Universidad Michoacana de San Nicolás de Hidalgo
Instituto de Física y Matemáticas

Propagadores y vértices de 3 puntos no perturbativos en la teoría de Yang-Mills

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Por

Pietro Dall' Olio

Asesor

Dr. Axel Weber
(IFM-UMSNH)

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Resumen

En esta tesis se estudia el comportamiento de los propagadores de gluón y de fantasma, y de los vértices de fantasma-gluón y de 3 gluones, en una descripción de la teoría de Yang-Mills que involucra un término de masa para el gluón, introducido en el intento de tomar en cuenta el efecto de la restricción a la región de Gribov a bajas energías. Para mejorar los resultados a altas energías, lejos del horizonte de Gribov, se resuman las funciones de correlación, evaluadas al orden de un lazo, mediante el grupo de renormalización en la formulación de Callan-Symanzik. En el análisis se consideran distintos esquemas de renormalización para encontrar los resultados que mejor se peguen a los datos calculados en la Red en la norma de Landau.

Palabras claves: Yang-Mills theory, Gribov problem, renormalization group, Curci-Ferrari model, Callan-Symanzik equations.

Abstract

In this thesis we study the behaviour of the gluon and ghost propagators and of the ghost-gluon and three-gluon vertices in a description of Yang-Mills theory that involves a gluonic mass term, introduced in the attempt to take into account the effect of the restriction to the Gribov region at low energies. In order to improve the results at high energies, far from the Gribov horizon, the correlation functions evaluated at one-loop order in perturbation theory, are resummed with the Callan-Symanzik equations of the renormalization group. In this analysis we considered different renormalization schemes in order to achieve a better quantitative agreement with the Lattice data calculated in Landau gauge.

Chapter 1

Introduction

In the seminal work of 1954 [1] C-N. Yang and R. L. Mills introduced for the first time the generalization of the gauge symmetry known in electrodynamics, to a non-Abelian group $SU(2)$, representing the isospin symmetry within the strong interaction between a proton and a neutron. They were motivated by the principle of locality, arguing that the arbitrariness in choosing which particle to call a proton or a neutron should be a space-time dependent property of the theory.

Since then the gauge principle has acquired a prominent role in physics. Every interaction at its fundamental level is in fact understood as a gauge theory. The electroweak and strong interactions, described in the Standard Model of particles, are mediated by fields that can be redefined, without changing the physical content of the theory, by a local transformation of the group $U(1)\times SU(2)\times SU(3)$. Also the gravitational interaction is described at the classical level by a theory, General Relativity, that is invariant under the local redefinition by diffeomorphisms of its degrees of freedom, hence it can be viewed as a gauge theory, although a proper way to quantize it is still lacking.

The presence of the local gauge symmetry implies that the field content of the theory is not in a biunivocal correspondence with physical quantities, otherwise the classical theory would not be deterministic. This can be understood in a Lagrangian formulation, because the equations of motion for the fields are not independent, but interrelated by a number of identities equal to the number of generators of the local symmetry (*second Noether theorem*). In the Hamiltonian formulation is understood by the fact the theory is a constrained one, with *first-class* constraints generating the local symmetry and spanning a hypersurface in the symplectic space whose points crossed by the fields evolutions are not determined by the equations of motion.

The physical quantities that save the determinism at the classical level are in fact the equivalence classes (*orbits*) of the fields, where two fields belong to the same class if they are related by a local gauge transformation, i.e. if they lie on the same orbit. Being very challenging to define a consistent quantum field theory whose degrees of freedom are the equivalence classes, the next best strategy is to select a single field configuration element from any conjugacy class, through the procedure known as *gauge fixing*. This procedure, which consists in a modification of the classical Lagrangian and corresponding equations of motion, overcomes the inconsistencies that arise at the quantum level, both in the covariant canonical formalism, where the classical equations of the gauge field are inconsistent with the separating property of the vacuum (cf. [2]), as in the path integral formulation, where the redundant gauge field configurations that are not weighted in the directions of the gauge transformations generate infinities that need to be factorized from the path integral.

Unfortunately, the standard procedure to fix the gauge introduced by Faddeev and Popov [5], which consists in intersecting each gauge orbit with an hypersurface defined by a covariant gauge condition, presents a topological obstruction at low energies, as showed by Gribov [6]. This is understood by the fact that in a non abelian gauge theory, the gauge trajectories along an orbit that move away from the trivial configuration, hence far from the region were perturbation theory is valid, intersect the gauge surface on different points (Gribov copies), spoiling the validity of the uniqueness property of the gauge fixing procedure. Singer [7] then demonstrated that such ambiguity cannot be avoided by any local analytical gauge fixing condition.

This constitutes a serious problem for the non perturbative definition of a non abelian gauge theory in the continuum. QCD (and the pure gluodynamics Yang-Mills theory) has been studied so far at a non perturbative level in a lattice formulation [8], where the gauge fixing procedure is unnecessary due to the finite number of degrees of freedom in the discretized space-time, and by solving the functional *Dyson-Schwinger* equations (DSEs) and *Functional Renormalization Group* equations (FRG) [9]. In this work we study an alternative formulation introduced by Tissier and Wschebor [10] who showed how introducing an explicit massive term for the gluon, the evaluation of correlation functions at one-loop order well approximate the lattice results at low energy (IR) and by improving them with the renormalization group resummation, the approximation is qualitatively good for the whole range of momenta.

The theoretical motivation to introduce a gluon mass is the soft breaking [45] of the BRST symmetry [3, 4], the residual global symmetry that survive after gauge fixing and that represents the local gauge symmetry at the quantum level, due to the restriction of the functional integration to the first Gribov region in order to take care of the Gribov ambiguity, as shown by Zwanziger [44]. Here we push further the analysis done by Tissier and Wschebor for the two-point correlation functions and three-point vertices in Yang-Mills theory, by exploring the implementation of different renormalization schemes in order to achieve better quantitative results that fit the lattice data in Landau gauge.

This work is structured as following: in this introductory chapter I we fix the notations in the construction of the Yang-Mills Lagrangian, the gauge fixing procedure through the Faddeev-Popov method and the BRST symmetry. In Chapter II we briefly explain the Gribov problem, the Gribov-Zwanziger scenario that is constructed in order to overcome the Gribov ambiguity and how it heuristically justifies the introduction of the gluon mass. Chapter III is dedicated to the exhibition of the one-loop results for the two and three-point correlation functions and in Chapter IV we present the different results for the renormalization group improved correlation functions in different renormalization schemes.

1.1 Y-M path integral quantization

In a non abelian gauge theory the matter fields $\phi(x)$ [1] are sections of a principle G-bundle, where G is the structure group (here we consider only the semi-simple compact Lie group $SU(N)$) acting on the fibre, whose sections define in local coordinates the *gauge transformations* of the field:

$$\phi'(x) = U(x)\phi(x), \qquad U(x) = e^{ig\theta(x)}, \tag{1.1}$$

where $U(x)$ is the unitary matrix belonging to $SU(N)$, g is the coupling constant and $\theta(x) = \theta^a(x)T^a$ contains the functions that parametrize the transformation for each direction of the Lie algebra hermitian generators T^a that obey the commutation relation

$$[T^a, T^b] = if^{abc}T^c, \qquad a = 1, \ldots, N^2 - 1, \tag{1.2}$$

[1] Here it is irrelevant if they represent scalar or spinor particles.

where f^{abc} are the totally antisymmetric real structure constants of the group, and being N^2-1 the dimension of the group. The generators are normalized as follows:

$$\mathrm{Tr}\left[T^a T^b\right] = \frac{1}{2}\delta^{ab}. \tag{1.3}$$

The fibre bundle is equipped with a Lie algebra valued 1-form connection $\omega = gA_\mu(x)dx^\mu = gA^a_\mu(x)T^a dx^\mu$ that allows to differentiate the fiber sections. The connection defines a covariant derivative

$$D_\mu = \partial_\mu - igA_\mu, \tag{1.4}$$

that transforms covariantly under a gauge transformation, guaranteeing the local gauge invariance o the theory:

$$D'_\mu \phi'(x) = D'_\mu \left(U(x)\phi(x)\right) = U(x)D_\mu\phi(x). \tag{1.5}$$

This implies an affine transformation law for the gauge field:

$$A'_\mu(x) = U(x)A_\mu U^\dagger(x) + \frac{i}{g}U(x)\partial_\mu U^\dagger(x), \tag{1.6}$$

which reduces for the case of an infinitesimal transformation $U(x) \simeq 1 - ig\theta(x)$ to

$$A'_\mu(x) = \partial_\mu\theta(x) - ig[A_\mu, \theta(x)] = D^A_\mu\theta(x), \tag{1.7}$$

being D^A_μ the covariant derivative in the adjoint representation. In terms of the gauge field components A^a_μ (1.7) translates to:

$$A'^a_\mu(x) = D^{A\,ab}_\mu\theta^b(x) = \partial_\mu\theta^a(x) + gf^{abc}A(x)^b_\mu\theta^c(x). \tag{1.8}$$

The gauge invariant action is given in terms of the field strength curvature $F_{\mu\nu}(x) = F^a_{\mu\nu}(x)T^a$ defined by:

$$F_{\mu\nu}(x) \equiv \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - ig[A_\mu(x), A_\nu(x)], \tag{1.9}$$

which for the components reads:

$$F^a_{\mu\nu}(x) = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu(x)A^c_\nu(x). \tag{1.10}$$

. Since the field strength transforms covariantly under a gauge transformation

$$F'_{\mu\nu}(x) = U(x)F_{\mu\nu}(x)U^\dagger(x), \tag{1.11}$$

the following Yang- Mills action defined in Euclidean space is indeed gauge invariant:

$$S_{YM} = \frac{1}{2}\int d^D x \mathrm{Tr}\left[F_{\mu\nu}F_{\mu\nu}\right] = \frac{1}{4}\int d^D x F^a_{\mu\nu}F^a_{\mu\nu}. \tag{1.12}$$

The starting point to quantize the theory in a Feynman path integral approach is the generating functional $Z[J]$ that contains all the non perturbative quantum fluctuations

$$Z[J] \equiv \int DA_\mu e^{-S_{YM} + \int d^D x J_\mu(x)A_\mu(x)}, \tag{1.13}$$

which would generate the n-point correlation functions for the gauge field, by consecutive functional differentiations:

$$\langle A_{\mu_1}(x_1)\cdots A_{\mu_n}(x_n)\rangle = \left.\frac{\delta^n Z[J]}{\delta J_{\mu_1}(x_1)\cdots\delta J_{\mu_n}(x_n)}\right|_{J_i=0}, \tag{1.14}$$

where a suitable normalization that allows to omit the vacuum diagrams is understood. Unfortunately (1.13) is ill defined, since it contains the redundant gauge configurations that are connected by gauge transformations, i.e. that belongs to the same gauge orbit. In particular it integrates over the configurations connected to the trivial configuration $A_\mu = 0$, that are not weighted by the exponential and therefore generate a formal divergence. This is reflected on the fact that the quadratic term in the action contains zero modes and therefore is not invertible, meaning a gauge propagator is not well defined.

A way out of this is to factorize the unwanted divergence by imposing a local gauge condition on the gauge field

$$\mathcal{F}[A] = 0, \tag{1.15}$$

inserting in the functional integration, the identity

$$\Delta_{\mathcal{F}}[A] \int dU \delta\left(\mathcal{F}[A^U]\right) = 1, \tag{1.16}$$

where dU is the invariant Haar measure

$$dU = \prod_x \prod_a d\theta^a(x), \tag{1.17}$$

and the functional Dirac delta is to be understood as the product of deltas

$$\delta\left(\mathcal{F}[A^U]\right) = \prod_x \prod_a \delta\left(\mathcal{F}^a[A^U(x)]\right), \tag{1.18}$$

where $A^U(x)$ represents the gauge transformed field configuration under the gauge transformation $U(x)$. In (1.16) $\Delta_{\mathcal{F}}[A]$ is the absolute value of the gauge invariant Faddeev-Popov determinant:

$$\Delta_{\mathcal{F}}[A] =\mid \det \mathcal{M}^{ab}(x,y) \mid, \qquad \mathcal{M}^{ab}(x,y) = \left.\frac{\delta \mathcal{F}^a[A^U(x)]}{\delta\theta^b(y)}\right|_{\mathcal{F}[A^U]=0}. \tag{1.19}$$

(1.16) is the functional generalisation of the one dimensional identity

$$\mid f'(x_0) \mid \int dx \delta(f(x)) = 1, \tag{1.20}$$

where x_0 is the only solution of $f(x) = 0$. Inserting (1.16) into (1.13) and exploiting the gauge invariance of the functional measure dA_μ, of the action and of the Faddeev-Popov determinant, one can therefore factorize the integration over the gauge group measure and absorb it in the normalization of the generating functional, yielding (omitting the source term):

$$Z = \int DA_\mu \delta\left(\mathcal{F}[A]\right) \Delta_{\mathcal{F}}[A] e^{-S_{YM}}. \tag{1.21}$$

Since the Faddeev-Popov determinant is gauge invariant it can be evaluated around the identity $\theta^a = 0$, implying for the Faddev-Popov operator $\mathcal{M}^{ab}(x,y)$

$$\mathcal{M}^{ab}(x,y) = \int dz \frac{\delta \mathcal{F}^a[A_\mu(x)]}{\delta A^c_\nu(z)} \left.\frac{\delta A^{c\,U}_\nu(z)}{\delta\theta^b(y)}\right|_{\theta=0,\mathcal{F}[A]=0} = \int dz \frac{\delta \mathcal{F}^a[A_\mu(x)]}{\delta A^c_\nu(z)} \left. D^{A\,bc}_\nu \delta^D(z-y)\right|_{\theta=0,\mathcal{F}[A]=0}, \tag{1.22}$$

where in the last line we substituted the infinitesimal gauge transformation [2] of the gauge field (1.8). Using the covariant Lorentz gauge condition $\mathcal{F}[A] = -\partial_\mu A^a_\mu(x) + \omega^a(x)$, with $\omega^a(x)$ an arbitrary function, the functional integration results:

$$Z = \int DA_\mu \mid \det\left(-\partial_\mu D_\mu\right) \mid \delta\left(\partial_\mu A^a_\mu - \omega^a\right) e^{-S_{YM}}. \tag{1.23}$$

If one assumes that the Faddeev-Popov determinant is positive, condition that is satisfied nearby the trivial gauge field configuration, the absolute value can be omitted and the functional determinant can be lifted into the action by the introduction of a couple of Grassmannian scalar fields $c^a(x)$ and $\bar{c}^a(x)$, using the functional identity for the determinant of an operator $\mathcal{O}$:

$$\det\mathcal{O} = \int DcD\bar{c}\, \exp\left[\iint dxdy\, \bar{c}^a(x)\mathcal{O}^{ab}(x,y)c^b(y)\right]. \tag{1.24}$$

Moreover, the Dirac delta that imposes the gauge condition can also be lifted into the action, by the following trick: since the physics must not depend on the arbitrariness of the auxiliary function $\omega^a(x)$, we can average it with a Gaussian factor:

$$\delta\left(\partial_\mu A^a_\mu - \omega^a\right) \to \int D\omega\delta\left(\partial_\mu A^a_\mu - \omega^a\right) \exp\left[-\frac{1}{2\xi}\int d^Dx\,(\omega^a(x))^2\right] = \exp\left[-\frac{1}{2\xi}\int d^Dx\,(\partial_\mu A^a_\mu(x))^2\right] \tag{1.25}$$

By inserting (1.25) and (1.24) with $\mathcal{O}^{ab}(x,y) = -\partial_\mu D^{ab}_\mu \delta^D(x-y)$, the generating functional becomes:

$$Z = \int D[A_\mu, c, \bar{c}] \exp\left[-S_{YM} - \int d^Dx \left(\partial_\mu \bar{c}^a (D_\mu c)^a + \frac{1}{2\xi}(\partial_\mu A^a_\mu)^2\right)\right]. \tag{1.26}$$

The term added to the original action is referred to as the gauge-fixing action S_{GF}. We furthermore rewrite the last term, containing the arbitrary gauge parameter ξ, by inserting in the functional integration an auxiliary scalar field $B^a(x)$, known as the *Nakanishi-Lautrup* field:

$$Z = \int D[A_\mu, c, \bar{c}, B] e^{-S}, \tag{1.27}$$

where the total action S is given by:

$$S = S_{YM} + S_{GF} = \int d^Dx \left[\frac{1}{4}(F^a_{\mu\nu})^2 + \partial_\mu \bar{c}^a (D_\mu c)^a + iB^a\partial_\mu A^a_\mu + \frac{\xi}{2}B^aB^a\right]. \tag{1.28}$$

Integrating (1.27) w.r.t the non dynamical auxiliary field $B(x)$ we indeed recover the form in (1.26). We stress how, in the Landau gauge considered from now on, corresponding to the choice of the gauge parameter $\xi = 0$, the terms containing the auxiliary field in the action reduce to the exponential representation of the functional Dirac delta $\delta(\partial_\mu A_\mu)$, strongly enforcing the gauge condition.

The Grassmannian scalar fields introduced in the gauge-fixing process are called *ghosts*, since their excitation modes cannot appear in the asymptotic physical spectrum of the theory, due to their nature of fermionic Lorentz scalar particle that is in explicit contradiction with the famous *spin-statistic* theorem. However, their presence in the closed loops of the perturbative expansion of correlation functions is essential in order to cancel the unphysical degrees of freedom of the gauge field, the longitudinal and temporal components that appear as intermediate states, and thus ensure the preservation of unitarity.

[2] From now on we will suppress the index A on the covariant derivative, being clear it is in the adjoint representation

1.2 BRST symmetry

The action (1.28) is not locally gauge invariant anymore, because of the gauge-fixing term. Nevertheless it still possess a global symmetry, residual of the broken local gauge symmetry, where the ghost field takes the place of the parameter function of the gauge transformation. The action is indeed invariant by the following transformations that constitute the so called BRST symmetry:

$$\begin{aligned} sA^a_\mu &= (D_\mu c)^a, \qquad & sc^a &= -\frac{g}{2} f^{abc} c^b c^c, \\ s\bar{c}^a &= iB^a, \qquad & sB^a &= 0. \end{aligned} \tag{1.29}$$

The variation s is a Grassmannian nilpotent operator, i.e. satisfies $s^2 = 0$. The reason to introduce the auxiliary field $B(x)$ was in fact to have the nilpotency property satisfied without the use of the equations of motion (off-shell). The nilpotency property of BRST on the gauge field, in particular, encodes the information of a symmetry that mimics the infinitesimal local gauge transformation but that cannot be extended to a finite transformation by repeated iterations, due to the gauge-fixing.

The BRST symmetry is a powerful tool to prove the renormalizability at all orders of Yang-Mills theory, through the use of the Slavnov-Taylor identities that derive from it. Even more importantly, it constitutes the key element to define a subsidiary condition on the Fock space of the asymptotic states that select only positive norm states. It is well known in fact, that a gauge theory covariantly quantized inevitably yield to a Fock space with an indefinite metric. A mechanism to remove form the physical Hilbert space negative norm states that would break the unitarity of the theory is therefore required. Kugo and Ojima [46] defined the physical Hilbert space as the subspace of states that live in the cohomolgy of the BRST operator Q_B,

$$\mathcal{H}_{phys} = \mathrm{Ker}Q_B/\mathrm{Im}Q_B, \qquad \mathcal{H}_{phys} = \{ |\Psi\rangle \,; \quad Q_B |\Psi\rangle = 0 \wedge |\Psi\rangle \neq Q_B |\phi\rangle \} , \tag{1.30}$$

Q_B being the conserved charge operator associated to the BRST symmetry. By studying the representations of the supersymmetric algebra generated by Q_B and Q_c, the charge associated to the ghost-number conservation, Kugo and Ojima showed how unphysical states group together to form quartets (doublets under BRST transformation that come in pairs with opposite ghost number) that can appear in the asymptotic physical space, through time evolution, only in zero norm combinations, therefore without jeopardizing the unitarity of the theory.

We now define the effective action that generates the proper full correlation functions. Adding to the action a source term that includes sources corresponding to each field and to the non linear BRST variations,

$$S_{sources} = \int d^D x \left[J^a_\mu A^a_\mu + \bar{\eta}^a c^a + \bar{c}^a \eta^a + R^a B^a + K^a_\mu (sA_\mu)^a + L^a (sc)^a \right] , \tag{1.31}$$

the generating functional is given by

$$Z[J_\mu, R, \bar{\eta}, \eta, K_\mu, L] = \int D[A_\mu, c, \bar{c}, B] e^{-S+S_{sources}} \equiv e^{G[J_\mu, R, \bar{\eta}, \eta, K_\mu, L]}, \tag{1.32}$$

where we defined the functional $G[J_\mu, R, \bar{\eta}, \eta, K_\mu, L]$ that generates the connected n-point correlation functions. The effective action is given by the Legendre transform of this latter w.r.t the classical fields:

$$\begin{aligned} A^a_\mu(x) &= \frac{\delta G}{\delta J^a_\mu(x)}, \qquad & B^a(x) &= \frac{\delta G}{\delta R^a(x)}, \\ c^a(x) &= \frac{\delta G}{\delta \bar{\eta}^a(x)}, \qquad & \bar{c}^a(x) &= -\frac{\delta G}{\delta \eta^a(x)}. \end{aligned} \tag{1.33}$$

The effective action is defined by

$$\Gamma[A_\mu, B, c, \bar{c}, K_\mu, L] = -G[J_\mu, R, \bar{\eta}, \eta, K_\mu, L] + \int d^D x \left(J^a_\mu A^a_\mu + R^a B^a + \bar{\eta}^a c^a + \bar{c}^a \eta^a \right) , \quad (1.34)$$

where the sources are to be considered functions of the classical fields by inverting (1.33). This functional, which at tree-level coincides with the classical action, generates all the 1PI (one particle irreducible) diagrams, i.e. diagrams with amputated external legs that cannot be split in two parts by cutting an internal propagator, and that constitute the quantum corrections to the vertices.

The notation for the two-point function in the space of momenta generated by the effective action is, for the ghosts:

$$(2\pi)^{2D} \frac{\delta^2 \Gamma}{\delta c^a(-p) \delta \bar{c}^b(q)} \bigg|_{(A_\mu, B, c, \bar{c})=0} = \Gamma_{c\bar{c}}(p^2) \delta^{ab} (2\pi)^D \delta^D(p-q), \quad (1.35)$$

and similarly for other combinations of fields. We stress that in passing to the momentum space the following convention for the Fourier transform is used

$$f(x) = \int \frac{d^D p}{(2\pi)^D} f(p) e^{ipx}. \quad (1.36)$$

Chapter 2

Yang-Mills with a massive gluon

2.1 Gribov problem

Albeit the nilpotent BRST symmetry is essential for the construction of the physical Hilbert space of positive norm states and is at the basis of the Kugo-Ojima criterion for removing from the physical space non-singlet colour states [46, 32], thus explaining the confinement of coloured particles, it has been very challenging to preserve BRST symmetry in a non perturbative framework. This is mainly due to the Neuberger's no-go theorem [47] which asserts that in a BRST invariant lattice formulation, the expectation value of any gauge invariant physical observable takes the form of the indeterminate ratio 0/0. In the continuum formulation this can be understood by the fact that the gauge-fixing factor in the Faddeev-Popov action (1.28) is a coboundary term for the BRST transformation (1.29):

$$S_{GF} = \int d^Dx \left[\partial_\mu \bar{c}^a (D_\mu c)^a + iB^a \partial_\mu A^a_\mu + \frac{\xi}{2} B^a B^a\right] = -s \int d^Dx \left[\partial_\mu \bar{c}^a A^a_\mu + i\frac{\xi}{2}\bar{c}^a B^a\right]. \quad (2.1)$$

This property, which guarantees the invariance of any physical quantity on the gauge parameter, incidentally also causes the generating functional to formally vanish. In fact, if we consider the gauge-fixing part of this latter with the first term in (2.1) multiplied by a real parameter t

$$Z_{GF}(t) = \int D[c, \bar{c}, B] e^{s\int d^Dx \left[t\partial_\mu \bar{c}^a A^a_\mu + i\frac{\xi}{2}\bar{c}^a B^a\right]}, \quad (2.2)$$

its derivative w.r.t t is zero, because of the nilpotency of the BRST transformation and the vanishing of the functional integral of a total BRST variation (if the vacuum is a BRST invariant physical state as it is supposed to be):

$$\begin{aligned} \frac{d}{dt} Z_{GF}(t) &= \int D[c, \bar{c}, B] s\left(\partial_\mu \bar{c}^a A^a_\mu\right) e^{s\int d^Dx \left[t\partial_\mu \bar{c}^a A^a_\mu + i\frac{\xi}{2}\bar{c}^a B^a\right]} \\ &= \int D[c, \bar{c}, B] s\left(\partial_\mu \bar{c}^a A^a_\mu \, e^{s\int d^Dx \left[t\partial_\mu \bar{c}^a A^a_\mu + i\frac{\xi}{2}\bar{c}^a B^a\right]}\right) = 0. \end{aligned} \quad (2.3)$$

Since Z_{GF} does not depend on t it can therefore be evaluated at $t = 0$, but then, because of the rules of Grassmannian integration, we have

$$Z_{GF}(0) = \int D[c, \bar{c}, B] e^{-\int d^Dx \left[\frac{\xi}{2} B^a B^a\right]} = 0. \quad (2.4)$$

The Neuberger problem is in turn deeply connected with a topological obstruction discovered by Gribov [6] which is found when one tries to extend the Faddeev-Popov gauge-fixing procedure described in the previous chapter, at a non perturbative level. Through the derivation of the

Faddeev-Popov action, in fact, two assumptions were made that turn out to be wrong when one moves away from the perturbative region: in using the identity (1.16) it was implicitly assumed that the gauge condition has a unique solution along each orbit of the gauge field, and in removing the absolute value of the functional determinant in (1.23) it was assumed that the Faddeev-Popov operator $-\partial_\mu D_\mu[A]$ is positive definite.

For what concerns the first assumption, the gauge condition $\mathcal{F}[A] = 0$ has more than one solution (*Gribov copy*), i.e. the hypersurface represented by the gauge condition intersects the gauge orbits in more than one point, if for a certain gauge configuration that satisfies it there is a gauge transformation $U(x)$ such that the transformed field also satisfies it $\mathcal{F}[A^U] = 0$. For the Lorentz covariant gauge $\partial_\mu A_\mu = 0$ this translates to the condition for the gauge transformation:

$$\begin{aligned}\partial_\mu A^U_\mu &= (\partial_\mu U)\, A_\mu U^\dagger + U A_\mu \partial_\mu U^\dagger + \frac{i}{g}\partial_\mu U \partial_\mu U^\dagger + \frac{i}{g} U \partial^2 U^\dagger \\ &= \frac{i}{g}\partial_\mu \left(U D_\mu[A] U^\dagger\right) = 0,\end{aligned} \tag{2.5}$$

where we used $\partial_\mu A_\mu = 0$. Gribov explicitly derived a solution for such copy, known as the *Gribov pendolum*, in the case of Coulomb gauge for the three dimensional group $SU(2)$. For an infinitesimal gauge transformation (2.5) reduces to

$$-\partial_\mu D_\mu[A] = (-\partial^2 + ig\, \partial_\mu \left[A_\mu, \cdot\right]) = 0, \tag{2.6}$$

meaning that two points on a gauge orbit infinitesimally closed are Gribov copies to each other, as soon as the gauge field is sufficiently strong to generate zero modes in the Faddeev-Popov operator (the free part $-\partial^2$ is positive definite), in the same way as the *Schrödinger* equation generates bound states modes if the potential is strong enough.

Gribov therefore proposed to overcome both of the inconsistencies in the Faddeev-Popov formulation, i.e. the presence of Gribov Copies and the presence of negative eigenvalues of the Faddeev-Popov operator for large non perturbative gauge configurations, by restricting the functional integration over the region of those gauge field configurations that keep the Faddeev-Popov operator positive definite. This region Ω, known as the *first Gribov region* is thus defined as

$$\Omega \equiv \{A_\mu,\ \partial_\mu A_\mu = 0,\ -\partial_\mu D_\mu[A] > 0\}\,. \tag{2.7}$$

The boundary of the Gribov region, called *Gribov horizon* therefore represents the set of gauge configurations that turn the lowest eingevalue of the Faddeev-Popov operator into a negative value. In the same manner one can then define the second Gribov region as the one where the second lowest eigenvalue becomes negative, and so on.

Incidentally one can understand the origin of the Neuberger problem as caused by the fact that ignoring the absolute value of the functional determinant inside the functional integration is equivalent to sum over all the Gribov copies with a factor corresponding to the alternate sign of the determinant, that switches every time a subsequent Gribov region is crossed. These alternate signs cancel out among each other [33]. This zero has also a deeper topological meaning, being related through the Gauss-Bonnet theorem [34] to the computation of the Euler's character, a topological invariant that vanishes for the compact group $SU(N)$.

It can be easily proven that the Gribov region is convex and bounded in every direction. It is also equivalent to the set of local minima of the following functional

$$\|\ A^U\ \|^2 = \mathrm{Tr}\int d^D x A^U_\mu(x) A^U_\mu(x) = \frac{1}{2}\int d^D x A^{a\,U}_\mu(x) A^{a\,U}_\mu(x), \tag{2.8}$$

which ca be seen as a norm in the space of gauge configurations. In fact, the stationary points that cancel the first variation of (2.8) w.r.t an infinitesimal gauge transformation (1.7), obey the Landau gauge condition

$$
\begin{aligned}
0 = \delta \mid\mid A^U \mid\mid^2 &= \int d^D x \delta A^a_\mu(x) A^a_\mu(x) = - \int d^D x D^{ab}_\mu \theta^b(x) A^a_\mu(x) \\
&= \int d^D x\, \theta^a(x) \partial_\mu A^a_\mu(x) \implies \partial_\mu A^a_\mu(x) = 0.
\end{aligned} \tag{2.9}
$$

The local minima which make the second variation to be positive definite are the ones that make the Faddeev-Popov operator to be positive definite

$$
\begin{aligned}
\delta^2 \mid\mid A^U \mid\mid^2 &= \int d^D x\, \theta^a(x) \partial_\mu \delta A^a_\mu(x) = \int d^D x\, \theta^a(x) \left(-\partial_\mu D_\mu\right)^{ab} \theta^b(x) > 0 \\
&\implies -\partial_\mu D_\mu > 0.
\end{aligned} \tag{2.10}
$$

The Gribov region therefore contains all the local minima of the functional (2.8) and it can be showed that every orbit in the space of gauge configurations possess at least a local minimum, thus every orbit intersects at least once the Gribov region. Generally the minima are more than one though, meaning that the Gribov region still contains redundant Gribov copies and that in order to pick a single representative configuration for each orbit one should further restrict the functional integration to the region containing only the global minima of (2.8), known as the *Fundamental Modular Region* Λ. Unfortunately it is very challenging to construct a procedure that effectively restrict the integration only to Λ, both analytically as in a lattice formulation, but there are arguments, due to Zwanziger, that suggest that the dominant configurations in the functional integration lie in the common boundary $\delta\Omega \cap \delta\Lambda$ of the Gribov region and the Fundamental region, meaning that one might safely compute correlation functions restricting only to the Gribov region.

Gribov also understood the implications of such a restriction for the IR behaviour of the ghost and gluon propagator, which he calculated in a semiclassical approximation. The positive definiteness of the Faddeev-Popov operator, in particular, is closely connected to the analytical structure of the ghost propagator. This latter can in fact be written as

$$
\begin{aligned}
\langle c^a(x) \bar{c}^b(y) \rangle &= \left. \frac{\delta^2 Z}{\delta \eta^b(y) \bar{\eta}^a(x)} \right|_{sources=0} \\
&= \int_\Omega DA_\mu \det\left(-\partial_\mu D_\mu\right) \left[-\partial_\mu D^{ab}_\mu(x,y)\right]^{-1} e^{-S_{YM}}.
\end{aligned} \tag{2.11}
$$

The ghost propagator is therefore given by the expectation value of the inverse of the Faddeev-Popov operator, multiplied by its determinant, and since this latter is positive definite inside the Gribov region, the ghost propagator, in the space of momenta, cannot have a pole except at vanishing momentum (*no-pole* condition), which therefore corresponds to the Gribov horizon. Calculating the ghost propagator at one-loop order in a semiclassical approximation, i.e. keeping the gauge field as an external background field, Gribov found

$$
\langle c^a(p) \bar{c}^b(-p) \rangle \approx \delta^{ab} \frac{1}{p^2} \frac{1}{1 - \sigma(p, A)}, \tag{2.12}
$$

where

$$
\sigma(p, A) = \frac{1}{V} \frac{1}{p^2} \frac{N g^2}{N^2 - 1} \int \frac{d^D k}{(2\pi)^D} A^a_\mu(-k) A^a_\nu(k) \frac{(p-k)_\mu k_\nu}{(p-k)^2}, \tag{2.13}
$$

where V is the volume factor to be taken to infinity in the thermodynamic limit, where the saddle point approximation becomes exact. Since one can prove that $\sigma(p, A)$ is a decreasing function of momentum, the no-pole condition, thought as equivalent to the restriction over the Gribov region, can be settled by introducing inside the path integral the constraint

$$\theta\left(1-\sigma(0,A)\right)=\int_{-i\infty}^{+i\infty}\frac{d\beta}{2\pi i\beta}e^{\beta(1-\sigma(0,A))}, \tag{2.14}$$

where $\theta(x)$ is the Heaviside function and $\sigma(0,A)$ is given by

$$\sigma(0,A)=\frac{1}{V\,D}\frac{Ng^2}{N^2-1}\int\frac{d^Dk}{(2\pi)^D}A^a_\mu(-k)A^a_\mu(k)\frac{1}{k^2}. \tag{2.15}$$

It is noteworthy that the factor inside the integral in (2.15) corresponds to the inverse of the usual tree-level kinetic term. Inserting this constraint inside the path integral to evaluate the gluon propagator at tree-level and using a saddle point approximation for the integration over the parameter β in (2.14) one gets

$$\langle A^a_\mu(p)A^b_\nu(-p)\rangle\approx\delta^{ab}\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\frac{p^2}{p^4+\lambda^4}, \tag{2.16}$$

where λ, called the Gribov mass, is given by the gap equation

$$1=\frac{D-1}{D}Ng^2\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^4+\gamma^4}. \tag{2.17}$$

The gluon propagator, transverse since we are in Landau gauge, recovers it's bare behaviour for high values of momentum, but it drastically changes its form in the IR, vanishing at vanishing momentum, due to the presence of the Gribov horizon. One can interpret (2.16) as a massive tree-level propagator with a mass that is a function of momentum and that diverges quadratically with the momentum in the IR and vanishes quadratically in the UV.

Inserting this expression for the gluon propagator back to the one-loop calculation of the ghost propagator, Gribov found that this latter is IR enhanced compared to its tree-level form. For small values of momentum it behaves like

$$\langle c^a(p)\bar{c}^b(-p)\rangle\approx\delta^{ab}\frac{1}{p^4}. \tag{2.18}$$

2.2 Gribov-Zwanziger scenario

After Gribov's analysis, Zwanziger managed to generalize Gribov's result to all orders [44], transferring the restriction over the Gribov region to the exponential inside the path integral, i.e. to a new extended action, as compared to the standard Faddeev-Popov action (1.28). Instead of implementing the constraint (2.14) on the no-pole condition, Zwanziger inserted in the path integral the constrain $\theta(\lambda(A))$, where $\lambda(A)$ is the lowest eigenvalue, depending on the gauge field, of the Faddeev-Popov operator

$$\left(-\partial_\mu D_\mu[A]\right)^{ab}=-\delta^{ab}\partial^2+gf^{abc}A^c(x)\partial_\mu, \tag{2.19}$$

where the Landau gauge condition $\partial_\mu A_\mu=0$ is explicitly imposed (on-shell). Zwanziger therefore considered the Faddeev-Popov operator as a free operator $-\partial^2$ plus a perturbation, applied degenerate perturbation theory and managed to sum at all orders, in the thermodynamic limit, the different factors in which the lowest degenerate eigenvalue splits because of the interaction term in (2.19). He then lifted the constraint to the exponential factor inside the path integral and performed the saddle point approximation, which at the end it translates to adding to the Lagrangian the following non-local term

$$h(x) = \gamma^4 \int d^D y g f^{bal} A^a_\nu(x) \left[(-\partial_\mu D_\mu)^{lm} \right]^{-1} \delta^D(x-y) g f^{bkm} A^k_\nu(y), \tag{2.20}$$

where the parameter γ is given by the gap equation

$$\langle h(x) \rangle = \gamma^4 D \left(N^2 - 1 \right). \tag{2.21}$$

In order to localize the non-local term given by the inverse of the Faddeev-Popov operator, Zwanziger then introduced new fields in the path integral, making use of the sketched identity for an operator $\mathcal{O}$

$$\begin{aligned} \exp\left[J \mathcal{O}^{-1} \bar{J} \right] &= \det \mathcal{O} \int D\varphi D\bar{\varphi} \exp\left[-\bar{\varphi} \mathcal{O} \varphi + \varphi J + \bar{\varphi} \bar{J} \right] \\ &= \int D\omega D\bar{\omega} \varphi D\bar{\varphi} \exp\left[\bar{\omega} \mathcal{O} \omega - \bar{\varphi} \mathcal{O} \varphi + \varphi J + \bar{\varphi} \bar{J} \right], \end{aligned} \tag{2.22}$$

where $(\varphi, \bar{\varphi})$ are commuting scalar fields and $(\omega, \bar{\omega})$ are Grassmannian scalar fields. Eventually the generating functional takes the form:

$$Z = \int D[A, c, \bar{c}, B, \varphi, \bar{\varphi}, \omega, \bar{\omega}] e^{-S_{GZ}}, \tag{2.23}$$

where the Gribov-Zwanziger action S_{GZ} is given by

$$S_{GZ} = S_{YM} + S_{GF} + S_0 + S_\gamma, \tag{2.24}$$

where

$$S_0 = \int d^D x \left[\bar{\varphi}^{ac}_\mu \partial_\mu D^{ab}_\mu \varphi^{bc}_\mu - \bar{\omega}^{ac}_\mu \partial_\mu D^{ab}_\mu \omega^{bc}_\mu - g f^{abc} \partial_\mu \bar{\omega}^{ae}_\nu D^{bd}_\mu c^d \varphi^{ce}_\nu \right], \tag{2.25}$$

$$S_\gamma = -\gamma^2 g \int d^D x \left[f^{abc} A^a_\mu \varphi^{bc}_\mu + f^{abc} A^a_\mu \bar{\varphi}^{bc}_\mu \right]. \tag{2.26}$$

The gap equation now reads:

$$\langle g f^{abc} A^a_\mu (\varphi^{bc}_\mu + \varphi^{bc}_\mu) \rangle = -2\gamma^2 D (N^2 - 1). \tag{2.27}$$

The last term in S_0 is generated by performing a non-local shift on the field ω^{ab}_μ

$$\omega^{ab}_\nu(x) \to \omega^{ab}_\nu + \int d^D z \left(-\partial_\mu D^{ad}_\mu \right)^{-1} (x, z) g f^{dkl} \partial_\mu \left(D^{ke}_\mu c^e(z) \varphi^{lb}_\nu \right), \tag{2.28}$$

which is implemented in order to preserve the renormalizability of the theory. The BRST symmetry (1.29) can be trivially extended to the extra fields, organized in doublets according to

$$\begin{aligned} s\varphi^{ab}_\mu &= \omega^{ab}_\mu, \qquad s\omega^{ab}_\mu = 0, \\ s\bar{\omega}^{ab}_\mu &= \bar{\varphi}^{ab}_\mu, \qquad s\bar{\varphi}^{ab}_\mu = 0. \end{aligned} \tag{2.29}$$

It can be verified that the action S_0 is BRST invariant under such transformations combined with the ones in (1.29), but S_γ breaks the BRST symmetry because of the presence of the γ parameter that encodes the presence of the Gribov horizon:

$$sS_\gamma = -g\gamma^2 \int d^D x f^{abc} \left[A^a_\mu \omega^{bc}_\mu - (D^{ae}_\mu c^e (\bar{\varphi}^{bc}_\mu + \varphi^{bc}_\mu)) \right]. \tag{2.30}$$

We also stress that the behaviour of the gluon and ghost propagators in this Gribov-Zwanziger scenario is the same, at least at the first perturbative order, as the one already found by Gribov in the semiclassical approximation, i.e. one finds an IR vanishing gluon propagator and an IR enhanced ghost propagator. Such behaviour was initially believed to be in agreement with the lattice results at low energy and also to the so called *scaling* solution, found solving the DSE for the propagators [23, 24, 27], solution corresponding to a power scaling behaviour in the IR and to a non-trivial IR fixed point for the coupling. It also confirmed the Kugo-Ojima criterion [32] for the confinement of colour charge that in Landau gauge predicts an IR enhancement of the ghost propagator, albeit it must be stressed that the Kugo-Ojima argument is based on the maintenance of the BRST symmetry at the non-perturbative level, condition that as it has been showed, is not fulfilled in this scenario.

Since 2007, however, numerical data from the largest lattices [54, 55, 56] seemed to univocally point to an IR finite behaviour of the gluon propagator and to a non enhancement of the ghost propagator, which in the IR behaves therefore like its tree-level counterpart. This kind of IR behaviour was also found solving the DSE [57, 58, 59] and it is known as the *decoupling* solution, and corresponds to a trivial IR fixed-point.

We stress that also in the Gribov-Zwanziger scenario this IR behaviour for the propagators can be reproduced, in particular the IR gluon massive behaviour, if one takes into account the presence of the two dimensional gluonic condensate and of the BRST invariant condensate $\langle\bar{\varphi}\varphi - \bar{\omega}\omega\rangle$, in what is referred to as the *refined* Gribov-Zwanziger scenario [45].

2.3 Curci-Ferrari action

In this work, instead of scrutinizing the implications to the correlation functions from the complicated Gribov-Zwanziger action, we take a more heuristically point of view, initiated by Tissier and Wschebor [10], based on the attempt to include the effect of the presence of the Gribov horizon by simply adding to the Faddeev-Popov action (1.28) a massive term for the gluon, thus working with the following action

$$S = \int d^D x \left[\frac{1}{4}(F^a_{\mu\nu})^2 + \partial_\mu \bar{c}^a (D_\mu c)^a + iB^a \partial_\mu A^a_\mu + \frac{m^2}{2}(A^a_\mu)^2\right]. \tag{2.31}$$

In [10] was in fact shown that the renormalized gluon and ghost propagators calculated at one-loop order in perturbation theory starting from (2.31) well reproduce the lattice data in the IR. This suggests that Yang-Mills theory is well described in the IR by (2.31). Furthermore, in [51] was shown, through an ϵ-expansion analysis, how such action can generate an IR attractive trivial fixed point for $D > 2$, accounting for the *decoupling* solution, and through a suitable *fine-tuning* that enhances the ghost propagator in the IR, it can also generate the non-trivial fixed point corresponding to the *scaling* solution, and even closely reproduce the IR exponents of the power-like behaviour found in the DSE solutions.

However, as already pointed out in [10], plain perturbation theory with a massive gluon will not be able to reproduce the behaviour of the propagators in the whole range of momentum, since the effect of the mass parameter must vanish in the UV perturbative region. In fact, the reason why the presence of a massive operator for the gluon seems to provide a good description of the non-perturbative IR region, is thought to be the breaking of BRST symmetry, explained in the last section, caused by the restriction to the Gribov region, breaking that cannot therefore prevent the generation of such relevant operator in the IR. But the BRST breaking is contained in the term (2.30), proportional to the dimensionful Gribov parameter γ^2 that accounts for the presence of the Gribov horizon and should not be "seen" by the perturbative region closed to the trivial gauge field configuration which is far from the horizon. In this sense the BRST symmetry breaking is called "soft", meaning that it affects only the IR region and in fact does not spoil the renormalizability of the theory. The vanishing of the effect of the mass parameter in the UV will be taken care of, in this framework, by the flow equations of the renormalization group machinery explained in the next chapter.

We point out that (2.31) is a particular case of the so called Curci-Ferrari model [12, 13], which was constructed in the search of the most general extension of the Faddeev-Popov action that could preserve both BRST and anti-BRST symmetries. This latter is introduced by exchanging the roles of ghost and anti-ghost. In particular, in Landau gauge, the following conjugation transformation C_{FP} is a symmetry of the action (1.28)

$$\begin{aligned} C_{FP}A_\mu^a &= A_\mu^a, \qquad C_{FP}B^a = B^a - igf^{abc}\bar{c}^b c^c, \\ C_{FP}c^a &= \bar{c}^a, \qquad\quad C_{FP}\bar{c}^a = -c^a. \end{aligned} \tag{2.32}$$

The anti-BRST transformation $\bar{s}$ is then defined by

$$\bar{s} = C_{FP} s C_{FP}^{-1}, \tag{2.33}$$

and acts on the fields according to

$$\begin{aligned} \bar{s}A_\mu^a &= (D_\mu \bar{c})^a, \qquad \bar{s}c^a = -gf^{abc}\bar{c}^b c^c - iB^a, \\ \bar{s}\bar{c}^a &= -\frac{g}{2}f^{abc}\bar{c}^b\bar{c}^c, \qquad \bar{s}B^a = -gf^{abc}\bar{c}^b B^c. \end{aligned} \tag{2.34}$$

The anti-BRST transformation can therefore be seen as the specular version of the BRST-transformation where the roles of the ghost and anti-ghost fields are interchanged. This specularity can be completely accomplished if one furthermore introduces the anti-auxiliary field $\bar{B}$, defined as

$$\bar{B}^a \equiv -B^a + igf^{abc}\bar{c}^b c^c. \tag{2.35}$$

In fact under the conjugation transformation this anti-field satisfies

$$C_{FP}B^a = -\bar{B}^a, \qquad C_{FP}\bar{B}^a = -B^a, \tag{2.36}$$

and the anti-BRST transformations on the RHS of (2.34) can be written in the more (anti-) symmetric way

$$\bar{s}c^a = i\bar{B}^a, \qquad \bar{s}\bar{B}^a = 0. \tag{2.37}$$

It is noteworthy that albeit the conjugation transformation C_{FP} is a symmetry of the Faddeev-Popov action only in Landau gauge, the anti-BRST transformation leaves the general covariant gauge action (1.28) invariant. One can now extend the action (1.28) by adding to the gauge-fixing part S_{GF} the following quadratic term in the anti-auxiliary field

$$\begin{aligned} S_{GF} \to S_{GF} &= \int d^D x \left[\partial_\mu \bar{c}^a (D_\mu c)^a + iB^a \partial_\mu A_\mu^a + \frac{\xi}{2}B^a B^a + \frac{\eta}{2}\bar{B}^a\bar{B}^a\right] \\ &= -s\int d^D x \left[\partial_\mu \bar{c}^a A_\mu^a + i\frac{\xi}{2}\bar{c}^a B^a - i\frac{\eta}{2}\bar{c}^a\bar{B}^a\right] \\ &= \bar{s}\int d^D x \left[\partial_\mu c^a A_\mu^a + i\frac{\xi}{2}c^a B^a - i\frac{\eta}{2}c^a\bar{B}^a\right]. \end{aligned} \tag{2.38}$$

Adding such factor to the action therefore does not affect the expectation value of physical gauge-invariant quantities, since the gauge-fixing term still retains its coboundary form w.r.t. the BRST transformation and the anti-BRST transformation, which is also nilpotent and anticommutes with the BRST one

$$s\bar{s} + \bar{s}s = 0. \tag{2.39}$$

Note that the quadratic term in the auxiliary field contains a quartic ghost anti-ghost interaction, due to its definition (2.35). If the two gauge parameters are now identified $\xi = \eta$, one can further extend the gauge-fixing action by adding a massive term for the gluon and for the ghost in the following way

$$S_{GF} \to S_{GF} + \int d^D x \left[\frac{m^2}{2} A^a_\mu A^a_\mu + 2i\xi m^2 \bar{c}^a c^a \right]. \tag{2.40}$$

This massive extended action preserves the BRST and anti-BRST symmetry if one modifies the transformation laws of the auxiliary field by the following

$$sB^a = -im^2 c^a, \qquad \bar{s}B^a = -im^2 \bar{c}^a - g f^{abc} c^b B^c. \tag{2.41}$$

Exploiting the preservation of such symmetry at a quantum level one can prove the renormalizability of the massive Curci-Ferrari model at all orders. Due to the modifications in (2.41), however, the transformations lose their nilpotency property, that was crucial to properly define the Hilbert space of physical states. This is the main reason why the massive Curci-Ferrari model has not been taken very seriously so far. Indeed, if one insists in defining the physical Hilbert space as the set of states contained in the kernel of this not nilpotent BRST charge operator, one can show that this space contains asymptotic states of negative norm that break the unitarity of the theory [14, 15].

In addition, the Curci-Ferrari model was used in a lattice formulation in order to bypass the Neuberger problem [16], but it was pointed out that in such formulation the gauge-fixing partition function is orbit dependent, meaning that the expectation value of a gauge invariant quantity now depends on the gauge-fixing process and in particular on the gauge parameter ξ (the dependency vanishes only at $m^2 = 0$).

However here we take the point of you that the Curci-Ferrari model provides a good description of non-perturbative Yang-Mills theory only in the Landau gauge $\xi = 0$, where the ghost remains massless and there is no quartic ghost anti-ghost interaction. The action considered here is therefore given by (2.31) which is invariant under the extended not nilpotent BRST transformation

$$\begin{aligned} sA^a_\mu &= (D_\mu c)^a, \qquad & sc^a &= -\frac{g}{2} f^{abc} c^b c^c, \\ s\bar{c}^a &= iB^a, \qquad & sB^a &= -im^2 c^a. \end{aligned} \tag{2.42}$$

2.4 Transversality of the gluon propagator

We show here how, despite the gluon mass term that breaks the usual nilpotent BRST symmetry, the gluon propagator remains transverse in Landau gauge. This is easily derived using the Dyson-Schwinger equation for the auxiliary field. The vanishing of the integral of a total field derivative, gives, for the auxiliary field

$$0 = \int D[B^a \cdots] \frac{\delta}{\delta B^a(x)} e^{-S + S_{sources}} = \int D[B^a \cdots] \left(-i\partial_\mu A^a_\mu + R^a \right) e^{-S + S_{sources}}, \tag{2.43}$$

where $S_{sources}$ was given in (1.31). This vanishing expectation value can be written in terms of the generating functionals defined in the previous chapter as

$$-i\partial_\mu \frac{\delta G}{\delta J^a_\mu(x)} + R^a(x) = -i\partial_\mu A^a_\mu(x) + \frac{\delta \Gamma}{\delta B^a(x)} = 0,. \tag{2.44}$$

By taking another derivative of the left side of (2.44) with respect to a J source, the transversality of the connected gluon propagator is immediately obtained. Incidentally, by differentiating the same equation referred to the effective action, it is also proved that the two-point functions involving the auxiliary field do not receive any quantum corrections, i.e. (following the notation (1.35), although it would not be necessary to put the external sources equal to zero):

$$\Gamma_{BB}(p) = 0\,, \qquad \Gamma_{A_\mu B}(p) = p_\mu. \tag{2.45}$$

We can also extract information about the transverse and longitudinal part of the gluon two-point function (the latter, unlike the longitudinal part of the propagator, does not vanish in the presence of a mass term) by looking at the identity

$$\begin{pmatrix} \Gamma_{A_\mu A_\nu}(p) & \Gamma_{A_\mu B}(p) \\ \Gamma_{BA_\nu}(p) & \Gamma_{BB}(p) \end{pmatrix} \times \begin{pmatrix} G_{J_\nu J_\rho}(p) & G_{J_\nu R}(p) \\ G_{RJ_\rho}(p) & G_{RR}(p) \end{pmatrix} = \begin{pmatrix} \delta_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.46}$$

which gives a non trivial matrix relation, due to the non vanishing mixed terms involving both gauge and auxiliary fields. By decomposing the two-point function into its transverse and longitudinal parts

$$\Gamma_{A_\mu A_\nu}(p) = \Gamma^{\perp}_{AA}(p) P^{\perp}_{\mu\nu}(p) + \Gamma^{\parallel}_{AA}(p) P^{\parallel}_{\mu\nu}(p), \tag{2.47}$$

where

$$P^{\perp}_{\mu\nu}(p) = \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}, \qquad P^{\parallel}_{\mu\nu}(p) = \frac{p_\mu p_\nu}{p^2}, \tag{2.48}$$

and considering the transversality of the gluon propagator: $G_{J_\mu J_\nu}(p) = G_{JJ}(p) P^{\perp}_{\mu\nu}(p)$, the identity in (2.46) yields

$$\Gamma^{\perp}_{AA}(p) = (G_{JJ}(p))^{-1}\,, \qquad \Gamma^{\parallel}_{AA}(p) = p^2 G_{RR}(p). \tag{2.49}$$

The first identity in particular guarantees that the transverse part of the gluon two-point function is given by the inverse of the gluon propagator.

Chapter 3

One-loop corrections

3.1 Feynman rules

The perturbative series of the correlation functions are systematically organized in terms of Feynman diagrams. The building blocks of these diagrams are the propagators and vertices, whose expressions in the space of momentum can be derived from the classical action. Expanding the free quadratic and the interaction terms, the action (2.31) reads, neglecting the term with the auxiliary field[1]:

$$S = \int d^D x \left[\frac{1}{2} A^a_\mu(x) \left((-\partial^2 + m^2)\delta_{\mu\nu} + \partial_\mu \partial_\nu \right) A^a_\nu(x) + \bar{c}(x)(-\partial^2)c(x) + g\, f^{abc} \partial_\mu \bar{c}(x) A^b_\mu(x) c^c(x) \right.$$
$$\left. + g\, f^{abc} \partial_\mu A^a_\nu(x) A^b_\mu(x) A^c_\nu(x) + g^2 f^{abc} f^{ade} A^b_\mu(x) A^c_\nu(x) A^d_\mu(x) A^e_\nu(x) \right] . \tag{3.1}$$

The gluonic quadratic term proportional to $\delta_{\mu\nu}$ corresponds to the transverse part of the gluon two-point function at the classical level, therefore its inverse is the free gluon propagator which is transverse (vd. 2.4). In momentum space the free gluon and ghost propagator are therefore:

$$\langle A^a_\mu(p) A^b_\nu(-p) \rangle_0 = \frac{1}{p^2 + m^2} P^\perp_{\mu\nu}(p)\, \delta^{ab} , \qquad \langle c^a(p) \bar{c}^b(-p) \rangle_0 = \frac{1}{p^2} \delta^{ab}, \tag{3.2}$$

where we have already imposed the momentum conservation, omitting the Dirac delta, as part of the Feynman rules. To get the expressions for the vertices it is sufficient to functionally derive the Fourier transformed action w.r.t corresponding fields. For example, the Feynman rule for the ghost-gluon vertex is given by:

$$- \left. \frac{\delta^3 S}{\delta c^b(-k) \delta \bar{c}^a(p) \delta A^c_\mu(q)} \right|_{(A,\bar{c},c)=0} = i g f^{abc} p_\mu, \tag{3.3}$$

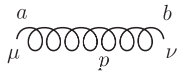
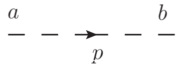
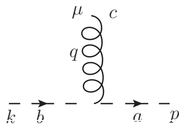
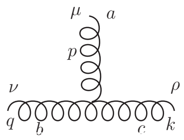
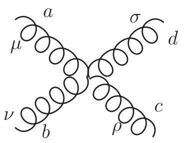
where again we have omitted the Dirac delta for the conservation of momentum. In the derivation of (3.3), the convention for the Fourier transform given in (1.36) is understood , where the momentum taken with a positive sign is meant as outgoing. The minus sign in the left side of (3.3) corresponds to the minus sign of the exponential weight inside the generating function Z. The other Feynman rules are given in Table 3.1.

3.2 Tensor reduction

The one-loop corrections to the two-point and three-point functions have been evaluated using the Mathematica package FeynCalc. This tool allows to perform the tensor reduction, also

[1] We could go back and integrate it out, but it is not necessary here in order to get the gluon propagator.

Table 3.1: Feynman rules

Feynman diagram	Feynman rule
	$\frac{1}{p^2+m^2}\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\delta^{ab}$
	$\frac{1}{p^2}\delta^{ab}$
	$igf^{abc}p_\mu$
	$-igf^{abc}\left[\delta_{\rho\nu}(k-q)_\mu+\delta_{\rho\mu}(p-k)_\nu+\delta_{\mu\nu}(q-p)_\rho\right]$
	$g^2\left[f^{abe}f^{cde}(\delta_{\mu\sigma}\delta_{\nu\rho}-\delta_{\mu\rho}\delta_{\nu\sigma})+f^{ace}f^{dbe}(\delta_{\mu\nu}\delta_{\rho\sigma}-\delta_{\mu\sigma}\delta_{\rho\nu})+f^{ade}f^{bce}(\delta_{\mu\rho}\delta_{\sigma\nu}-\delta_{\mu\nu}\delta_{\sigma\rho})\right]$

The gluon momenta are meant as outgoing and the ghost momenta as following the ghost flow.

known as the Passarino-Veltman reduction [64] of the Feynman integrals, i.e. to decompose the latter in terms of scalar integrals that do not contain powers of loop momentum in the numerator but only propagator denominators and whose expressions are given in the appendices. They are called scalar integrals because they represent the genuine Feynman integrals in a theory with scalar particles.

In order to sketch the general procedure of the tensor reduction let us define the integral with n propagators and a function of the loop momentum $f(l)$ in the numerator:

$$I_n[f(l)] \equiv \int \frac{d^Dl}{(2\pi)^D}\frac{f(l)}{(l^2+m_0^2)((l+q_1)^2+m_1^2)\cdots((l+q_{n-1})^2+m_{n-1}^2)}, \tag{3.4}$$

where $q_i \equiv p_1+\cdots p_i$, being p_i the n external incoming momentum. The scalar integrals correspond to the case $f(l)=1$. We illustrate the tensor reduction for one power of loop momentum in the numerator, $f(l)=l^\mu$. Since dimensional regularization preserves the Lorentz covariance, the integral must be a combination of the independent external momenta:

$$I_n[l^\mu] = \int \frac{d^Dl}{(2\pi)^D}\frac{l^\mu}{(l^2+m_0^2)((l+q_1)^2+m_1^2)\cdots((l+q_{n-1})^2+m_{n-1}^2)} = \sum_{i=1}^{n-1} A_{n,i}p_i^\mu. \tag{3.5}$$

Contracting each side with p_j^μ one gets:

$$I_n[p_j\cdot l] = \sum_{i=1}^{n-1} A_{n,i}\Delta_{ij}, \tag{3.6}$$

where $\Delta_{ij}=p_i\cdot p_j$ is the *Gram* matrix. Writing $p_j\cdot l=(q_j-q_{j-1})\cdot l$ as

$$p_j\cdot l = \frac{1}{2}\left(((l+q_j)^2+m_j^2)-((l+q_{j-1})^2+m_{j-1}^2)-m_j^2+m_{j-1}^2-q_j^2+q_{j-1}^2\right), \tag{3.7}$$

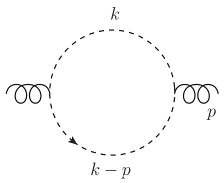
$I_n[p_j \cdot l]$ is reduced to a combination of scalar integrals:

$$I_n[p_j \cdot l] = \frac{1}{2}\left(I^{(j)}_{n-1}[1] - I^{(j-1)}_{n-1}[1] + (m_j^2 - m_{j-1}^2 - q_j^2 + q_{j-1}^2) I_n[1]\right) = \sum_{i=1}^{n-1} A_{n,i}\Delta_{ij}, \tag{3.8}$$

where we denote with $I^{(j)}_{n-1}[1]$ the scalar integral $I_n[1]$ without the $(j+1)^{th}$ propagator $((l+q_j)^2 + m_j^2)$. One can now invert (3.8) multiplying both sides with Δ^{-1}_{ij} and obtain the coefficients $A_{n,i}$ in terms of scalar integrals. For the case of two powers of loop momentum in the numerator $l^\mu l^\nu$ one can similarly decompose the integral in a combination of the tensor structures $\delta^{\mu\nu}$ and $p_i^\mu p_j^\nu$ and derive the coefficients by suitable contractions in terms of the previously calculated scalar integrals, in a sort of recursive procedure.

To better illustrate the tensor reduction, let us show it for the simple case of the ghost loop for the gluon self-energy:

$\Pi^{gh,ab}_{\mu\nu}(p) =$



$$= -g^2 N \delta^{ab} \int \frac{d^D k}{(2\pi)^D} \frac{k_\mu (k_\nu - p_\nu)}{k^2 (k-p)^2}. \tag{3.9}$$

We therefore need to reduce the integrals with two massless propagators and with one and two loop momenta in the numerator, $I_2[k_\mu]$ and $I_2[k_\mu k_\nu]$. Contracting $I_2[k_\mu] = A\, p_\mu$ with p_μ one gets:

$$I_2[p \cdot k] = \int \frac{d^D k}{(2\pi)^D} \frac{p \cdot k}{k^2 (k-p)^2} = A\, p^2. \tag{3.10}$$

Writing $p \cdot k$ as $\frac{1}{2}(k^2 - (k-p)^2 + p^2)$ and taking into account that in dimensional regularization

$$\int \frac{d^D k}{(2\pi)^D} \frac{l_\mu}{(l^2)^\kappa} = \int \frac{d^D k}{(2\pi)^D} \frac{1}{(l^2)^\kappa} = 0, \tag{3.11}$$

it turns out that

$$A = \frac{1}{2} I_2[1] = \frac{1}{2} \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 (k-p)^2}. \tag{3.12}$$

The term with two loop momenta can be decomposed in:

$$I_2[k_\mu k_\nu] = \int \frac{d^D k}{(2\pi)^D} \frac{k_\mu k_\nu}{k^2 (k-p)^2} = A_0 \delta_{\mu\nu} + A_1 p_\mu p_\nu. \tag{3.13}$$

Contracting both sides with $\delta_{\mu\nu}$ one gets:

$$0 = A_0\, D + A_1\, p^2. \tag{3.14}$$

Contracting with $(p_\mu p_\nu)/p^2$:

$$\frac{1}{p^2} I_2[(p \cdot k)^2] = \frac{1}{p^2} \int \frac{d^D k}{(2\pi)^D} \frac{(p \cdot k)^2}{k^2 (k-p)^2} = A_0 + A_1 p^2. \tag{3.15}$$

To obtain $I_2[(p \cdot k)^2]$ we substitute again one power of $p \cdot k$ with $\frac{1}{2}(k^2 - (k-p)^2 + p^2)$:

$$\frac{1}{p^2} I_2[(p \cdot k)^2] = \frac{1}{2p^2} \left[\int \frac{d^D k}{(2\pi)^D} \frac{p \cdot k}{(k-p)^2} - \int \frac{d^D k}{(2\pi)^D} \frac{p \cdot k}{k^2} + p^2 \int \frac{d^D k}{(2\pi)^D} \frac{p \cdot k}{k^2 (k-p)^2} \right]. \tag{3.16}$$

Due to (3.11) only the last term in parenthesis survives, which is just the $I_2[p\cdot k]$ previously obtained. We therefore get:

$$\frac{1}{p^2} I_2[(p\cdot k)^2] = \frac{1}{2} I_2[p\cdot k] = \frac{1}{2} A\, p^2 = A_0 + A_1 p^2. \tag{3.17}$$

Combining it with (3.14) we obtain

$$A_0 = -\frac{A\, p^2}{2(D-1)}, \qquad A_1 = \frac{A\, D}{2(D-1)}. \tag{3.18}$$

The ghost loop diagram is therefore reduced to:

$$\begin{aligned} \Pi^{gh,ab}_{\mu\nu}(p) &= -g^2 N \delta^{ab} \left[A_0 \delta_{\mu\nu} + (A_1 - A) p_\mu p_\nu\right] \\ &= g^2 N \delta^{ab} \frac{p^2 A}{2(D-1)} \left[\delta_{\mu\nu} + (D-2)\frac{p_\mu p_\nu}{p^2}\right] \\ &= g^2 N \delta^{ab} \frac{p^2}{4(D-1)} \left[\delta_{\mu\nu} + (D-2)\frac{p_\mu p_\nu}{p^2}\right] B_0(p^2,0,0), \end{aligned} \tag{3.19}$$

where we have introduced the traditional notation for the two-point scalar integrals, bookkeeper of the number of massive propagators:

$$B_0(p^2, m_0^2, m_1^2) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2+m_0^2)\left((k-p)^2+m_1^2\right)}, \tag{3.20}$$

whose expressions are given in the appendix. In all the diagrams evaluated in this work there is only one mass parameter present, the mass of the gluon, therefore only those corresponding scalar integrals are calculated. For future reference we also introduce the notation for the three-point scalar integrals:

$$C_0(p_1^2, (p_1-p_2)^2, p_2^2, m_0^2, m_1^2, m_2^2) = \int \frac{d^D l}{(2\pi)^D} \frac{1}{(l^2+m_0^2)\left((l+p_1)^2+m_1^2\right)\left((l+p_2)^2+m_2^2\right)}. \tag{3.21}$$

Since we will calculate the three-point vertices at the symmetric configuration of momenta, where the expressions depend only on the number of massive propagators and not on which one is massive, we also introduce the condensed notation (for D=4):

$$\begin{aligned} \tilde{J}_0 &\equiv \mu^2 (4\pi)^2 C_0(\mu^2,\mu^2,\mu^2,0,0,0), \\ \tilde{J}_1(t) &\equiv \mu^2 (4\pi)^2 C_0(\mu^2,\mu^2,\mu^2,m^2,0,0), \\ \tilde{J}_2(t) &\equiv \mu^2 (4\pi)^2 C_0(\mu^2,\mu^2,\mu^2,m^2,m^2,0), \\ \tilde{J}_3(t) &\equiv \mu^2 (4\pi)^2 C_0(\mu^2,\mu^2,\mu^2,m^2,m^2,m^2), \end{aligned} \tag{3.22}$$

where we defined the dimensionless integrals $\tilde{J}_i(t)$, functions of the dimensionless parameter $t = \mu^2/m^2$. The subindex i labels the number of massive scalar propagators.

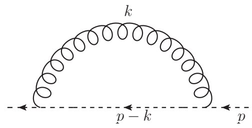
In the following illustration of Feynman diagrams we will not express the symbolic tensor reduction, implemented by FeynCalc, in terms of the scalar integrals, but we will directly show the final results where the scalar integrals are substituted with their expressions given in the appendices. Only the functions $\tilde{J}_i(t)$ will be kept in their symbolic notation, since they are given in terms of one dimensional integrals that will be eventually numerically evaluated.

3.3 Two-point functions

The gluon and ghost self energies, respectively denoted as $\Pi_{\mu\nu}(p)$ and $\Sigma(p)$ contain the quantum corrections of the corresponding two-point functions (the trivial color factor is not included):

$$\begin{aligned} \Gamma_{A_\mu A_\nu}(p) &= p^2\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + m^2\delta_{\mu\nu} - \Pi_{\mu\nu}(p), \\ \Gamma_{c\bar{c}}(p) &= p^2 - \Sigma(p). \end{aligned} \tag{3.23}$$

The gluon self energy splits into transverse and longitudinal parts $\Pi_{\mu\nu}(p) = \Pi^{\perp}(p)P^{\perp}_{\mu\nu}(p) + \Pi^{\parallel}(p)P^{\parallel}_{\mu\nu}(p)$. Below we give the explicit expressions for the Feynman diagrams that contribute at one-loop order to the ghost and gluon self energies, evaluated at dimension $D = 4 - \epsilon$. Because of the tensor structor of the gluonic part at tree-level, the expressions result simpler if written in terms of the combination $\left(\Pi^{\perp}(p) - \Pi^{\parallel}(p)\right)\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + \Pi^{\parallel}(p)\delta_{\mu\nu}$. The ghost self energy at one loop[2] is given by the following diagram:

$\Sigma(p) =$ 

$$= \frac{3}{4}\frac{Ng^2}{(4\pi)^2}p^2\,\overline{\frac{2}{\epsilon}} + \frac{1}{4}\frac{Ng^2}{(4\pi)^2}p^2\left(s^{-1} + 5 + s\ln s - s^{-2}(1+s)^3\ln(1+s)\right) \tag{3.24}$$

where $s = p^2/m^2$. In (3.24) we denoted the divergent term, that in dimensional regularisation include both eventual UV and IR divergences of the integral, as $\overline{\frac{2}{\epsilon}}$, given by:

$$\overline{\frac{2}{\epsilon}} = \frac{2}{\epsilon} - \gamma_E - \log(m^2/4\pi\kappa^2), \tag{3.25}$$

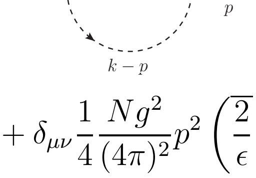
where γ_E is the Euler constant and κ is an arbitrary scale denoting the dimension of the coupling constant in $D = 4 - \epsilon$ ($g \to g\,\kappa^{\frac{\epsilon}{2}}$). In the IR limit ($s \ll 1$) the first terms of the diagram's expansion in powers of s are

$$\Sigma(p)/p^2 = \frac{3}{4}\frac{Ng^2}{(4\pi)^2}\left(\overline{\frac{2}{\epsilon}} + \frac{5}{6}\right) + \frac{1}{4}s\frac{Ng^2}{(4\pi)^2}\left(\ln s - \frac{11}{6}\right) + O(s^2), \tag{3.26}$$

while the UV limit ($s \gg 1$) is given by

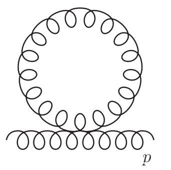
$$\Sigma(p)/p^2 = \frac{3}{4}\frac{Ng^2}{(4\pi)^2}\left(\overline{\frac{2}{\epsilon}} - \ln s + \frac{4}{3}\right) - \frac{3}{4}\frac{1}{s}\frac{Ng^2}{(4\pi)^2}\left(\ln s + \frac{1}{2}\right) + O(1/s^2). \tag{3.27}$$

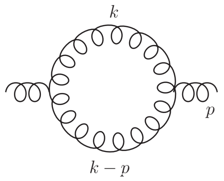
The gluon self energy at one loop contains the following diagrams. Among these only the one with a gluon loop has non trivial limits for low and high momentum.

$\Pi^{gh}_{\mu\nu}(p) =$ 

$$\begin{aligned} &= -\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{1}{6}\frac{Ng^2}{(4\pi)^2}p^2\left(\overline{\frac{2}{\epsilon}} - \ln s + \frac{5}{3}\right) \\ &+ \delta_{\mu\nu}\frac{1}{4}\frac{Ng^2}{(4\pi)^2}p^2\left(\overline{\frac{2}{\epsilon}} - \ln s + 2\right), \end{aligned} \tag{3.28}$$

[2]We omit the super index on the self energies that denote their one-loop approximation, being clear that this is the case.

$$\Pi^{tad}_{\mu\nu}(p) =$$  $$= \delta_{\mu\nu}\frac{9}{4}\frac{Ng^2}{(4\pi)^2}m^2\left(\frac{\overline{2}}{\epsilon}+\frac{1}{6}\right), \quad (3.29)$$

$$\Pi^{gl}_{\mu\nu}(p) =$$  $$= \left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\frac{Ng^2}{(4\pi)^2}p^2\left[\frac{7}{3}\left(\frac{\overline{2}}{\epsilon}\right)\right.$$

$$+\frac{1}{72s^2}\left(2(12-144s+131s^2)-6s^{-1}(1+s)^3(4-10s+s^2)\ln(1+s)\right.$$

$$\left.\left.+3s^4\ln s-3\sqrt{s}(4+s)^{3/2}(s^2-20s+12)\ln\left(\frac{\sqrt{4+s}-\sqrt{s}}{\sqrt{4+s}+\sqrt{s}}\right)\right)\right]$$

$$-\delta_{\mu\nu}\frac{Ng^2}{(4\pi)^2}p^2\left[\frac{1}{4}\left(\frac{\overline{2}}{\epsilon}\right)(12s^{-1}+1)+\frac{1}{8s^2}\left(2+13s+4s^2\right.\right.$$

$$\left.\left.-2s^{-1}(1+s)^3\ln(1+s)\right)\right]. \quad (3.30)$$

This last diagram has the following expansion in the deep IR ($s \ll 1$)

$$\Pi^{gl}_{\mu\nu}(p)/p^2 = \left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\frac{Ng^2}{(4\pi)^2}\left[\frac{7}{3}\left(\frac{\overline{2}}{\epsilon}\right)-\frac{11}{36}-\frac{217}{360}s+O(s^2)\right]$$
$$-\delta_{\mu\nu}\frac{Ng^2}{(4\pi)^2}\left[\frac{1}{4}\left(\frac{\overline{2}}{\epsilon}\right)(12s^{-1}+1)+\frac{1}{24}\left(24s^{-1}+1-\frac{3}{2}s\right)+O(s^2)\right] \quad (3.31)$$

and in the UV ($s \gg 1$)

$$\Pi^{gl}_{\mu\nu}(p)/p^2 = \left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\frac{Ng^2}{(4\pi)^2}\left[\frac{7}{3}\left(\frac{\overline{2}}{\epsilon}-\ln s\right)+\frac{107}{36}-\frac{1}{8s}(6\ln s+71)+O(1/s^2)\right]$$
$$-\delta_{\mu\nu}\frac{Ng^2}{(4\pi)^2}\left[\frac{1}{4}\left(\frac{\overline{2}}{\epsilon}-\ln s\right)(12s^{-1}+1)+\frac{1}{2}-\frac{1}{8s}(6\ln s-11)+O(1/s^2)\right] \quad (3.32)$$

We point out that in the deep IR the dominant contribution to the gluon self energy comes from the ghost loop diagram (3.28), which contains a term $\ln p^2$ while the other two diagrams approach a constant value in this limit. This had to be expected, since the diagram contains massless propagators which dominate in the IR over the massive gluonic ones.

For completeness we give the explicit expressions of the transverse and longitudinal parts of the gluon two-point function:

$$\Gamma^{\perp}_{AA}(p) = p^2+m^2+\frac{Ng^2}{(4\pi)^2}p^2\frac{\overline{2}}{\epsilon}\left(\frac{13}{6}-\frac{3}{4}s^{-1}\right)-\frac{Ng^2}{24(4\pi)^2}p^2\left[-\frac{242}{3}+126s^{-1}-2s^{-2}+(2-s^2)\ln s\right.$$
$$\left.+2(1+s^{-1})^3(1-10s+s^2)\ln(1+s)+(4s^{-1}+1)^{3/2}(12-20s+s^2)\ln\left(\frac{\sqrt{4+s}-\sqrt{s}}{\sqrt{4+s}+\sqrt{s}}\right)\right], \quad (3.33)$$

$$\Gamma^{\parallel}_{AA}(p) = m^2 - \frac{3}{4}\frac{Ng^2}{(4\pi)^2}m^2\frac{\overline{2}}{\epsilon} - \frac{Ng^2}{4(4\pi)^2}m^2\left[s^{-1} + 5 + s\ln s - s^{-1}(1+s)^3\ln(1+s)\right]. \quad (3.34)$$

In the deep IR ($p^2 \ll m^2$) they approach the following limits:

$$\Gamma^{\perp}_{AA}(p^2) = p^2 + m^2 + \frac{3}{4}m^2\frac{Ng^2}{(4\pi)^2}\left(\frac{\overline{2}}{\epsilon} + \frac{5}{6}\right) - \frac{13}{6}p^2\frac{Ng^2}{(4\pi)^2}\left(\frac{\overline{2}}{\epsilon} - \frac{1}{26}\ln\frac{p^2}{m^2} - \frac{3}{52}\right), \quad (3.35)$$

$$\Gamma^{\parallel}_{AA}(p^2) = m^2 + \frac{3}{4}m^2\frac{Ng^2}{(4\pi)^2}\left(\frac{\overline{2}}{\epsilon} + \frac{5}{6}\right) - \frac{1}{4}p^2\frac{Ng^2}{(4\pi)^2}\left(-\ln\frac{p^2}{m^2} + \frac{11}{6}\right). \quad (3.36)$$

From these expressions we can see how the transverse and longitudinal parts satisfy, at least up to the one-loop order, the locality condition:

$$\Gamma^{\perp}_{AA}(0) = \Gamma^{\parallel}_{AA}(0). \quad (3.37)$$

We also stress that in the IR the coefficients of the $\ln p^2$ do not coincide with the coefficients of the divergent terms $2/\epsilon$, as it happens in the UV. This is due to the presence of the gluonic mass. The asymptotic expressions in the UV ($p^2 \gg m^2$) are:

$$\Gamma^{\perp}_{AA}(p^2) = p^2 + m^2 + \frac{3}{4}m^2\frac{Ng^2}{(4\pi)^2}\left(\frac{\overline{2}}{\epsilon} + \frac{79}{6}\right) - \frac{13}{6}p^2\frac{Ng^2}{(4\pi)^2}\left(\frac{\overline{2}}{\epsilon} - \ln\frac{p^2}{m^2} + \frac{97}{78}\right), \quad (3.38)$$

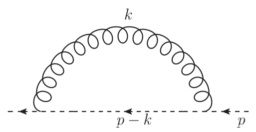
$$\Gamma^{\parallel}_{AA}(p^2) = m^2 + \frac{3}{4}m^2\frac{Ng^2}{(4\pi)^2}\left(\frac{\overline{2}}{\epsilon} - \ln\frac{p^2}{m^2} + \frac{4}{3}\right). \quad (3.39)$$

The divergent terms $2/\epsilon$ multiply only the local terms present at the classical level, which permits to absorb them in the definitions of the renormalized mass and field strengths, as we will examine in the next chapter. This was obviously to be expected being the theory renormalizable.

3.3.1 Two-point functions in D=3

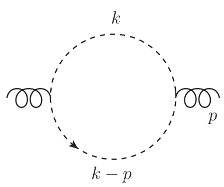
We also calculated the Feynman diagrams for the one-loop corrections to the two-point functions in dimension D=3, where such diagrams are finite.

$\Sigma(p) =$



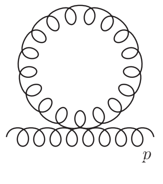
$$= \frac{Ng^2}{32\pi}p\,\left[2\sqrt{s}(s-1) - \pi s^2 + 2(s+1)^2\arctan\left(\sqrt{s}\right)\right] \quad (3.40)$$

$\Pi^{gh}_{\mu\nu}(p) =$

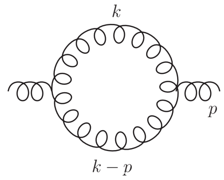


$$= -\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{Ng^2}{64}p + \delta_{\mu\nu}\frac{Ng^2}{32}p, \quad (3.41)$$

$\Pi^{tad}_{\mu\nu}(p) =$



$$= \delta_{\mu\nu}\frac{Ng^2}{3\pi}m, \quad (3.42)$$

$\Pi^{gl}_{\mu\nu}(p) =$ 

$$= \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) \frac{Ng^2}{128\pi} \frac{m}{s^{3/2}} \Big[4\sqrt{s}(5s^2 + 9s - 3) - \pi s^3$$
$$+4(s^2 - 6s + 3)(s+1)^2 \arctan\left(\sqrt{s}\right) - 2s(s+4)(s^2 - 12s + 8) \arctan\left(\frac{\sqrt{s}}{2}\right)\Big]$$
$$- \delta_{\mu\nu} \frac{Ng^2}{48\pi} \frac{m}{s^{3/2}} \left[\sqrt{s}(19s - 3) + 3(s+1)^2 \arctan\left(\sqrt{s}\right)\right] . \tag{3.43}$$

The transverse and longitudinal part of the gluon two-point function are therefore given by:

$$\Gamma^{\perp}_{AA}(p) = p^2 + m^2 - \frac{Ng^2}{128\pi} \frac{m}{s^{3/2}} \Big[4\sqrt{s}\,(5s^2 + 7s - 1) - \pi s^3 \tag{3.44}$$
$$+4(s^2 - 6s + 1)(s+1)^2 \arctan\left(\sqrt{s}\right) - 2s(s+4)(s^2 - 12s + 8) \arctan\left(\frac{\sqrt{s}}{2}\right)\Big] ,$$

$$\Gamma^{\parallel}_{AA}(p) = m^2 - \frac{Ng^2}{32\pi} \frac{m}{s^{3/2}} \left[-2\sqrt{s}(s-1) + \pi s^2 - 2(s+1)^2 \arctan\left(\sqrt{s}\right)\right] . \tag{3.45}$$

3.4 Three-point functions

Below we show the result for the one-loop corrections to the three-point vertices, i.e. the ghost-gluon vertex and the three-gluon vertex. The interest in these results is twofold: in the next chapter we will use them on one hand to estimate the behaviour of the vertices dressing functions at all energies through the integration of the renormalization group equations, and on the other to define the renormalized coupling constants in different renormalization schemes.

3.4.1 Ghost-gluon vertex

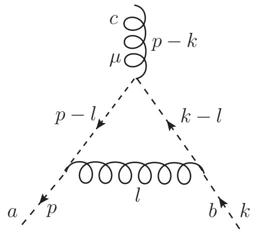
The ghost-gluon vertex has played a fundamental role in the previous analysis of the IR properties of the propagators in Landau gauge (and Coulomb gauge). This is due to the Taylor non renormalization theorem [28] which establishes how in these transverse gauges the quantum corrections of the ghost-gluon vertex vanish at all orders when the incoming ghost momentum vanishes. This has allowed to solve the coupled system of DSEs for the ghost and gluon propagators approximating the full vertex with its tree-level counterpart, since also in other momenta configurations its contribution stays close to the bare vertex, as it as been confirmed from lattice calculations and from its DSE [65]. This approximation allows to close the two-point system of DSEs if one is interested only in the IR behaviour, in a truncation that includes only the ghost-loop diagram, justified by the IR ghost dominance [30, 38, 39, 40]. With the addition of a proper ansatz for the three-gluon vertex introduced *ad hoc* in order to recover the correct UV behaviour of the propagators, it was possible to solve for whole range of momenta [23, 36, 37, 65].

The full ghost-gluon vertex is denoted as:

$$\Gamma^{abc}_{\bar{c}cA_\mu}(p, -k, k-p) = igf^{abc}\Gamma_\mu(p, -k, k-p)$$
$$= igf^{abc}\left[P^{\perp}_{\mu\nu}(k-p)p_\nu D^t_{\bar{c}cA}(p, k, k-p) + (k-p)_\mu D^l_{\bar{c}cA}(p, k, k-p)\right] , \tag{3.46}$$

where it has been decomposed into its transverse and longitudinal dressing functions $D^t_{\bar{c}cA}(p,k,k-p)$ and $D^l_{\bar{c}cA}(p,k,k-p)$, w.r.t the external gluon momentum. Since we work in Landau gauge we consider the external gluon attached to its transverse projector $P^{\perp}_{\mu\nu}(k-p)$ and therefore we consider only the transverse dressing function (omitting the super index t from now on).

Below we show the expressions at the symmetry point ($p^2 = k^2 = (p-k)^2 = \mu^2$) for the two diagrams contributing to the one-loop correction of the ghost-gluon vertex, at dimension $D = 4$ (the diagrams are UV convergent in Landau gauge):



$$= iP^{\perp}_{\mu\nu}(p-k)p_\nu \frac{Ng^3 f^{abc}}{24(4\pi)^2 t^2}\Big[3t + t(2-t^2)\ln t - (1+t)(5-t^2)\ln(1+t)$$

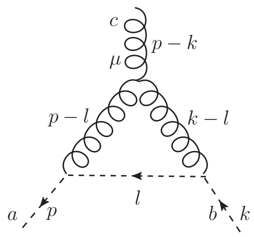
$$-t^3\tilde{J}_0 + (2+t)(1+t^2)\tilde{J}_1(t)\Big], \qquad t = \frac{\mu^2}{m^2} \tag{3.47}$$

where the $\tilde{J}_i(t)$ were defined in (3.22). This diagram in the IR ($t \ll 1$) limit behaves like

$$iP^{\perp}_{\mu\nu}(p-k)p_\nu \frac{Ng^3 f^{abc}}{288(4\pi)^2}\left[t(-30\ln t + 31 - 12\tilde{J}_0) + O(t^2)\right], \tag{3.48}$$

and in the UV ($t \gg 1$)

$$iP^{\perp}_{\mu\nu}(p-k)p_\nu \frac{Ng^3 f^{abc}}{96(4\pi)^2}\left[8\tilde{J}_0 + \frac{1}{t}(-18\ln t + 3 - 4\tilde{J}_0) + O(1/t^2)\right]. \tag{3.49}$$



$$= iP^{\perp}_{\mu\nu}(p-k)p_\nu \frac{Ng^3 f^{abc}}{48(4\pi)^2 t^2}\Big[2t(2+9t) - t^2(4+18t-3t^2)\ln t$$

$$-6(1+t)^2(2-t+t^2)\ln(1+t) - \sqrt{t(4+t)}(8-4t+18t^2+3t^3)\ln\left(\frac{\sqrt{4+t}-\sqrt{t}}{\sqrt{4+t}+\sqrt{t}}\right)$$

$$+2t^4\tilde{J}_0 - 4t(1+t)(1+4t+t^2)\tilde{J}_1(t) - 2(1+t)^2(4-8t-t^2)\tilde{J}_2(t)\Big], \tag{3.50}$$

whose limit in the IR is

$$iP^{\perp}_{\mu\nu}(p-k)p_\nu \frac{Ng^3 f^{abc}}{72(4\pi)^2}\left[51t + t^2(-31 + 3\tilde{J}_0) + O(t^3)\right], \tag{3.51}$$

and in the UV is

$$iP^{\perp}_{\mu\nu}(p-k)p_\nu \frac{Ng^3 f^{abc}}{24(4\pi)^2}\left[18 + 3\tilde{J}_0 - \frac{2}{t}(6 + 9\ln t + \tilde{J}_0) + O(1/t^2)\right]. \tag{3.52}$$

Summing the expressions for the two diagrams we get for the transverse dressing function at the symmetry point:

$$D_{\bar{c}cA}(\mu^2,\mu^2,\mu^2) = 1 + \frac{g^2N}{48(4\pi)^2t^2}\left[2t(9t+5) + t(3t^3-20t^2-4t+4)\ln t - 2(t+1)(3t^3-t^2+3t+11)\right.$$
$$\times \ln(1+t) - (3t^3+18t^2-4t+8)\sqrt{t(t+4)}\ln\left(\frac{\sqrt{4+t}-\sqrt{t}}{\sqrt{4+t}+\sqrt{t}}\right) + 2t^3(t-1)\tilde{J}_0$$
$$\left. -2(2t^4+9t^3+8t^2+t-2)\tilde{J}_1(t) + 2(t+1)^2(t^2+8t-4)\tilde{J}_2(t)\right], \tag{3.53}$$

which in the IR ($t \ll 1$) goes like:

$$D_{\bar{c}cA}(\mu^2,\mu^2,\mu^2) = 1 + \frac{g^2N}{288(4\pi)^2}\left[t(235 - 12\tilde{J}_0 - 30\ln t) + O(t^2)\right], \tag{3.54}$$

and in the UV ($t \gg 1$):

$$D_{\bar{c}cA}(\mu^2,\mu^2,\mu^2) = 1 + \frac{g^2N}{24(4\pi)^2}\left[18 + 5\tilde{J}_0 + O\left(\frac{1}{t}\right)\right]. \tag{3.55}$$

It can be thus seen that the ghost-gluon dressing function approaches a constant value at high momentum (it will actually approach unity when the coupling constant will be let run, due to the asymptotic freedom), and its one-loop correction is suppressed in the IR with a power of $t = \mu^2/m^2$.

The ghost-gluon dressing function has also been evaluated in the configuration of vanishing external gluon momentum. The three-point scalar integrals in this limit reduce to the expressions given in the appendix.

$$D_{\bar{c}cA}(p^2,p^2,0) = 1 + \frac{g^2N}{16(4\pi)^2t^2}\left[(t+2)(2t+5) + 2t^3(t-1)\ln t - 2(1+t)^2(t^2-3t+5)\ln(1+t)\right], \tag{3.56}$$

which shares the same qualitative asymptotic limits as in the case of the symmetry point configuration.

3.4.2 Three-gluon vertex

The three-gluon vertex has been studied so far using different approaches. Perturbatively, it has been calculated off-shell, in a covariant gauge, at one-loop level [66, 67, 68, 69, 70, 71] and in some special momentum configurations at two-loops [72, 73, 74]. In [75] a compact representation of the vertex at one-loop in Feynman gauge is obtained using the world-line formalism.

At a non-perturbative level the three-gluon vertex was mainly studied using DSEs. An IR analysis in Landau gauge was conducted, focusing on the scaling solution, using a power counting technique based on the skeleton expansion [76, 77, 78]. The IR divergent behaviour of the three-gluon vertex for this kind of solution was derived in both the uniform limit and in the soft momentum configuration. The role of these IR divergencies in the behaviour of the quark-gluon vertex and the role of the latter in the mechanisms of dynamical chiral symmetry breaking and confinement was studied in [79].

The IR behaviour of the three-gluon vertex was also studied in [82] in relation to the generation of massless composite excitations that might trigger the Schwinger mechanism and therefore generate a gluonic mass dynamically.

A full momentum dependence of the three-gluon vertex was derived in Landau gauge by numerically solving its DSE, for the dressing function corresponding to the tree-level tensor structure [80] and for the full transverse tensor basis [81].

The fully dressed three-gluon vertex is denoted as:

$$\Gamma_{A^a_\mu A^b_\nu A^c_\rho}(p_1, p_2, p_3) = -igf^{abc}\,\Gamma_{\mu\nu\rho}(p_1, p_2, p_3), \tag{3.57}$$

where we factored out the colour structure. The latter being totally antisymmetric, $\Gamma_{\mu\nu\rho}(p_1, p_2, p_3)$ must be totally antisymmetric in the exchange of two arbitrary momenta and corresponding Lorentz indices, since the full vertex must be totally symmetric, being obtained by the functional derivatives of the effective action w.r.t. bosonic gauge fields.

The full covariant basis for the three-gluon vertex was found by Ball and Chiu in [68] where in the decomposition of the vertex into its independent tensor structures and corresponding dressing functions, terms with mixed symmetry in the exchange of momenta appear. In [81] a fully antisymmetric tensor basis, with corresponding fully symmetric scalar dressing functions, was constructed by properly combining the irreducible representations of the permutation group S_3, i.e. its symmetric and antisymmetric singlets and its mixed-symmetric doublets. In Landau gauge it is sufficient to consider the transverse tensor structures, i.e. the ones that survive under contraction with the external transverse projectors. Such fully antisymmetric basis consists of four terms:

$$\begin{aligned}
\tau_1^{\mu\nu\rho} &= (p_2 - p_3)^\mu \delta^{\nu\rho} + (p_3 - p_1)^\nu \delta^{\rho\mu} + (p_1 - p_2)^\rho \delta^{\mu\nu}, \\
\mathcal{S}_0\, \tau_2^{\mu\nu\rho} &= (p_2 - p_3)^\mu (p_3 - p_1)^\nu (p_1 - p_2)^\rho, \\
\mathcal{S}_0\, \tau_3^{\mu\nu\rho} &= p_1^2 (p_2 - p_3)^\mu \delta^{\nu\rho} + p_2^2 (p_3 - p_1)^\nu \delta^{\rho\mu} + p_3^3 (p_1 - p_2)^\rho \delta^{\mu\nu}, \\
-\frac{\mathcal{S}_1}{\mathcal{A}} \mathcal{S}_0 \tau_4^{\mu\nu\rho} &= (p_3^2 - p_2^2)(p_2 - p_3)^\mu \delta^{\nu\rho} + (p_1^2 - p_3^2)(p_3 - p_1)^\nu \delta^{\rho\mu} + (p_2^2 - p_1^2)(p_1 - p_2)^\rho \delta^{\mu\nu},
\end{aligned} \tag{3.58}$$

where $\mathcal{S}_0$ and $\mathcal{S}_1$ are two of the three totally symmetric singlets under permutation of the scalar momenta, and $\mathcal{A}$ is the totally antisymmetric singlet:

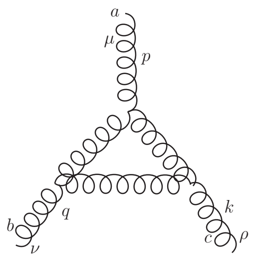
$$\begin{aligned}
\mathcal{S}_0 &= \frac{1}{6}(p_1^2 + p_2^2 + p_3^2), \\
\mathcal{S}_1 &= \frac{1}{9\,\mathcal{S}_0^2}(p_1^4 + p_2^4 + p_3^4 - p_1^2 p_2^2 - p_1^2 p_3^2 - p_2^2 p_3^2), \\
\mathcal{S}_2 &= \frac{1}{54\,\mathcal{S}_0^3}(p_1^2 + p_2^2 - 2p_3^2)(2p_1^2 - p_2^2 - p_3^2)(p_1^2 - 2p_2^2 + p_3^2), \\
\mathcal{A} &= -\frac{1}{6\sqrt{3}\,\mathcal{S}_0^3}(p_1^2 - p_2^2)(p_1^2 - p_3^2)(p_2^2 - p_3^2),
\end{aligned} \tag{3.59}$$

that are given in such a way that only S_0 has the dimension of momentum square, all the others are dimensionless. The first structure in (3.58) clearly corresponds to the tree-level vertex.

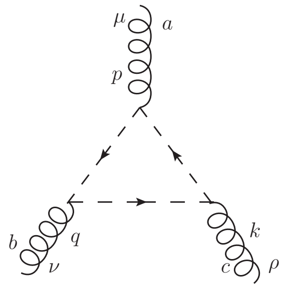
At the symmetry point where $p_1^2 = p_2^2 = p_3^2 = \mu^2$ only two out of the four independent structures in (3.58) remain, since in this momentum configuration the third one coincides with the first one (save a numeric multiplicative factor) and the fourth one vanishes. After contracting with the transverse projectors corresponding to the external legs $P^\perp_{\mu\alpha}(p1)$, $P^\perp_{\nu\beta}(p_2)$, $P^\perp_{\rho\gamma}(p_3)$, which does not change the symmetry properties of the tensor, imposing the momentum conservation ($p_1 + p_2 + p_3 = 0$) and renaming the momenta $p_1 \to p$, $p_2 \to q = -p - k$, $p_3 \to k$, the two tensor structures are given by:

$$\begin{aligned}
\mathcal{T}_1 \equiv P^{\perp}_{\mu\alpha}(p)P^{\perp}_{\nu\beta}(-k-p)P^{\perp}_{\rho\gamma}(k)\tau_{1\alpha\beta\gamma}\big|_{s.p.} &= (\delta_{\mu\rho}(k_\nu - p_\nu) + \delta_{\mu\nu}(k_\rho + 2p_\rho) - \delta_{\nu\rho}(2k_\mu + p_\mu)) \\
&+ \frac{1}{2\mu^2}\left[2k_\mu k_\rho(k_\nu + p_\nu) - p_\mu k_\rho(p_\nu - k_\nu) - 2p_\mu p_\rho(p_\nu + k_\nu)\right], \\
\mathcal{T}_2 \equiv P^{\perp}_{\mu\alpha}(p)P^{\perp}_{\nu\beta}(-k-p)P^{\perp}_{\rho\gamma}(k)\tau_{2\alpha\beta\gamma}\big|_{s.p.} &= \frac{2}{\mu^2}(p_\mu + 2k_\mu)(p_\nu - k_\nu)(2p_\rho + k_\rho).
\end{aligned} \tag{3.60}$$

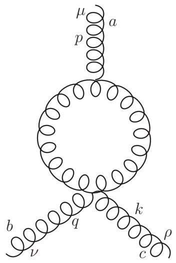
The one-loop corrections to the three-gluon vertex with external momenta at the symmetry point are given by the following diagrams (the momenta of the external legs are all outgoing and are contracted with the corresponding transverse projectors):



$$\begin{aligned}
&= \frac{ig^3 N f^{abc}}{(4\pi)^2}\left\{\frac{\mathcal{T}_1}{2}\frac{\overline{2}}{\epsilon} + \frac{\mathcal{T}_1}{144t^2}\left[6t(43t-21) + 3t^2(t^2+11t+6)\log(t)\right.\right. \\
&- 6(1+t)^2(t^2+42t-31)\log(1+t) - 3\sqrt{t(t+4)}(t^3+75t^2-32t-20)\log\left(\frac{\sqrt{t+4}-\sqrt{t}}{\sqrt{t+4}+\sqrt{t}}\right) \\
&+ 2t^5\tilde{J}_0 - 6t(1+t)^2(t^2-t-3)\tilde{J}_1(t) + 6(t-5)(1+t)^2(t^2+5t-2)\tilde{J}_2(t) \\
&\left. -2t(t^4+3t^3-66t^2-132t+72)\tilde{J}_3(t)\right] + \frac{\mathcal{T}_2}{864t^2}\left[3(31t^2+52t-14) + 3t(2-7t)(1+t)^2\log(t)\right. \\
&+ 6t^{-1}(1+t)^3(7t^2+9t+9)\log(1+t) + 3\sqrt{1+4t^{-1}}(7t^4+34t^3+57t^2+68t+4)\log\left(\frac{\sqrt{t+4}-\sqrt{t}}{\sqrt{t+4}+\sqrt{t}}\right) \\
&+ 2t^5\tilde{J}_0 - 3(1+t)^2(2t^3+3t^2+6t-2)\tilde{J}_1(t) + 6t^{-1}(1+t)^5(t+2)\tilde{J}_2(t) \\
&\left.\left. -t(2t+9)(t^3+6t^2+12t+24)\tilde{J}_3(t)\right]\right\}.
\end{aligned} \tag{3.61}$$



$$+ \circlearrowright = \frac{ig^3 N f^{abc}}{216(4\pi)^2}\left\{\mathcal{T}_1\left[-9\frac{\overline{2}}{\epsilon} - 27 + 9\log(t) + 3\tilde{J}_0\right] + \mathcal{T}_2\left[-\frac{11}{4} + \tilde{J}_0\right]\right\}. \tag{3.62}$$



$$+\,perm. = \frac{ig^3 N f^{abc}}{16(4\pi)^2}\mathcal{T}_1 \left\{ -30\,\overline{\frac{2}{\epsilon}} + 10t^{-1} - 48 - t^2\log(t) + 2t^{-2}(t-5)(1+t)^3\log(1+t) \right.$$

$$\left. +(t^2-6t-40)\sqrt{1+4t^{-1}}\log\left(\frac{\sqrt{t+4}-\sqrt{t}}{\sqrt{t+4}+\sqrt{t}}\right)\right\}. \tag{3.63}$$

The diagram with the ghost triangle is summed to the same diagram with the inverted ghost flow, which at the symmetry point is equal to the first diagram. The latter is the IR dominant diagram, because of its massless internal propagators containing a logarithmic divergence. The last diagram, called *swordfish*, is summed to other two diagrams with the external momenta permuted. The sum of these three swordish diagrams contains only the tree-level tensor structure. We also stress that the divergent terms in each diagram are contained only in the tree-level structure, as it should be in a renormalizable theory.

For the three-gluon vertex, what is actually calculated in the lattice and what will be used later to define a sensible renormalized coupling constant, is actually the dressing function obtained by contracting the vertex $\Gamma_{\mu\nu\rho}(p,q,k)$ with the external transverse projectors and the tree-level structure and normalized with the same contraction with only the tree-level structures:

$$D_{AAA}(p,q,k) = \frac{(\delta_{\rho\mu}(k-q)_\mu + perm.)\Gamma_{\sigma\xi\gamma}(p,q,k)P^{\perp}_{\mu\sigma}(p)P^{\perp}_{\nu\xi}(q)P^{\perp}_{\gamma\rho}(k)}{(\delta_{\rho\mu}(k-q)_\mu + perm.)(\delta_{\gamma\xi}(k-q)_\sigma + perm.)P^{\perp}_{\mu\sigma}(p)P^{\perp}_{\nu\xi}(q)P^{\perp}_{\gamma\rho}(k)}. \tag{3.64}$$

Summing the diagrams given above and performing the proper contractions we therefore get for the three-gluon dressing function at the symmetry point:

$$\begin{aligned} D_{AAA}(\mu^2,\mu^2,\mu^2) = {} & 1 - \frac{17}{12}\frac{g^2N}{(4\pi)^2}\overline{\frac{2}{\epsilon}} + \frac{g^2N}{1584(4\pi)^2t^3}\left\{-3t^2(8t^3-145t^2-94t+4)\log(t)\right. \\ & - 4t(569t^2+177t-21) + 3(8t^4-1091t^3-1082t^2+84t-8)\sqrt{t(t+4)} \\ & \times\log\left(\frac{\sqrt{t+4}-\sqrt{t}}{\sqrt{t+4}+\sqrt{t}}\right) + 12(1+t)^2(4t^3-313t^2+70t-9)\log(1+t) \\ & + 2t^3(9t^3+10)\tilde{J}_0 - 6t(1+t)^2(9t^3-14t^2-39t+2)\tilde{J}_1(t) \\ & \left. +6(1+t)^2(9t^4-10t^3-315t^2+96t-4)\tilde{J}_2(t) - 24t^2(t^3-67t^2-134t+48)\tilde{J}_3(t)\right\} \end{aligned} \tag{3.65}$$

The IR limit of the dressing function is:

$$D_{AAA}(\mu^2,\mu^2,\mu^2) = 1 - \frac{17}{12}\frac{g^2N}{(4\pi)^2}\overline{\frac{2}{\epsilon}} + \frac{g^2N}{1584(4\pi)^2}\Big(66\log(t) + 187 + 20\tilde{J}_0 + O(t)\Big), \tag{3.66}$$

where the $\log(t)$ comes from the ghost triangle diagram. In UV it goes like:

$$D_{AAA}(\mu^2,\mu^2,\mu^2) = 1 - \frac{17}{12}\frac{g^2N}{(4\pi)^2}\left(\overline{\frac{2}{\epsilon}} - \log(t)\right) + \frac{g^2N}{396(4\pi)^2}\left(-167 + 128\tilde{J}_0 + O\left(\frac{1}{t}\right)\right). \tag{3.67}$$

We can again ascertain that the coefficient of the logarithmic term coincides with the coefficient of the $2/\epsilon$ term only in the high energy limit, where the effect of the gluon mass becomes negligible.

In the limit when one gluon external momentum vanishes ($p_\mu \to 0$), before contracting with the external transverse projections, there are three independent tensor structures. This can be seen because, by Lorentz covariance, there are four structures involving the only momentum k: $\delta_{\mu\nu}k_\rho$, $\delta_{\mu\rho}k_\nu$, $\delta_{\rho\nu}k_\mu$, $k_\mu k_\nu k_\rho$, but imposing the Bose-symmetry property to the vertex makes two of these terms, $k_\rho\delta_{\mu\nu}$ and $k_\nu\delta_{\rho\mu}$, group together. In this limit, without transverse projections, the vertex reads (now $t = k^2/m^2$):

$$\begin{aligned}\Gamma_{\mu\nu\rho}(0,-k,k) = (-k_\nu\delta_{\rho\mu} - k_\rho\delta_{\mu\nu})\Big\{1 - \frac{17}{12}\frac{g^2N}{(4\pi)^2}\frac{\overline{2}}{\epsilon} - \frac{g^2N}{144(4\pi)^2t^3}\big[t(6t^3+355t^2-594t+36)\\
+3t^3(3t^2-2)\log(t) - 3(t(t+4))^{3/2}(t^2-30t+24)\log\left(\frac{\sqrt{t+4}-\sqrt{t}}{\sqrt{t+4}+\sqrt{t}}\right)\\
-12(1+t)^3(t^2-9t+3)\log(1+t)\big]\Big\} + 2k_\mu\delta_{\rho\nu}\Big\{1 - \frac{17}{12}\frac{g^2N}{(4\pi)^2}\frac{\overline{2}}{\epsilon} + \frac{g^2N}{144(4\pi)^2t^3}\big[\\
t(24t^3-131t^2-321t+18) + 3t^3(t-1)(t^2-2t-2)\log(t)\\
-6(1+t)^2(t-1)(t^3-7t^2+7t-3)\log(1+t)\\
-3t\sqrt{t(t+4)}(t^4-11t^3+22t^2+16t+48)\log\left(\frac{\sqrt{t+4}-\sqrt{t}}{\sqrt{t+4}+\sqrt{t}}\right)\big]\Big\}\\
-(k_\mu k_\nu k_\rho)\frac{g^2N}{(4\pi)^2}\frac{1}{72m^2t^4}\big\{2t(15t^3+112t^2-417t+54) + 3t^6\log(t)\\
-6(1+t)^2(t^4-6t^3-2t^2-22t+18)\log(1+t)\\
-3t(t^4-10t^3-4t^2-80t+144)\sqrt{t(t+4)}\log\left(\frac{\sqrt{t+4}-\sqrt{t}}{\sqrt{t+4}+\sqrt{t}}\right)\big\}.\end{aligned} \tag{3.68}$$

The dressing function D_{AAA} in this momentum configuration is given by:

$$\begin{aligned}D_{AAA}(0,k^2,k^2) = 1 - \frac{17}{12}\frac{g^2N}{(4\pi)^2}\frac{\overline{2}}{\epsilon} + \frac{g^2N}{144(4\pi)^2t^3}\big[3t^3(t-1)(t^2-2t-2)\log(t)\\
-6(1+t)^2(t-1)(t^3-7t^2+7t-3)\log(1+t) + t(24t^3-95t^2-321t+18)\\
-3t(t^4-11t^3+22t^2+16t+48)\sqrt{t(t+4)}\log\left(\frac{\sqrt{t+4}-\sqrt{t}}{\sqrt{t+4}+\sqrt{t}}\right)\big],\end{aligned} \tag{3.69}$$

which also has the same qualitative asymptotic expansions as in the symmetry point configuration.

Chapter 4

Renormalization group equations

4.1 Callan-Symanzik equations

The renormalization group is a powerful tool used in Quantum Field Theory as well as in Statistical Mechanics in order to extract non perturbative information about the effective behaviour of the theory at low energies. Its original formulation due to Wilson [18, 19] is based on the functional integration formulation and consists in integrating small shells of momenta in order to assess the influence of the quantum (or statistical) fluctuations throughout the flow in the space of theories, generally towards a fixed point.

Although this formulation exhibits the whole power of the renormalization group, it is rather cumbersome to handle, especially dealing with a gauge theory, since it defines the theory with the use of a cut-off UV regulator, therefore its flow in the space of possible Lagrangians does not preserve the Ward (or Slavnov-Taylor) identities without the introduction of proper counter terms.

Technically it is simpler to work with flow equations that involve the renormalized quantities, for which the cut-off scale has already been sent to infinity (or in dimensional regularisation the poles in ϵ have already been absorbed in the parameters of the theory). This implies that it has already been assumed *a priori* that the theory approaches a fixed point around which it takes the form of a renormalizable theory. In order to study the effect of non renormalizable interactions (*irrelevant* operator) in a vicinity of the fixed point, one can add them to the renormalizable theory and study their flow through the renormalization group equations (cf. [49, 50]). In this work we are interested in studying the flow of Yang-Mills theory between two different fixed point (*crossover*): the well known UV trivial one (*Gaussian* fixed point), where the gluon mass vanishes, and an IR fixed point where the gluon mass operator dominates over the kinetic one [51] ($m^2 \gg p^2$), the so called *high temperature* fixed point, which, as shown in [51], is IR attractive, hence of physical relevance.

The renormalization group equations for renormalized correlation functions were originally formulated by Callan and Symanzik [20, 22], although in a different form than the one used here, since they were originally given in terms of variations w.r.t. the mass parameter, whose renormalized value was fixed on-shell, instead of variations w.r.t to the scale at which the renormalized functions and parameters are defined. This latter formulation used here, suitable for off-shell renormalization schemes, was actually introduced by Georgi and Politzer [21], but the designation "Callan-Symanzik equations" has historically stuck.

In the so called MOM (momentum subtraction) renormalization scheme the renormalized correlation functions and parameters are in fact defined at an arbitrary space-like momentum scale μ by suitable normalization conditions that fix the Z_i renormalization constants that multiplicatively relate the bare parameters and fields with their corresponding renormalized ones:

$$\begin{aligned} A_\mu^{a\,B}(x) &= Z_A^{1/2} A_\mu^a(x), & c^{a\,B}(x) &= Z_c^{1/2} c^a(x), \\ g_B &= Z_g g, & m_B^2 &= Z_{m^2} m^2, \end{aligned} \tag{4.1}$$

where we use the index B to denote the bare quantities. The Callan-Symanzik equations derive from the fact that the original bare theory, and hence the physics, does not depend on the arbitrary scale μ at which we choose to renormalize. The renormalized $(n_A + n_c)$-point function generated by the effective action, containing n_A external gluons and n_c ghosts, is related to its bare version by:

$$\Gamma_{n_A\, n_c}\left(\{p_i\}, \mu, g(\mu), m^2(\mu)\right) = Z_A^{n_A/2}(\mu) Z_c^{n_c/2}(\mu) \Gamma^B_{n_A\, n_c}\left(\{p_i\}, \epsilon, g_B, m_B^2\right). \tag{4.2}$$

The LHS represents the finite renormalized 1PI vertex, that depends explicitly on μ and implicitly through its dependence on the renormalized parameters. In the RHS the dependence on the renormalization scale μ is only in the renormalized constants Z_A and Z_c, while the bare vertex depends only on bare quantities and on a regularization parameter, here denoted ϵ, referring to dimensional regularization. Deriving both sides w.r.t. μ we therefore obtain:

$$\mu \frac{d}{d\mu} \Gamma_{n_A\, n_c}\left(\{p_i\}, \mu, g(\mu), m^2(\mu)\right) = \left(\frac{n_A}{2}\gamma_A(\mu) + \frac{n_c}{2}\gamma_c(\mu)\right) \Gamma_{n_A\, n_c}\left(\{p_i\}, \mu, g(\mu), m^2(\mu)\right), \tag{4.3}$$

where $\gamma_A(\mu)$ and $\gamma_c(\mu)$ are the anomalous dimension functions defined by:

$$\gamma_A(\mu) \equiv \frac{1}{Z_A} \mu \frac{d}{d\mu} Z_A \bigg|_{g_B, m_B}, \qquad \gamma_c(\mu) \equiv \frac{1}{Z_c} \mu \frac{d}{d\mu} Z_c \bigg|_{g_B, m_B}, \tag{4.4}$$

where we made explicit that the derivatives have to be taken keeping the bare quantities fixed. However, after the derivation is performed, the bare parameters must be substituted with the finite renormalized ones. Since the anomalous dimensions are dimensionless functions, their dependence on μ is explicitly contained in the dimensionless parameter $t = \mu^2/m^2$ and implicitly in its dependence on $g(\mu)$ and $m^2(\mu)$. It is important to stress that, although the definition of the anomalous dimensions involves the formally divergent renormalization constants Z_i, they are finite, because the divergent terms contained in the Z_i (the factors $2/\epsilon$) are not multiplied by any function of μ and hence they disappear once the derivative is performed.

Integrating (4.3) one gets:

$$\Gamma_{n_A\, n_c}\left(\{p_i\}, \mu, g(\mu), m^2(\mu)\right) = \Gamma_{n_A\, n_c}\left(\{p_i\}, \mu_0, g(\mu_0), m^2(\mu_0)\right) e^{-\int_\mu^{\mu_0} \frac{d\mu'}{\mu'}\left[\frac{n_A}{2}\gamma_A(\mu') + \frac{n_c}{2}\gamma_c(\mu')\right]}. \tag{4.5}$$

This equation connects the $(n_A + n_c)$-point function at two difference renormalization scales μ and μ_0. Inside the function, and inside the anomalous dimensions, the running coupling and running mass are given by the solutions of the flow equations defined by the beta functions:

$$\beta_g(\mu) \equiv \mu \frac{d}{d\mu} g(\mu) \bigg|_{g_B, m_B}, \qquad \beta_{m^2}(\mu) \equiv \mu \frac{d}{d\mu} m^2(\mu) \bigg|_{g_B, m_B}. \tag{4.6}$$

The beta functions, like the anomalous dimensions, are finite. Once they are given at a certain loop order, as functions of the renormalized parameters, they can be integrated, at least numerically, as the coupled system of ordinary differential equations (4.6) (coupled because they both contain both g and m^2). If the Callan-Symanzik equations gave us information only about the dependence of the correlation function on the renormalization scale, as showed in (4.3), they would not provide a lot of physical insight, since as we have stressed, physics does not depend on the arbitrary scale μ at which we renormalize. But if we choose as the arbitrary scale μ_0 the physical momentum scale p of the same order of the external momenta of the vertex we get:

$$\Gamma_{n_A\, n_c}\left(\{p_i\}, \mu, g(\mu), m^2(\mu)\right) = \Gamma_{n_A\, n_c}\left(\{p_i\}, p, g(p), m^2(p)\right) e^{-\int_\mu^p \frac{d\mu'}{\mu'}\left[\frac{n_A}{2}\gamma_A(\mu')+\frac{n_c}{2}\gamma_c(\mu')\right]}. \tag{4.7}$$

The point of this is that the factor multiplying the exponential in the RHS is now under control by perturbation theory. In fact, if in the collection of the external momenta $\{p_i\}$ only a relevant scale of order p appears (like in the case of the symmetry point or soft momentum configuration for the three-point vertices), this factor can simply coincide with its tree-level counterpart, usually given as the normalization condition at the renormalization scale, or be well approximated by the first loop correction, because the presence of the running parameters at the scale p makes sure that the expressions in the perturbative expansion remain relatively small, despite the presence of logarithmic terms that go like $\ln(p/\mu)$ and therefore become large when the momentum move away from the renormalization scale μ. We therefore see how, knowing (at least up to a certain approximation under control) the factor multiplying the exponential, (4.7) provides us with an expression for the full non perturbative correlation function, for the whole range of momenta. This is explained by the fact that, as one can show (cf. [50]), the insertion of the running parameters at the scale of momentum under interest, correspond to include the contribution of all the powers of the leading logarithmic terms of the whole perturbation expansion (or of further subleading logarithmic terms if we approximate the beta functions and the anomalous dimensions beyond the one-loop level).

Of course the whole procedure would break down if at some scale the parameters became too large, as it happens in the usual treatment of QCD where the running coupling constant flows towards a Landau pole. Fortunately, as we will show, the presence of the mass term allows to find some renormalization schemes in which the running values of both the coupling and the mass parameters remain finite and relatively small.

4.2 Slavnov-Taylor identity

In the renormalization schemes that we will shortly present, the following identity for the bare quantities plays a crucial role:

$$\Gamma^B_{c\bar{c}}(p^2)\,\Gamma^{\parallel\,B}_{AA}(p^2) = p^2 m_B^2. \tag{4.8}$$

We stress how this identity implies that the longitudinal part of the gluon two-point function vanishes only when $m = 0$, as expected. The identity can be obtained by exploiting the invariance of the action and of the integration measure under the BRST variation (2.42), from which it is easy to derive the Slavnov-Taylor identity for the effective action:

$$\int d^D x \left[\frac{\delta\Gamma}{\delta A^a_\mu(x)}\frac{\delta\Gamma}{\delta K^a_\mu(x)} + \frac{\delta\Gamma}{\delta c^a(x)}\frac{\delta\Gamma}{\delta L^a(x)} - iB^a(x)\frac{\delta\Gamma}{\delta\bar{c^a}(x)} + im^2 c^a(x)\frac{\delta\Gamma}{\delta B^a(x)}\right] = 0. \tag{4.9}$$

We will also make use of the antighost equation:

$$\partial_\mu\frac{\delta G}{\delta K^a(x)} + \eta^a(x) = -\partial_\mu\frac{\delta\Gamma}{\delta K^a(x)} + \frac{\delta\Gamma}{\bar{c}^a(x)} = 0, \tag{4.10}$$

where G is the generating functional of the connected diagrams previously introduced. (4.10) is derived by taking a total derivative w.r.t. antighost field inside the integral of the generating functional. In momentum space it reads

$$ip_\mu \frac{\delta\Gamma}{\delta K^a(p)} + \frac{\delta\Gamma}{\delta\bar{c}^a(p)} = 0. \tag{4.11}$$

Taking another derivative with respect to $c^b(-q)$ we obtain:

$$\Gamma_{c\bar{c}}(p) = -ip_\mu \Gamma_{cK_\mu}(p), \tag{4.12}$$

which, by Lorentz covariance, implies that $\Gamma_{cK_\mu}(p) = i\Gamma_{c\bar{c}}(p)p_\mu/p^2$. Deriving (4.9) with respect to $A^b_\nu(y)$ and $c^c(z)$ and going to momentum space, we get, at vanishing sources, considering the conservation of the ghost number at quantum level:

$$\Gamma_{A_\mu A_\nu}(p)\Gamma_{cK_\mu}(p) = im^2\Gamma_{A_\nu B}(p). \tag{4.13}$$

Substituting (4.12) and the expression for $\Gamma_{A_\nu B}(p) = p_\nu$ derived in (2.45), into (4.13) it follows:

$$p_\mu \Gamma_{A_\mu A_\nu}(p)\Gamma_{c\bar{c}}(p) = m^2 p^2 p_\nu. \tag{4.14}$$

Finally, decomposing the gluon two-point function in its transverse and longitudinal part, (4.8) is obtained. By taking the expressions of the gluon and ghost two-point functions calculated in section 3.3 one can check that the identity is indeed satisfied at one-loop order.

4.3 Renormalization schemes

Physical quantities, like scattering amplitudes, cannot depend on the arbitrary renormalization scale or on the normalization conditions we impose to the renormalized quantities at this scale. Even at a given order of expansion in perturbation theory, renormalized correlation functions defined in different renormalization schemes differ at most for an irrelevant multiplicative factor. Nevertheless in the integration of the Callan-Symanzik equations (4.7), since we are approximating the flow functions at one-loop order, different renormalization schemes can give different results. We therefore explore different normalization conditions in order to see which one can give better results as compare with lattice results.

It is important to stress that the expression for the beta function of the coupling constant changes according to different renormalization schemes. Its usual independence of the renormalization scheme up to two-loop orders is not valid here, because of the presence of the mass parameter. As we will see in fact, different renormalization schemes can even bring deep qualitative changes to the beta function, e.g. changing its sign at low energies, determining the appearance (or disappearance) of a Landau pole in the IR. If the Landau pole is not generated and the coupling runs towards an IR fixed point (which is trivial in $D = 4$ as we will see), the scheme is referred to as "IR safe".

4.3.1 Tissier-Wschebor scheme

In the "IR safe scheme" proposed by Tissier and Wschebor in Ref. [10], the following normalization conditions are imposed on the renormalized two-point functions at the renormalization scale μ:

$$\Gamma^{\perp}_{AA}(\mu^2) = \mu^2 + m^2\,, \tag{4.15}$$

$$\Gamma^{\parallel}_{AA}(\mu^2) = m^2\,, \tag{4.16}$$

$$\Gamma_{c\bar{c}}(\mu^2) = \mu^2\,. \tag{4.17}$$

The conditions (4.15) and (4.17) fix the values for the renormalization constants Z_A and Z_c, at one-loop order, to:

$$Z_c(\mu) = 1 + \frac{\Sigma^B(\mu^2)}{\mu^2}, \tag{4.18}$$

$$Z_A(\mu) = 1 + \frac{1}{\mu^2}\left(\Pi^{\perp\,B}(\mu^2) - \Pi^{\parallel\,B}(\mu^2)\right) = 1 + \frac{1}{\mu^2}\left(\Pi^{\perp\,B}(\mu^2) + \frac{m_B^2}{\mu^2}\Sigma^B(\mu^2)\right), \tag{4.19}$$

where Σ^B, $\Pi^{\perp\,B}$ and $\Pi^{\parallel\,B}$ are respectively the ghost, gluon transverse and gluon longitudinal self energies defined in (3.23). In the last identity in (4.19) we have used the Slavnov-Taylor identity (4.8) which for the self energies translate, at one-loop order, into:

$$m_B^2\Sigma^B(p^2) = -p^2\Pi^{\parallel\,B}(p^2). \tag{4.20}$$

By substituting the Z_A and Z_c, using the expressions for the self energies calculated in section 3.3, in the definitions of the anomalous dimensions (4.4) we get for these latter ($t = \mu^2/m^2$):

$$\gamma_c(t) = \frac{g^2N}{2(4\pi)^2t^2}\left[-2t(1+t) + t^3\ln t - (1+t)(t^2-t-2)\ln(1+t)\right], \tag{4.21}$$

$$\begin{aligned}\gamma_A(t) = &\frac{g^2N}{6(4\pi)^2t^2}\Big[-17t^2 + 74t - 12 + t^4\ln t - t^{-1}(2t-3)(t-2)^2(t+1)^2\ln(1+t) \\ &-\sqrt{t(t+4)}(t^3 - 9t^2 + 20t - 36)\ln\left(\frac{\sqrt{t+4}-\sqrt{t}}{\sqrt{t+4}+\sqrt{t}}\right)\Big].\end{aligned} \tag{4.22}$$

The Slavnov-Taylor identity (4.8) can be written in terms of the renormalized quantities as

$$\Gamma_{c\bar{c}}(p^2)\,\Gamma^{\parallel}_{AA}(p^2) = (Z_A Z_c Z_{m^2})p^2m^2\,. \tag{4.23}$$

If we specialize to $p^2 = \mu^2$ and apply the normalization condition (4.17), the Slavnov-Taylor identity implies that

$$\Gamma^{\parallel}_{AA}(\mu^2) = (Z_A Z_c Z_{m^2})m^2\,. \tag{4.24}$$

It is hence apparent that the normalization condition (4.16) is equivalent to $Z_A Z_c Z_{m^2} = 1$, provided that $\Gamma_{c\bar{c}}(p^2)$ is normalized as in Eq. (4.17), in which form it was originally proposed in Ref. [10]. We furthermore conclude from Eq. (4.23) that the normalization condition implies the renormalized counterpart of the Slavnov-Taylor identity,

$$\Gamma_{c\bar{c}}(p^2)\,\Gamma^{\parallel}_{AA}(p^2) = p^2m^2\,. \tag{4.25}$$

Since one of the principles in renormalizing a quantum field theory is to preserve the symmetries of the classical theory (whenever possible, i.e., in the absence of anomalies), the fact that the renormalization scheme (4.15)–(4.17) implies the identity (4.25) for the renormalized quantities is certainly satisfactory.

A second renormalization scheme presented in Ref. [10] replaces the normalization condition (4.16) with

$$\Gamma^{\parallel}_{AA}(0) = m^2\,. \tag{4.26}$$

In fact, in Ref. [10] the normalization condition was imposed on $\Gamma^{\perp}_{AA}(0)$, however, in three and four space-time dimensions we find that

$$\Gamma^{\perp\,B}_{AA}(0) = \Gamma^{\parallel\,B}_{AA}(0) \tag{4.27}$$

to any order in perturbation theory, meaning that the gluon two-point function $\Gamma^B_{A_\mu A_\nu}(p)$ is *local*, a property that carries over to the renormalized two-point function. Since $\Gamma^{\|}_{AA}(0) \neq \Gamma^{\|}_{AA}(\mu^2)$ in general, the argument above shows that the Slavnov-Taylor identity (4.25) for the renormalized quantities is not fulfilled in the renormalization scheme with condition (4.26). It was also shown in Ref. [10] that this scheme is not IR safe, i.e., the integration of the renormalization group equations leads to a Landau pole. We shall come back to the property of IR safeness below.

4.3.2 IR safeness

In order to establish the "IR safe" property of this renormalization scheme we need to investigate the IR behaviour of the beta function, which depends on the definition of renormalized coupling constant. Tissier and Wschebor defined it at the so called "Taylor point", meaning the configuration of momentum for which the ghost-gluon vertex does not receive quantum corrections in Landau gauge, i.e. at vanishing incoming ghost momentum:

$$g(\mu) \equiv g D_{\bar{c}cA}(\mu, 0, \mu) = g_B\, Z_A^{1/2}(\mu) Z_c(\mu). \tag{4.28}$$

Therefore the beta function is simply given in therm of the anomalous dimensions:

$$\beta_g(\mu) = g\left(\frac{\gamma_A(\mu)}{2} + \gamma_c(\mu)\right). \tag{4.29}$$

The behaviour of the beta function in the deep IR is therefore given by the limits of the anomalous dimensions in this region. For $p^2 \ll m^2$ the ghost anomalous dimension goes like p^2/m^2, as can be seen by looking at the IR expansion of the ghost self energy (3.26), hence it does not contribute to the beta function in the limit of vanishing momentum. The sign of the beta function in this limit is determined solely by the gluon anomalous dimension, which, for $t \ll 1$ goes like:

$$\gamma_A(t) = \frac{g^2 N}{3(4\pi)^2} + O(t), \tag{4.30}$$

which implies that in the deep IR the beta function is positive:

$$\beta_g = \mu\frac{dg}{d\mu} = \frac{g^3 N}{6(4\pi)^2}, \qquad \mu^2 \ll m^2. \tag{4.31}$$

Its IR positivity makes sure that the coupling diminishes at low momenta, running towards a trivial IR fixed point. Integrating (4.31) we can see the behaviour of the running coupling near its fixed point:

$$g^2(\mu) = \left(\frac{N}{6(4\pi)^2}\right)^{-1}\left(\ln\left(\frac{\Lambda_{IR}^2}{\mu^2}\right)\right)^{-1}, \qquad \Lambda_{IR}^2 \equiv \mu_0^2 \exp\left[\left(\frac{N g^2(\mu_0)}{6(4\pi)^2}\right)^{-1}\right], \tag{4.32}$$

where μ_0 is an arbitrary reference scale. Hence the coupling constant logarithmically vanishes in the IR, in the same way as it logarithmically vanishes in the UV, where it also reaches a trivial fixed point. In the limit where $\mu^2 \gg m^2$, in fact, the beta function recovers its traditional asymptotic form that causes the famous *asymptotic freedom* in QCD:

$$\beta_g = -\frac{11}{3}\frac{g^3 N}{(4\pi)^2} \qquad \mu^2 \gg m^2, \tag{4.33}$$

which implies a flow for the running coupling constant in the UV like:

$$g^2(\mu) = \left(\frac{11N}{3(4\pi)^2}\right)^{-1}\left(\ln\left(\frac{\mu^2}{\Lambda_{UV}^2}\right)\right)^{-1}, \qquad \Lambda_{UV}^2 \equiv \mu_0^2 \exp\left[-\frac{3}{11}\left(\frac{Ng^2(\mu_0)}{(4\pi)^2}\right)^{-1}\right]. \quad (4.34)$$

We therefore see how the crucial property that the beta function needs to satisfy in order to yields an "IR safe" scheme, is to change sign at a certain intermediate scale and to remain positive in the deep IR. The fact that the coupling runs to an IR trivial fix point makes thus the perturbative treatment reliable.

However, In order to rely on the asymptotic expressions, both in the IR as in UV, we need to ensure that the limits $\mu^2 \ll m^2$ and $\mu^2 \gg m^2$ do actually make sense, i.e. that the mass parameter does not grow faster than μ^2 at high energy and does not vanish faster in the IR. This is guaranteed by the asymptotic expressions of β_{m^2} in these limits. Because of the definition of the renormalized mass parameter given in (4.16) that we saw to imply the identity $Z_{m^2}Z_AZ_c = 1$ the beta function for the mass is given by:

$$\beta_{m^2}(\mu) = \mu\frac{dm^2}{d\mu} = \mu\frac{d}{d\mu}\left(Z_A(\mu)Z_c(\mu)m_B^2\right) = m^2(\mu)\left(\gamma_A(\mu) + \gamma_c(\mu)\right). \quad (4.35)$$

Its limit in the IR is positive, guaranteeing the vanishing of the running mass at low energy:

$$\beta_{m^2}(\mu) = m^2(\mu)\frac{g^2(\mu)N}{3(4\pi)^2} \qquad \mu^2 \ll m^2, \quad (4.36)$$

and in the UV is negative, guarantying the vanishing of the running mass at high energy:

$$\beta_{m^2}(\mu) = -m^2(\mu)\frac{35}{6}\frac{g^2(\mu)N}{(4\pi)^2} \qquad \mu^2 \gg m^2. \quad (4.37)$$

Integrating these asymptotic equations, using the expressions for the running coupling in this limits given in (4.32) and (4.34), one finds that the mass logarithmically vanishes in the IR:

$$m^2(\mu) = m^2(\mu_0)\ln\left(\frac{\Lambda_{IR}^2}{\mu_0^2}\right)\left(\ln\left(\frac{\Lambda_{IR}^2}{\mu^2}\right)\right)^{-1}, \quad (4.38)$$

and vanishes like a power of the logarithm in the UV:

$$m^2(\mu) = m^2(\mu_0)\left(\ln\left(\frac{\mu_0^2}{\Lambda_{UV}^2}\right)\right)^{\frac{35}{44}}\left(\ln\left(\frac{\mu^2}{\Lambda_{UV}^2}\right)\right)^{-\frac{35}{44}}. \quad (4.39)$$

These two limit expressions therefore guarantee that it is effectively sensible to talk about $\mu^2 \ll m^2$ and $\mu^2 \gg m^2$. The last one, in particular, guarantees that in the UV we recover the traditional massless Yang-Mills theory and BRST symmetry.

Now, going back to the IR limit of the beta function for the coupling constant, it is interesting to check the different contributions to this limit from the transverse part and the longitudinal part of the gluon self energy. Since we stressed that the ghost anomalous dimension does vanishes in the IR, the positivity of the beta function in the IR only depends on the gluon anomalous dimension, which in this renormalization scheme involves in fact the difference between the transverse and longitudinal parts of the gluon self energy, cf. (4.19). Taking a look at the IR limits of these latter in (3.35) we can rewrite the IR asymptotic expression for the beta function separating the two contributions:

$$\beta_g(\mu) = \frac{g^3N}{(4\pi)^2}\left[-\frac{1}{12}^{\perp} + \frac{1}{4}^{\parallel}\right]. \quad (4.40)$$

The positivity of the beta function is therefore due to the contribution of the longitudinal part that needs to overcome the negative contribution from the transverse part. This important fact implies the impossibility of having a renormalization scheme that is "IR safe" and that involves only the transverse part of the gluon two-point function $\Gamma^{\perp}_{AA}(p^2)$.

The idea behind such a scheme would be that, in the Landau gauge, only the transverse part of the gluon propagator which is the inverse of $\Gamma^{\perp}_{AA}(p^2)$, ever contributes to correlation functions which are completely transverse with respect to the external gluon momenta, to all orders in perturbation theory. One would then expect normalization conditions like

$$\frac{\partial}{\partial p^2}\,\Gamma^{\perp}_{AA}(p^2)\Big|_{p^2=\mu^2} = 1\,, \tag{4.41}$$

$$\Gamma^{\perp}_{AA}(\mu^2) = \mu^2 + m^2\,, \tag{4.42}$$

to lead to an optimal approximation of the full gluon propagator at momentum scales of the order of μ^2, and thus include as many higher-loop contributions as possible in the one-loop renormalization group-improved transverse correlation functions. However, as we have showed, such normalization conditions lead to a negative beta function, and hence to a Landau pole, for the coupling constant in the IR.

The integration of the renormalization group equations (4.7) for the two-point functions in this Tissier-Wschebor renormalization scheme gives:

$$\begin{aligned}\Gamma_{c\bar{c}}(p^2,\mu^2,g(\mu),m^2(\mu)) &= \Gamma_{c\bar{c}}(p^2,p^2,g(p),m^2(p))\exp\left\{-\int_{\mu}^{p}\frac{d\mu'}{\mu'}\gamma_c(\mu')\right\} \\ &= p^2\exp\left\{-\int_{\mu}^{p}\frac{d\mu'}{\mu'}\gamma_c(\mu')\right\}\end{aligned} \tag{4.43}$$

$$\begin{aligned}\Gamma^{\perp}_{AA}(p^2,\mu^2,g(\mu),m^2(\mu)) &= \Gamma^{\perp}_{AA}(p^2,p^2,g(p),m^2(p))\exp\left\{-\int_{\mu}^{p}\frac{d\mu'}{\mu'}\gamma_A(\mu')\right\} \\ &= (p^2+m^2(p))\exp\left\{-\int_{\mu}^{p}\frac{d\mu'}{\mu'}\gamma_A(\mu')\right\},\end{aligned} \tag{4.44}$$

where in the last identities we substituted the normalization conditions (4.15) and (4.17). These equations have been numerically integrated, using the *Runge-Kutta* method up to fourth order. We point out how, in order to achieve a good numerical precision in both the IR and UV limits we needed to explicitly glue the expressions for the flow functions beta and anomalous dimensions with their corresponding asymptotic expressions, because subtle analytical cancellations between terms of the same order take place in these limits. The expressions are derived for the ghost and gluon propagators $G_c(p^2)$ and $G_A(p^2)$, the inverse of the two-point functions

$$G_c(p^2) = \Gamma^{-1}_{c\bar{c}}(p^2), \tag{4.45}$$

$$G_A(p^2) = \left(\Gamma^{\perp}_{AA}(p^2)\right)^{-1}. \tag{4.46}$$

The propagators have been evaluated for the $SU(2)$ case ($N=2$ in all the expressions) in order to compare them with the data of the largest lattice available, i.e. the data that go deeper in the IR [52] . For the ghost we plot its dressing function $p^2G_c(p^2)$, the function that encodes how much the propagator moves away from its tree-level form, since it remains finite and different from zero in the whole range of momenta. For the gluon we plot both the propagator and the dressing function, since the propagator vanishes at high energy and approaches a constant value in the IR and the dressing function vanishes in the IR and decreases as a power of logarithm in the UV, therefore in order to appreciate the goodness of the fit with the lattice data is better to look at both.

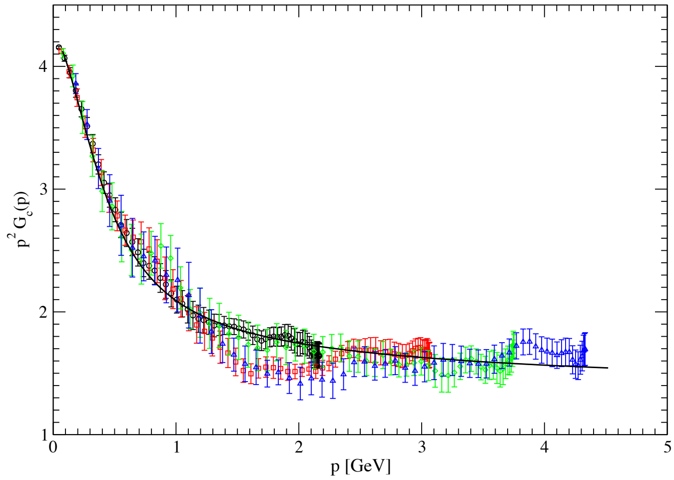


Figure 4.1: Ghost dressing function in the TW scheme.

The strategy for fitting, regarding this and the next presented renormalization schemes, is the following: we adjust the coupling and mass parameters in such a way that the ghost dressing function well fits the lattice data, lying within its error bars. In this way, in order to estimate the goodness of the scheme one has only to look at the gluonic diagrams. We therefore show the ghost dressing function plot only once, for this Tissier-Wschebor (TW) scheme. The best fit has been obtained by setting $g = 2.92$ and $m = 0.31\,\text{GeV}$ at the renormalization scale $\mu = 3\,\text{GeV}$.

Tissier-Wschebor scheme in $D = 3$

We also resummed the two-point functions in the three dimensional case, $D = 3$, in the TW scheme. Imposing the normalization conditions given in (4.15) and (4.17) the following expressions for the anomalous dimensions are obtained in $D = 3$ (note that now they have a dimension of $[m]^{-1}$)

$$\gamma_c(t) = \frac{Ng^2}{32\pi}\frac{1}{mt^{3/2}}\left[2\sqrt{t}(t+3) - \pi t^2 + 2(t-3)(t+1)\arctan\left(\sqrt{t}\right)\right], \tag{4.47}$$

$$\begin{aligned}\gamma_A(t) = &\frac{Ng^2}{128\pi}\frac{1}{mt^{5/2}}\left[4\sqrt{t}(7t^2 - 29t + 15) - 3\pi t^4 + 4(t+1)(3t^3 - 7t^2 + 15t - 15)\arctan\left(\sqrt{t}\right)\right.\\ &\left.-2t(3t^3 - 8t^2 + 40t - 96)\arctan\left(\frac{\sqrt{t}}{2}\right)\right].\end{aligned} \tag{4.48}$$

The best fit in this scheme for the ghost dressing function and for the gluon propagator is shown in Fig. 4.4. It is noteworthy the IR raising of the gluon propagator, which is reproduced by the renormalization group resummation. We stress that such raising in the deep IR also is found in the four dimensional, albeit it is not clear in the plots since the lattice data do not reach such deep IR values.

4.3.3 Derivative schemes

We also explored the different results for schemes in which the renormalized two-point functions break the Slavnov-Taylor identity (4.25), meaning the identity $Z_{m^2}Z_AZ_c = 1$ is not satisfied

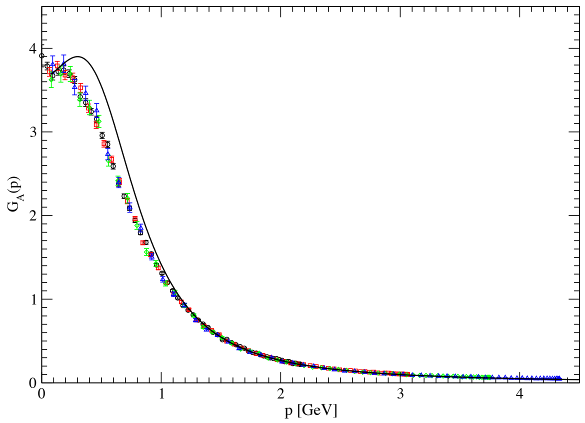


Figure 4.2: Gluon propagator (left) and gluon dressing function (right) in the TW scheme. $g = 2.92$ and $m = 0.31$ GeV at $\mu = 3$ GeV.

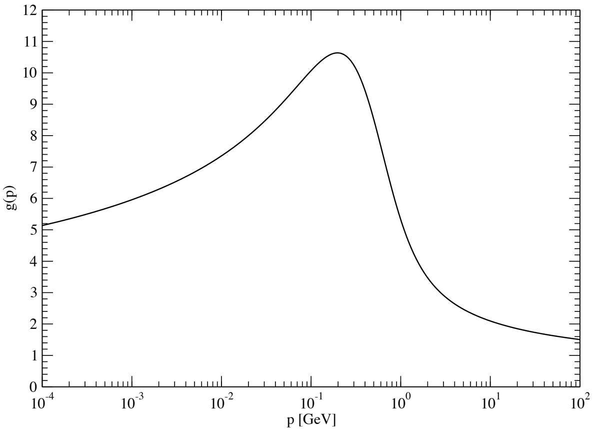


Figure 4.3: Running coupling (left) and running mass (right) in the TW scheme.

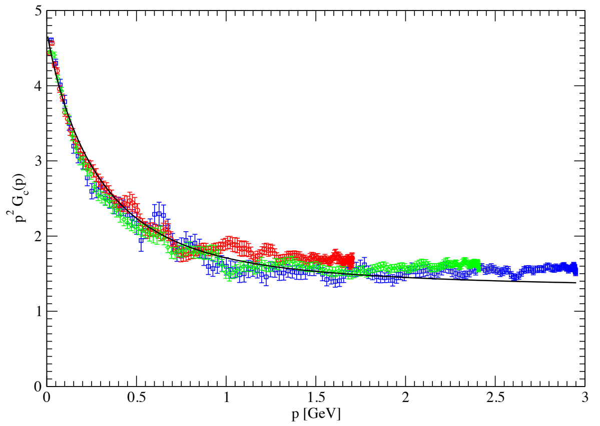


Figure 4.4: Ghost dressing function (left) and gluon propagator (right) in the TW scheme. $g = 2.68$ and $m = 0.64$ GeV at $\mu = 1$ GeV.

. These schemes are given by imposing renormalization conditions that fix the values of the derivatives of the two-point functions, instead of the two-point functions themselves. The first scheme we tried is given by the following conditions:

$$\frac{\partial}{\partial p^2}\left(\Gamma^{\perp}_{AA}(p^2)-\Gamma^{\parallel}_{AA}(p^2)\right)\bigg|_{p^2=\mu^2}=1 \tag{4.49}$$

and, by analogy,

$$\frac{\partial}{\partial p^2}\,\Gamma_{c\bar{c}}(p^2)\bigg|_{p^2=\mu^2}=1\,. \tag{4.50}$$

The condition (4.49) looks more natural when one considers decomposing the gluonic two-point function not in its transverse and longitudinal parts, but rather as

$$\Gamma_{A_\mu A_\nu}(p)=\left(\Gamma^{\perp}_{AA}(p^2)-\Gamma^{\parallel}_{AA}(p^2)\right)\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)+\Gamma^{\parallel}_{AA}(p^2)\delta_{\mu\nu}\,, \tag{4.51}$$

which mimics the grouping of terms in the classical action (2.31), as already pointed out in section 3.3. It is clear that the new condition for the ghost two-point function (4.50) alone spoils the Slavnov-Taylor identity for the renormalized quantities, since for this to be satisfied $\Gamma_{c\bar{c}}(\mu^2)=\mu^2$ had to be fulfilled. The conditions (4.49) and (4.50) imply new expressions for the renormalization constants Z_A and Z_c:

$$Z_c(\mu)=1+\frac{\partial\Sigma^B(p^2)}{\partial p^2}\bigg|_{p^2=\mu^2}\,, \tag{4.52}$$

$$Z_A(\mu)=1+\frac{\partial}{\partial p^2}\left(\Pi^{\perp\,B}(p^2)-\Pi^{\parallel\,B}(p^2)\right)\bigg|_{p^2=\mu^2}\,, \tag{4.53}$$

which in turn give rise to slightly different expressions for the anomalous functions:

$$\gamma_c(t)=\frac{g^2N}{2(4\pi)^2t^2}\left[-t(t-2)+2t^3\ln t-2(t^3+1)\ln(1+t)\right], \tag{4.54}$$

$$\gamma_A(t)=\frac{g^2N}{6(4\pi)^2t^2}\left[-2(3t^2+22t-12)+3t^4\ln t-2t^{-1}(1+t)(t-2)(3t^3-4t^2+2t-6)\ln(1+t)\right. \tag{4.55}$$

$$\left.-(t-2)(3t^3-2t^2-38t-36)\sqrt{t(t+4)^{-1}}\ln\left(\frac{\sqrt{t+4}-\sqrt{t}}{\sqrt{t+4}+\sqrt{t}}\right)\right]. \tag{4.56}$$

These anomalous dimensions share the same UV asymptotic limit with the previous one in the TW scheme, since at high energies the presence of the mass is irrelevant and we recover the universality of the anomalous dimensions at one-loop. The limits, for this particular scheme, are also the same in the IR. In particular, the beta function does not change its behaviour at low and high momentum. We also stress how the UV behaviour of the gluon anomalous dimension is purely determined by the transverse part of the self energy, since in any sensible scheme the longitudinal part must vanish at high energy, in order to restore the original BRST symmetry.

A third normalization condition is required to complete the renormalization scheme given by Eqs. (4.49) and (4.50) so far. One might be inclined to use the condition (4.42) to this end, however, as we shall now show, such a scheme runs into trouble in the UV limit $\mu^2\gg m^2$. In fact, using the normalization condition (4.42) the renormalized mass is given, in terms of bare quantities by:

$$m^2(\mu)=Z_A(\mu)\left(\mu^2+m_B^2-\Pi^{\perp\,B}(\mu^2)\right)-\mu^2, \tag{4.57}$$

implying for the beta function of the mass parameter:

$$\begin{aligned}\beta_{m^2}(\mu) &= \mu\frac{d}{d\mu}m^2(\mu) = \gamma_A(\mu)(\mu^2+m^2(\mu)) + 2\mu^2(Z_A(\mu)-1) - \mu\frac{d}{d\mu}\Pi^{\perp\,B}(\mu) \\ &\underset{\mu^2\gg m^2}{\longrightarrow} -\frac{13}{3}\frac{g^2N}{(4\pi)^2}\mu^2.\end{aligned} \tag{4.58}$$

It is therefore seen how the beta function and then the running mass become arbitrarily negative in the UV and the usual Yang-Mills theory and BRST symmetry are not recovered in this limit. Instead of using Eq. (4.42), we hence complement the normalization conditions (4.49) and (4.50) with the condition (4.16),

$$\Gamma^{\parallel}_{AA}(\mu^2) = m^2\,, \tag{4.59}$$

in order to complete the renormalization scheme. We thus see how the longitudinal part of the gluon self energy is of crucial importance in any renormalization scheme. It must be included in the condition that defines the gluon anomalous dimension in order to have an IR positive beta function for the coupling constant ("IR safe") and it must be used to define the renormalized mass in order for it to behave properly in the UV.

From the definition of the renormalized mass (4.59) the mass beta function is given by:

$$\begin{aligned}\beta_{m^2} &= \mu\frac{dm^2}{d\mu} = m^2\gamma_A(\mu) - \mu\frac{d\,\Pi^{\parallel\,B}(\mu)}{d\mu} \\ &= m^2\frac{g^2N}{6(4\pi)^2t^2}\Big[3t^3(1+t)\ln t - (t-2)(t+1)(6t^2-5t+7-12t^{-1})\ln(1+t) \\ &\quad -2(6t^2+25t-12) - (t-2)(3t^3-2t^2-38t-36)\sqrt{t(t+4)^{-1}}\ln\left(\frac{\sqrt{t+4}-\sqrt{t}}{\sqrt{t+4}+\sqrt{t}}\right)\Big].\end{aligned} \tag{4.60}$$

We point out how now this beta function cannot be written as a simple sum of the gluon and ghost anomalous dimensions, as in (4.35), since this latter was derived by the identity $Z_{m^2}Z_AZ_c = 1$ which is not valid here. However its IR limit does not change, since this depends only on the limit of the gluon anomalous dimension, and this, as we stressed before, does not change from the previous one calculated in the TW scheme.

The integration of the renormalization group equations is slightly more complicated in this derivative renormalization scheme. In order to implement the normalization conditions (4.49) and (4.50), in fact, one has to take the derivative of (4.44) w.r.t p^2 (before setting $\mu_0^2 = p^2$). For the ghost two-point function we have:

$$\begin{aligned}\frac{\partial}{\partial p^2}\Gamma_{c\bar{c}}(p^2,\mu^2,g(\mu),m^2(\mu)) &= \left\{\frac{\partial}{\partial p^2}\Gamma_{c\bar{c}}(p^2,\mu_0^2,g(\mu_0),m^2(\mu_0))\exp\left[-\int_\mu^{\mu_0}\frac{d\mu'}{\mu'}\gamma_c(\mu')\right]\right\}\Bigg|_{\mu_0^2=p^2} \\ &= \exp\left\{-\int_\mu^{p}\frac{d\mu'}{\mu'}\gamma_c(\mu')\right\},\end{aligned} \tag{4.61}$$

where in the last line we imposed the condition (4.50). Integrating in p^2 from a vanishing momentum we obtain:

$$
\begin{aligned}
\Gamma_{c\bar{c}}(p^2,\mu^2,g(\mu),m^2(\mu)) &= \Gamma_{c\bar{c}}(p^2=0,\mu^2,g(\mu),m^2(\mu)) + \int_0^{p^2} dp'^2 \exp\left\{-\int_\mu^{p'} \frac{d\mu'}{\mu'}\gamma_c(\mu')\right\} \\
&= \int_0^{p^2} dp'^2 \exp\left\{-\int_\mu^{p'} \frac{d\mu'}{\mu'}\gamma_c(\mu')\right\},
\end{aligned}
\tag{4.62}
$$

where in the last line we put $\Gamma_{c\bar{c}}(p^2=0)=0$, since the ghost remains massless at the quantum level in Landau gauge. This can be seen looking again at the Slavnov-Taylor identity (4.8) for the bare quantities: putting $p^2=0$ implies the vanishing of the product $\Gamma^B_{c\bar{c}}(p^2=0)\Gamma^{\parallel B}_{AA}(p^2=0)$. Since the bare longitudinal part of the gluon two-point function is different from zero, this implies the vanishing of the bare ghost two-point function at vanishing momentum, and therefore the vanishing of its renormalized counter part, due to multiplicative renormalizability.

For the resummation of the gluon two-point function, due to its normalization condition (4.50) that involves the difference between the transverse and longitudinal parts, we consider the derivative of the corresponding renormalization group equations (we omit the dependence of the two-point functions on the renormalized parameters):

$$
\begin{aligned}
&\frac{\partial}{\partial p^2}\left(\Gamma^{\perp}_{AA}(p^2,\mu^2)-\Gamma^{\parallel}_{AA}(p^2,\mu^2)\right) \\
&= \left\{\frac{\partial}{\partial p^2}\left(\Gamma^{\perp}_{AA}(p^2,\mu_0^2)-\Gamma^{\parallel}_{AA}(p^2,\mu_0^2)\right)\exp\left[-\int_\mu^{\mu_0}\frac{d\mu'}{\mu'}\gamma_A(\mu')\right]\right\}\Bigg|_{\mu_0^2=p^2} \\
&= \exp\left\{-\int_\mu^{p}\frac{d\mu'}{\mu'}\gamma_A(\mu')\right\},
\end{aligned}
\tag{4.63}
$$

which after integration yields:

$$
\begin{aligned}
\Gamma^{\perp}_{AA}(p^2,\mu^2) &= \Gamma^{\parallel}_{AA}(p^2,\mu^2) + \int_0^{p^2} dp'^2 \exp\left\{-\int_\mu^{p'}\frac{d\mu'}{\mu'}\gamma_A(\mu')\right\} \\
&= m^2(p)\exp\left\{-\int_\mu^{p}\frac{d\mu'}{\mu'}\gamma_A(\mu')\right\} + \int_0^{p^2} dp'^2 \exp\left\{-\int_\mu^{p'}\frac{d\mu'}{\mu'}\gamma_A(\mu')\right\},
\end{aligned}
\tag{4.64}
$$

where in the last line we substituted the expression for the resummed longitudinal part:

$$
\Gamma^{\parallel}_{AA}(p^2,\mu^2) = \Gamma^{\parallel}_{AA}(p^2,p^2)\exp\left\{-\int_\mu^{p}\frac{d\mu'}{\mu'}\gamma_A(\mu')\right\} = m^2(p)\exp\left\{-\int_\mu^{p}\frac{d\mu'}{\mu'}\gamma_A(\mu')\right\}. \tag{4.65}
$$

In both (4.64) and (4.65) is clearly assumed that the transverse and the longitudinal parts are renormalized by the same renormalization constant $Z_A(\mu)$. We also stress how the constant of integration in (4.64) vanishes due to the local condition $\Gamma^{\perp}_{AA}(p^2=0)=\Gamma^{\parallel}_{AA}(p^2=0)$.

The gluon propagator and gluon dressing function for this derivative scheme (denoted by D) are plotted in Fig. 4.5. The best fit has been found fixing $g=2.6$ and $m=0.33$ GeV at $\mu=3$ GeV.

The use of the p^2-derivative actually allows for a greater flexibility in the normalization conditions: one may interpolate between the conditions (4.41) and (4.49) by introducing a parameter ζ and replacing the condition (4.49) with

$$
\frac{\partial}{\partial p^2}\left(\Gamma^{\perp}_{AA}(p^2)-\zeta\,\Gamma^{\parallel}_{AA}(p^2)\right)\Bigg|_{p^2=\mu^2} = 1\,, \tag{4.66}
$$

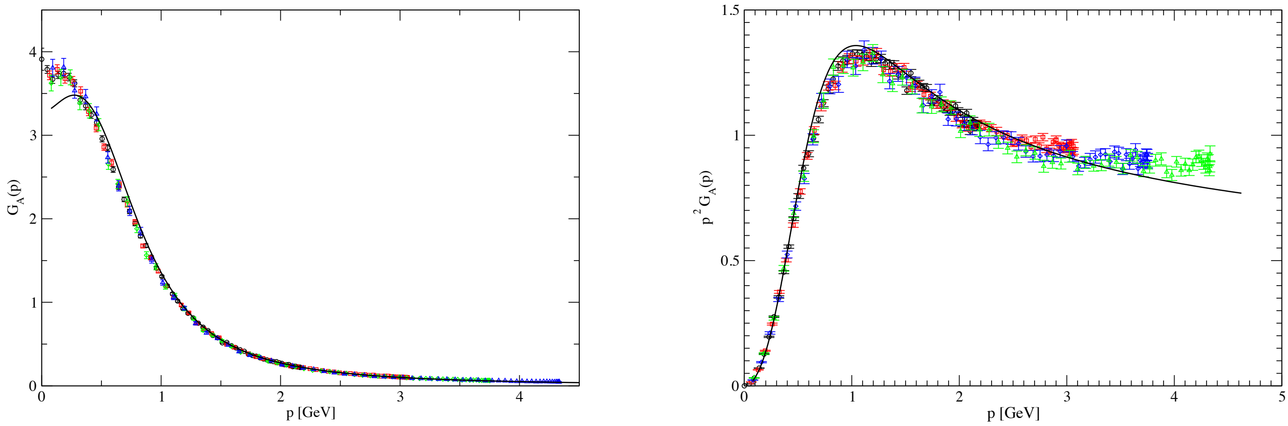


Figure 4.5: Gluon propagator (left) and gluon dressing function (right) in the derivative (D) scheme. $g = 2.6$ and $m = 0.33$ GeV at $\mu = 3$ GeV.

while keeping the conditions (4.50) and (4.59) (the condition (4.42) cannot be used, for the same reasons as before). Note that a condition analogous to (4.66) but without the p^2-derivative, like

$$\Gamma^{\perp}_{AA}(\mu^2) - \zeta\,\Gamma^{\|}_{AA}(\mu^2) = \mu^2 + (1-\zeta)m^2\,, \tag{4.67}$$

would be completely equivalent to the condition (4.15) in the original IR safe scheme of Tissier and Wschebor as can explicitly be seen by adding

$$\zeta\,\Gamma^{\|}_{AA}(\mu^2) = \zeta m^2 \tag{4.68}$$

to Eq. (4.67). The normalization condition (4.66) corresponds to the decomposition

$$\Gamma_{A_\mu A_\nu}(p) = \left(\Gamma^{\perp}_{AA}(p^2) - \zeta\,\Gamma^{\|}_{AA}(p^2)\right)\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + \Gamma^{\|}_{AA}(p^2)\left(\zeta\,\delta_{\mu\nu} + (1-\zeta)\frac{p_\mu p_\nu}{p^2}\right) \tag{4.69}$$

of the gluonic two-point function which interpolates linearly between the standard decomposition for $\zeta = 0$

$$\Gamma_{A_\mu A_\nu}(p) = \Gamma^{\perp}_{AA}(p^2)\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + \Gamma^{\|}_{AA}(p^2)\frac{p_\mu p_\nu}{p^2} \tag{4.70}$$

and the decomposition (4.51) analogous to the classical action for $\zeta = 1$. Because of the normalization condition (4.66), the IR limits of the coupling and mass beta functions now become:

$$\beta_g = \frac{g^3 N}{(4\pi)^2}\left(-\frac{1}{12} + \frac{\zeta}{4}\right) \tag{4.71}$$

$$\beta_{m^2} = m^2\frac{g^3 N}{(4\pi)^2}\left(-\frac{1}{6} + \frac{\zeta}{2}\right), \tag{4.72}$$

so that the renormalization scheme is IR safe, with a trivial IR fixed point of the coupling constant, for $\zeta > 1/3$. For $\zeta < 1/3$, on the other hand, the integration of the renormalization group equations generates a Landau pole. The case $\zeta = 1/3$ is particularly interesting, since for this "critical" value, we find that the renormalized coupling constant $g(\mu^2)$ and the renormalized mass $m(\mu^2)$ converge to a nontrivial value in the limit $\mu^2 \to 0$ (Fig. 4.6). However, the actual

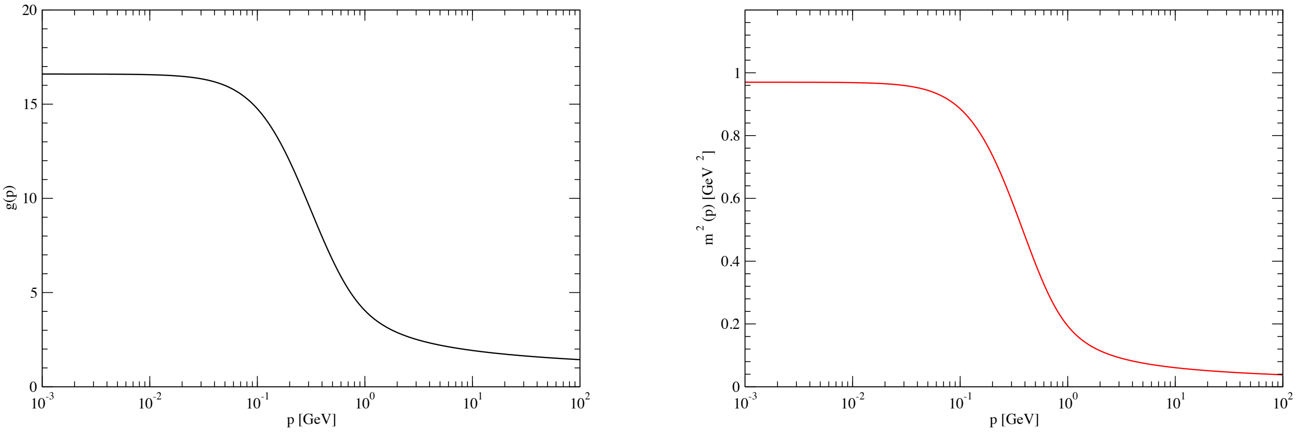


Figure 4.6: Running coupling (left) and running mass (right) in the critical derivative scheme (Dc) corresponding to $\zeta = 1/3$. $g = 2.52$ and $m = 0.305$ GeV at $\mu = 3$ GeV.

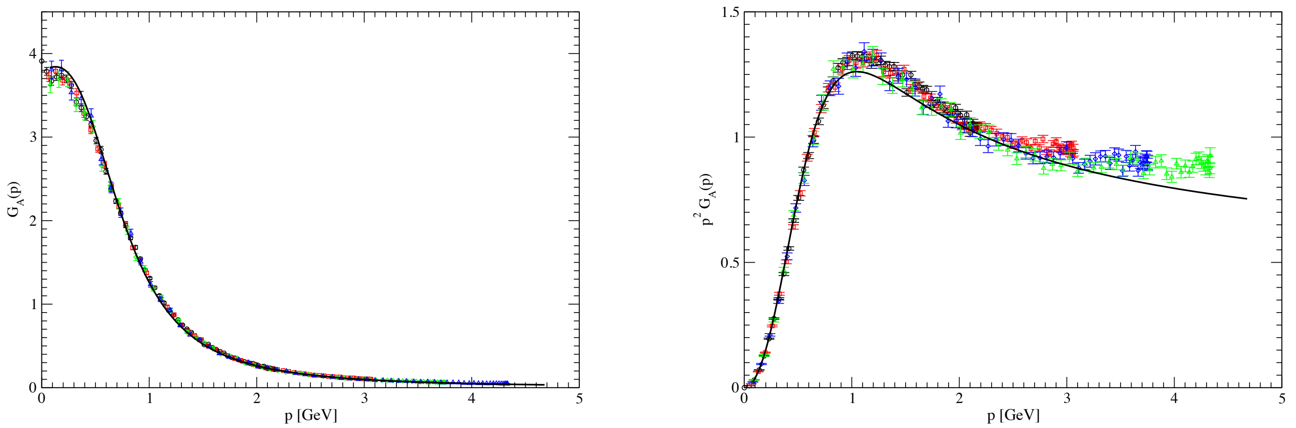


Figure 4.7: Gluon propagator (left) and gluon dressing function (right) in the critical derivative scheme (Dc), corresponding to $\zeta = 1/3$. $g = 2.52$ and $m = 0.305$ GeV at $\mu = 3$ GeV.

values of g and m in this limit depend on the initial conditions, i.e. on their value fixed at the renormalization scale.

Manipulating the renormalization group equation as we did in (4.64), we get for the transverse part in this critical renormalization scheme:

$$\begin{aligned}\Gamma^{\perp}_{AA}(p^2,\mu^2) =& \frac{2}{3}m^2(p^2\to 0)\exp\left\{\int_0^{\mu}\frac{d\mu'}{\mu'}\gamma_A(\mu')\right\} + \frac{1}{3}m^2(p^2)\exp\left\{-\int_{\mu}^{p}\frac{d\mu'}{\mu'}\gamma_A(\mu')\right\} \\ &+ \int_0^{p^2} dp'^2 \exp\left\{-\int_{\mu}^{p'}\frac{d\mu'}{\mu'}\gamma_A(\mu')\right\}.\end{aligned} \tag{4.73}$$

The plots for the gluon propagator and gluon dressing function in this scheme are showed in Fig. 4.7. Their fits to lattice data represent an improvement compared to the previous schemes.

This analysis brought us to consider another normalization condition from the observation that the standard decomposition (4.70) of the gluonic two-point function is particularly adequate in the UV limit where its longitudinal part vanishes as a consequence of the reinstatement of the proper (not extended) BRST symmetry in this limit, while the decomposition (4.51) is rather natural in the IR limit where the difference $(\Gamma^{\perp}_{AA}(p^2) - \Gamma^{\parallel}_{AA}(p^2))$ goes to zero, see Eq. (4.27). It might then seem to be a good idea to allow the parameter ζ that interpolates between the two decompositions in Eq. (4.69) to depend on the renormalization scale μ. The simplest choice is to replace ζ in the normalization condition (4.66) with

$$\zeta = \frac{1}{1 + (\mu^2/m^2)}, \tag{4.74}$$

so that $\zeta \to 1$ for $\mu^2 \ll m^2$ and $\zeta \to 0$ for $\mu^2 \gg m^2$ (the other normalization conditions (4.50) and (4.59) go unchanged). We shall refer to this renormalization scheme as the dynamical ζ-scheme.

For this normalization condition the renormalization constant Z_A is given by:

$$Z_A(\mu) = 1 + \frac{\partial}{\partial p^2}\left(\Pi^{\perp B}(p^2) - \frac{1}{1 + (\mu^2/m^2(\mu^2))}\Pi^{\parallel B}(p^2)\right)\bigg|_{p^2=\mu^2}, \tag{4.75}$$

which yields the following expression for the anomalous dimension

$$\begin{aligned}\gamma_A(t) =& \frac{g^2 N}{12(4\pi)^2 t^3 (1+t)^2}\big[-2t(t+1)(6t^3 + 59t^2 + 32t - 24) + 6t^4(t^3 + 2t^2 + t - 1)\ln t \\ & - 2(t+1)^2(6t^5 - 14t^4 - 3t^2 - 8t + 24)\ln(1+t) - 2t(t+1)^2(t-2)(3t^3 - 2t^2 - 38t - 36) \\ & \times (1 + 4t^{-1})^{-1/2} \ln\left(\frac{\sqrt{t+4} - \sqrt{t}}{\sqrt{t+4} + \sqrt{t}}\right)\bigg].\end{aligned} \tag{4.76}$$

The asymptotic limits of γ_A in the IR and in the UV are, as expected, the same as before. In order to get the renormalization group equation for the transverse part of the gluon two-point function we consider the following:

$$\begin{aligned}&\frac{\partial}{\partial p^2}\Gamma^{\perp}_{AA}(p^2,\mu^2) - \left(\frac{1}{1 + \mu_0^2/m^2(\mu_0)}\right)\frac{\partial}{\partial p^2}\Gamma^{\parallel}_{AA}(p^2,\mu^2) \\ &= \frac{\partial}{\partial p^2}\left(\Gamma^{\perp}_{AA}(p^2,\mu_0^2) - \left(\frac{1}{1 + \mu_0^2/m^2(\mu_0)}\right)\Gamma^{\parallel}_{AA}(p^2,\mu_0^2)\right)\exp\left[-\int_{\mu}^{\mu_0}\frac{d\mu'}{\mu'}\gamma_A(\mu')\right].\end{aligned} \tag{4.77}$$

Setting on both sides $\mu_0 = p$, so that the term multiplying the exponential turns into unity because of the normalization condition, and integrating over p^2, we get

$$\begin{aligned}\Gamma^{\perp}_{AA}(p^2,\mu^2) =& \;\Gamma^{\perp}_{AA}(p^2 \to 0,\mu^2) + \int_0^{p^2} dp'^2 \left(\frac{1}{1 + p'^2/m^2(p')}\right)\Gamma^{\parallel}_{AA}(p'^2,\mu^2) \\ & + \int_0^{p^2} dp'^2 \exp\left[-\int_{\mu}^{p'}\frac{d\mu'}{\mu'}\gamma_A(\mu')\right] \\ =& \; m(p^2 \to 0)\exp\left[\int_0^{\mu}\frac{d\mu'}{\mu'}\gamma_A(\mu')\right] + \int_0^{p^2} dp'^2 \left\{\left(\frac{1}{1 + p'^2/m^2(p')}\right)\right. \\ & \left.\times m^2(p'^2)\exp\left[-\int_{\mu}^{p'}\frac{d\mu'}{\mu'}\gamma_A(\mu')\right]\right\} + \int_0^{p^2} dp'^2 \exp\left[-\int_{\mu}^{p'}\frac{d\mu'}{\mu'}\gamma_A(\mu')\right],\end{aligned} \tag{4.78}$$

where we imposed the locality condition (4.27) on the transverse part at vanishing momentum and substituted the expression for the resummed longitudinal part, given in (4.65). The best fits to the lattice data for this dynamical ζ-scheme are showed in Fig. 4.8. The plots are almost identical to the previous ones calculated in the critical scheme $\zeta = 1/3$.

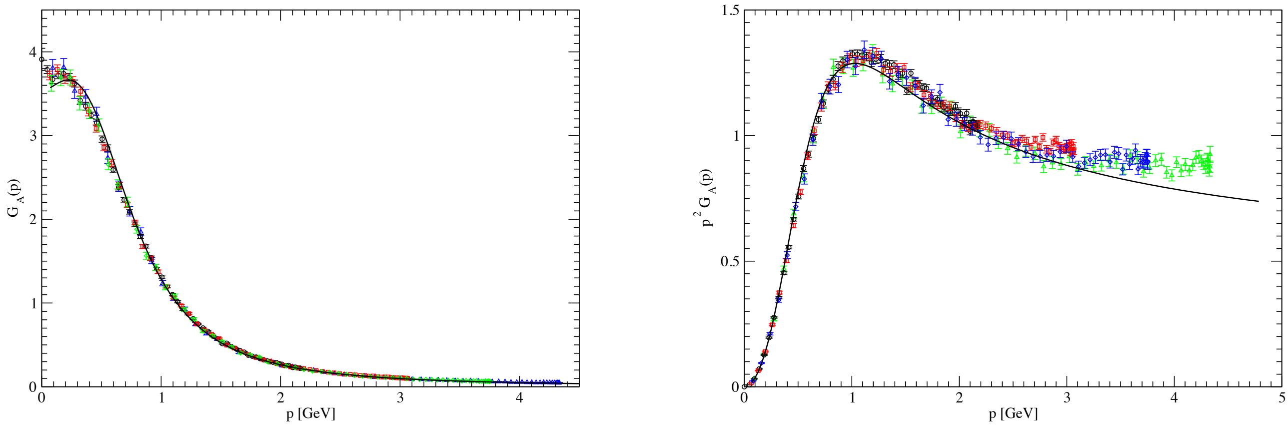


Figure 4.8: Gluon propagator (left) and gluon dressing function (right) in the dynamical ζ-scheme. $g = 2.56$ and $m = 0.315$ GeV at $\mu = 3$ GeV.

4.3.4 Symmetry point scheme

We now explore a scheme that involves a different definition of the renormalized coupling constant at the renormalization scale. In this scheme the coupling, instead of being defined at the asymmetric momentum configuration of the ghost-gluon vertex, is defined at the symmetry point configuration (SP)

$$g(\mu) \equiv g D_{\bar{c}cA}(\mu^2, \mu^2, \mu^2) = g_B Z_A^{1/2}(\mu) Z_c(\mu) D^B_{\bar{c}cA}(\mu^2, \mu^2, \mu^2). \tag{4.79}$$

Since, as we showed in section 3.4.1, the one-loop correction to the ghost-gluon dressing function does not vanish in this momentum configuration, the beta function acquire an extra term in this scheme, as compared to the expression (4.29), where it was given only in terms of the anomalous dimensions:

$$\beta_g(\mu) = g \left(\frac{\gamma_A(\mu)}{2} + \gamma_c(\mu) + \mu \frac{\partial}{\partial \mu} D^B_{\bar{c}cA}(\mu^2, \mu^2, \mu^2) \right). \tag{4.80}$$

From the expression for the vertex dressing function at the symmetry point in (3.53) and from the derivatives of the scalar three-point functions given in the appendix, we have:

$$\begin{aligned}
\mu \frac{\partial}{\partial \mu} D^B_{\bar{c}cA}(\mu^2, \mu^2, \mu^2) = {} & 2t \frac{d}{dt} D^B_{\bar{c}cA}(t) \\
= & \frac{g^2 N}{12(4\pi)^2 t^2} \Bigg[-t^2(7t+10) + \frac{t(3t^5 - 5t^4 + 2t^3 - 4t^2 - t - 4)}{t^2+t+1} \ln t \\
& - 2\, \frac{(1+t)(3t^5 + 3t^4 + 10t^3 - 8t^2 - 9t - 11)}{t^2+t+1} \ln(1+t) \\
& - 3(t^2+2t+2)(t^2+4t-4)\sqrt{t(t+4)^{-1}} \ln\left(\frac{\sqrt{t+4} - \sqrt{t}}{\sqrt{t+4} + \sqrt{t}} \right) \\
& + (t+1)(t^2+8t-4)\sqrt{t(t+4)^{-1}} \ln\left(\frac{1}{4}\left(t + \sqrt{t(t+4)} + 2 \right)^2 \right) \\
& + t^3(2t-1)\tilde{J}_0 - (4t^4 + 9t^3 - t + 4)\tilde{J}_1(t) + 2(t^4 + 5t^3 + 4)\tilde{J}_2(t) \Bigg].
\end{aligned} \tag{4.81}$$

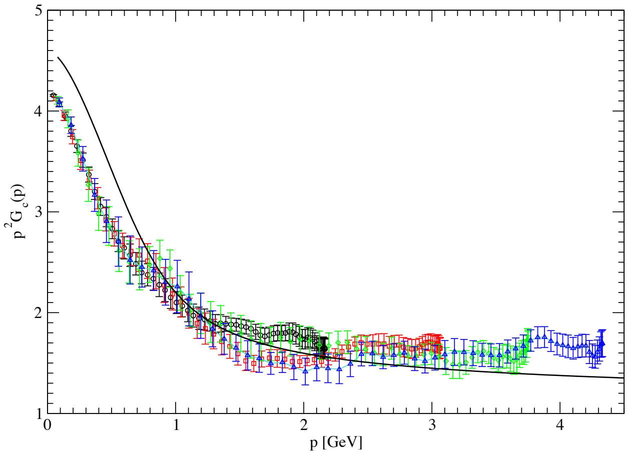


Figure 4.9: Ghost dressing function in the symmetry point scheme (SP). $g = 3.3$ and $m = 0.28$ GeV at $\mu = 3$ GeV.

We already know from the behaviour of the dressing function at low and high energies (cf. (3.54) and (3.55)), that this extra factor does not contribute in these limits to the beta function. However, it quantitatively affects the integration of the Callan-Symanzik equations for the propagators, as it can be seen in Fig. 4.9 and 4.10. The fits are clearly worse than the previously obtained ones (for the other flow functions we have used the original TW scheme). This can be intuitively understood by the following: the vertex momentum configuration at which the renormalized coupling is defined should be chosen in such a way that where the coupling appears within the Feynman diagrams, the vertex Feynman rule joined with the propagators attached to it, give the most relevant contribution to the Feynman integral at that particular configuration of momenta. In this way the running coupling will be able to encapsulate the largest contribution of the higher orders in the perturbative expansion. Therefore, since the ghost propagator is massless, the configuration of vanishing incoming momentum should be more dominant compared to the symmetric one, despite the fact that the ghost-gluon vertex contributes with a power of outgoing ghost momentum, which, because of the transversality of the gluon propagator, is equivalent to the incoming ghost momentum. It is still a power of momentum in the numerator from the vertex and two powers of momentum from the propagators.

This is for what concerns the ghost-gluon vertex. As we stressed before, the diagrams where it mostly appears are the dominant ones at low energies, because of the ghost dominance. But up to this point the renormalized coupling has been made run also in the diagrams where it is attached to the three and four gluon vertices. Because of the Bose-symmetry of these vertices and since they are attached to massive propagators, it is clear that a renormalized coupling defined at the symmetry point would be more suitable for these vertices. This consideration brought us to consider a scheme involving different couplings defined at different momenta configuration.

4.3.5 Two running couplings

In this section we consider as distinct the coupling associated to the ghost-gluon vertex, denoted by g, and the coupling associated to the three-gluon vertex, denoted by g_3. The reason for this, as stated in the end of the previous section, is to have more chance to absorb the largest amount of momentum contribution at higher orders through the flow integration of the running

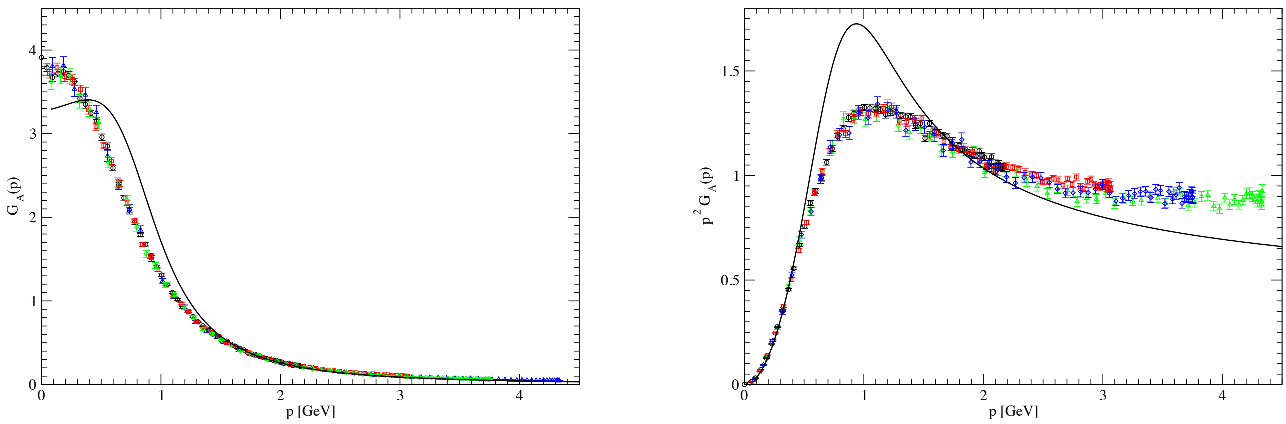


Figure 4.10: Gluon propagator (left) and gluon dressing function (right) in the symmetry point scheme (SP). $g = 3.3$ and $m = 0.28$ GeV at $\mu = 3$ GeV.

couplings related to the corresponding vertices. This procedure, that could be interpreted as an *over-renormalization* of the coupling, has actually a deeper physical justification. Because of the presence of the gluon mass scale, in fact, the couplings associated to different vertices are not prevented to separately flow towards the IR fixed point. This was in particular pointed out in [51] where it is shown how the IR *high temperature* fixed point changes the canonical dimension of the gluon field compared to the UV *Gaussian* fixed point, due to the mass parameter, causing different scalings for the different coupling constants. In particular, only the ghost-gluon coupling is shown to be *marginal* in the upper critical dimension of the IR fixed point $D = 2$, while the other couplings are *irrelevants*, another manifestation of the IR ghost dominance.

We therefore intend to study the effect of the separate running of the three different couplings, but for the time being we consider only two of them, identifying the coupling associated to the four-gluon vertex with the one corresponding to the three-gluon vertex g_3. The normalization condition for this latter is defined at the symmetry point of the three-gluon vertex:

$$g_3(\mu) \equiv \Gamma_{AAA}(\mu^2, \mu^2, \mu^2, g, g_3) = Z_A^{3/2}(\mu)\Gamma^B_{AAA}(\mu^2, \mu^2, \mu^2, g_B, g_{3B}), \tag{4.82}$$

where with $\Gamma_{AAA}(\mu^2, \mu^2, \mu^2, g, g_3)$ we denote the full vertex after the proper contractions at the symmetry point, where we made explicit its dependence on the two couplings. All the diagrams that have been calculated in Chapter 3, in fact, are now to be thought with the right factors of g or g_3 according to the type of vertex present in each diagram. Thus, if we set $g = g_3$ we would have

$$\Gamma_{AAA}(\mu^2, \mu^2, \mu^2, g, g) = g D_{AAA}(\mu^2, \mu^2, \mu^2), \tag{4.83}$$

where $D_{AAA}(\mu^2, \mu^2, \mu^2)$ is the three-gluon dressing function defined in (3.64). The renormalized coupling g is defined as before at the Taylor limit of the ghost-gluon vertex. The anomalous dimensions and the beta function for the mass parameter are defined using the original Tissier-Wschebor scheme, given by the conditions (4.15), (4.16), (4.17), re-expressed by putting g and g_3 in their corresponding places.

We now have a coupled system of three flow equations to numerically integrate, constituted by the two beta functions of the two different couplings and the beta function of the mass parameter. Special care has to be taken in the UV limit, though, in order to guarantee the vanishing of the running mass and the merging of the two running couplings, thus restoring the original BRST symmetry at high energy. In this limit in fact, now β_{m^2} takes the the asymptotic form:

$$\beta_{m^2} = m^2 \frac{N}{(4\pi)^2}\left(-\frac{37}{6}g_3^2 + \frac{1}{3}g^2\right) + \frac{\mu^2}{2}\frac{N}{(4\pi)^2}\left(1 - \ln\left(\frac{\mu^2}{m^2}\right)\right)\left(g_3^2 - g^2\right) + O(m^2/\mu^2). \quad (4.84)$$

We see from the last term that in the deep UV the difference between the squares of the couplings needs to vanish faster than $1/\mu^2$ in order to recover the asymptotic limit of the beta function that guarantees the vanishing of the mass parameter as a power of the logarithm of the scale (cf. (4.39)). To make sure that this is the case we need to investigate the analytical behaviour of the couplings in the UV. The asymptotic expressions of the other two beta functions are:

$$\begin{aligned}
\beta_g =& g\left(\frac{\gamma_A}{2} + \gamma_c\right) = \frac{N}{(4\pi)^2}\left[-\frac{4}{3}g^3 - \frac{7}{3}g g_3^2 + g_0^3\frac{m^2}{\mu^2}\left(\frac{59}{8} + \frac{9}{4}\ln\left(\frac{\mu^2}{m^2}\right)\right)\right] + O((m^2/\mu^2)^2), \\
\beta_{g_3} =& \frac{3}{2}g_3\,\gamma_A + \mu\frac{d}{d\mu}\left[g_3^3\left(\Lambda^{gl}(\mu^2) + \Lambda^{sw}(\mu^2)\right) + g^3\Lambda^{gh}(\mu^2)\right] \qquad (4.85) \\
=& \frac{N}{(4\pi)^2}\left[-\frac{17}{4}g_3^3 + \frac{1}{2}g_3 g^2 + \frac{1}{12}g^3 + g_0^3\frac{m^2}{\mu^2}\left(\frac{1863}{88} + \frac{71}{33}\tilde{J}_0 + \frac{9}{4}\ln\left(\frac{\mu^2}{m^2}\right)\right)\right] + O\left((m^2/\mu^2)^2\right),
\end{aligned}$$

where in the expression for the beta function of g_3 we indicated the different contributions from the one-loop Feynman diagrams with the coupling constants factored out. In the terms that decrease with $1/\mu^2$ we neglected the difference between g and g_3, indicating with g_0 the asymptotic solution (4.34)

$$g_0^2(\mu) = \frac{3}{11}\left(\frac{N}{(4\pi)^2}\right)^{-1}\frac{1}{\ln\left(\frac{\mu^2}{\Lambda_{UV}^2}\right)}. \quad (4.86)$$

Neglecting for the moment these terms in the beta functions and approximating the two different couplings by linear deviations from the asymptotic solution (4.86), i.e. putting $g = g_0 + \delta g$ and $g_3 = g_0 + \delta g_3$, we obtain a coupled system of differential equations for the linear deviations:

$$\mu\frac{d}{d\mu}\,\delta g = \frac{N g_0^2}{(4\pi)^2}\left[-\frac{19}{3}\delta g - \frac{14}{3}\delta g_3\right] \quad (4.87)$$

$$\mu\frac{d}{d\mu}\,\delta g_3 = \frac{N g_0^2}{(4\pi)^2}\left[\frac{5}{4}\delta g - \frac{49}{4}\delta g_3\right], \quad (4.88)$$

which can be solved by diagonalizing it. We obtain for the following combinations:

$$\begin{aligned}
&\mu\frac{d}{d\mu}\,\delta\bar{g} \equiv \mu\frac{d}{d\mu}\left[-\frac{15}{41}\delta g + \frac{56}{41}\delta g_3\right] = -11\frac{N g_0^2}{(4\pi)^2}\delta\bar{g} \\
&\Longrightarrow \delta\bar{g} = \frac{\bar{C}}{\left(\ln\left(\frac{\mu^2}{\Lambda_{UV}^2}\right)\right)^{3/2}}, \qquad (4.89)
\end{aligned}$$

$$\begin{aligned}
&\mu\frac{d}{d\mu}\,\delta\tilde{g} \equiv \frac{12}{41}\mu\frac{d}{d\mu}\left[\delta g - \delta g_3\right] = -\frac{91}{12}\frac{N g_0^2}{(4\pi)^2}\delta\tilde{g} \\
&\Longrightarrow \delta\tilde{g} = \frac{\tilde{C}}{\left(\ln\left(\frac{\mu^2}{\Lambda_{UV}^2}\right)\right)^{91/88}}, \qquad (4.90)
\end{aligned}$$

where $\bar{C}$ and $\tilde{C}$ are proper constant factors. In solving (4.90) we substituted the expression (4.86) for g_0. By re-inserting into the beta functions the terms that go with $1/\mu^2$, the inhomogenous equations for the combinations $\delta\bar{g}$ and $\delta\tilde{g}$ can be solved by the method of variation of constants applied to the homogeneous solutions. $\bar{C}$ and $\tilde{C}$ thus become functions of μ that satisfy the following equations:

$$\mu \frac{d}{d\mu}\bar{C}(\mu) = \frac{4\pi}{N^{1/2}}\left(\frac{3}{11}\right)^{3/2}\frac{m^2}{\mu^2}\left(\frac{94593}{3608}+\frac{3976}{1352}\tilde{J}_0+\frac{9}{4}\ln\left(\frac{\mu^2}{m^2}\right)\right), \tag{4.91}$$

$$\mu \frac{d}{d\mu}\tilde{C}(\mu) = -\frac{4\pi}{N^{1/2}}\left(\frac{3}{11}\right)^{3/2}\frac{1}{\left(\ln\left(\frac{\mu^2}{m^2}\right)\right)^{41/88}}\frac{m^2}{\mu^2}\left(\frac{1821}{451}+\frac{284}{451}\tilde{J}_0\right). \tag{4.92}$$

We can thus see that, if m^2 approaches its asymptotic values m_0^2 given in (4.39), the functions $\bar{C}$ and $\tilde{C}$ indeed vanish as $1/\mu^2$, and so they do the variations δg and δg_3. By rewriting the UV expression of the mass beta function (4.84), considering that

$$g_3^2 - g^2 \simeq 2g_0(\delta g_3 - \delta g) = -\frac{41}{6}\delta\tilde{g}, \tag{4.93}$$

$$\beta_{m^2} \simeq -\frac{35}{6}\frac{Ng_0^2}{(4\pi)^2}m^2 + \frac{41}{12}\frac{Ng_0}{(4\pi)^2}\delta\tilde{g}\,\mu^2\left(\ln\left(\frac{\mu^2}{m^2}\right)-1\right), \tag{4.94}$$

we can therefore assure the well behaviour of the mass at high energies, due to the decay of $\delta\tilde{g}$ faster than $1/\mu^2$. Operationally what it's done is integrating the asymptotic equations (4.94) and (4.92) from infinity (numerically from a very high scale), where the constants of integration are set to zero (which can be viewed as the proper fine tuning to recover BRST), down to the renormalization scale, which is chosen at the large momentum $\mu = 10$ GeV in order to rely on the approximate expressions for the flow functions used and on the linear approximation of the couplings, fixing the values of g and g_3 by inverting (4.90), and from the renormalization scale down to the IR integrating the full expressions for the flow functions.

It is worthy to stress that, although we consider the separate running of the two couplings, we fix only one parameter g_0 at the renormalization scale. The running of the couplings and of the mass is plotted in Fig. 4.11. We see that the ghost-gluon coupling g dominates in the IR the three-gluon coupling g_3. Moreover, in the deep IR the three-gluon coupling becomes negative. This can be also assessed analytically by looking at the IR asymptotic expression for β_{g_3}:

$$\beta_{g_3} = \mu\frac{dg_3}{d\mu} = \frac{N}{(4\pi)^2}\left[\frac{1}{12}g^3 + \frac{1}{2}g^2 g_3\right] + O(\mu^2/m^2), \tag{4.95}$$

which constitutes a first order inhomogenous differential equation for g_3. The IR behaviour of g is already known from (4.32):

$$\frac{Ng^2}{(4\pi)^2} = \frac{6}{\ln\left(\frac{\Lambda_{IR}^2}{\mu^2}\right)}, \qquad \mu^2 \ll \Lambda_{IR}^2. \tag{4.96}$$

This allows to solve the inhomogenous equation by the method of variation of constants. The result is:

$$g_3(\mu) = \left(\frac{3}{8}\frac{4\pi}{N}\right)^{1/2}\frac{\ln\left(\frac{\mu^2}{\Lambda^2}\right)}{\left(\ln\left(\frac{\Lambda_{IR}^2}{\mu^2}\right)\right)^{3/2}}, \tag{4.97}$$

where Λ in another scale containing the constant of integration. Therefore, for sufficient small values of μ, the coupling becomes negative (the denominator in (4.97) remains always positive and the numerator changes sign when $\mu < \Lambda$), and it then approaches zero as μ tends to zero.

The best fits for the propagators are plotted in Fig. 4.12 and 4.13. Since they are the best fits to the lattice data obtained so far, we plot also the result for the ghost dressing function.

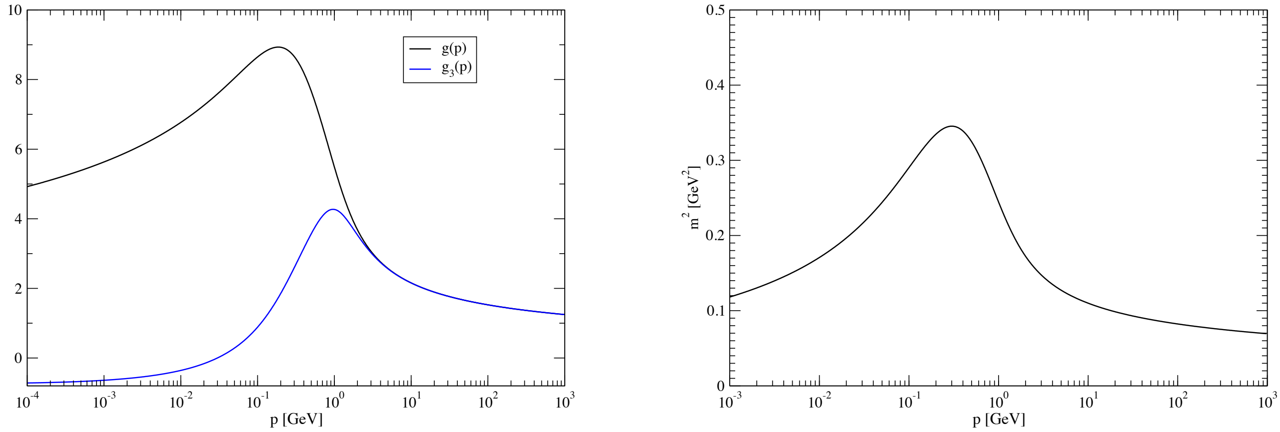


Figure 4.11: Running couplings g and g_3 (left) and running mass (right) in the TW scheme. $g = 2.16$ and $m = 0.11$ GeV at $\mu = 10$ GeV.

This latter does not change compared to the original fit in the TW scheme plotted in Fig. 4.1, since, as we explained, the parameters are fixed in such a way to optimize the ghost dressing fit (although for the symmetry point scheme (SP) this was not possible). The fits to the gluon propagator and gluon dressing function, however, have considerably improved compared to Fig. 4.2. Considering that the normalization conditions used for the two-point functions are the same in the two cases, the only difference being the inclusion of the separate running of the two coupling constants, it would therefore seem that the considerations that lead us to explore this scenario were indeed correct. This could be viewed as a further evidence of the soft BRST symmetry breaking.

4.4 Three-point dressing functions

We shall now integrate the Callan-Symanzik equations for the vertices dressing functions, in some of the renormalization schemes introduced before. In the TW renormalization scheme the calculations have already been performed in [35].

Let us consider the three-gluon dressing function at the symmetry point configuration of momenta, meaning the external gluon propagators have the same momentum p^2. The renormalization group equation (4.7) for this case reads:

$$\Gamma_{AAA}(p^2,\mu^2,g(\mu^2),m(\mu^2)) = \Gamma_{AAA}(p^2,p^2,g(p^2),m(p^2)) \exp\left\{-\frac{3}{2}\int_\mu^p \frac{d\mu'}{\mu'}\gamma_A(\mu',g(\mu'),m(\mu'))\right\}, \tag{4.98}$$

where, as explained before, $\Gamma_{AAA}(p^2)$, is the three-gluon vertex after the proper contractions, the ones used to define the dressing function in (3.64), related to the vertex through:

$$\Gamma_{AAA}(p^2,\mu^2,g(\mu^2),m(\mu^2)) = g(\mu)D_{AAA}(p^2,\mu^2,g(\mu^2),m(\mu^2)). \tag{4.99}$$

In the case where we considered the running of the two couplings g and g_3, in principle there would be an ambiguity on which coupling to put in (4.99), but since we renormalized at a scale in the UV (10 GeV) where the two couplings coincide (cf. Fig 4.11), it is indifferent if we choose $g(\mu)$ or $g_3(\mu)$.

For this renormalization scheme the factor multiplying the exponential in (4.98) is just the

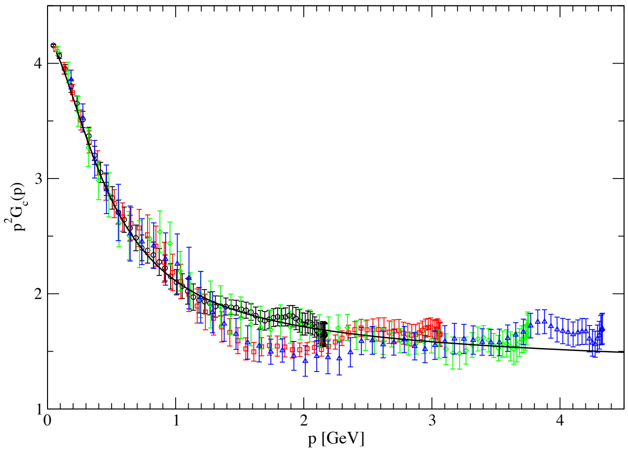


Figure 4.12: Ghost dressing function in the TW scheme with two couplings. $g = 2.16$ and $m = 0.11$ GeV at $\mu = 10$ GeV.

running coupling $g_3(p)$ defined in (4.82). The equation for the dressing function thus reads:

$$D_{AAA}(p^2, \mu^2, g(\mu^2), m(\mu^2)) = \frac{g_3(p)}{g(\mu)} \exp\left\{-\frac{3}{2}\int_\mu^p \frac{d\mu'}{\mu'}\gamma_A(\mu', g(\mu'), g_3(\mu'), m(\mu'))\right\}. \quad (4.100)$$

For the renormalization schemes that use only one coupling defined through the ghost-gluon vertex, the factor multiplying the exponential is approximated by its perturbative expression at one-loop order, as explained in the introduction of this chapter. We have:

$$\begin{aligned} \Gamma_{AAA}(p^2, p^2, g(p^2), m(p^2)) &= g(p) D_{AAA}(p^2, p^2, g(p^2), m(p^2)) \\ &= g_B\, Z_A^{3/2}(p) D^B_{AAA}(p^2, \epsilon, g(p^2), m(p^2)) \\ &= g(p)\frac{Z_A(p)}{Z_c(p)} D^B_{AAA}(p^2, \epsilon, g(p^2), m(p^2)), \end{aligned} \quad (4.101)$$

where we used the definition of the renormalized coupling at the Taylor point

$$g(p) = g_B Z_A^{1/2}(p) Z_c(p). \quad (4.102)$$

In (4.101) $D^B_{AAA}(p^2, \epsilon, g(p^2), m(p^2))$ is given by the bare perturbative dressing function in (3.65), putting $t = p^2/m(p^2)$ and substituting the coupling constant with its running function. These, in fact, are the conditions that allow us to rely on the perturbative approximation. It is noteworthy that the factor Z_A/Z_c is indeed required in order to cancel the divergent pole present in the bare dressing function. The renormalization group equation for the three-gluon dressing function for the schemes that use one coupling defined at the Taylor point, is therefore reduced to

$$\begin{aligned} D_{AAA}(p^2, \mu^2, g(\mu^2), m(\mu^2)) =& \frac{g(p^2)}{g(\mu^2)}\frac{Z_A(p)}{Z_c(p)} D^B_{AAA}(p^2, \epsilon, g(p^2), m(p^2)) \\ & \times \exp\left\{-\frac{3}{2}\int_\mu^p \frac{d\mu'}{\mu'}\gamma_A(\mu', g(\mu'), m(\mu'))\right\}. \end{aligned} \quad (4.103)$$

The integrated three-gluon dressing function is plotted in Fig. 4.14 for the renormalization scheme that exploits the running of the two couplings (2g) and for two schemes that use only one

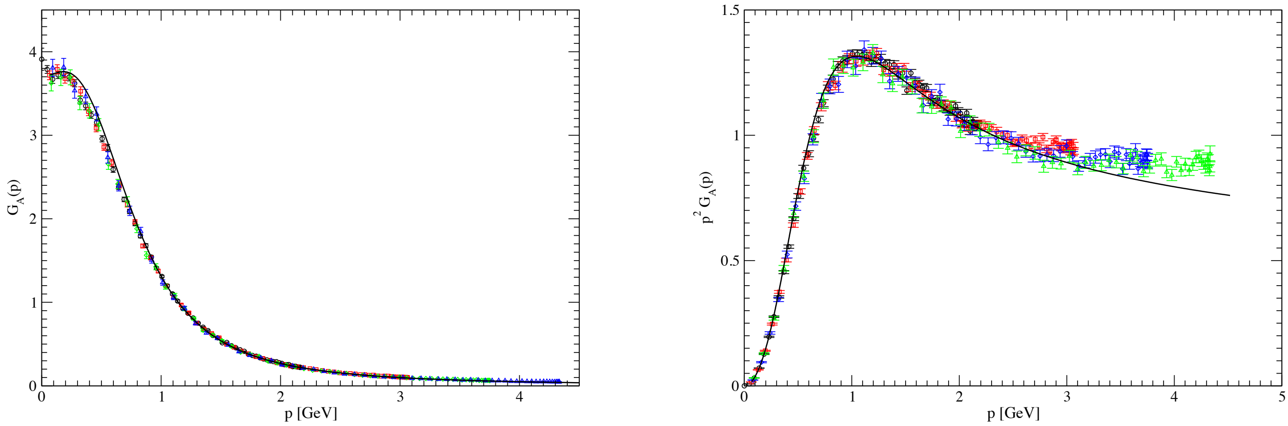


Figure 4.13: Gluon propagator (left) and gluon dressing function (right) in the TW scheme with two couplings. $g = 2.16$ and $m = 0.11$ GeV at $\mu = 10$ GeV.

coupling, the original Tissier-Wschebor scheme (TW) and the critical ζ-scheme corresponding to $\zeta = 1/3$. As one can see the dressing function integrated with the two running coupling behaves considerably differently in the IR, with a milder logarithmically behaviour. In particular, its behaviour is much closer to the one found by solving its DSE [80], as it can be seen in Fig. 4.15. The zero crossing scale at which the dressing function becomes negative is also closer to the DSE solution, corresponding to about 80 MeV, being 30 MeV for the solution with two couplings and about 500 MeV for the solutions with one coupling.

It is important to stress that once the parameters g and m^2 are fixed at the renormalization scale by fitting the propagators with the lattice data, the dressing functions for the vertices are completely determined, according to the renormalization scheme used. In this description of Yang-Mills theory, thus, the whole dimensional physical content is contained in two parameters, the coupling constant, that implicitly fixes the unit of momentum scale, and the mass parameter that describes the whole IR behaviour and should somehow be related to Λ_{QCD}, although in this description is a free parameter.

In a similar fashion we can obtain the Callan-Symanzik equation for the ghost-gluon dressing function at the symmetry point:

$$\Gamma_{\bar{c}cA}(p^2, \mu^2, g(\mu^2), m(\mu^2)) = \Gamma_{\bar{c}cA}(p^2, p^2, g(p^2), m(p^2)) \exp\left\{ -\int_\mu^p \frac{d\mu'}{\mu'} \left(\frac{1}{2}\gamma_A(\mu') + \gamma_c(\mu') \right) \right\}, \tag{4.104}$$

which for the dressing function reads:

$$D_{\bar{c}cA}(p^2, \mu^2, g(\mu^2), m(\mu^2)) = \frac{g(p)}{g(\mu)} D^B_{\bar{c}cA}(p^2, g(p^2), m(p^2)) \exp\left\{ -\int_\mu^p \frac{d\mu'}{\mu'} \left(\frac{1}{2}\gamma_A(\mu') + \gamma_c(\mu') \right) \right\}, \tag{4.105}$$

where the finite bare dressing function at the symmetry point is the one calculated at one-loop order, given in (3.53). Note that in the Taylor scheme the bare dressing function is equal to the renormalized one. We have in fact:

$$\begin{aligned} D_{\bar{c}cA}(p^2, \mu^2) &= \frac{1}{g(\mu)} \Gamma_{\bar{c}cA}(p^2, \mu^2) = \frac{Z_A^{1/2}(\mu) Z_c(\mu)}{g(\mu)} \Gamma^B_{\bar{c}cA}(p^2, \mu^2) \\ &= Z_A^{1/2}(\mu) Z_c(\mu) \frac{g_B}{g(\mu)} D^B_{\bar{c}cA}(p^2) = D^B_{\bar{c}cA}(p^2). \end{aligned} \tag{4.106}$$

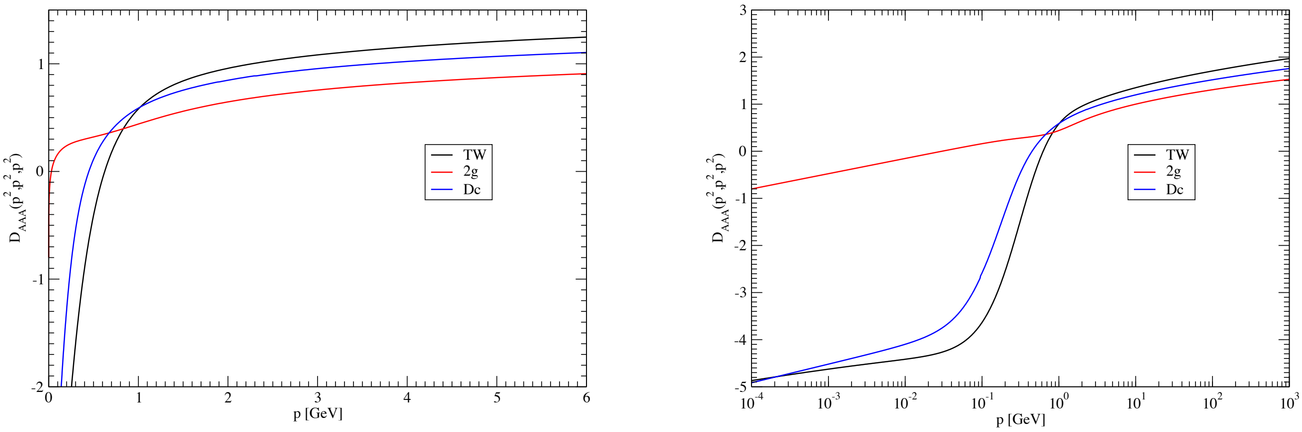


Figure 4.14: Three-gluon dressing function at the symmetry point in linear (left) and logarithmic (right) scale, calculated for the renormalization schemes with two couplings (2g) and with one coupling at the Taylor point in the Tissier-Wschebore scheme (TW) and in the critical derivative scheme (Dc).

The integrated ghost-gluon dressing function is shown in Fig. 4.16, where it is plotted for the configuration of momenta at the symmetry point and at vanishing gluon momentum. As it can be seen, it shows little deviation from its tree-level value, as assessed by the Taylor argument.

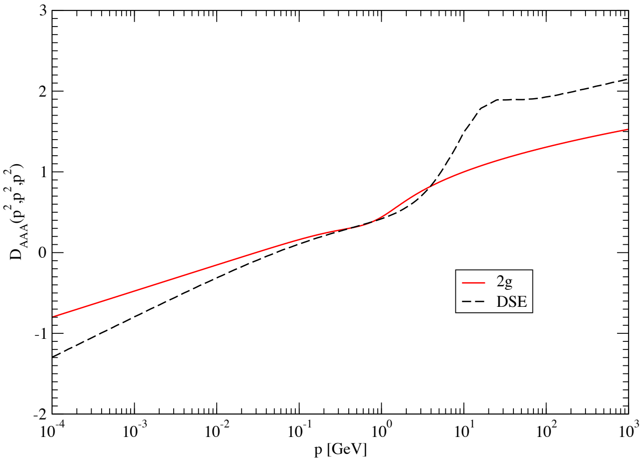


Figure 4.15: Three-gluon dressing function at the symmetry point in the scheme with two couplings (2g) compared with the DSE solution.

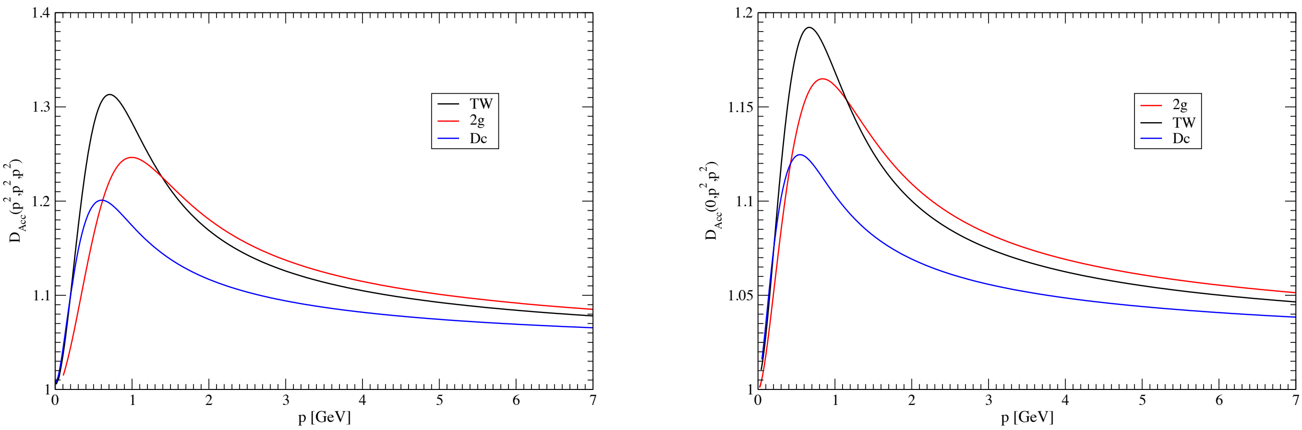


Figure 4.16: Ghost-gluon dressing function at the symmetry point (left) and at vanishing gluon momentum (right), calculated for the renormalization schemes with two couplings (2g) and with one coupling at the Taylor point in the Tissier-Wschebore scheme (TW) and in the critical derivative scheme (Dc).

Chapter 5

Conclusions

In this work we have taken a closer look at the analysis carried out by Tissier and Wschebor [10, 35] who first calculated the one-loop corrections of the Yang-Mills two and three point functions in Landau gauge with the insertion of a massive gluon. This additional mass term was included in order to account for the effect of the Gribov horizon in the IR. In fact, in the general framework of the refined version of the Gribov-Zwanziger action constructed in order to overcome the problem of the Gribov ambiguity, a massive IR behaviour for the gluon propagator is found, in agreement with the latest lattice data and with a particular solution of the DSEs. We repeated the calculations done in [10, 35], exploring other possible renormalization schemes in order to capture the largest non perturbative contributions throughout the resummation with the renormalization group equations, performed to better connect the IR massive fixed point to the massless UV one. We understood in particular the crucial role in every renormalization scheme of the longitudinal part of the gluon two-point function, that must be included, despite in Landau gauge the gluon propagator is transverse, in the definition of the renormalized mass parameter and in the normalization condition for the gluon two-point function, otherwise the scheme would not be IR safe, i.e. a Landau pole would be generated invalidating the perturbative approach, and/or the flow equations would not reproduce the correct behaviour in the UV, restoring the original BRST symmetry through the vanishing of the running mass parameter at high energies.

We then played with the freedom offered by renormalization schemes that involve the derivative of the two-point functions in the normalization conditions, and found better quantitative agreement with the lattice data. The best fit, however, was found by letting separately run the coupling constants corresponding to the ghost-gluon vertex, renormalized at the Taylor configuration of momenta (vanishing incoming ghost momentum) and the one corresponding to the three-gluon vertex, renormalized at the symmetry point configuration of momenta. This result, in particular, suggests that this framework is the correct one to describe Yang-Mills theory at all scales within a perturbative approach, and that the soft breaking of the BRST symmetry, allegedly originated by the presence of the Gribov horizon, has to be taken seriously, since it causes the different couplings to scale differently in the IR.

Appendix A

Two-points scalar integrals

Following the Passarino-Veltman notation [64] we define the general two-point scalar integral as:

$$B_0(p^2, m_0^2, m_1^2) = \int \frac{d^D l}{(2\pi)^D} \frac{1}{(l^2 + m_0^2)\left((l-p)^2 + m_1^2\right)}\,, \tag{A.1}$$

and the one-point (tadpole) integral:

$$A_0(m^2) = \int \frac{d^D l}{(2\pi)^D} \frac{1}{(l^2 + m^2)}. \tag{A.2}$$

These are easily evaluated in dimensional regularisation by introducing Feynman parameters. For the ones that are needed in the calculations of this thesis only a single mass parameter appears and the corresponding expressions are given, in 4 dimensions ($D = 4 - \epsilon$):

$$B_0(p^2, m^2, m^2) = \frac{1}{(4\pi)^2} \left\{ \overline{\frac{2}{\epsilon}} + 2 + \sqrt{\frac{4m^2}{p^2} + 1} \log \left[\frac{\sqrt{4 + \frac{p^2}{m^2}} - \sqrt{\frac{p^2}{m^2}}}{\sqrt{4 + \frac{p^2}{m^2}} + \sqrt{\frac{p^2}{m^2}}} \right] \right\}, \tag{A.3}$$

$$B_0(p^2, m^2, 0) = \frac{1}{(4\pi)^2} \left\{ \overline{\frac{2}{\epsilon}} + 2 - \left(1 + \frac{m^2}{p^2}\right) \log\left(1 + \frac{p^2}{m^2}\right) \right\}, \tag{A.4}$$

$$B_0(p^2, 0, 0) = \frac{1}{(4\pi)^2} \left\{ \overline{\frac{2}{\epsilon}} + 2 - \log\left(\frac{p^2}{m^2}\right) \right\}, \tag{A.5}$$

$$A_0(m^2) = -\frac{m^2}{(4\pi)^2} \left\{ \overline{\frac{2}{\epsilon}} + 1 \right\}, \tag{A.6}$$

where $\overline{\frac{2}{\epsilon}} = \frac{2}{\epsilon} - \gamma_E - \log(m^2/4\pi\kappa^2)$, being γ_E the Euler constant and κ an arbitrary scale given by the dimension of the coupling constant, $g = \overline{g}\,\kappa^{\frac{\epsilon}{2}}$ ($\overline{g}$ is dimensionless).

For example, in order to obtain the expression in (A.3) we introduce a Feynman parameter, writing:

$$
\begin{aligned}
B_0(p^2,m^2,m^2) &= \int \frac{d^D l}{(2\pi)^D} \frac{1}{(l^2+m^2)\left((l-p)^2+m^2\right)} = \int_0^1 dx \int \frac{d^D l}{(2\pi)^D} \frac{1}{[l^2+x(1-x)p^2+m^2]^2} \\
&= \frac{\Gamma(2-D/2)}{(4\pi)^{D/2}} \int_0^1 dx \frac{1}{(x(1-x)p^2+m^2)^{2-D/2}} \\
&\underset{D=4-\epsilon}{=} \frac{\Gamma(\frac{\epsilon}{2})}{(4\pi)^{2-\epsilon/2}} \int_0^1 \left[1 - \frac{\epsilon}{2} \log\left[x(1-x)p^2+m^2\right]\right] \\
&= \frac{1}{(4\pi)^2} \left\{ \frac{2}{\epsilon} - \gamma_E + \log(4\pi) - \int_0^1 \log\left[x(1-x)p^2+m^2\right] \right\} \\
&= \frac{1}{(4\pi)^2} \left\{ \frac{2}{\epsilon} + 2 + \sqrt{\frac{4m^2}{p^2}+1} \log\left[\frac{\sqrt{4+\frac{p^2}{m^2}} - \sqrt{\frac{p^2}{m^2}}}{\sqrt{4+\frac{p^2}{m^2}} + \sqrt{\frac{p^2}{m^2}}} \right] \right\},
\end{aligned}
\tag{A.7}
$$

where in the first line we used the Feynman's formula:

$$
\frac{1}{A_1^{\alpha_1} \cdots A_n^{\alpha_n}} = \frac{\Gamma(\sum_i \alpha_1)}{\prod_i \alpha_i} \int_0^1 dx_1 \cdots dx_n \, \delta(x_1 + \cdots x_n - 1) \frac{\prod_i x_i^{\alpha_i - 1}}{(\sum_i x_i A_i)^{\sum_i \alpha_i}}, \tag{A.8}
$$

in the second line we integrated the D dimensional loop integral using:

$$
\int \frac{d^D l}{(2\pi)^D} \frac{(l^2)^a}{(l^2+\Delta)^b} = \frac{\Gamma(b-a-D/2)\Gamma(a+D/2)}{(4\pi)^{D/2}\Gamma(b)\Gamma(D/2)} \, \Delta^{D/2+a-b}, \tag{A.9}
$$

and in the third line we expanded the Γ function around its pole, neglecting terms of order ϵ:

$$
\Gamma\left(-n + \frac{\epsilon}{2}\right) = \frac{(-1)^n}{n!} \left[\frac{2}{\epsilon} - \gamma_E + \sum_{k=1}^{n} \frac{1}{k} + O(\epsilon) \right]. \tag{A.10}
$$

In three dimensions the same scalar integrals, now convergent, are given by the following expressions (D=3):

$$
B_0(p^2,m^2,m^2) = \frac{1}{4\pi} \frac{\arctan\left(\frac{p}{2m}\right)}{p}, \tag{A.11}
$$

$$
B_0(p^2,m^2,0) = \frac{1}{4\pi} \frac{\arctan\left(\frac{p}{m}\right)}{p}, \tag{A.12}
$$

$$
B_0(p^2,0,0) = \frac{1}{8p}, \tag{A.13}
$$

$$
A_0(m^2) = -\frac{m}{(4\pi)}. \tag{A.14}
$$

Appendix B

Three-point scalar integrals

The notation for the three-point scalar integral is:

$$C_0(p_1^2,(p_1-p_2)^2,p_2^2,m_0^2,m_1^2,m_2^2) = \int\frac{d^Dl}{(2\pi)^D}\frac{1}{(l^2+m_0^2)\,((l+p_1)^2+m_1^2)\,((l+p_2)^2+m_2^2)}\,. \quad \text{(B.1)}$$

In order to reduce (B.1) in the simplest form it is useful tu use the Cheng-Wu trick [11] that consists in loosing two of the three Feynman parameters from the Dirac delta in (A.8), letting them go to infinity:

$$C_0(p_1^2,(p_1-p_2)^2,p_2^2,m_0^2,m_1^2,m_2^2) = 2\int_0^1 dx\int_0^1 dy\int_0^1 dz\,\delta(x+y+z-1) \quad \text{(B.2)}$$
$$\times\int\frac{d^Dl}{(2\pi)^D}\frac{1}{\left[x(l^2+m_0^2)+y((l+p_1)^2+m_1^2)+z((l+p_2)^2+m_2^2)\right]^3}$$
$$= 2\int_0^\infty dy\int_0^\infty dz\int\frac{d^Dl}{(2\pi)^D}\frac{1}{\left[(l^2+m_0^2)+y((l+p_1)^2+m_1^2)+z((l+p_2)^2+m_2^2)\right]^3}$$
$$= 2\int_0^\infty dy\int_0^\infty dz\,\frac{1}{(1+y+z)^3}\int\frac{d^Dl}{(2\pi)^D}\frac{1}{\left[l^2+\frac{y(1+z)p_1^2+z(1+y)p_2^2-2yzp_1\cdot p_2+(1+y+z)(m_0^2+ym_1^2+zm_2^2)}{(1+y+z)^2}\right]^3}$$
$$= \frac{\Gamma(3-D/2)}{(4\pi)^{D/2}}\int_0^\infty dy\int_0^\infty dz\,(1+y+z)^{3-D} \quad \text{(B.3)}$$
$$\times\left[y(1+z)p_1^2+z(1+y)p_2^2-2yzp_1\cdot p_2+(1+y+z)(m_0^2+ym_1^2+zm_2^2)\right]^{D/2-3}.$$

B.1 Symmetry point

At the symmetry point configuration $p_1^2=p_2^2=(p_1-p_2)^2=\mu^2$ in four dimensions and with only one mass parameter, it is possible to perform an integration over one Feynman parameter. For example, the scalar integral with one massive propagator, at the symmetry point with $D=4$

reads:

$$\begin{aligned}
C_0(\mu^2,\mu^2,\mu^2,m^2,0,0) &= \frac{1}{(4\pi)^2}\int_0^\infty dy\int_0^\infty dz\frac{1}{(1+y+z)\left[\mu^2(y+z+yz)+(1+y+z)m^2\right]}\\
&= -\frac{1}{(4\pi)^2\mu^2}\int_0^\infty dy\int_0^\infty dz\frac{1}{1+y+y^2}\left(\frac{1}{1+y+z}-\frac{1}{z+\frac{y(\mu^2+m^2)}{(1+y)\mu^2+m^2}}\right)\\
&= \frac{1}{(4\pi)^2\mu^2}\int_0^\infty dy\,\frac{1}{1+y+y^2}\log\left(\frac{(1+y)(\mu^2(1+y)+m^2)}{y(\mu^2+m^2)+m^2}\right)\\
&= \frac{1}{(4\pi)^2\mu^2}\int_0^1 dx\,\frac{1}{1-x+x^2}\log\left(\frac{x+t}{x(1+xt)}\right), \qquad t=\frac{\mu^2}{m^2}
\end{aligned} \tag{B.4}$$

where in the last line we substituted $x = 1/(1+y)$, in order to have a compact domain of integration, which is suitable for numerical evaluation. In a similar fashion one can reduce the three-point scalar integrals at the symmetry point with two or three massive propagators:

$$C_0(\mu^2,\mu^2,\mu^2,m^2,m^2,0) = \frac{1}{(4\pi)^2\mu^2}\int_0^1 dx\,\frac{1}{1-x+x^2}\log\left(\frac{1+t}{1+xt(1-x)}\right), \tag{B.5}$$

$$\begin{aligned}
C_0(\mu^2,\mu^2,\mu^2,m^2,m^2,m^2) = \frac{1}{2(4\pi)^2\mu^2}\int_0^1 dx\,\frac{1}{1-x+x^2}&\Big\{-\log\left(1+x(1-x)t\right)\\
&+\frac{t}{\sqrt{t^2+4t(1-x+x^2)}}\log\left(\frac{t+2+\sqrt{t^2+4t(1-x+x^2)}}{t+2-\sqrt{t^2+4t(1-x+x^2)}}\right)\Big\},
\end{aligned} \tag{B.6}$$

and the massless three-point scalar integral:

$$\begin{aligned}
C_0(\mu^2,\mu^2,\mu^2,0,0,0) &= -2\frac{1}{(4\pi)^2\mu^2}\int_0^1 dx\,\frac{1}{1-x+x^2}\log(x)\\
&= \frac{1}{18(4\pi)^2\mu^2}\left(\psi'\left(\frac{1}{6}\right)+\psi'\left(\frac{1}{3}\right)-\psi'\left(\frac{2}{3}\right)-\psi'\left(\frac{5}{6}\right)\right),
\end{aligned} \tag{B.7}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function.

Since at the symmetry point the result for the integrals depends only on the number of massive scalar propagators and not on which one is massive and which one is not, we condense the notation and call the dimensionless part of these integrals $\tilde{J}_i(t)$, where the lower index stands for the number of massive propagators, i.e. $\tilde{J}_1 = \mu^2(4\pi)^2 C_0(\mu^2,\mu^2,\mu^2,m^2,0,0)$. It is noteworthy that the derivatives of $\tilde{J}_i(t)$ w.r.t. their dimensionless parameter $t = \mu^2/m^2$ have compact expressions:

$$\frac{d\tilde{J}_1}{dt} = -\frac{\log(t)-2\log(1+t)}{t^2+t+1}, \tag{B.8}$$

$$\frac{d\tilde{J}_2}{dt} = \frac{\log\left(\frac{1}{4}\left(2+t+\sqrt{t(t+4)}\right)^2\right)}{(1+t)\sqrt{t(t+4)}}, \tag{B.9}$$

$$\frac{d\tilde{J}_3}{dt} = \frac{\log\left(\frac{\sqrt{t+4}+\sqrt{t}}{\sqrt{t+4}-\sqrt{t}}\right)+\log\left(\frac{t+2+\sqrt{t(t+4)}}{t+2-\sqrt{t(t+4)}}\right)}{(t+3)\sqrt{t(t+4)}}. \tag{B.10}$$

Here below we give the asymptotic expressions for these scalar integrals in the IR limit ($t \ll 1$) and in the UV limit ($t \gg 1$), which are used in the evaluation of the asymptotic behaviour of the Feynman diagrams that is needed for the numerical integration of the renormalization group equations. Since these limits correspond to branch points for some of these integrals, the fastest way to get the asymptotic behaviours is to expand the derivative expressions given above and then integrate. In the IR we have:

$$\tilde{J}_1(t) = t(1-\log(t)) + t^2\left(\frac{\log(t)}{2} + \frac{3}{4}\right) - t^3 + t^4\left(\frac{23}{48} - \frac{\log(t)}{4}\right) + t^5\left(\frac{\log(t)}{5} + \frac{19}{150}\right) + O(t^6), \tag{B.11}$$

$$\tilde{J}_2(t) = t - \frac{7}{12}t^2 + \frac{2}{5}t^3 - \frac{169}{560}t^4 + \frac{1523}{6300}t^5 + O(t^6), \tag{B.12}$$

$$\tilde{J}_3(t) = \frac{t}{2} - \frac{t^2}{8} + \frac{t^3}{30} - \frac{31}{3360}t^4 + \frac{11}{4200}t^5 + O(t^6). \tag{B.13}$$

The expansions in the UV are:

$$\tilde{J}_1(t) = \tilde{J}_0 - \frac{\log(t)+1}{t} + \frac{\frac{1}{2}\log(t) - \frac{3}{4}}{t^2} + \frac{1}{t^3} - \frac{\frac{1}{4}\log(t) + \frac{23}{48}}{t^4} + O\left(\frac{1}{t^5}\right), \tag{B.14}$$

$$\tilde{J}_2(t) = \tilde{J}_0 - \frac{2\log(t)+2}{t} + \frac{3\log(t) - \frac{1}{2}}{t^2} - \frac{6\log(t) - 4}{t^3} + \frac{\frac{29}{2}\log(t) - \frac{317}{24}}{t^4} + O\left(\frac{1}{t^5}\right), \tag{B.15}$$

$$\tilde{J}_3(t) = \tilde{J}_0 - \frac{3\log(t)+3}{t} + \frac{\frac{15}{2}\log(t) + \frac{3}{4}}{t^2} - \frac{21\log(t) - 6}{t^3} + \frac{\frac{249}{4}\log(t) - \frac{515}{16}}{t^4} + O\left(\frac{1}{t^5}\right). \tag{B.16}$$

In three dimensions one can perform both of the integrals over the Feynman parameters, due to the simplification occurring in (B.2) (the factor $(1+y+z)^{3-D}$ becomes 1). For $D = 3$ we define three-point scalar integrals at the symmetry point $\tilde{J}_i(t)$ as before, the only difference being the factor multiplying the C_0 functions is now $\mu^3(4\pi)$, in order for them to be dimensionless. Their expressions are:

$$\tilde{J}_0 = \frac{\pi}{2}, \tag{B.17}$$

$$\tilde{J}_1(t) = \frac{t}{\sqrt{1+t+t^2}} \arctan\left(\sqrt{\frac{1+t+t^2}{t}}\right), \tag{B.18}$$

$$\tilde{J}_2(t) = \frac{t}{1+t} \arctan\left(\frac{\sqrt{t}}{2}\right), \tag{B.19}$$

$$\tilde{J}_3(t) = \sqrt{\frac{t}{t+3}}\left[\arctan(\sqrt{t+3}) + \arctan\left(\frac{t+2}{2\sqrt{t+3}}\right) - \frac{\pi}{2}\right]. \tag{B.20}$$

The expressions for the three-point scalar integrals with one and two massive propagators are also given in [17].

B.2 Soft momentum

In the evaluation of the three-point vertices diagrams in the limit of an external vanishing gluon momentum, the following three-point scalar integrals are required ($D = 4 - \epsilon$):

$$C_0(k^2,k^2,0,0,0,0) = \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^4\,(l+k)^2} = \frac{1}{(4\pi)^2 m^2 t}\left[-\frac{\overline{2}}{\epsilon} + \log(t)\right], \qquad t = \frac{k^2}{m^2} \tag{B.21}$$

$$C_0(k^2,k^2,0,0,m^2,0) = \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^4\,((l+k)^2+m^2)} \tag{B.22}$$

$$= \frac{1}{(4\pi)^2 m^2 (1+t)}\left[-\frac{\overline{2}}{\epsilon} + t^{-1}(t-1)\log(1+t)\right], \tag{B.23}$$

$$C_0(k^2,k^2,0,m^2,0,m^2) = C_0(0,k^2,k^2,m^2,m^2,0) = \int \frac{d^D l}{(2\pi)^D} \frac{1}{(l^2+m^2)^2\,(l+k)^2} \tag{B.24}$$

$$= \frac{1}{(4\pi)^2 m^2 t}\log(1+t), \tag{B.25}$$

$$C_0(0,k^2,k^2,m^2,m^2,m^2) = \int \frac{d^D l}{(2\pi)^D} \frac{1}{(l^2+m^2)^2\,((l+k)^2+m^2)} \tag{B.26}$$

$$= \frac{1}{(4\pi)^2 m^2 \sqrt{t(t+4)}}\log\left(\frac{\sqrt{t+4}+\sqrt{t}}{\sqrt{t+4}-\sqrt{t}}\right). \tag{B.27}$$

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