.
By eq.(3.51) we can write the Noether's theorem

$$m(\varphi_1^- - \varphi_1 + 2\pi z_1^-) + m(\varphi_n^+ - \varphi_n + 2\pi z_n^+) = 0. \tag{3.54}$$

Hence what is conserved here is $m(\varphi_k^- - \varphi_k + 2\pi z_k^-)$.

Heuristically the conserved quantity is the angular momentum of each mini interval (q_k^-, q_k) or (q_k^+, q_k).

3.4 Discrete 2-D Gravity Model as a Constrained BF Theory

To write the action for general relativity in a 4 dimensional configuration space which coordinatized by gauge fields, we obtain Plebanski action that can be called as constrained BF theory. Michael Reisenberger in his paper presents a model to discretize this action. He ends up with 4d simplicial lattice model with $SU(2)$ gauge fields in the boundary which converges to the continuum limit[12].
In this section first we start with BF theory in 2 dimensions and write the action in our discrete model then go to discrete 2d general relativity by adding a constraint term to this action. We see that our action is in fact the 2 dimensional Reisenberger's action.
Reisenberger starts with orientable simplicial complex, Δ. In each 4-simplex ν of Δ, there is one c_ν and there are five vertices C_τ's in the boundary of ν and six vertices C_σ's in the boundary of τ's. We introduced these vertices in the beginning of last section.
Reisenberger gives a $su(2)$ Lie algebra element to each surface,$s(\nu,\sigma)$, which are formed by the center of ν, σ and the centers of two τ's which share σ. Also he gives a Lie group element h_l to each edge $l(\nu,\tau)$ from C_ν to C_τ where $\tau < \nu$

and a Lie group element k_r to each edge $r(\tau,\sigma)$ from C_τ to C_σ where $\sigma < \tau$. Therefore, there is a closed holonomy around the boundary of $s(\nu,\sigma)$, $g_{\partial s}$. With all these variables, Reisenberger gives his discrete version of the Plebanski action as

$$I_\Delta(e,h,k,\varphi) = \sum_{\nu<\Delta}(\sum_{s<\nu} e_{si}\theta_s^i - \frac{1}{60}\varphi_\nu^{ij}\sum_{s,\bar{s}<\nu} e_{si}e_{\bar{s}_j}sgn(s,\bar{s})), \tag{3.55}$$

where φ_ν is a spin 2 $SU(2)$ tensor that represented by a symmetric, traceless matrix, φ_ν^{ij}, and $\theta_s^i = tr[J^i g_{\partial s}]$.
Note that s is given orientation from 4-simplex ν. $sgn(s,\bar{s}) = 0$ if s and $\bar{s}$ are in a same τ and $sgn(s,\bar{s}) = 1$ or -1 depends on the orientations match.

3.4.1 2-D $SO(2)$ BF Theory

The action eq.(3.55) in 2 dimensional spacetime becomes

$$S(\Phi,e,z) = \sum_{\nu<\Delta}\sum_{s<\nu}[e_\nu(\Phi_s + 2\pi z_s)] = \sum_{i=1}^{4} L_\nu^{s_i} \tag{3.56}$$

Here we define $L_\nu^{s_i}$'s for simplicity of writing equations of motion. Also we have to mention that since we are in 2 dimensional spacetime, we have to assign a form of degree 0 to each ν which means there is a function e_ν to each ν. Also we have a simple field for each s:

$$\Phi_s = h_{l_1} + h_{l_2} + k_{r_1} + k_{r_2},$$

where, now, h_l's and k_r's are angles. Again we can see the factor $z_s \in \mathbb{Z}$ in the holonomy in the action eq.(3.56), which is necessary to determine the number of the complete rounds of the parallel transport around ∂s.

We have five sets of Euler lagrange equation which only one of them is non-trivial.

$$\begin{aligned}
&\frac{\partial L_\nu^{s_i}}{\partial h_{l_2}} + \frac{\partial L_\nu^{s_{i+1}}}{\partial h_{\lambda_1}} = 0 \qquad \Rightarrow e_\nu - e_\nu = 0 \\
&\frac{\partial L_\nu^{s_i}}{\partial k_{r_1}} + \frac{\partial L_\nu^{s_{i+1}}}{\partial k_{r_2}} = 0 \qquad \Rightarrow e_\nu - e_{\nu_i} = 0 \\
&\frac{\partial L_\nu^{s_i}}{\partial k_{r_2}} + \frac{\partial L_\nu^{s_{i-1}}}{\partial k_{r_1}} = 0 \qquad \Rightarrow e_\nu + e_{\nu_{i+1}} = 0 \\
&\frac{\partial L_\nu^{s_i}}{\partial h_{l_1}} + \frac{\partial L_\nu^{s_{i-1}}}{\partial h_{l_2}} = 0 \qquad \Rightarrow e_\nu - e_\nu = 0 \\
&\sum_{i=1}^{4} \frac{\partial L_\nu^{s_i}}{\partial e_\nu} = 0 \qquad \Rightarrow \sum_{s<\nu} \Phi_{s_i} + 2\pi z_{s_i} = 0
\end{aligned} \tag{3.57}$$

Note that each h_l^i is in two s's but with opposite sign, therefore the non-trivial equation of motion, gives us

$$\sum_{s<\nu} \Phi_{s_i} + 2\pi z_{s_i} = 0 \longrightarrow \sum_{i=1}^{4} k_r^i + 2\pi z_{s_i} = 0 \tag{3.58}$$

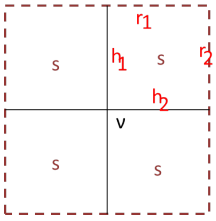


Figure 3.3:

It is clear that the Lagrange 1-form is

$$\theta = \sum_{i=1}^{4} (e_\nu dk_{r_1}^i + e_\nu dk_{r_2}^i) \tag{3.59}$$

From eq.(3.59) we can write the symplectic structure

$$\omega = \sum_{i=1}^{4}(de_\nu \wedge dk^i_{r_1} + de_\nu \wedge dk^i_{r_2}) = de_\nu \wedge \sum_{i=1}^{4}(dk^i_{r_1} + dk^i_{r_2})$$

Using equation of motion, we have

$$\omega = de_\nu \wedge \sum_{i=1}^{4}(dk^i_{r_1} + dk^i_{r_2}) = de_\nu \wedge d\sum_{i=1}^{4}(k^i_{r_1} + k^i_{r_2}) = 0$$

3.4.2 $SO(2)$ Gauge Theory for 2-D Gravity

We obtain the 2d-gravity action from BF theory by adding a constraint to the BF action.

$$S_\nu(\Phi, e) = \sum_{s<\nu}[e_\nu(\Phi_{s_i} + 2\pi z_{s_i}) - \lambda_\nu(e^2 - 1)] = \sum_{i=1}^{4} L_\nu^{s_i} \tag{3.60}$$

and

$$\Phi_s = h_{l_1} + h_{l_2} + k_{r_1} + k_{r_2}$$

The same as BF theory, we have only one set of non trivial equation

$$\sum_{i=1}^{4} \frac{\partial L_\nu^{s_i}}{\partial e_\nu} = 0 \Rightarrow \sum_{s<\nu}[\Phi_{s_i} + 2\pi z_{s_i}] - 4 \times 2\lambda e_\nu = 0 \tag{3.61}$$

Which gives

$$e^2 - 1 = 0 \qquad \Rightarrow \qquad de = 0$$

We can obtain the symplectic form from lagrange 1-form:

$$\theta = \sum_{i=1}^{4}(e_\nu dk_{r_1}^i + e_\nu dk_{r_2}^i)$$

$$\omega = -d\theta$$

$$\omega = \sum_{i=1}^{4} de_\nu \wedge (dk_{r_1}^i + dk_{r_2}^i) \tag{3.62}$$

Using the solution of lagrange equation, we can see the conservation of ω

$$de = 0 \Rightarrow \omega = 0$$

Chapter 4

Summary of part I

Our formalism provides a classical counterpart to spin foam models, in which spacetime atoms í are autonomous minimal regions of spacetime that enjoy a complete version of the formalism. In our framework, the variation of the action decomposes into bulk and boundary terms.

The form Θ_L provides a covariant momentum map linking symmetries and conserved quantities, and it generates the multisymplectic form $\Omega_L = -d\Theta_L$ which is conserved in covariant evolution. On the other hand, the form Ω_L is responsible for the equations of motion in the interior of spacetime atoms and for the simple gluing conditions that let us find solutions on larger regions of spacetime amalgamating solutions over atoms. The clean separation between bulk and boundary parts makes gluing transparent: neighboring regions share boundary data and gluing conditions asking for momentum matching appear when regions are fused. These structures have (sometimes implicit) quantum counterparts in spin foam models. We hope that our contribution, making available the classical counterparts of such structural properties, makes it possible for some research lines to advance further or enter a new level. One example is the study of the semiclassical limit of quantum gravity spin foam models, in which most of the research has been carried on at the level of a single atom; this study may now progress to study larger regions of spacetime aided with compatible the structures of quantum and classical gluing. In the case of lattice gauge theories, the transfer matrix for lattice gauge the-

ory can be constructed gluing cellular propagators (or 'atomic propagators' in the terminology used here); in this paper we wrote the discrete field equations for the classical atomic propagator which is the classical counterpart of the mentioned atomic propagator. We presented a Lagrangian picture and a Hamiltonian picture with their respective structures and compatibility with the mentioned classical cellular propagator. Along this line, comparing our formalism with the classical Hamiltonian formalism of Kogut and Susskind, would be interesting. Another line suggested by our framework is investigating the classical counterpart of the heat kernel action, and study the resulting theory because it is the most elegant spin foam model for lattice gauge theory.

Part II

New Helicity Formalism In Born-Infeld Theory

Chapter 5

Introduction to part II

5.1 Nonlinear Electrodynamics

5.1.1 Maxwell Theory

In the second half of the 19th century, James Clerk Maxwell unified the theories of electricity, magnetism, and light. Today Maxwell's equations are the essential tools to work with electromagnetism and engineering.

Maxwell's equations refer to a set of four relations that describe the properties and interrelations of electric and magnetic fields.

$$\begin{aligned} \nabla \cdot B &= 0 && \text{Gauss}' \text{ law for magnetism}, \\ \nabla \times E &= -\frac{\partial B}{\partial t} && \text{Faraday}' \text{s law}, \\ \nabla \cdot D &= \rho && \text{Gauss}' \text{ law}, \\ \nabla \times H &= \frac{\partial D}{\partial t} + J && \text{Ampre} - \text{Maxwell}' \text{s law}. \end{aligned} \tag{5.1}$$

E is the electric field and $D = \epsilon E$ is the electric displacement; ϵ indicates how the electrical charges in a material become polarized in an electric field. The magnetic force fields are described by H, the magnetic field, and $B = \mu H$, the

magnetic flux density with μ accounting for the magnetization of a material.

The lagrangian for Maxwell's theory can be either written in terms of the electric and magnetic fields components,

$$L = \frac{1}{2}\left(E^2 - B^2\right), \tag{5.2}$$

or in terms of the field strength tensor$F_{\mu\nu}$

$$L = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{5.3}$$

Maxwell's theory and his lagrangian lead to difficulties with the self energy of a point particle. In this theory the self energy of a point particle becomes infinite and this violates the principle that physical quantities never become infinite. This principle is called the "principle of finiteness" by Born[11].

5.1.2 Born-Infeld Theory

In Born's view, special relativity avoids the violation of the "principle of finiteness", by imposing an upper limit of velocity c replacing the Newtonian action $\frac{1}{2}mv^2$ with the relativistic expression $mc^2\left(1-\sqrt{1-v^2/c^2}\right)$. Analogously, Born assumed in his paper [11],1934, an upper limit of the field strength and modified the electrodynamic action to obtain his action which now respects the principle of finiteness,

$$L = b^2\left(\sqrt{1+\frac{1}{b^2}(E^2-B^2)}-1\right). \tag{5.4}$$

Note that Maxwell theory is obtained in the limit $b \to \infty$.

Born and Infeld generalized (5.4) making use also of the second Lorentz invariant$\vec{E}.\vec{B}$. There are two independent nontrivial Lorentz invariant objects

that we can build using $F_{\mu\nu}$ but not its derivatives

$$s = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}\left(E^2 - B^2\right) , \tag{5.5}$$

$$p = -\frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} = E \cdot B . \tag{5.6}$$

We can express Born-Infeld lagrangian using these Lorentz invariants.

$$L_{\mathrm{BI}} = b^2\left(1 - \sqrt{1 - \frac{2s}{b^2} - \frac{p^2}{b^4}}\right) . \tag{5.7}$$

or, expanding in powers of the field,

$$L_{\mathrm{BI}} = s + \frac{1}{2b^2}\left(s^2 + p^2\right) + \frac{1}{2b^4}\left(s^3 + sp^2\right) + \ldots . \tag{5.8}$$

Thus for $b \to \infty$ the Born-Infeld Lagrangian converges against the Maxwell one,

$$\lim_{b\to\infty} L = s = -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} \tag{5.9}$$

Note that in eq.(5.8), p appears squared, since p is only a pseudo-Lorentz invariant. Although Born proposed his lagrangian in terms of the fields E and B, he mentioned that there are four different but equivalent ways of writing this Lagrangian. In fact we need two independent vectors to write this Lagrangian and there are different possible ways to choose these two vectors from (E, B) or (D, H). A combination of the electric vectors E and D with a combination of the magnetic field$B$ and H could be a candidate of our variables in this lagrangian.

Following this, Born and Infeld showed in their article that the group of transformations of their lagrangian has an invariant spacetime integral[13]. This group is the group of special relativity and is larger than the group of the

old lagrangian. Although this lagrangian was not written to describe electromagnetism in the presence of a material medium, it is easy to see that with a nontrivial relation between (D, H) and (E, B) we can obtain Maxwell's equation. In fact the vacuum behaves as some kind of material in nonlinear electrodynamics.

In 1935, Schrödinger proposed two complex combinations of these fields in such a way that their conjugate are equal to their complex conjugate[14].

$$\begin{aligned} \mathfrak{F} &= B - iD \,, \\ \mathfrak{G} &= E + iH \,, \end{aligned} \tag{5.10}$$

where

$$\begin{aligned} \mathfrak{F}^* &= \frac{\partial L}{\partial \mathfrak{G}} \,, \\ \mathfrak{G}^* &= \frac{\partial L}{\partial \mathfrak{F}} \,. \end{aligned}$$

In terms of these variables, the Born-Infeld lagrangian is

$$L = b^2 \frac{\mathfrak{F}^2 - \mathfrak{G}^2}{\mathfrak{F}\,\mathfrak{G}} \,. \tag{5.11}$$

Note that with the new variables, the square root which makes the Born-Infeld Lagrangian so specific, has disappeared.

5.2 Alternative derivation of the Born-Infeld Theory

5.2.1 BI and String Theory

In some systems the low energy states (compared to an energy scale) are effectively independent of high-energy states which means that, for those systems, what goes on at high energies is irrelevant for studying the low energy sector. Therefore, one can describe the system at low energies by an effective field theory which is a theory of the dynamics of the system below some energy scale. In some cases the full theory is known and using effective field theory is not necessary but it makes the calculations more easier. However, in many cases the full theory (and high energy theory) is unknown. In those cases, one can use effective field theory to made predictions at low energies without the necessity of knowing the details of high energy theory.

Although we can see the Born-Infeld theory as a fundamental theory substituting for the Maxwell theory, as a quantum theory is not renormalizable. Therefore, it is natural to think of the Born-Infeld Lagrangian as an effective field theory.

By ***effective action*** we mean the action which is obtained by integrating out all massive degrees of freedom. The ***low-energy effective action*** we get by omitting terms in the effective action that are small compared to the mass scale defined by those degrees of freedom. In the present context, this means that we keep only the terms involving field strength tensor but not its derivatives. These terms are the ones which vary slowly at the string scale. In other words, we have to keep all momenta small compared to the cutoff $1/\alpha'$, as we do in other effective field theories.

Fradkin and Tseytlin in 1985 computed the effective action for an abelian vector field coupled to the open bosonic string [15]. For $D = 26$ string theory, they gave an exact solution for a constant field strength in the tree and one loop approximation. The resulting tree-level effective Lagrangian was shown to coincide with the Born-Infeld Lagrangian.

Later in 1999, Rocek and Tseytlin studied the connection between the break-

ing of supersymmetry and nonlinear actions[16]. They explained the relation between the BI action ($U(1)$ duality) and $N = 2$ partial supersymmetry (S-duality).

5.2.2 BI and D-branes

In 1995, the Born-Infeld action found a new interpretation as an action of D-branes in the static gauge.

The Dirac-Born-Infeld Lagrangian, which describes the behavior of electromagnetic fields on D-branes is

$$L_{\mathrm{DBI}} = -T_p(g)\sqrt{-\det\Big(\eta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}\Big)} \tag{5.12}$$

We can compare this Lagrangian with Born-Infeld Lagrangian density which is written in terms of the field F

$$L_{BI} = -b^2\sqrt{-\det\Big(\eta_{\mu\nu} + \frac{1}{b}F_{\mu\nu}\Big)} + b^2 \tag{5.13}$$

Note that in (5.12) we do not subtract the leading constant term in contrast to (5.13), since it has natural interpretation as the total mass energy of the D-brane.

In the theory of D-branes, ***T-duality*** links the finiteness of the velocity of light with the finiteness of the electric field strength. We can use T-duality and show that a D-brane with a constant electric field along a compact direction as a background is physically equivalent to a D(p-1)-brane without any electric field which is moving with constant velocity along the dual circle. It is clear that the velocity of the D(p-1)-brane can not reach more than the light velocity. Using this property, we understand that the strength of an electric field on D-branes is bounded. This result which we obtain using T-duality, leads us to think about the Born-Infeld theory as an appropriate candidate for electromagnetism in string theory[17].

5.3 New Aspects on The Field Theory Side

5.3.1 Helicity Conservation

As is well-known, the Maxwell equations are invariant under the duality transformation :

$$\vec{E} \to \vec{B}$$

$$\vec{B} \to -1/c^2 \vec{E}$$

This is a discrete transformation, however there exist a continuous version of this invariance:

$$\vec{E} \to \cos\alpha \vec{E} + \sin\alpha \vec{B}$$

$$\vec{B} \to -\sin\alpha \vec{E} + \cos\alpha \vec{B}$$

This is a $U(1)$ ***symmetry***. In the string theory contests, this duality to ***S-duality***.

Recently, Aschieri et al. have studied the formulation of $U(1)$ duality symmetry on shell and off shell and proved that the Born-Infeld action is invariant under this symmetry off-shell.

Rosly and Selivanov [18] have used the same symmetry to argue that at tree level, the helicity is preserved in the scattering of photons in Born-Infeld. They used $U(1)$ duality symmetry and defined a basis of self-dual and anti-self-dual plane waves and assigned a quantum number s, which they called self-duality number, to a plane wave. They argued that the tree-level contribution of the amplitude is zero unless sum of the self-duality numbers of all the scattering states vanishes. Therefore, we have helicity conservation at tree level of this theory.

$$
\begin{aligned}
&A_{\text{tree}}(+,+,...,+) = 0,\\
&A_{\text{tree}}(-,+,...,+) = 0,\\
&\vdots\\
&A_{\text{tree}}(\underbrace{-,-,...,-}_{n}\underbrace{+,+,...,+}_{n}) \neq 0 \qquad ''Middle\ \ component'',\\
&\vdots,\\
&A_{\text{tree}}(-,-,...,-) = 0.
\end{aligned}
\tag{5.14}
$$

Rosly and Selivanov mentioned that this helicity conservation may indicate that the Born-Infeld theory is, in a sense ,solvable.

5.4 Born-Infeld theory, Connection between field theory and string theory

Born-Infeld theory has interesting properties in both quantum field theory and string theory. On the field theory side, Born-Infeld theory is a model case of a highly nonlinear field theory, and in some aspects resembles nonabelian gauge theory, particularly in that it provides a nontrivial example for both the CSW [19] and BCFW [20] recursion relations. Let us shortly explain these concepts. Cachazo, Svrcek and Witten use tree-level amplitudes as building blocks for more complex amplitudes.

The usual field theory methods remain valid for complex momenta. To see this, note that a tree-amplitude is a rational function of momenta and polarization vectors. If we now consider complex momenta then tree-amplitudes become analytic functions but for some isolated poles.Using this fact, Britto, Cachazo, Feng and Witten [20] construct on-shell recursion relations in the number of legs, leads to a very compact formulas. Any tree-level amplitude of gluons is expressed as a sum over terms constructed from the product of two subamplitudes with fewer gluons times a Feynman propagator.

Boels et al [22]studied the tree level amplitudes in BI theory and verified the helicity conservation, predicted by Rosly and Selivanov [18], up to the 6-point amplitude. Also they noted that the BI theory is an example for BCFW.

BI amplitudes are of interest for string theory because they appear in the field theory limit of the open string. As was already mentioned, Fradkin and Tseytlin [15] have shown that the tree-level effective action for a constant field strength coupled to the open bosonic string coincides with Born-Infeld Lagrangian

Previously one primarily used to go only from string amplitudes to field theory amplitudes (as in the seminal work of Bern and Kosower[23]), but nowadays there is traffic in both directions; there is a big effort going on to systematically construct the string α' expansion starting from its lowest order term - the field theory limit - and the hope is to find structures that actually do not change (much) when you go from the leading to higher orders (such hopes are also fueled by similar things happening in Yang-Mills theory in the usual

perturbation expansion)[22, 41].

In particular, the string theory MHV amplitude, defined by two negative helicity legs, must be proportional to the field theory MHV amplitude[22]. This means that the two amplitudes differ only by a scalar function of the string coupling.

BI theory possesses a great amount of symmetry, of which some is visible already in field theory, such as duality and helicity conservation, while some is hidden and has been discovered only recently through strings and D-branes, such as S-duality and T-duality. Thus it is a very interesting question how much simplification is implied by the existence of these symmetries for the structure of the S-matrix. The work of Rosly and Selivanov[18], as well as Boels et al[22], already points to a level of simplicity that is very unusual for an interacting field theory in four dimension.

Here we wish to build a highly optimized formalism for the calculation of helicity amplitudes in BI theory, extending the work of Martin et al[28] and Boels et al[22].

5.5 The Structure of Part 2

From eq.(4.9) we can see that the Born-Infeld theory is quite complicated since its Lagrangian is an infinite power series in $F^2 = F^{\mu\nu}F_{\mu\nu}$, therefore, a priori one would expect amplitude calculations to be quite difficult, even at tree-level.

With this motivation, in chapter 5, we present a new and highly efficient formalism for the calculation of helicity-decomposed amplitudes in BI theory based on (off-shell) self-duality and (on-shell) helicity.

In chapter 6 we present our result for the 6 point amplitude and compare them with the ones which Boels et al calculated in their article. Also for the first time, we calculated the middle amplitude for the 8 point diagram. Boels et al were able to obtain only the contribution of the elementary 8-point vertex to this amplitude[22].

In chapter 7, we give a summary of our formalism. With this formalism it is possible to compute any diagrams at tree levels in a more efficient way than was previously possible.

Chapter 6

New self duality based formalism

Our goal here is to build an efficient formalism for the computation of helicity amplitudes in Born-Infeld theory. At the tree-level, our notation for the on-shell amplitude with M negative and N positive helicity legs is simply $A^{M,N}$. We will omit the global power of b, which is fixed by the mass dimension anyway, and it will be understood that the momenta $k_1, \ldots, k_M$ are associated to the negative helicity photons, and the remaining ones to the positive helicity ones. Although the Rosly-Selivanov theorem [18] predicts that only the "middle" component with $M = N$ can be nonzero, we will study other cases, too, since it will be instructive to see how the corresponding cancellations arise diagrammatically.

For the case of the QED effective Lagrangian, it is well-known [24–26] that, once it is written in terms of the field strength tensor $F_{\mu\nu}$, the corresponding low-energy N - photon amplitudes can be obtained from it by introducing a field strength tensor for each photon,

$$F_i^{\mu\nu} \equiv k_i^\mu \varepsilon_i^\nu - k_i^\nu \varepsilon_i^\mu \tag{6.1}$$

writing the background field as the sum of the photon field strength tensors,

$$\mathcal{F}_{\mu\nu} = \sum_{i=1}^{N} F_i^{\mu\nu} \tag{6.2}$$

plugging this into the effective Lagrangian, and keeping only the terms involving all N tensors $F_1, \ldots, F_N$. For the Euler-Heisenberg Lagrangian [41], the QED effective Lagrangian in the one-loop and constant field approximation, at the four-photon level this procedure is a textbook exercise (see, e.g., [25]), and has been carried out in closed form even at the N - point level [28]. Here it turned out to be advantageous to use the helicity basis for the polarizations, and to decompose the field strength tensors (6.1) correspondingly. The resulting tensors

$$F_i^{\pm\mu\nu} = \frac{1}{2}(F_i^{\mu\nu} \pm i\tilde{F}_i^{\mu\nu}) \tag{6.3}$$

Where

$$\tilde{F}_i^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{i\alpha\beta}$$

are self-duality eigenstates [18, 29–32] and have nice algebraic properties, as will be seen in detail below using the spinor helicity formalism. At the two-loop level, the corresponding calculation has been carried out up to the ten-point case for general polarizations in [28], and for arbitrary N for the case of the "all plus" (or "all minus") amplitudes in [33].

Since the Euler-Heisenberg Lagrangian is missing derivative corrections, the result gives only the leading low-energy approximation, where all photon energies ω_i are assumed small in comparison to the electron mass m. In the Born-Infeld case there are no such derivative corrections, and the same procedure gives the exact contributions of the elementary vertices contained in the expansion (5.8) of the Lagrangian. On the other hand, contrary to the QED case we now have to also include nontrivial tree-level diagrams, the first such diagram appearing at the six-point level.

Since such diagrams involve also internal photons, we will generally need to split the field strength tensor $\mathcal{F}$ as

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} + f_{\mu\nu} \tag{6.4}$$

where $F_{\mu\nu}$ will generate the external and $f_{\mu\nu}$ the internal photons. The field strength tensors $f_I^{\mu\nu}$ for the internal photons will have to be kept off-shell, and cannot be treated in the spinor helicity formalism. Nevertheless, it will still be useful to apply to them the same decomposition into self-duality eigenstates as in the on-shell case, (6.3),

$$f_I^{\pm\mu\nu} = \frac{1}{2}(f_I^{\mu\nu} \pm i\tilde{f}_I^{\mu\nu}) \tag{6.5}$$

This suggests to use a corresponding decomposition already for the BIL itself. Namely, we introduce $s_\pm$ by

$$s_\pm = -\frac{1}{4}\mathcal{F}^\pm_{\mu\nu}\mathcal{F}^{\pm\mu\nu} \tag{6.6}$$

Using that $\mathcal{F}_{\pm\mu\nu}\mathcal{F}^{\mp\mu\nu} = 0$, we have

$$\begin{aligned} s &= s_+ + s_- \\ p &= -i(s_+ - s_-) \end{aligned} \tag{6.7}$$

It follows that the expansion (5.8) of the BIL can also be rewritten in the form [35]

$$L = s_+ + s_- + \sum_{l=1}^{\infty} \frac{1}{b^{2l}} \sum_{\substack{m,n \\ m+n=l+1}} L_{mn}\, s_-^m s_+^n \tag{6.8}$$

Thus we will use this self-duality/helicity decomposition from the beginning (a similar approach was pursued in [22]).

The coefficients of this expansion L_{mn} can be given in closed form; namely they are essentially the *Narayana numbers*, $T(n,k)$[36], defined as

$$T(n,k) := \frac{1}{k}\binom{n-1}{k-1}\binom{n}{k-1} \tag{6.9}$$

Namely, one has

$$L_{mn} = 2T(m+n-1,n) \tag{6.10}$$

The numbers $T(m,n)$ are integers and appear in many combinatorial problems[40],[39], which fact we take as an indication of the naturalness of the expansion of L in terms of $s_{\pm}$. A table of the first few of these numbers is also given in Appendix C. While for the Euler-Heisenberg Lagrangian it is well-known that the corresponding coefficients are Bernoulli numbers[37], for the BI lagrangian we introduce the relation between these coefficients and Narayana numbers for the first time.

6.1 The formalism

We will now explain our computational approach in detail. As has already been said, we will generally write all field strength tensors in the decomposition

$$F_{\mu\nu} = F^{+}_{\mu\nu} + F^{-}_{\mu\nu}, \quad F^{\pm}_{\mu\nu} \equiv \frac{1}{2}(F_{\mu\nu} \pm i\tilde{F}_{\mu\nu}) \tag{6.11}$$

and identical formulas for $f_{\mu\nu}$. Diagrams will have vertices involving powers of s_+ and s_- as defined in (6.6). A vertex in general has external photons, represented by on-shell field strength tensors $F^{\pm}_i$ generated by $F^{\pm}$, and internal photons represented by off-shell field strength tensors $f^{\pm}_I$ generated by $f^{\pm}$. An internal photon with momentum k_I is represented by a photon propagator $D^{\mu\nu}_I$ (we will use Feynman gauge throughout)

$$D_I^{\mu\nu} = -\frac{\eta_{\mu\nu}}{k_I^2} \tag{6.12}$$

Thus before the $\pm$ decomposition an internal photon will yield a factor involving two off-shell field strength tensors from the two vertices connected by the momentum k_I,

$$\langle f_I^{\alpha\beta} f_I^{\mu\nu} \rangle := -\frac{k_I^{[\alpha}\eta^{\beta][\mu}k_I^{\nu]}}{k_I^2} \tag{6.13}$$

(here and in the following we symmetrize or antisymmetrize without a factor, e.g. $T_{[\alpha,\beta]} \equiv T_{\alpha\beta} - T_{\beta\alpha}$). From this basic formula one easily derives the corresponding formulas for the $\pm$ decomposition:

$$\begin{aligned} 4\langle f_I^{\pm\alpha\beta} f_I^{\pm\mu\nu} \rangle &= \eta^{\mu[\alpha}\eta^{\beta]\nu} \mp i\hat{k}_{I\rho}\Big(\varepsilon^{\alpha\beta\rho[\mu}\hat{k}_I^{\nu]} + \varepsilon^{\mu\nu\rho[\alpha}\hat{k}_I^{\beta]}\Big) \\ 4\langle f_I^{\pm\alpha\beta} f_I^{\mp\mu\nu} \rangle &= -\eta^{\mu[\alpha}\eta^{\beta]\nu} - 2\hat{k}_I^{[\alpha}\eta^{\beta][\mu}\hat{k}_I^{\nu]} \mp i\hat{k}_{I\rho}\Big(\varepsilon^{\alpha\beta\rho[\mu}\hat{k}_I^{\nu]} - \varepsilon^{\mu\nu\rho[\alpha}\hat{k}_I^{\beta]}\Big) \end{aligned} \tag{6.14}$$

where we have introduced the further notation

$$\hat{k}_I := \frac{k_I}{\sqrt{k_I^2}} \tag{6.15}$$

Further useful identities are obtained from (6.14) considering that each $f_I^{\pm\alpha\beta}$ must appear together with either an $f_{J\alpha\beta}^{\pm}$ or an $F_{i\alpha\beta}^{\pm}$. In the all off-shell case, one finds

$$
\begin{aligned}
f^{+}_{J\alpha\beta}\langle f_I^{+\alpha\beta} f_I^{+\mu\nu}\rangle f^{+}_{L\mu\nu} &= f^{+}_{J\alpha\beta} f_L^{+\alpha\beta} && (6.16)\\
f^{-}_{J\alpha\beta}\langle f_I^{-\alpha\beta} f_I^{-\mu\nu}\rangle f^{-}_{L\mu\nu} &= f^{-}_{J\alpha\beta} f_L^{-\alpha\beta} && (6.17)\\
f^{+}_{J\alpha\beta}\langle f_I^{+\alpha\beta} f_I^{-\mu\nu}\rangle f^{-}_{L\mu\nu} &= -2\hat{k}_I\{f_J^{+}, f_L^{-}\}\hat{k}_I && (6.18)\\
f^{-}_{J\alpha\beta}\langle f_I^{-\alpha\beta} f_I^{+\mu\nu}\rangle f^{+}_{L\mu\nu} &= -2\hat{k}_I\{f_J^{-}, f_L^{+}\}\hat{k}_I && (6.19)
\end{aligned}
$$

Here in writing the right hand sides we have used the identity

$$
\tilde{f}_I^{\pm} = \mp i f_I^{\pm} \tag{6.20}
$$

which follows from (6.5). The identities (6.16-6.19) hold, of course, also if one or both of the external field strength tensors are replaced by on-shell ones. If both are on-shell, however, we can achieve further simplification using some identities which can be derived for the $F_i^{\pm}$ in the spinor helicity formalism. Those are [28]

$$
\begin{aligned}
\{F_i^{+}, F_j^{+}\}^{\mu\nu} &= -\frac{1}{2}[ij]^2\eta^{\mu\nu} && (6.21)\\
\{F_i^{-}, F_j^{-}\}^{\mu\nu} &= -\frac{1}{2}\langle ij\rangle^2\eta^{\mu\nu} && (6.22)\\
[F_i^{+}, F_j^{-}] &= 0 && (6.23)
\end{aligned}
$$

Using these identities (6.16-6.19) become, in this case of two on-shell tensors,

$$
\begin{aligned}
F^{+}_{i\alpha\beta}\langle f_I^{+\alpha\beta} f_I^{+\mu\nu}\rangle F^{+}_{j\mu\nu} &= [ij]^2 && (6.24)\\
F^{-}_{i\alpha\beta}\langle f_I^{-\alpha\beta} f_I^{-\mu\nu}\rangle F^{-}_{j\mu\nu} &= \langle ij\rangle^2 && (6.25)\\
F^{+}_{i\alpha\beta}\langle f_I^{+\alpha\beta} f_I^{-\mu\nu}\rangle F^{-}_{j\mu\nu} &= -2\hat{k}_I\{F_i^{+}, F_j^{-}\}\hat{k}_I && (6.26)\\
F^{-}_{i\alpha\beta}\langle f_I^{-\alpha\beta} f_I^{+\mu\nu}\rangle F^{+}_{j\mu\nu} &= -2\hat{k}_I\{F_i^{-}, F_j^{+}\}\hat{k}_I && (6.27)
\end{aligned}
$$

For tree-level diagrams we can, as a final step, further simplify the resulting expressions by writing each internal (off-shell) photon momentum k_I as a sum

of external (on-shell) momenta k_i.

At the tree level, the final result for any amplitude will then have the structure

$$A^{M,N} = \sum_r \frac{N_r}{k^2_{I_{r1}} k^2_{I_{r2}} \cdots k^2_{I_{rp}}} \tag{6.28}$$

with (possibly) some uncancelled poles in the denominators, and numerators N_r that are completely written in terms of spinor products.

6.2 Elementary vertices

Let us start with the computation of the contributions to the helicity amplitudes coming from the elementary vertices. As was mentioned above, this part of the calculation is analogous to the computation of the low-energy limit of the one-loop QED photon amplitudes from the Euler-Heisenberg lagrangian, carried out explicitly in [28]. The elementary vertices contribute only to the amplitudes with M and N even, $M = 2m$ and $N = 2n$. Let us denote this contribution to $A^{2m,2n}$ by $V^{m,n}$. To get it from the term in (6.8) carrying a $s_-^m s_+^n$, we write for, say, s_-,

$$s_- \;=\; -\frac{1}{4}\mathcal{F}^-_{\mu\nu}\mathcal{F}^{-\mu\nu} = -\frac{1}{4}\sum_{i,j=1}^{M} F^-_{i\mu\nu}F_j^{-\mu\nu} = \frac{1}{4}\sum_{i<j}\mathrm{tr}\big(\{F_i^-, F_j^-\}\big) = -\frac{1}{2}\sum_{i,j}\langle ij\rangle^2 \tag{6.29}$$

where (6.22) was used in the last step. Defining [28]

$$\chi_- := \frac{1}{2}\sum_{i<j}\langle ij\rangle^2 \tag{6.30}$$

and

$$\chi^-_M \;:=\; (\chi_-)^m\Big|_{\text{all different}} = \frac{m!}{2^m}\Big\{\langle 12\rangle^2\langle 34\rangle^2\cdots\langle(M-1)M\rangle^2 + \text{ all inequivalent permutations}\Big\} \tag{6.31}$$

we have obtained the conversion

$$s_-^m \to (-1)^m\chi^-_M \tag{6.32}$$

Defining χ_+ and χ^+_N analogously to (6.30),(6.31), just with $\langle ij\rangle$ replaced by $[ij]$, we can write the elementary vertex contribution as

$$V^{m,n} = L_{mn}(-1)^{m+n}\chi^-_M\chi^+_N \tag{6.33}$$

Chapter 7

Helicity Amplitudes, the Middle Amplitude

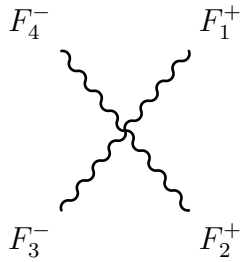
In this section we calculate the 4 point diagrams then we compare our result with the Boels et al's one.

Also for the first time we calculate the middle (non-vanishing) amplitude for the 8 point diagrams.

7.1 4-points

7.1.1 2- 2+

There is no reducible diagram for this amplitude and we have only the main vertex.



The basic vertex which we call V for this diagram (four point) is

$$V^{2,2} = L_{11}\chi_2^-\chi_2^+ = 2\chi_2^-\chi_2^+, \tag{7.1}$$

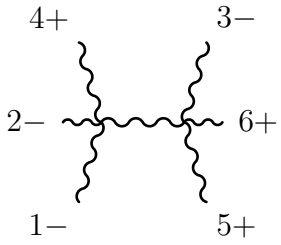
which comes from (6.33), with $m, n = 1$.

7.2 6-points

7.2.1 3- 3+

Unlike the four point diagram, there is no elementary vertex, V^{33}, contribution of 6-points since we need odd number of $F^{\pm}$'s.

However there is a reducible diagram.



$$C^{3,3} = \frac{1}{b^4}\frac{1}{2!} \times 2L_{11}^2 s_{i-}s_{i+}s_{j-}s_{j+} = \frac{1}{b^4}s_{i+}s_{j-}L_{11}^2 s_{i-}s_{j+}. \tag{7.2}$$

Here we labeled the s's only for clarity of calculation.

$$\begin{aligned} s_{i-}s_{j+} &\rightarrow (\frac{-1}{4})^2 \times 2^2 \times F_i^{-\mu\nu}\langle f^-_{i\mu\nu} f^+_{j\alpha\beta}\rangle F_j^{+\alpha\beta} = (\frac{-1}{4})^2 \times 2^2 \times (-2)\hat{k_I}\{F_i^-, F_j^+\}\hat{k_I} \\ &= (\frac{-1}{4})^2 \times 2^2 \times (-2) \times \frac{(k_{i_1}+k_{i_2}+k_{j_3})\{F_{i_3}^-, F_{j_3}^+\}(k_{i_1}+k_{i_2}+k_{j_3})}{(k_{i_1}+k_{i_2}+k_{j_3})^2} \\ &= -\frac{1}{2}\frac{(k_{i_1}+k_{i_2}+k_{j_3})\{F_{i_3}^-, F_{j_3}^+\}(k_{i_1}+k_{i_2}+k_{j_3})}{(k_{i_1}+k_{i_2}+k_{j_3})^2}. \end{aligned} \tag{7.3}$$

We call these $s^{\pm}$'s which participate in off-shell contractions as an ***active***.

$$s_{i+}s_{j-} \to (\frac{-1}{4})^2 F_{i_1}^{-\mu\nu} F^{-}_{i_2\mu\nu} F_{j_1}^{+\alpha\beta} F^{+}_{j_2\alpha\beta} = \frac{1}{16}\langle i_1\, i_2\rangle^2 [j_1, j_2]^2. \tag{7.4}$$

The $s^{\pm}$'s which never contribute in contractions, we call ***passive*** ; This kind of $s^{\pm}$ always appears in a $\chi^{\pm}$

$$\begin{aligned} C^{3,3} &= \frac{1}{b^4} L_{11}^2 \times \frac{-1}{32} \sum_{i_1,i_2.i_3=1}^{3} \sum_{j_1,j_2.j_3=4}^{6} \langle i_1\, i_2\rangle^2 [j_1\, j_2]^2 \frac{(k_{i_1}+k_{i_2}+k_{j_3})\{F^{-}_{i_3}, F^{+}_{j_3}\}(k_{i_1}+k_{i_2}+k_{j_3})}{(k_{i_1}+k_{i_2}+k_{j_3})^2} \\ &= -\frac{1}{b^4} \times \frac{1}{8} \sum_{i_1,i_2.i_3=1}^{3} \sum_{j_1,j_2.j_3=4}^{6} \langle i_1\, i_2\rangle^2 [j_1\, j_2]^2 \frac{(k_{i_1}+k_{i_2}+k_{j_3})\{F^{-}_{i_3}, F^{+}_{j_3}\}(k_{i_1}+k_{i_2}+k_{j_3})}{(k_{i_1}+k_{i_2}+k_{j_3})^2}. \end{aligned} \tag{7.5}$$

With some calculation we can see that

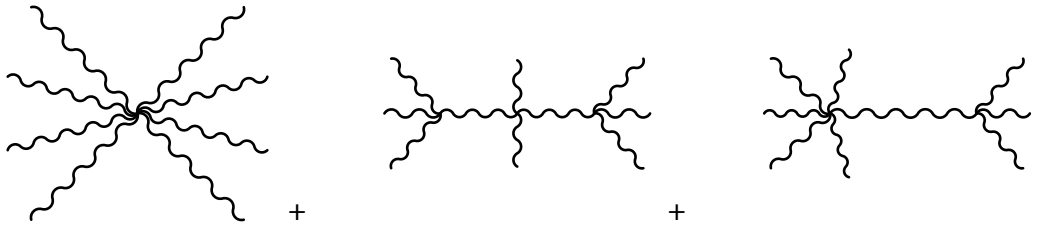
$$\begin{aligned} C^{3,3} &= -\frac{1}{b^4} \times \frac{4}{8} \langle 1\, 2\rangle^2 [5\, 6]^2 \frac{k_1\{F_3^{-}, F_4^{+}\}k_2}{(k_1+k_2+k_4)^2} - \text{ all inequivalent permutation of} \\ &\quad (1,2,3)\text{and}(4,5,6) \\ &= -\frac{1}{b^4} \times \frac{1}{2} \times \frac{1}{4} \frac{1}{(k_1+k_2+k_4)^2} \langle 13\rangle [34] \langle 42\rangle [21] - \text{ all inequivalent permutation.} \end{aligned} \tag{7.6}$$

This is equal with Boels et al after replacing $\frac{1}{b} = \frac{\pi\alpha'}{2}$ and some changes in notation.

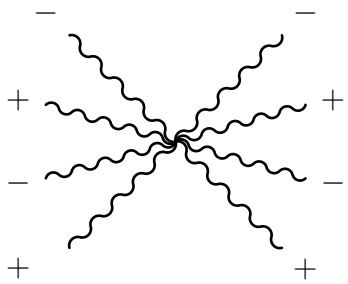
$$\begin{aligned} C^{3,3} &= -(\frac{\pi\alpha'}{2})^4 \times (\frac{1}{2})^3 \frac{1}{(k_1+k_2+k_4)^2} \langle 13\rangle [34] \langle 42\rangle [21] - \text{ all inequivalent permutation} \\ &= -\frac{(\pi\alpha')^4}{2^7} \frac{1}{(k_1+k_2+k_4)^2} \langle 13\rangle [34] \langle 42\rangle [21] - \text{ all inequivalent permutation.} \end{aligned} \tag{7.7}$$

7.3 8-Points, the Middle Amplitude

To compute the middle amplitude of 8-points, we have to consider three types of diagrams.



7.3.1 Main vertex



In our formalism it is easily calculated from the general formula which is given in eq.(6.8).

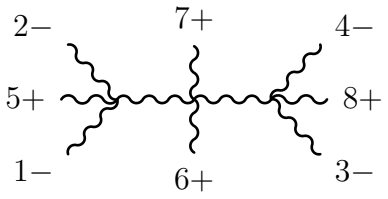
$$V^{4,4} = L_{22}\, s_-^2 s_+^2 = (-)^4 L_{22} \chi_4^+ \chi_4^- = 6\chi_4^+ \chi_4^- \tag{7.8}$$

$L_{22} = 6$.

7.3.2 Diagram A, Diagram 1

We call the reducible diagram with three branches as Diagram A. We have two types of this diagram which we denote by Diagram one and two.

The diagram type one, $C_{A1}^{4,4}$, is



$$C_{A1}^{4,4} = \frac{1}{3!}(L_{11})^3 s_{1-}s_{1+}s_{2-}s_{2+}s_{3-}s_{3+} + \Big(+ \leftrightarrow -\Big) \tag{7.9}$$

We have two passive s_- :

$$s_{1-}s_{3-} \to \chi_4^+ = \frac{1}{2}\Big(\langle 1,2\rangle^2\langle 3,4\rangle^2 + \text{ all inequivalent permutation}\Big) \tag{7.10}$$

$$s_{1+}s_{2-}s_{3+} \to (\frac{-1}{4})^3 \times 2^3 \sum_{i_3=1}^{4}\sum_{j_5,j_8=5}^{8} F_{j_5}^{+\alpha\beta}\langle f_{j_5\alpha\beta}^{+}\, f_{i_3\rho\sigma}^{-}\rangle\langle f_{i_3}^{-\rho\sigma}\, f_{j_8\mu\nu}^{+}\rangle F_{j_8}^{+\mu\nu}, \tag{7.11}$$

As a special case of (6.18), we can have

$$F_{j_5}^{+\alpha\beta}\langle f_{j_5\alpha\beta}^{+}\, f_{i_3\rho\sigma}^{-}\rangle f_{i_3}^{-\rho\sigma} = -2\hat{k_I}\{F_{j_5}^{+}, f_{i_3}^{-}\}\hat{k_I} \tag{7.12}$$

This leads to

$$s_{1+}s_{2-}s_{3+} \to (\frac{-1}{4})^3 \times 2^3 \times -4\hat{k_{I\alpha}}F_{j_5}^{+\alpha\beta}\langle f_{i_3\beta\gamma}^{-} f_{j_8\mu\nu}^{+}\rangle F_{j_8}^{+\mu\nu}\hat{k_I^{\gamma}}, \tag{7.13}$$

$$
\begin{aligned}
&\hat{k}_{I\alpha}F^{+\alpha\beta}_{j_5}\Big(\langle f^-_{i_3\beta\gamma}f^+_{j_8\mu\nu}\rangle\Big)F^{+\mu\nu}_{j_8}\hat{k}^{\gamma}_{I}\\
&=\frac{1}{4}\hat{k}_{I\alpha}F^{+\alpha\beta}_{j_5}\Bigg(-\eta^{\mu[\beta}\eta^{\gamma]\nu}-2\hat{k}^{[\beta}_{J}\eta^{\gamma][\mu}\hat{k}^{\nu]}_{J}+i\hat{k}_{J\rho}\Big(\varepsilon^{\beta\gamma\rho[\mu}\hat{k}^{\nu]}_{J}-\varepsilon^{\mu\nu\rho[\beta}\hat{k}^{\gamma]}_{J}\Big)\Bigg)F^{+\mu\nu}_{j_8}\hat{k}^{\gamma}_{I}\\
&=\frac{1}{4}\Bigg(-\hat{k_{I\alpha}}F^{+\alpha\mu}_{j_5}F^{+\mu\gamma}_{j_8}\hat{k}^{\gamma}_{I}+\hat{k_{I\alpha}}F^{+\alpha\nu}_{j_5}F^{+\gamma\nu}_{j_8}\hat{k}^{\gamma}_{I}\\
&-2\hat{k_{I\alpha}}\hat{k_{J\beta}}\hat{k_{J\nu}}F^{+\alpha\beta}_{j_5}F^{+\gamma\nu}_{j_8}\hat{k}^{\gamma}_{I}+2\hat{k_{I\alpha}}\hat{k_{J\gamma}}\hat{k_{J\nu}}F^{+\alpha\mu}_{j_5}F^{+\mu\nu}_{j_8}\hat{k}^{\gamma}_{I}\\
&+2\hat{k_{I\alpha}}\hat{k_{J\beta}}\hat{k_{J\mu}}F^{+\alpha\beta}_{j_5}F^{+\mu\nu}_{j_8}\hat{k}^{\nu}_{I}-2\hat{k_{I\alpha}}\hat{k_{J\gamma}}\hat{k_{J\mu}}F^{+\alpha\beta}_{j_5}F^{+\mu\beta}_{j_8}\hat{k}^{\gamma}_{I}\\
&+i\hat{k}_{I\alpha}F^{+\alpha\beta}_{j_5}\hat{k}_{J\rho}\varepsilon^{\beta\gamma\rho\mu}\hat{k}^{\nu}_{J}F^{+\mu\nu}_{j_8}\hat{k}^{\gamma}_{I}-i\hat{k_{I\alpha}}F^{+\alpha\beta}_{j_5}\hat{k}_{J\rho}\varepsilon^{\beta\gamma\rho\nu}\hat{k}^{\mu}_{J}F^{+\mu\nu}_{j_8}\hat{k}^{\gamma}_{I}\\
&-i\hat{k_{I\alpha}}F^{+\alpha\beta}_{j_5}\hat{k}_{J\rho}\varepsilon^{\mu\nu\rho\beta}\hat{k}^{\gamma}_{J}F^{+\mu\nu}_{j_8}\hat{k}^{\gamma}_{I}+i\hat{k_{I\alpha}}F^{+\alpha\beta}_{j_5}\hat{k}_{J\rho}\varepsilon^{\mu\nu\rho\gamma}\hat{k}^{\beta}_{J}F^{+\mu\nu}_{j_8}\hat{k}^{\gamma}_{I}\Bigg).
\end{aligned}
\tag{7.14}
$$

To continue our calculation we need to deal with the imaginary terms. If we use different identities for $\tilde{F}$:

$$
\begin{aligned}
\tilde{F}^{\pm\mu\nu}&=\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F^{\pm}_{\alpha\beta},\\
\tilde{F}^{\pm\mu\nu}&=\mp iF^{\pm\mu\nu}.
\end{aligned}
\tag{7.15}
$$

Then we obtain:

$$
\begin{aligned}
&\hat{k}_{I\alpha}F^{+\alpha\beta}_{j_5}\Big(\langle f_{i_3-\beta\gamma}f^{+}_{j_8\mu\nu}\rangle\Big)F^{+\mu\nu}_{j_8}\hat{k}^{\gamma}_{I}\\
&=\frac{1}{4}\Bigg(-2(\hat{k}_I F^{+}_{j_5}F^{+}_{j_8}\hat{k}_I)-4(\hat{k}_I F^{+}_{j_5}\hat{k}_J)(\hat{k}_I F^{+}_{j_8}\hat{k}_J)+4(\hat{k}_I\cdot\hat{k}_J)(\hat{k}_I F^{+}_{j_5}F^{+}_{j_8}\hat{k}_J)\\
&-2\tilde{F}^{+\alpha\beta}_{j_5}\varepsilon^{\beta\gamma\rho\mu}\hat{k}_{I\alpha}\hat{k}_{J\rho}\hat{k}^{\nu}_{J}F^{+\mu\nu}_{j_8}\hat{k^{\gamma}_{I}}-2i\hat{k_{I\alpha}}F^{+\alpha\beta}_{j_5}\hat{k}_{J\rho}\hat{k}^{\gamma}_{J}\tilde{F}^{+\rho\beta}_{j_8}\hat{k^{\gamma}_{I}}+2i\hat{k_{I\alpha}}F^{+\alpha\beta}_{j_5}\hat{k}_{J\rho}\hat{k}^{\beta}_{J}\tilde{F}^{+\rho\gamma}_{j_8}\hat{k^{\gamma}_{I}}\Bigg)\\
&=\frac{1}{4}\Bigg(-2(\hat{k}_I F^{+}_{j_5}F^{+}_{j_8}\hat{k}_I)-4(\hat{k}_I F^{+}_{j_5}\hat{k}_J)(\hat{k}_I F^{+}_{j_8}\hat{k}_J)+4(\hat{k}_I\cdot\hat{k}_J)(\hat{k}_I F^{+}_{j_5}F^{+}_{j_8}\hat{k}_J)\\
&-2\times\frac{1}{2}F^{+\lambda\sigma}_{j_5}\varepsilon^{\alpha\beta\lambda\sigma}\varepsilon^{\beta\gamma\rho\mu}\hat{k}_{I\alpha}\hat{k}_{J\rho}\hat{k}^{\nu}_{J}F^{+\mu\nu}_{j_8}\hat{k^{\gamma}_{I}}+2\hat{k_{I\alpha}}F^{+\alpha\beta}_{j_5}\hat{k}_{J\rho}\hat{k^{\gamma}_{J}}F^{+\rho\beta}_{j_8}\hat{k^{\gamma}_{I}}-2\hat{k_{I\alpha}}F^{+\alpha\beta}_{j_5}\hat{k}_{J\rho}\hat{k}^{\beta}_{J}F^{+\rho\gamma}_{j_8}\hat{k^{\gamma}_{I}}\Bigg)\\
&=\frac{1}{4}\Bigg(-2(\hat{k}_I F^{+}_{j_5}F^{+}_{j_8}\hat{k}_I)-2(\hat{k}_I F^{+}_{j_5}\hat{k}_J)(\hat{k}_I F^{+}_{j_8}\hat{k}_J)+2(\hat{k}_I\cdot\hat{k}_J)(\hat{k}_I F^{+}_{j_5}F^{+}_{j_8}\hat{k}_J)\\
&-2\times\frac{1}{2}F^{+\lambda\sigma}_{j_5}\varepsilon^{\alpha\beta\lambda\sigma}\varepsilon^{\beta\gamma\rho\mu}\hat{k}_{I\alpha}\hat{k}_{J\rho}\hat{k}^{\nu}_{J}F^{+\mu\nu}_{j_8}\hat{k^{\gamma}_{I}}\Bigg).
\end{aligned}
\tag{7.16}
$$

We see that still we have two ε's in the last term:

$$
\begin{aligned}
&s_{1+}s_{2-}s_{3+}\rightarrow(\frac{-1}{4})^3\times 2^3\times -4\times\frac{1}{4}\sum_{j_5,j_8=5}^{8}\Bigg(-2(\hat{k}_I F^{+}_{j_5}F^{+}_{j_8}\hat{k}_I)-2(\hat{k}_I F^{+}_{j_5}\hat{k}_J)(\hat{k}_I F^{+}_{j_8}\hat{k}_J)\\
&+2(\hat{k}_I\cdot\hat{k}_J)(\hat{k}_I F^{+}_{j_5}F^{+}_{j_8}\hat{k}_J)+2\times\frac{1}{2}F^{+\lambda\sigma}_{j_5}\varepsilon^{\beta\alpha\lambda\sigma}\varepsilon^{\beta\gamma\rho\mu}\hat{k}_{I\alpha}\hat{k}_{J\rho}\hat{k}^{\nu}_{J}F^{+\mu\nu}_{j_8}\hat{k^{\gamma}_{I}}\Bigg).
\end{aligned}
\tag{7.17}
$$

Note that the two ε's in that term share in the index β therefore we could use this identity

$$
\varepsilon_{\gamma\rho\mu\beta}\,\varepsilon^{\alpha\lambda\sigma\beta}=\eta^{\sigma}_{\gamma}\,\eta^{\lambda}_{\rho}\,\eta^{\alpha}_{\mu}-\eta^{\lambda}_{\gamma}\,\eta^{\sigma}_{\rho}\,\eta^{\alpha}_{\mu}-\eta^{\sigma}_{\gamma}\,\eta^{\alpha}_{\rho}\,\eta^{\lambda}_{\mu}+\eta^{\alpha}_{\gamma}\,\eta^{\sigma}_{\rho}\,\eta^{\lambda}_{\mu}+\eta^{\lambda}_{\gamma}\,\eta^{\alpha}_{\rho}\,\eta^{\sigma}_{\mu}-\eta^{\alpha}_{\gamma}\,\eta^{\lambda}_{\rho}\,\eta^{\sigma}_{\mu}.
\tag{7.18}
$$

We obtain

$$
\begin{aligned}
&F^{+\lambda\sigma}_{j_5}\varepsilon^{\beta\alpha\lambda\sigma}\varepsilon^{\beta\gamma\rho\mu}\hat{k}_{I\alpha}\hat{k}_{J\rho}\hat{k}^{\nu}_{J}F^{+\mu\nu}_{j_8}\hat{k^{\gamma}_{I}} \\
&= F^{+\lambda\sigma}_{j_5}\hat{k}_{I\alpha}\hat{k}_{J\rho}\hat{k}^{\nu}_{J}F^{+\mu\nu}_{j_8}\hat{k^{\gamma}_{I}}\Big(\eta^{\sigma}_{\gamma}\,\eta^{\lambda}_{\rho}\,\eta^{\alpha}_{\mu}-\eta^{\lambda}_{\gamma}\,\eta^{\sigma}_{\rho}\,\eta^{\alpha}_{\mu}-\eta^{\sigma}_{\gamma}\,\eta^{\alpha}_{\rho}\,\eta^{\lambda}_{\mu}+\eta^{\alpha}_{\gamma}\,\eta^{\sigma}_{\rho}\,\eta^{\lambda}_{\mu}+\eta^{\lambda}_{\gamma}\,\eta^{\alpha}_{\rho}\,\eta^{\sigma}_{\mu}-\eta^{\alpha}_{\gamma}\,\eta^{\lambda}_{\rho}\,\eta^{\sigma}_{\mu}\Big) \\
&= 2F^{+\rho\gamma}_{j_5}\hat{k}_{I\mu}\hat{k}_{J\rho}\hat{k}^{\nu}_{J}F^{+\mu\nu}_{j_8}\hat{k^{\gamma}_{I}}-2F^{+\mu\gamma}_{j_5}\hat{k}_{I\rho}\hat{k}_{J\rho}\hat{k}^{\nu}_{J}F^{+\mu\nu}_{j_8}\hat{k^{\gamma}_{I}}+2F^{+\mu\rho}_{j_5}\hat{k}_{I\gamma}\hat{k}_{J\rho}\hat{k}^{\nu}_{J}F^{+\mu\nu}_{j_8}\hat{k^{\gamma}_{I}} \\
&= -2(\hat{k}_I F^{+}_{j_5}\hat{k}_J)(\hat{k}_I F^{+}_{j_8}\hat{k}_J)+2(\hat{k}_I\cdot\hat{k}_J)(\hat{k}_I F^{+}_{j_5}F^{+}_{j_8}\hat{k}_J)-2(\hat{k}_I F^{+}_{j_5}F^{+}_{j_8}\hat{k}_I).
\end{aligned}
\tag{7.19}
$$

Finally

$$
\begin{aligned}
&s_{1+}s_{2-}s_{3+}\to(\frac{-1}{4})^3\times 2^3\times -4\times\frac{1}{4}\sum_{j_5,j_8=5}^{8}\Bigg(-4(\hat{k}_I F^{+}_{j_5}F^{+}_{j_8}\hat{k}_I)-4(\hat{k}_I F^{+}_{j_5}\hat{k}_J)(\hat{k}_I F^{+}_{j_8}\hat{k}_J) \\
&+4(\hat{k}_I\cdot\hat{k}_J)(\hat{k}_I F^{+}_{j_5}F^{+}_{j_8}\hat{k}_J)\Bigg) \\
&=\frac{1}{2}\Big(-(\hat{k}_I F^{+}_{j_5}F^{+}_{j_8}\hat{k}_I)-(\hat{k}_I F^{+}_{j_5}\hat{k}_J)(\hat{k}_I F^{+}_{j_8}\hat{k}_J)+(\hat{k}_I\cdot\hat{k}_J)(\hat{k}_I F^{+}_{j_5}F^{+}_{j_8}\hat{k}_J)\Big).
\end{aligned}
\tag{7.20}
$$

Therefore the amplitude for this diagram is

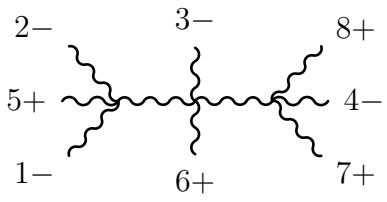
$$
\begin{aligned}
&C^{4,4}_{A1}=\frac{1}{3!}2^3\chi^{+}_{4}\times\frac{1}{2}\Big(-(\hat{k}_I F^{+}_{j_5}F^{+}_{j_8}\hat{k}_I)-(\hat{k}_I F^{+}_{j_5}\hat{k}_J)(\hat{k}_I F^{+}_{j_8}\hat{k}_J)+(\hat{k}_I\cdot\hat{k}_J)(\hat{k}_I F^{+}_{j_5}F^{+}_{j_8}\hat{k}_J) \\
&+\text{all inequivalent permutation}\Big)+\Big(+\leftrightarrow-\Big)
\end{aligned}
\tag{7.21}
$$

or

$$
\begin{aligned}
&C^{4,4}_{A1}=\frac{4}{3!}\chi^{+}_{4}\Big(-(\hat{k}_I F^{+}_{j_5}F^{+}_{j_8}\hat{k}_I)-(\hat{k}_I F^{+}_{j_5}\hat{k}_J)(\hat{k}_I F^{+}_{j_8}\hat{k}_J)+(\hat{k}_I\cdot\hat{k}_J)(\hat{k}_I F^{+}_{j_5}F^{+}_{j_8}\hat{k}_J) \\
&+\text{all inequivalent permutation}\Big) \\
&+\frac{4}{3!}\chi^{-}_{4}\Big(-(\hat{k}_I F^{-}_{i_1}F^{-}_{i_4}\hat{k}_I)-(\hat{k}_I F^{-}_{i_1}\hat{k}_J)(\hat{k}_I F^{-}_{i_4}\hat{k}_J)+(\hat{k}_I\cdot\hat{k}_J)(\hat{k}_I F^{-}_{i_1}F^{-}_{i_4}\hat{k}_J) \\
&+\text{all inequivalent permutation}\Big).
\end{aligned}
\tag{7.22}
$$

7.3.3 Diagram A, Diagram 2

The second type of diagram A, $C_{A2}^{4,4}$, is



Note that in this diagram we have two choices. We can choose f^+ to the left side of the middle branch and f^- to the right ($C_{A21}^{4,4}$) or vice versa ($C_{A22}^{4,4}$).

Therefore the complete amplitude of this diagram is

$$C_{A2}^{4,4} = C_{A21}^{4,4} + C_{A22}^{4,4}$$

First let us give the f^+ to the left :

$$C_{A21}^{4,4} = 2 \times \frac{1}{b^4}\frac{1}{3!}(L_{11})^3 s_{1-}s_{1+}s_{2-}s_{2+}s_{3-}s_{3+}, \tag{7.23}$$

The passive s's give

$$s_{1-}s_{3+} \to (\frac{-1}{4})^2 F_{i_1}^{-\mu\nu} F_{i_2\mu\nu}^{-} F_{j_7}^{+\alpha\beta} F_{j_8\alpha\beta}^{+} = \frac{1}{16}\langle i_1\, i_2\rangle^2 [j_7, j_8]^2, \tag{7.24}$$

For the active ones we have

$$s_{1+}s_{2+} \to (\frac{-1}{4})^2 \times 2^2 F_{j_5\mu\nu}^{+} \langle f_{j_5}^{\mu\nu+}\, f_{j_6\alpha\beta}^{+}\rangle F_{j_6}^{\alpha\beta+} = \frac{1}{4}\langle j_5, j_6\rangle^2, \tag{7.25}$$

$$s_{1-}s_{2-} \to (\frac{-1}{4})^2 \times 2^2 F_{i_3\mu\nu}^{-} \langle f_{i_3}^{\mu\nu-}\, f_{i_4\alpha\beta}^{-}\rangle F_{i_4}^{\alpha\beta-} = \frac{1}{4}\langle i_3, i_4\rangle^2, \tag{7.26}$$

$$C_{A21}^{4,4} = 2 \times \frac{1}{b^4}\frac{1}{3!}(L_{11})^3 \times (\frac{1}{16})^2 \sum_{i_1,i_2,i_3,i_4=1}^{4} \sum_{j_5,j_6,j_7,j_8=5}^{8} \langle i_1\, i_2\rangle^2 [j_7, j_8]^2 \langle j_5, j_6\rangle^2 \langle i_3, i_4\rangle^2, \tag{7.27}$$

$$C_{A21}^{4,4} = 2 \times \frac{1}{b^4}\frac{1}{3!} \times (\frac{1}{2})^5 \times 2^4 \langle 1\, 2\rangle^2 [7, 8]^2 \langle 5, 6\rangle^2 \langle 3, 4\rangle^2 + \text{ all inequivalent permutation}, \tag{7.28}$$

$$C_{A21}^{4,4} = \frac{1}{b^4}\frac{1}{3!} \langle 1\, 2\rangle^2 [7, 8]^2 \langle 5, 6\rangle^2 \langle 3, 4\rangle^2 + \text{ all inequivalent permutation}. \tag{7.29}$$

Now we start to calculate the second type of this diagram. If we give the f^- to the left, then

$$C_{A22}^{4,4} = 2 \times \frac{1}{b^4}\frac{1}{3!}(L_{11})^3 s_{1-} s_{1+} s_{2-} s_{2+} s_{3-} s_{3+}, \tag{7.30}$$

For passive s's we obtain

$$s_{1-} s_{3+} \to (\frac{-1}{4})^2 F_{i_1}^{-\mu\nu} F_{i_2\mu\nu}^{-} F_{j_7}^{+\alpha\beta} F_{j_8\alpha\beta}^{+} = \frac{1}{16} \langle i_1\, i_2\rangle^2 [j_7, j_8]^2, \tag{7.31}$$

We have two contractions for the active s's

$$s_{1+} s_{2-} \to (\frac{-1}{4})^2 \times 2^2 F_{j_5\mu\nu}^{+} \langle f_{j_5}^{\mu\nu+}\, f_{i_3\alpha\beta}^{-} \rangle F_{i_3}^{\alpha\beta-} = -\frac{1}{2} \frac{(k_{i_1} + k_{i_2} + k_{j_5})\{F_{j_5}^{+}, F_{i_3}^{-}\}(k_{i_1} + k_{i_2} + k_{j_5})}{(k_1 + k_2 + k_5)^2}, \tag{7.32}$$

$$s_{2+} s_{3-} \to (\frac{-1}{4})^2 \times 2^2 F_{j_6\mu\nu}^{+} \langle f_{j_6}^{\mu\nu+}\, f_{i_4\alpha\beta}^{-} \rangle F_{i_4}^{\alpha\beta-} = -\frac{1}{2} \frac{(k_{j_7} + k_{j_8} + k_{i_4})\{F_{j_6}^{+}, F_{i_4}^{-}\}(k_{j_7} + k_{j_8} + k_{i_4})}{(k_7 + k_8 + k_4)^2}, \tag{7.33}$$

$$C^{4,4}_{A22} = 2 \times \frac{1}{b^4}\frac{1}{3!}(L_{11})^3 \times (\frac{1}{2})^6 \sum_{i_1,i_2,i_3,i_4=1}^{4} \sum_{j_5,j_6,j_7,j_8=5}^{8} \langle i_1\, i_2\rangle^2 [j_7, j_8]^2$$
$$\times \frac{(k_{i_1} + k_{i_2} + k_{j_5})\{F^+_{j_5}, F^-_{i_3}\}(k_{i_1} + k_{i_2} + k_{j_5})}{(k_1 + k_2 + k_5)^2} \times \frac{(k_{j_7} + k_{j_8} + k_{i_4})\{F^+_{j_6}, F^-_{i_4}\}(k_{j_7} + k_{j_8} + k_{i_4})}{(k_7 + k_8 + k_4)^2}, \tag{7.34}$$

$$C^{4,4}_{A22} = \frac{1}{b^4}\frac{1}{3!} \times (\frac{1}{2})^2 \times 4\langle 1\, 2\rangle^2 [7, 8]^2 \frac{k_1\{F^-_3, F^+_5\}k_2}{(k_1 + k_2 + k_5)^2} \times \frac{k_7\{F^-_4, F^+_6\}k_8}{(k_7 + k_8 + k_4)^2}$$
$$+ \text{all inequivalent permutation}, \tag{7.35}$$

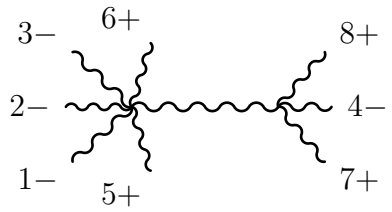
$$C^{4,4}_{A22} = \frac{1}{b^4}\frac{1}{3!} \times \langle 1\, 2\rangle^2 [7, 8]^2 \times \frac{1}{16}\frac{\langle 13\rangle[35]\langle 52\rangle[21]}{(k_1 + k_2 + k_5)^2} \times \frac{\langle 74\rangle[46]\langle 68\rangle[87]}{(k_7 + k_8 + k_4)^2}$$
$$+ \text{all inequivalent permutation}. \tag{7.36}$$

The total diagram type 2 is

$$C^{4,4}_{A2} = C^{4,4}_{A21} + C^{4,4}_{A22}$$
$$= \frac{1}{b^4}\frac{1}{3!}\Big(\langle 1\, 2\rangle^2 [7, 8]^2 \langle 5, 6\rangle^2 \langle 3, 4\rangle^2 + \frac{1}{16}\langle 1\, 2\rangle^2 [7, 8]^2 \frac{\langle 13\rangle[35]\langle 52\rangle[21]}{(k_1 + k_2 + k_5)^2} \times \frac{\langle 74\rangle[46]\langle 68\rangle[87]}{(k_7 + k_8 + k_4)^2}$$
$$+ \text{all inequivalent permutation}\Big). \tag{7.37}$$

7.3.4 Diagram B

For the 8-point amplitude, we call the reducible diagram with two branches diagram B.



$$C_B^{4,4} = 2 \times \frac{1}{b^4}\frac{1}{3!} L_{21} \times L_{11} s_{1-} s_{2-} s_{2+} s_{3-} s_{3+} + \Big(+ \leftrightarrow - \Big), \tag{7.38}$$

$$2 \times s_{1-} \to -2\chi_2^- = -\langle i_1, i_2 \rangle^2, \tag{7.39}$$

$$s_{2+} s_{3+} \to \chi_4^+ = \frac{1}{2}\Big([j_5, j_6]^2 [j_7, j_8]^2 + \text{ all inequivalent permutation} \Big), \tag{7.40}$$

$$s_{2-} s_{3-} \to 2^2 \times (\frac{-1}{4})^2 F_{i_3}^{-\mu\nu} \langle f_{i_3 \mu\nu}^- f_{i_4 \alpha\beta}^- \rangle F_{i_4}^{-\alpha\beta} = \frac{1}{4} \langle i_3, i_4 \rangle^2, \tag{7.41}$$

$$\begin{aligned} C_B^{4,4} = &-2 \times \frac{1}{b^4}\frac{1}{3!} L_{21} \times L_{11} \times 2\langle i_1, i_2 \rangle^2 \times \frac{1}{2}[j_5, j_6]^2 [j_7, j_8]^2 \times 2\frac{1}{4}\langle i_3, i_4 \rangle^2 \\ &+ \text{ all inequivalent permutation} + \Big(+ \leftrightarrow - \Big), \end{aligned} \tag{7.42}$$

$$C_B^{4,4} = -\frac{1}{b^4}\frac{4}{3!}\langle 1,2\rangle^2[5,6]^2[7,8]^2\langle 3,4\rangle^2 + \text{ all inequivalent permutation} + \Big(+ \leftrightarrow -\Big), \tag{7.43}$$

$$C_B^{4,4} = -2 \times \frac{1}{b^4}\frac{4}{3!}\langle 1,2\rangle^2[5,6]^2[7,8]^2\langle 3,4\rangle^2 + \text{ all inequivalent permutation}. \tag{7.44}$$

Total amplitude for the middle 8 point diagram is

$$A^{4,4} = V^{4,4} + C_A^{4,4} + C_B^{4,4}. \tag{7.45}$$

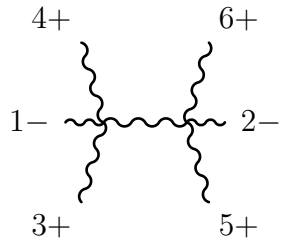
We compute this amplitude for the first time.

Chapter 8

Example of a vanishing amplitude

Finally, let us also give an example for the vanishing of the non-middle amplitude, as predicted by Rosly and Selivanov[18].

This amplitude is the MHV amplitude at 6-points. We show that this amplitude vanishes.



The basic vertex which we call V, for this diagram is

$$V^{2,4} = -L_{12}\chi_2^-\chi_4^+ = -2\chi_2^-\chi_4^+, \tag{8.1}$$

which comes from (6.33)

The reducible diagram is

$$C^{2,4} = L_{11}^2 s_{1-}s_{1+}s_{2-}s_{2+}, \tag{8.2}$$

$$s_{1+}s_{2+} \to \chi_4^+,$$

(8.3)

$$s_{1-}s_{2-} \to (\frac{-1}{4})^2 F^-_{\alpha\beta}\langle f^{-\alpha\beta}\, f^-_{\mu\nu}\rangle F^{-\mu\nu} \times 2 \times 2. \tag{8.4}$$

The factor 4 comes from the choices which we have for the internal f's.

$$\begin{aligned} 4 \times F^-_{\alpha\beta}\langle f^{-\alpha\beta}\, f^-_{\mu\nu}\rangle F^{-\mu\nu} &= \frac{1}{k^2}\Big(2k^2 F^-_{1\mu\nu}F_2^{-\mu\nu} + 4ik\tilde{F}_1^- F_2^- k + 4ikF_1^-\tilde{F}_2^- k\Big) \\ &= \frac{1}{k^2}\Big(2k^2 F^-_{1\mu\nu}F_2^{-\mu\nu} - 8kF_1^- F_2^- k\Big), \end{aligned} \tag{8.5}$$

$$2k^2 F^-_{1\mu\nu}F_2^{-\mu\nu} = -k^2 tr\{F_1^- F_2^-\} = \frac{1}{2}k^2 < 1\,2 >^2 tr(\eta_{\mu\nu}) = 2 < 1\,2 >^2 \tag{8.6}$$

$$-8kF_1^- F_2^- k = -4k\{F_1^-, F_2^-\}k = -4k^2 \times (\frac{-1}{2}) < 1\,2 >^2 . \tag{8.7}$$

Finally we have

$$4 \times F^-_{\alpha\beta}\langle f^{-\alpha\beta}\, f^-_{\mu\nu}\rangle F^{-\mu\nu} = 4k^2 < 1\,2 >^2, \tag{8.8}$$

$$C^{2,4} = L_{11}^2 \chi_4^+ (\frac{-1}{4})^2 \times 4 < 1\,2 >^2 = 2\chi_4^+\chi_2^-, \tag{8.9}$$

where $L_{11} = 2$

$$A^{2,4} = V^{2,4} + C^{2,4} = -2\chi_4^+\chi_2^- + 2\chi_4^+\chi_2^- = 0. \tag{8.10}$$

Chapter 9

Summary of part II

After calculating the 8-point diagrams, we now understand the structure of the BI amplitudes sufficiently well to build an efficient formalism for the calculation of any tree level diagram. Our formalism has the power of calculating any diagram at tree level.

At tree level, we can see that any diagram can be decomposed into a ***core*** and an ***ornament***. Here the ornament is defined as the product of all the passive factors of $s^{\pm}$ (that are not involved in the contractions defined by the propagators of the graph). The remained is called core, and it factorizes into a product of ***closed chains***. Here a closed chain is defined by

$$F^{\pm}\langle ff\rangle ...\langle ff\rangle F^{\pm} \tag{9.1}$$

Similarly, we define an open chain by ***open chain***,

$$f^{\pm}\langle ff\rangle ...\langle ff\rangle f^{\pm} \tag{9.2}$$

and a ***half-open chain*** by

$$f^{\pm}\langle ff\rangle ...\langle ff\rangle F^{\pm} \tag{9.3}$$

Also we have to types of chain, odd or even. The ***even chains*** are end with the same "helicity" F's(or f's) and the odd ones with F's(f's) of opposite sign.

Also we have two kind of chains, we call these chains "**alternating**" chain and "**non alternating**" one.

The alternating chains are made from a contraction of some active s^+ and s^-, but do not includes $\langle f^+ f^+\rangle$ or $\langle f^- f^-\rangle$.

$$\langle f^+ f^-\rangle\langle f^- f^+\rangle\langle f^+ f^-\rangle\langle f^- f^+\rangle....\langle f^+ f^-\rangle\langle f^- f^+\rangle \tag{9.4}$$

A priori it is not obvious that a non alternating chain is reducible to an alternating one, but due to the eqs.(6.16, 6.17)we can easily show this equality. For example,

$$\langle f^+ f^-\rangle\langle f^- \underbrace{f^+\rangle\langle f^+ f^+\rangle\langle f^+}\, f^-\rangle....\langle f^+ f^-\rangle\langle f^- f^+\rangle \equiv \langle f^+ f^-\rangle\langle f^- f^+\rangle\langle f^+ f^-\rangle....\langle f^+ f^-\rangle\langle f^- f^+\rangle \tag{9.5}$$

Using these chains and the identities which we have seen in chapter 5, we can calculate any diagram at tree level.

At the end, let us use our experience with these low-point calculations to define a step-by-step procedure for the calculation of any tree level diagram in BI theory:

Step 1: Strip off any factors of s_i that are not involved in contractions.

Step 2: Factorize the remainder (the core) of the graph into closed chains.

Step 3: Use eqs.(6.16, 6.17) to reduce all chains to alternating ones.

Step 4: Use momentum conservation to write all internal (off-shell) momenta in terms of external (on-shell) ones.

Step 5: With all momenta now on-shell, write the alternating chains in terms of spinor helicity variables.

Step 6: Sum over permutations over the external legs (separately for the $+$ and the $-$ ones).

Step 7: Write the result in terms of the variables χ (as far as is possible).

Let us mention also that steps one to three are not specific to the tree level and can be applied to any diagram in BI theory.

For the first time we will show these calculations, including also first result one-loop BI amplitude, in our paper[38].

Appendices

Appendix A

Veselov Discretization

Marsden et al. derive Vesselov's discrete mechanics from a variational approach. The cornerstone is Vesselov's model for the discrete tangent bundle $Q \times Q$, heuristically for each time interval δt, the point (q_F, q_I) has information regarding the position and velocity of the particle during this lapse of time, then the discrete Lagrangian density can be defined as a smooth map of the initial and final copies of Q, $L : Q \times Q = (q_F, q_I) \mapsto \mathbb{R}$ and the discrete action is

$$S(q) = \sum_{k=0}^{n} L(q_{k+1}, q_k), \tag{A.1}$$

where $q_k + 1$ is the final point and the q_k is the initial one.
We may vary the discrete action with arbitrary, not fixed, endpoints and has the discrete analogue of the Lagrange 1-form and the Euler-Lagrange equations.

$$\begin{aligned} dS(q_0, ..., q_n) \cdot (\vec{\delta} q_0, ..., \vec{\delta} q_n)) = \sum_{k=1}^{n-1} \big(\frac{\partial L}{\partial q_1}(q_k, q_{k-1}) + \frac{\partial L}{\partial q_0}(q_{k+1}, q_k) \big) \delta q_k \\ + \frac{\partial L}{\partial q_0}(q_1, q_0) \delta q_0 + \frac{\partial L}{\partial q_1}(q_n, q_{n-1}) \delta q_n. \end{aligned} \tag{A.2}$$

By definition we have the discrete Euler-Lagrange equations

$$\frac{\partial L(q_1,q_0)}{\partial q_1}+\frac{\partial L(q_2,q_1)}{\partial q_0}=0, \tag{A.3}$$

then in the space of solutions the last two terms in eq.(A.2) which are the boundary terms are defined as the discrete analogue of Lagrange 1-form

$$\theta_L^-(q_1,q_0)\cdot(\delta q_1,\delta q_0)\equiv\frac{\partial L}{\partial q_0}(q_1,q_0)\delta q_0,$$

and

$$\theta_L^+(q_1,q_0)\cdot(\delta q_1,\delta q_0)\equiv\frac{\partial L}{\partial q_1}(q_1,q_0)\delta q_1. \tag{A.4}$$

If we use the solutions of the equations of motion to define a discrete flow

$$F:Q\times Q\longrightarrow Q\times Q$$

$$(q_1,q_0)\mapsto(q_2,q_1), \tag{A.5}$$

and if we use the initial condition to parameterize solutions, we can define

$$S:Q\times Q\to\mathbb{R},$$

by

$$S(q_1,q_0)=S(q(q_1,q_0)),$$

then eq.(A.2) becomes

$$dS(q_1,q_0)=\theta_L^-(q_1,q_0)+(F^n)^*\big(\theta_L^+(q_n,q_{n-1})\big). \tag{A.6}$$

By using the fact that $ddS=0$, eq.(A.6) becomes

$$(F^n)^*(d\theta_L^+)=-d\theta_L^-. \tag{A.7}$$

Note that

$$\theta_L^-+\theta_L^+=dL,$$

thus

$$d\theta_L^- + d\theta_L^+ = 0. \tag{A.8}$$

If we define the symplectic form as

$$\omega_L \equiv \theta_L^- = -d\theta_L^+, \tag{A.9}$$

eq.(A.7) becomes

$$(F^n)^*\omega_L = \omega_L, \tag{A.10}$$

which is exactly the symplecticity condition of the flow.
Marsden et al. also show that if L is G-invariant and thus the map F is G-equivariant in the space of solutions then defining a discrete momentum

$$J_\xi \equiv \xi_{Q\times Q} \lrcorner\, \theta_L^+, \tag{A.11}$$

leads to a discrete version of Noether's theorem.

$$J_\xi = F^*(J_\xi)$$

Appendix B

Comparison of Our Model with the Marsden et al's model

Before starting to compare these two frameworks for the field theory, it is good to have a review of Marsden et al's framework in 2 dimensions.

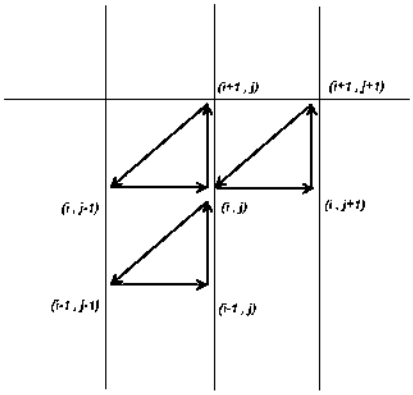


Figure B.1: *The triangles which share vertex (i , j)*

Fig.B.1 shows the triangulation of the space in their framework. *(Note that this is not a triangulation since there are many holes in it and this is not allowed topo-*

logically in triangulation. Working with these triangles is correct since there is one triangle per square which is the cell in this model.)

The discrete Lagrangian for the triangle $\triangle = \big((i,j),(i,j+1),(i+1,j+1)\big)$ is

$$L_{\triangle} = \frac{1}{2}\big(\frac{y_{ij+1}-y_{ij}}{h}\big)^2 - \frac{1}{2}\big(\frac{y_{i+1j+1}-y_{ij+1}}{k}\big)^2 + N\big(\frac{y_{i+1j+1}+y_{ij+1}+y_{ij}}{3}\big), \quad \text{(B.1)}$$

where y_{ij} is the element of Y over the point$(i,j) \in \triangle^{n+1}$.
Then the action in Marsden et. al.'s framework is summation over the Lagrangian of triangles in $\triangle^{n+1}$

$$S(\phi) = \sum_{\triangle < \triangle^{n+1}} L_{\triangle} hk \,.$$

Now lets start to compare two frameworks with writing the discrete action in our framework but with the potential of a cell as the average of all points in that cell.

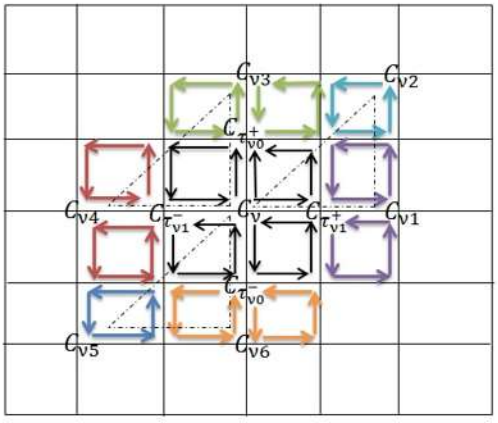


Figure B.2: $s(\nu,\sigma)$

Fig.B shows the cells in our discrete spacetime X^{disc}.

$$\begin{aligned} S(\phi) &= \sum_{\tau<\partial U}\Big(-\frac{1}{2}\big(\frac{\phi(C_{\tau_{\nu 0}})-\phi(C_\nu)}{k'}\big)^2+\frac{1}{2}\big(\frac{\phi(C_{\tau_{\nu 1}})-\phi(C_\nu)}{h'}\big)^2 \\ &\quad -N\big(\frac{\phi(C_\nu)+\phi(C_{\nu_1)}+\phi(C_{\nu_2})}{3}\big)\Big)h'k' \\ &\quad +\sum_{\nu<U^\circ}\Big(-\frac{1}{2}\big(\frac{\phi(C_{\tau^+_{\nu 0}})-\phi(C_\nu)}{k'}\big)^2-\frac{1}{2}\big(\frac{\phi(C_\nu)-\phi(C_{\tau^-_{\nu 0}})}{k'}\big)^2 \\ &\quad +\frac{1}{2}\big(\frac{\phi(C_{\tau^+_{\nu 1}})-\phi(C_\nu)}{h'}\big)^2+\frac{1}{2}\big(\frac{\phi(C_\nu)-\phi(C_{\tau^-_{\nu 1}})}{h'}\big)^2 \\ &\quad +4N\big(\frac{\phi(C_\nu)+\phi(C_{\nu_1)}+\phi(C_{\nu_2})}{3}\big)\Big)h'k'. \end{aligned} \tag{B.2}$$

By C_{τ_ν} we mean the center of $\tau<\nu$ in the boundary of U. Note that there is just one such ν and τ which can be τ^+ or τ^- .

In the last three equations, ν_1 and ν_2 are the neighbors of ν which are shown in fig.B.

Note that this action is written in the Veselov's interval so we choose U in a way that there is no C_ν in the boundary.

We can write the equations of motion, for this, we have to vary the action $S(\phi)$ respect to $\phi(\tau_0)$, $\phi(\tau_1)$ and $\phi(\nu)$'s. Then there are three sets of field equations.

$$\begin{aligned} &\sum_{\tau<\nu}\big(-\frac{\phi(C_\nu)-\phi(C_{\tau_{\nu 0}})}{k'^2}+\frac{\phi(C_\nu)-\phi(C_{\tau_{\nu 1}})}{h'^2}\big)+\frac{1}{3}N'\big(\frac{\phi(C_\nu)+\phi(C_{\nu_1)}+\phi(C_{\nu_2})}{3}\big) \\ &+\frac{1}{3}N'\big(\frac{\phi(C_{\nu_3})+\phi(C_\nu)+\phi(C_{\nu_4})}{3}\big)+\frac{1}{3}N'\big(\frac{\phi(C_{\nu_5})+\phi(C_{\nu_6})+\phi(C_\nu)}{3}\big) \\ =\ &-\frac{\phi(C_{\tau^-_{\nu 0}})-2\phi(C_\nu)+\phi(C_{\tau^+_{\nu 0}})}{k'^2}+\frac{\phi(C_{\tau^-_{\nu 1}})-2\phi(C_\nu)+\phi(C_{\tau^+_{\nu 1}})}{h'^2} \\ &+\frac{1}{3}N'\big(\frac{\phi(C_\nu)+\phi(C_{\nu_1)}+\phi(C_{\nu_2})}{3}\big)+\frac{1}{3}N'\big(\frac{\phi(C_{\nu_3})+\phi(C_\nu)+\phi(C_{\nu_4})}{3}\big) \\ &+\frac{1}{3}N'\big(\frac{\phi(C_{\nu_5})+\phi(C_{\nu_6})+\phi(C_\nu)}{3}\big)=0, \end{aligned} \tag{B.3}$$

for all $\nu < U$ and

$$\frac{\phi(C_{\tau_{\nu 0}}) - \phi(C_\nu)}{k'^2} = \frac{\phi(C_{\tau_{\nu' 0}}) - \phi(C_{\nu'})}{k'^2}, \tag{B.4}$$

$$\frac{\phi(C_{\tau_{\nu 1}}) - \phi(C_\nu)}{k'^2} = \frac{\phi(C_{\tau_{\nu' 1}}) - \phi(C_{\nu'})}{k'^2}, \tag{B.5}$$

for two ν's which share τ,all $\tau < U^\circ$.

If we substitute the solution of the eqs.(B.4,B.5) in the second summation in the action eq.(B.2) which is the bulk part of our action, and consider the difference between Reimann sums, we obtain

$$\sum_{\nu<U^\circ} \left(-\frac{1}{2}\big(\frac{\phi(C_{\nu_2}) - \phi(C_{\nu_1})}{k}\big)^2 + \frac{1}{2}\big(\frac{\phi(C_{\nu_1}) - \phi(C_\nu)}{h}\big)^2 + N\big(\frac{\phi(C_\nu) + \phi(C_{\nu_1}) + \phi(C_{\nu_2})}{3}\big) \right) hk, \tag{B.6}$$

which is the bulk part of Marsden et al's action. We see that our framework has more terms in the boundary.

Appendix C

First Few Coefficients in the expansion of the Born-Infeld Lagrangian

Here is a table of the first few coefficients L_{mn}:

Table

L_{mn}	1	2	3	4	5	6	7	8	9	10
1	2	2	2	2	2	2	2	2	2	2
2	2	6	12	20	30	42	56	72	90	110
3	2	12	40	100	210	392	672	1080	1650	2420
4	2	20	100	350	980	2352	5040	9900	18150	31460
5	2	30	210	980	3528	10584	27720	65340	141570	286286
6	2	42	392	2352	10584	38808	121968	339768	858858	2004002
7	2	56	672	5040	27720	121968	453024	1472328	4294290	11451440
8	2	72	1080	9900	65340	339768	1472328	5521230	18404100	55621280
9	2	90	1650	18150	141570	858858	4294290	18404100	69526600	236390440
10	2	110	2420	31460	286286	2004002	11451440	55621280	236390440	898283672

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