UNIVERSIDAD MICHOACANA DE
DE SAN NICOLÁS DE HIDALGO

INSTITUTO DE FÍSICA Y MATEMÁTICAS

APLICACIÓN DEL FORMALISMO LÍNEA DE MUNDO A ESTADOS LIGADOS

TESIS

QUE PARA OBTENER EL GRADO DE

DOCTOR EN CIENCIAS EN EL ÁREA DE FÍSICA

PRESENTA

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APLICACIÓN DEL FORMALISMO LÍNEA DE MUNDO A ESTADOS LIGADOS

Resumen

En este estudio utilizamos el formalismo línea de mundo para reunir todas las escaleras y escaleras cruzadas en la teoría $\phi^2\chi$ con masa cero para la partícula χ. El Lagrangiano euclideano para esta teoría es $L = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{2}(\partial_\mu\chi)^2 + \frac{1}{2}\mu^2\chi^2 + \frac{\lambda}{2!}\phi^2\chi$. Derivamos representaciones integrales para tres clases de amplitudes en la teoría del campo escalar: (i) propagador escalar intercambiando N momentos con un campo escalar de fondo, (ii) La "media" escalera con N peldaños en el espacio x, (iii) en la escalera de cuatro puntos tanto con N peldaños en el espacio x, como en el espacio de momentos (fuera de la capa de masa). En cada caso se da una expresión compacta que combina los $N!$ diagramas de Feynman que contribuyen a la amplitud. Como nuestra aplicación principal, se reconsidera el caso bien conocido de dos escalares masivos que interactúan a través del intercambio de un escalar sin masa. Aplicando estimaciones asintóticas y una aproximación de punto silla a las escaleras con N peldaños más diagramas de escalera cruzados, derivamos una fórmula de aproximación semi-analitica para el estado ligado de menor masa en este modelo.

APPLICATION OF THE WORLDLINE FORMALISM TO BOUND STATES

Abstract

In this study we use the worldline formalism to sum up all ladders and cross-ladders in the $\varphi^2\chi$ theory with zero mass of the χ particle. The (euclidean) Lagrangian for this theory is $L = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{2}(\partial_\mu\chi)^2 + \frac{1}{2}\mu^2\chi^2 + \frac{\lambda}{2!}\phi^2\chi$. We derive an integral representations for three classes of amplitudes in scalar field theory: (i) the scalar propagator exchanging N momenta with a scalar background field (ii) the "half-ladder"with N rungs in $x-$space (iii) the four-point ladder with N rungs in $x-$space as well as in (off-shell) momentum space. In each case we give a compact expression combining the $N!$ Feynman diagrams contributing to the amplitude. As our main application, we reconsider the well-known case of two massive scalars interacting through the exchange of a massless scalar. Applying asymptotic estimates and a saddle point approximation to the $N-$rung ladders plus crossed ladder diagrams, we derive a semi-analytic approximation formula for the lowest bound state mass in this model.

CONTENTS

CHAPTER 1

INTRODUCTION

The history of our study goes back to the non-relativistic Schrodinger eq. (1926), followed by the work of Dirac in 1928 and Gregory in 1929, who treated interactions of elementary particles relativistically, still without using separate time formalism and Feynman rules. Thereafter, relativistic bound states and quantum electrodynamics had a long wait of two or more decades. In 1947 Bethe gave an approximate but adequate prescription to carry out renormalization in order to deal with divergencies, and to theoretically calculate the Lamb shift for the first time. Tomonaga (1948), Schwinger (1948), Dyson (1949) and Feynman (1949) published a series of papers proposing different methods for computing the $S-$matrix in QFT. However, Feynman's formalism had great practical advantages as it could be characterized by Feynman diagrams and graphs. The Bethe Salpeter equation was given at an American Physical Society meeting at the beginning of 1951 and then published in more detail later in 1951 Bethe-Salpeter (1951). A proton and an electron can move separately, the total center of mass energy greater than the rest masses, and such a pair of particles can be described as an ionized atom. Once the electron of an atom starts to "orbit" the proton, the energy becomes smaller than the rest masses, and a bound state namely the hydrogen atom is formed. At most the lowest energy bound state, the ground state, is stable. The other excited states are unstable and will decay into a bound states with less energy.

In chapter 2 of the thesis, we discuss the bound-state problem of Quantum Field Theory in detail, this problem of QFT has been a field of active research for many years. Making exact predictions using the correct Lagrangian for a system is not an easy task. Therefore, we have to make approximations, the most common approximation is known as perturbation theory. Perturbation theory involves making an expansion in the coupling strength of the interaction and works for small couplings. Perturbation theory cannot explain bound states, irrespective of how small the coupling strength is. Therefore, bound states are always fully non-perturbative or we can say non-perturbative methods are required where standard perturbative methods are not applicable. Bound states are identified by poles in Green's function. The n-body bound state is defined by the pole of the n-body propagator. The bound state singularity comes from an infinite summation of perturbation series. A perturbative approximation of an n-body propagator does not produce the bound state pole location. Hence it is essential that reliable non-perturbative methods that take all orders of interaction into account are developed. Building blocks of matter, quarks and gluons only exist in bound states. Any reaction involving quarks definitely involves bound states in the initial and final states and necessitates a non-perturbative treatment.

The fact that not much work is done at present on this problem reflects its complexity rather than a lack of importance. The present-day description of (light) hadrons, which are intrinsically relativistic bound states of quarks and gluons, is not satisfactory from a theoretical standpoint. Not only a precise description of the effective interaction of quarks and gluons is missing, but also a convenient formalism for the calculation of the hadronic states once an appropriate description of the interaction is established.

A fully relativistic equation for the masses and structure of the bound states of two constituents has been established in quantum field theory a long time ago by Salpeter and Bethe [1, 2]. Unfortunately, the practical application of this equation suffers from all kinds of difficulties [3]. In particular, despite the fact that the equation is exact in principle, applications can hardly go beyond the ladder approximation to the equation which amounts to replacing the totality of diagrams contributing to the four-point function with the ladder graphs, *excluding* all crossed ladder graphs. The inclusion of the crossed ladder graphs, however, is essential for the consistency of the one-body limit where one of the constituents becomes infinitely heavy, and for maintaining gauge invariance (in gauge theories).

Alternatives to the Bethe-Salpeter equation have been devised that partially include the crossed ladder graphs, the best-known being the Blankenbecler-Sugar equation [4], the Gross (or spectator) equation [5] and the equal-time equation [6]. In order to assess how well these so-called quasipotential equations are doing in incorporating the effects of the crossed-ladder graphs, and to establish some benchmark values for the relativistic bound state problem, Nieuwenhuis and Tjon [7] have numerically evaluated the path integrals of the "worldline representation" to be defined below, for the same scalar model field theory that we will study in this thesis, thus including all ladder *and* crossed ladder graphs. The results, if the numerical evaluation is to be trusted, are not reassuring: while the predictions of the quasi-potential equations are closer to the numerical values for the lowest bound state mass than the solution of the Bethe-Salpeter equation, they still differ substantially from the worldline values (and from one another). On the other hand, the predictions of the quasipotential equations for the equal-time wave function of the ground state are *worse* than the ones of the Bethe-Salpeter equation.

While it is not possible here to give comprehensive review of bound state theory, in chapter 2 we focus on demonstrating the interplay between the box and the crossed box diagrams in scalar field theory.

In chapter 3, the worldline formalism is discussed in detail starting with definitions and various derivations. A Worldline is a curve in space time that traces out the time history of a particle. The worldline formalism, in which the $S-$matrix is constructed as a relativistic path integral over particle trajectories [8] is an alternative [9] to the usual second quantized formalism in Quantum Field Theory.

This formalism which goes under various names, e.g. "Feynman-Schwinger representation", "quantum mechanical path integral formalism", "first-quantized formalism" or "worldline formalism" (which we will adopt here) has been studied by many authors (see [9] for an extensive bibliography), but for several decades was considered as mainly of conceptual interest. However, partly as a consequence of developments

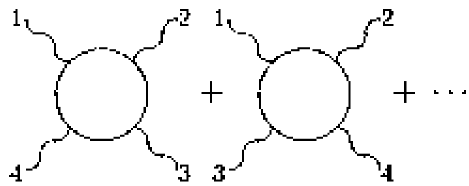

Fig. 1.1: Diagrams contributing to QED photon-photon scattering.

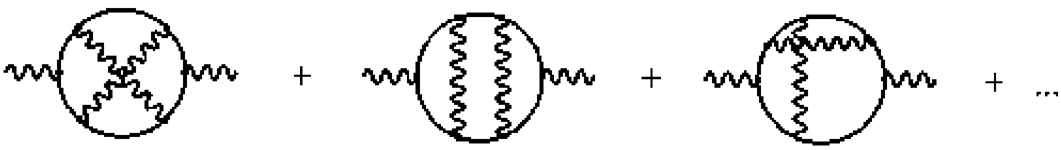

Fig. 1.2: Diagrams contributing to the three loop QED photon propagator.

in string theory [10], it has in recent years emerged also as a powerful practical tool for the computation of a wide variety of quantities in quantum field theory. This includes one-loop on-shell [11, 12, 13, 14] and off-shell [15, 16] gluon amplitudes, one- and two-loop Euler-Heisenberg-Weisskopf Lagrangians [14, 17], heat-kernel coefficients [18, 19], Schwinger pair creation in constant and non-constant fields [20, 21, 22], Casimir energies [23] and various types of anomalies (see [11], QED/QCD bound states [7, 24, 25], heavy-quark condensates [26], and QED/QCD instantaneous Hamiltonians [27], Extensions to curved space [28] and quantum gravity [29] have also been considered.

Here we concentrate on one specific advantage of the worldline formalism. One of the interesting aspects of this approach is that often it combines into a single expression contributions from a large number of Feynman diagrams. For example, in the QED case it generally allows one to combine into one integral all contributions from Feynman diagrams which can be identified by letting photon legs slide along scalar/fermion loops or lines. Thus e.g. the well-known sum of six permuted diagrams for one-loop QED photon-photon scattering (see fig. 1.1) here naturally appears combined into a single integral [9].

While in this case the summation involves graphs that differ only by permutations of the external legs, at higher loop orders the summation will generally involve topologically different diagrams; as an example, we show in fig. 1.2 the “quenched” contributions to the three-loop photon propagator.

This property is particularly interesting in view of the fact that it is just this type of summation which in QED often leads to extensive cancellations, and to final results which are substantially simpler than intermediate ones (see, e.g., [30, 31]). More recently, similar cancellations have been found also for graviton amplitudes (see, e.g., [32]).

Our two main goals here are, firstly to obtain the expression for the ground integral formulas combining sum of many Feynman diagrams and secondly, semi-analytically calculating the lowest bound state mass in the Wick-Cutosky model, using the specific example of the ladder graphs.

In chapter 4 we discuss the work of Tjon and Nieuwenhuis, mentioned above, where the authors already tried to explore this property for the case of the scalar ladder where exchange mass was 0.15 times the mass of the constituents. Here a brute-force Monte Carlo calculation of the path integral was employed.

All our investigations in this thesis will be based on the well known $\varphi^2\chi-$theory with two scalars interacting through a cubic vertex is

$$S[\phi,\chi] = \int d^D x \left(\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{2}(\partial_\mu \chi)^2 + \frac{1}{2}\mu^2 \chi^2 + \frac{\lambda}{2!}\phi^2 \chi \right) .$$

Here we consider three classes of Green's functions: the first one depicted in fig. 1.3, is the $x-$ space propagator for one scalar interacting with the second one through the exchange of N given momenta.

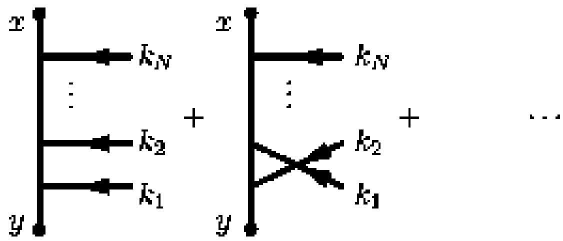

Fig. 1.3: Sum of diagrams contributing to the N - propagator

This "$N-$ propagator" is given by a set of $N!$ simple tree-level graphs in section 3.2 of chapter 3. We use the worldline formalism to combine them into a single integral. We also obtain a momentum space version of this result. More precisely, in the background field formalism ($\phi \to \phi_q + \phi$) we have an expression for the full propagator in the worldline formalism represented as a two point function containing the interaction. The integral for the full propagator is computed by summing over all possible paths including the quantum fluctuations between the two points.

In chapter 5, in going from the two point to the four point function, we take the product of two copies of the N-propagator, identifying k_i of one N- propagator with the -k_i of second to obtain N-ladder graphs: Here we are able to combine, for a fixed number of rungs, the uncrossed ladder graph with all $N! - 1$ crossed ones see fig. 1.5.

In chapter 6, we start on the application of this representation of the four-point func-

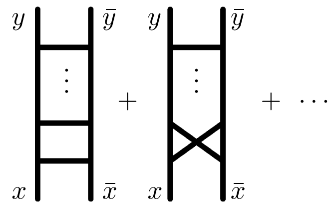

Fig. 1.4: ladder graph

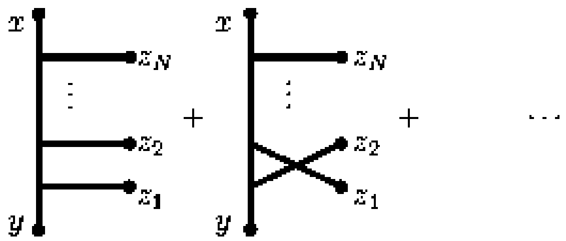

Fig. 1.5: Sum of diagrams contributing to the half-ladder

tion in the extraction of the lowest bound state mass. We use the fact that only the large time limit is needed for this task to achieve a number of simplifications.

In chapter 7, we use a gaussian approximation around the saddle point in the large time limit. This expression is then summed to obtain the full Green's function given in terms of coefficient c_n whose asymptotic behaviour we study.

In chapter 8, we extract the lowest bound state mass contained in the expression for the binding energy

$$e^{(2m-E_b)\frac{2\hat{t}}{m}} = e^{(-4+\frac{1}{2}\pi^2 g^2)\hat{t}} \tag{1.1}$$

obtained in the nonrelativistic limit of the Schrodinger equation. Using this in the full representation of the Green's function we get an exponential expression in terms of the coupling constant. The coefficient $\bar{c}_N$ appearing in the full Green's function has a specific asymptotic behaviour. From the resummed expression we get an analytic expression for the lowest bound state mass by a second saddle point approximation (8.18) or expanded in the coupling.

Hence we have our main result:

$$\frac{m_0}{m} = 2 - \frac{\pi^2 g^2}{4} - \frac{9}{64}(\pi^2 g^2)^2 - \frac{81}{512}(\pi^2 g^2)^3 - \dots \tag{1.2}$$

where m_0 is the lowest bound state mass, m is the mass of the free φ particle and $g = (\frac{\lambda}{4\pi m})^2$.

Finally, we compare the result of the lowest bound state mass with the relativistic eikonal approximation or Todorov's equation [33, 34] and we also make a comparison of the maximal value of the coupling constant, to the critical value of the variational worldline approximation [35]. The latter value without self- energy and vertex corrections, for a massless exchanged particle, comes out be somewhat larger than our value. The existence of a critical coupling constant is attributed to the instability of the vacuum in a scalar field theory [35].

Chapter 9, deals with the $N-$half ladders. This third class are as shown in fig 1.5, defined by a line connecting the points x and y and N further points $z_1, \dots, z_N$ connecting to this line in an arbitrary order.

Chapter 10, presents the conclusions and suggestions for further work. Thereafter there are three appendices, appendix A which includes conventions, appendix B which includes more details on the variables c_N, $\mathcal{M}_N$ and $\mathcal{R}_N$ and appendix C which includes a comparison with Feynman diagrams.

The results decribed in this thesis will be published in ref. [36].

CHAPTER 2

BOUND STATES IN QUANTUM FIELD THEORY

The bound-state problem in Quantum Field Theory has been a field of active research for many years. Making exact predictions using the correct Lagrangian for a system is not an easy task. Therefore, one has to make approximations, one common approximation is known as perturbation theory. Perturbation theory involves making an expansion in the coupling strength of the interaction and works for small couplings. Perturbation theory cannot explain bound states, irrespective of how small the coupling strength is. Therefore, bound states are always fully non-perturbative or we can say non-perturbative method is required where standard perturbative methods are not applicable. Bound states problems are an important example of it. Bound states are identified by poles in Green's function. The n-body bound state is defined by the pole of the n-body propagator. Bound state singularity comes from an infinite summation of perturbation series. A perturbative approximation of n-body propagator does not produce the bound state pole location. Hence it is essential that reliable non-perturbative methods that takes all orders of interaction into account are developed. Building blocks of matter, quarks and gluon only exist in bound states. Any reaction involving quarks definitely involve bound states in the initial and final states and necessitates a non-perturbative treatment.

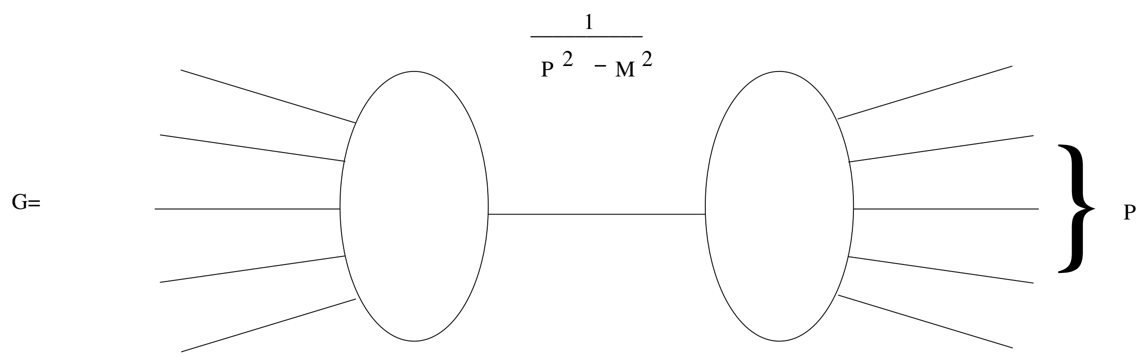

Fig. 2.1: Bound states mass is determined by the pole of Green's function.

2.1 Bethe-Salpeter equation

The Bethe-Salpeter (B-S) equation [1, 2] is the most orthodox tool for discussing the relativistic two-body problem in quantum field theory. It was proposed more than sixty years ago. The Bethe-Salpeter equation describes the bound state of two particles. Though the equation appears in many forms however the most commonly used form is

$$\Gamma(P,p) = \int \frac{d^4k}{(2\pi)^4} K(P,p,k) S(K - \frac{P}{2}) \Gamma(P,k) S(k + \frac{P}{2}) \tag{2.1}$$

where Γ is the Bethe-Salpeter amplitude, K is the interaction and S the propagators of the two propagating constituents.

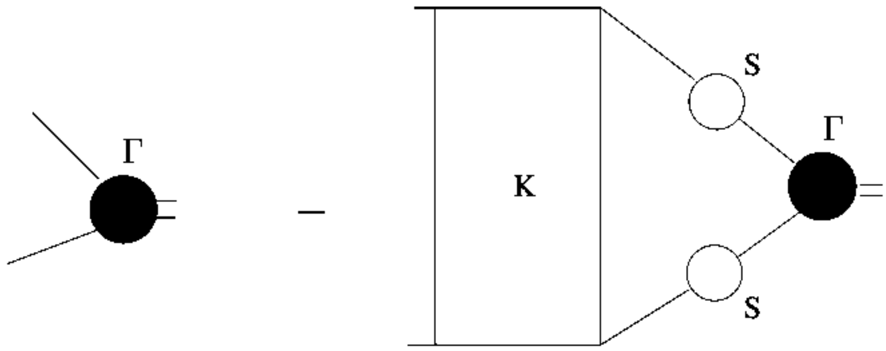

Fig. 2.2: A graphical representation of the Bethe-Salpeter equation.

2.2 Derivation of the B-S equation

Consider the scattering of two non-identical scalar particles a and b. Let $\varphi_a(x)$ and $\varphi_b(x)$ be the field operators of a and b, respectively in the Heisenberg representation. The scattering Green's function $G(x_a, x_b; y_a, y_b)$ is defined by

$$G(x_a, x_b; y_a, y_b) \equiv \langle 0|T[\varphi_a(x_a)\varphi_b(x_b)\varphi_a^\dagger(y_a)\varphi_b^\dagger(y_b)]|0\rangle \tag{2.2}$$

On expanding G into a perturbation series, expressed in terms of connected Feynman graphs corresponding to the process $a + b \to a + b$.

$$G(x_a, x_b; y_a, y_b) = \mathbf{\Delta'_{Fa}(x_a - y_a)\Delta'_{Fb}(x_b - y_b)}$$

$$+ \int d^4z_a \int d^4z_b \int d^4z_a' \int d^4z_b' \mathbf{\Delta'_{Fa}(x_a - z_a)\Delta'_{Fb}(x_b - z_b)}$$

$$\mathbf{I(z_a, z_b; z_a', z_b')G(z_a, z_b; y_a, y_b)}, \tag{2.3}$$

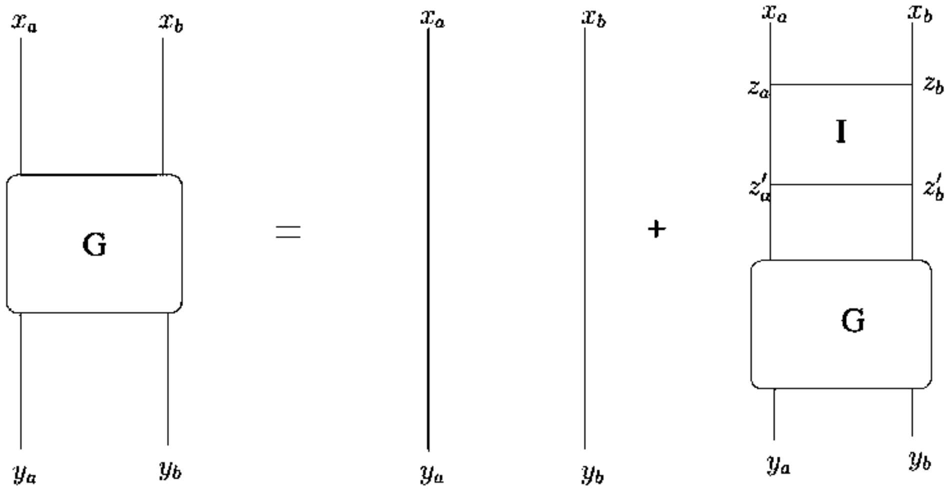

Fig. 2.3: Graphical representation of the position-space B-S equation (2.3)

where **G** and **I** are functions of $x_a - x_b, -y_a + y_b$ and $\eta_a(x_a - y_a) + \eta_b(x_b - y_b)$, where η_a and η_b are arbitrary real quantities such that $\eta_a + \eta_b = 1$. If p,q and P are the conjugate momenta. Then

$$[\mathbf{\Delta'_{Fa}}(\eta_\mathbf{a}\mathbf{P} + \mathbf{p})\mathbf{\Delta'_{Fb}}(\eta_\mathbf{b}\mathbf{P} - \mathbf{p})]^{-\mathbf{1}}\mathbf{G}(\mathbf{p}, \mathbf{q}, ; \mathbf{P}) \tag{2.4}$$

$$= \delta^4(p - q) + \int d^4p' I(p, p'; P)G(p', q; P) \tag{2.5}$$

where Δ'_{Fa}, G and I are the fourier transforms of $\mathbf{\Delta'_{Fa}}$, **G** and **I**.

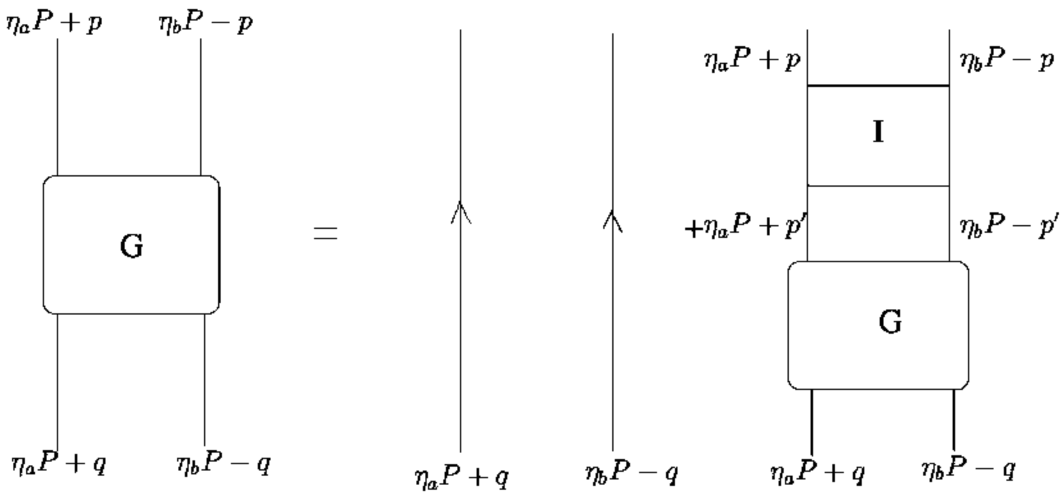

Fig. 2.4: Graphical representation of the momentum-space B-S equation eq. (2.5)

To simplify

$$\mathbf{K}(\mathbf{p},\mathbf{q},;\mathbf{P}) \equiv [\mathbf{\Delta}'_{\mathbf{Fa}}(\eta_{\mathbf{a}}\mathbf{P}+\mathbf{p})\mathbf{\Delta}'_{\mathbf{Fb}}(\eta_{\mathbf{b}}\mathbf{P}-\mathbf{p})]^{-1}\delta^4(\mathbf{p}-\mathbf{q}) \tag{2.6}$$

using operator notation:

$$\int d^4p' A(p,p')B(p',q) \equiv (AB)(p,q). \tag{2.7}$$

Then eq. (2.5) can be written as

$$KG = 1 + IG \tag{2.8}$$

$$G = (K-I)^{-1}; \tag{2.9}$$

hence

$$GK = 1 + GI \tag{2.10}$$

The time-reversed equation of (2.8) can be written as

$$\bar{G}\bar{K} = 1 + \bar{G}\bar{I} \tag{2.11}$$

If theory is invariant under time reversal, then $\bar{G} = G$, $\bar{K} = K$ and $\bar{I} = I$ eq. (2.10) and eq. (2.11) are identical.
If the invariant squares of momenta are denoted as below and if, $P^2 = \mathbf{P}^2 - P_0^2$:

$$-P^2 = s, -(p-q)^2 = t,$$

$$-[(\eta_a - \eta_b)P + p + q)]^2 = u$$

$$-(\eta_a P + p)^2 = v, -(\eta_b - p)^2 = w,$$

$$-(\eta_a P + q)^2 = v_0, -(\eta_b - q)^2 = w_0. \tag{2.12}$$

If m_1,m_2 be the masses of a and b respectively. Then the mass shells are defined by

$$v = m_1^2,\ w = m_2^2 \tag{2.13}$$

$$v_0 = m_1^2,\ w_0 = m_2^2 \tag{2.14}$$

The Feynman amplitude $F(p,P)$ equals the residue of $-(G - K^{-1})$ at (2.14) and the

scattering amplitude is defined as residue of $G - K^{-1}$ at both (2.11) and (2.14).
For practical considerations we are concerned with the ladder approximation. Moreover, $\mathbf{\Delta_F'}$ is replaced by a free Feynman propagator.

$$\Delta_F(k,m) \equiv -i(m^2 + k^2 - i\varepsilon)^{-1}, \tag{2.15}$$

where ε is always an infinitesimal positive quantity. The integral kernel $\mathbf{I}$ contains a single- particle-exchange contribution, so $\mathbf{I}$ is independent of $\mathbf{P}$ and is proportional to a coupling parameter $\Lambda \equiv \frac{g_a g_b}{(4\pi)^2}$, where $g_j(j = a, b)$ denotes the coupling constant between the particle j and the exchanged particle (for $g_a = g_b$, we denote it by g)

We now discuss the homogeneous B-S equation for bound states. Let $\mid B,1\rangle, \mid B,2\rangle, \cdots \mid B,n\rangle$ be degenerate bound states having the 4-momentum P_B with $\mathbf{P_B} = \mathbf{P}$ and $P_B^2 = s_B$. The B-S amplitude for $\mid B,r\rangle$ and its conjugate is defined to be

$$\phi_{\mathbf{Br}}(\mathbf{x_a}, \mathbf{x_b}; \mathbf{P_B}) \equiv \langle \mathbf{0} \mid \mathbf{T}[\varphi_{\mathbf{a}}(\mathbf{x_a})\varphi_{\mathbf{b}}(\mathbf{x_b})] \mid \mathbf{B}, \mathbf{r}\rangle,$$

$$\bar{\phi}_{\mathbf{Br}}(\mathbf{x_a}, \mathbf{x_b}; \mathbf{P_B}) \equiv \langle \mathbf{B}, \mathbf{r} \mid \mathbf{T}[\varphi^\dagger_{\mathbf{a}}(\mathbf{x_a})\varphi^\dagger_{\mathbf{b}}(\mathbf{x_b})] \mid \mathbf{0}\rangle$$

$$= \langle B, r \mid \bar{T}[\varphi_{\mathbf{a}}(\mathbf{x_a})\varphi_{\mathbf{b}}(\mathbf{x_b})] \mid \mathbf{0}\rangle^*, \tag{2.16}$$

respectively, where $\bar{T}$ and $*$ denote anti-chronological operator and complex conjugation, respectively. Because of translational invariance, we can write

$$\phi_{\mathbf{Br}}(\mathbf{x_a}, \mathbf{x_b}; \mathbf{P_B}) = (\mathbf{2}\pi)^{-\frac{3}{2}} \mathbf{e}^{-\mathbf{i P_B X}} \phi_{\mathbf{Br}}(\mathbf{x}, \mathbf{P_B})$$

$$\bar{\phi}_{\mathbf{Br}}(\mathbf{x_a}, \mathbf{x_b}; \mathbf{P_B}) = (\mathbf{2}\pi)^{-\frac{3}{2}} \mathbf{e}^{-\mathbf{i P_B X}} \bar{\phi}_{\mathbf{Br}}(\mathbf{x}, \mathbf{P_B}) \tag{2.17}$$

where

$$X = \eta_a x_a + \eta_b x_b,$$

$$X = x_a - x_b \tag{2.18}$$

The reduced amplitude $\phi_{\mathbf{Br}}(\mathbf{x}, \mathbf{P_B})$ is called the B-S amplitude.
If we insert the complete set of states[1] into the middle of the right-hand side of (2.2), then the contribution to $\mathbf{G}(\mathbf{x_a}, \mathbf{x_b}; \mathbf{y_a}, \mathbf{y_b})$ from the intermediate states $\langle B, r \mid (r01, 2, \cdots n)$ may be written as

$$\sum_{r=1}^{n} \int d^4 P \phi_{\mathbf{Br}}(\mathbf{x_a}, \mathbf{x_b}; \mathbf{P}) \bar{\phi}_{\mathbf{Br}}(\mathbf{y_a}, \mathbf{y_b}; \mathbf{P}) \theta(\mathbf{P_0}) \delta(\mathbf{P^2} - \mathbf{s_B}) \theta(\mathbf{X_0} - \mathbf{Y_0})$$

[1]We here assume that all states have positive norm.

$$= (2\pi)^{-3}\sum_r \int \frac{d^4\mathbf{P}}{2\omega_B}\phi_{\mathbf{Br}}(\mathbf{x};\mathbf{P_B})\bar{\phi}_{\mathbf{Br}}(\mathbf{y};\mathbf{P_B})$$

$$\times exp[-i\omega_B(X_0-Y_0)+i\mathbf{P}(\mathbf{X}-\mathbf{Y})]\theta(\mathbf{X_0}-\mathbf{Y_0}), \tag{2.19}$$

where

$$\omega_B = (P_B)_0 = (\mathbf{P^2}+\mathbf{s_B})^{\frac{1}{2}} \tag{2.20}$$

Y and y are defined analogous to (2.18). using an identity

$$\theta(z) = -(2\pi i)^{-1}\int dk\, e^{-ikz}(k+i\varepsilon)^{-1} \tag{2.21}$$

after a transformation $k = P_0 - \omega_B$ eq.(2.19) can be rewritten as

$$-i(2\pi)^{-4}\sum_r \int d^4P\,\phi_{\mathbf{Br}}(\mathbf{x},\mathbf{P_B})\,\bar{\phi}_{\mathbf{Br}}(\mathbf{y},\mathbf{P_B})\frac{\mathbf{exp}[-\mathbf{iP}(\mathbf{X}-\mathbf{Y})]}{\mathbf{2\omega_B}(\mathbf{P_0}-\omega_{\mathbf{B}}+\mathbf{i}\varepsilon)} \tag{2.22}$$

Fourier transform of (3.22)

$$\frac{i\sum_r \phi_{Br}(p,P)\bar{\phi}_{Br}(p,P)}{2\omega_B(P_0-\omega_B+i\varepsilon)} \tag{2.23}$$

aprat from a term regular at $P_0 = \omega_B$. By adding the contribution from anti-particle states of $\mid B, r\rangle$, we find that $G(p,q;P)$ has a pole term at

$$\frac{i\sum_r^n \phi_{Br}(p,P)\bar{\phi}_{Br}(p,P)}{(s-s_B+i\varepsilon)} \tag{2.24}$$

Substituting (5.14) for the pole term of G in (2.5), and comparing the residues at $s = s_B$ of both sides, due to the linear independence of $\phi_{B1}, \phi_{B2}, \cdots \phi_{Bn}$, we find

$$[\mathbf{\Delta}'_{\mathbf{Fa}}(\eta_{\mathbf{a}}\mathbf{P_B}+\mathbf{p})\mathbf{\Delta}'_{\mathbf{Fb}}(\eta_{\mathbf{b}}\mathbf{P_B}-\mathbf{p})]^{-\mathbf{1}}\mathbf{G}(\mathbf{p},\mathbf{P_B})$$

$$= \int d^4p'\,I(p,p';P_B)\phi_{Br}(p',P_B) \tag{2.25}$$

$$K_B\phi_{Br} = I_B\phi_{Br} \tag{2.26}$$

where the subscript B means to put $s = s_B$. Eq. (2.25) or (2.26) is usually called the B-S equation.

2.3 Ladder Approximation

The interaction kernel contains all possible two-particle interactions that occur between the constituents. In the ladder approximation we use only a single interaction in the kernel. This over simplification of the ladder approximation caused a lot of problems and thus crossed ladders have to be included

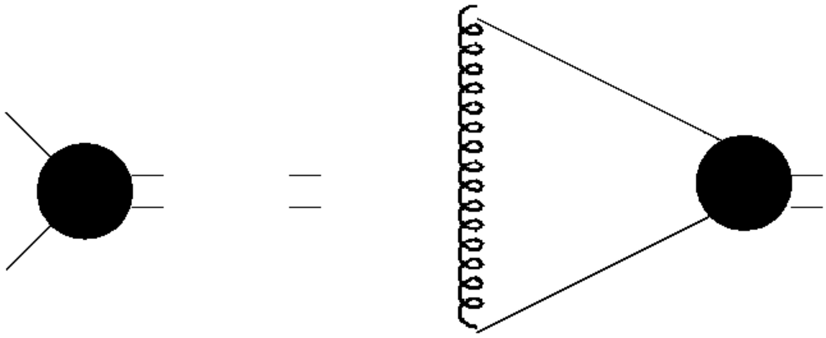

Fig. 2.5: A graphical representation of the Bethe-Salpeter equation in ladder approximation.

2.4 Non-relativistic Bound State

In non- relativistic Quantum Mechanics bound states have a fairly simple structure. One dimensional Schrodinger equations for bound states in various potentials have simple and exact solutions. Scattering problems can be solved exactly in many cases, especially in one-dimensional situations. Two-body bound states could be solved by reducing them to one body problems by introducing relative coordinates. For systems with more particles few exact solutions are not known but principles are.

For a bound state made up of two scalar partices of equal masses m that interact via the exchange of another, the massless scalar nonrelativistic Schrodinger equation is given by

$$(-\frac{1}{2m_r}\bigtriangledown^2 - \frac{\alpha}{r})\Psi(r) = (E-2m)\Psi(r) \tag{2.27}$$

In the center of mass system (c.m.s). where m_r is the reduced mass and $m_r = m/2$. $\Psi(r)$ is the non relativistic wave function depending on the relative coordinate. E is the energy of the rest mass (in the c.m.s.) of the bound state. $E_b = 2m - E$ is the binding energy.

For our scalar model, in the nonrelativistic limit, the binding energy in the ground state would be [37],

$$E_b = \frac{1}{2} m_r \alpha^2 = \frac{1}{4} m\alpha^2 \tag{2.28}$$

where

$$\alpha = \frac{\lambda^2}{16\pi m^2} = \pi g \tag{2.29}$$

λ is the coupling constant.

2.5 Relativistic Bound State

In an exact relativistic treatment, one has to deal with an infinite number of particles due to pair production and radiative corrections. Thus, there are infinitely many degrees of freedom. Relativistic equations for bound states have to be formulated in the framework of relativistic field theory.

Describing relativistic dynamics is an important issue in the study of composite hadronic systems at higher energies. Our knowledge of the relativistic two-body bound state problem in field theory is mostly based on application of the ladder approximation to the Bethe-Salpeter equation (BSE). The general applicability of ladder theory can be questioned on physical grounds. The so called one-body limit (where, in the case of different constituent masses, the one-body limit means that one of the masses becomes infinite and the other remains fixed.) does not lead to the Klein-Gordon equation as it should. Moreover gauge invariance cannot be satisfied within this approximaton. To recover these properties, all cross ladder contributions are needed additionally. So-far, however, the study of the two-body Green function beyond the ladder theory has not been considered feasible in practice. Therefore, several quasipotential equations have been proposed and studied as possible candidate for an effective theory. Both the ladder BSE as well as several quasipotential equations (QPEs) have been used in numerous studies throughout a wide range of systems, including mesons, small nuclei, few electron atoms and positronium.

In constructing the QPEs, one usually chooses the approximations leading to them such that the above mentioned problems are, at least partially, solved. However, due to our ignorance of the behaviour of full BSE solutions, it is presently unclear which of the possibly infinite number of QPEs provides the best effective desciption. In this connection, it is actually of interest to have actual solutions available for cases where a larger class of graphs than the ladder series is included in the BSE and that do not suffer from the difficulties inherent in the latter approximations. Such solutions may serve as a testing ground for various QPE descriptions.

There are at least three areas of physics in which the relativistic bound state problem needs to be addressed: the first one is bound-state Quantum Electrodynamics (QED) where ultra-precise experimental data are available.

Another area for relativistic bound-state calculations where strong coupling is required: the electromagnetic structure of nuclear few-body systems at intermediate and high energies. Example, relativistic effects in the electromagnetic form factor of the deutron is important at low momentum transfers.

Finally, there is the area of hadronic physics where (light) relativistic quarks and gluons bind to form the low energy mesons and baryons.

To describe bound states let us consider the scattering process below

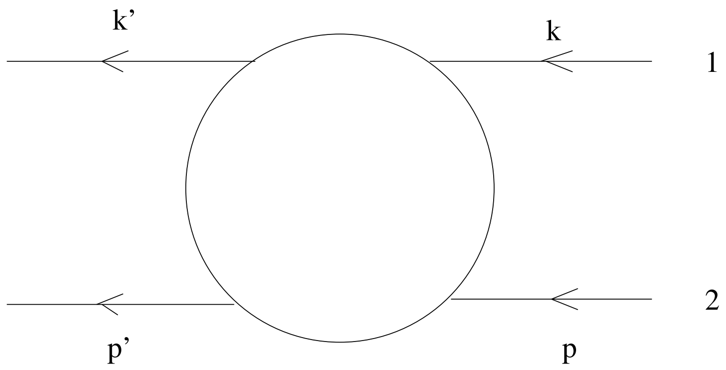

Fig. 2.6: Scattering of particle of type 1 and 2

The $S-$matrix for the process is given by

$$S = \langle k'p'|U_I|kp\rangle \tag{2.30}$$

where U_I is the time translation operator to second order expressed in terms of hamiltonian densities.

$$U_I = 1 - i\int d^4x\mathcal{H}_{int}(x) + \frac{(-i)^2}{2}\int d^4x_1 d^4x_2 T(\mathcal{H}_{int}(x_1)\mathcal{H}_{int}(x_2)) + \ldots \tag{2.31}$$

$$S = -i\frac{(2\pi)^4\delta(k'+p'-k-p)}{(2\pi)^6\sqrt{16E_1(k)E_1(k')E_2(p)E_2(p')}}\mathcal{M} \tag{2.32}$$

and $\mathcal{M}$ is the amputated four-point function.

2.6 Ladder diagrams

For the box diagram in the $\phi^2\chi$ theory (here, we have two ϕ fields with different masses), We are looking at contribution to $\mathcal{M}$.

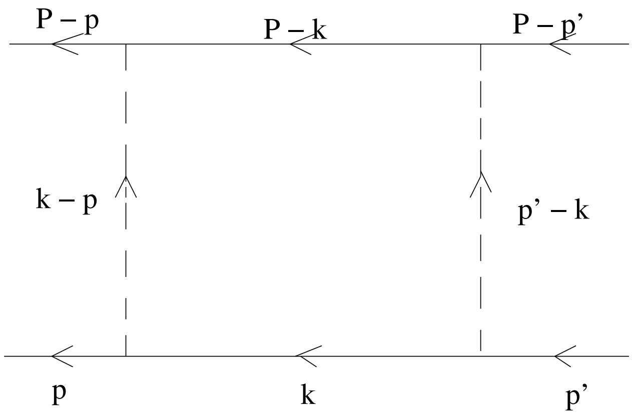

Fig. 2.7: Box diagram

$$\mathcal{M} = i\lambda_1^2\lambda_2^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{D_1 D_2 D_0 D_0^{'}} \tag{2.33}$$

where $\lambda_1\lambda_2 \to \lambda$

$$\mathcal{M} = -i\lambda_1^2\lambda_2^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(m_1^2 + (P-k)^2 - i\epsilon)(m_2^2 + k^2 - i\epsilon)(\mu^2 + (k-p)^2 - i\epsilon)(\mu^2 + (k-p')^2 - i\epsilon)} \tag{2.34}$$

$m_1 < m_2$ are the masses of two heavy particles being scattered,$\mu \ll m_1$ mass of the light meson being exchanged.

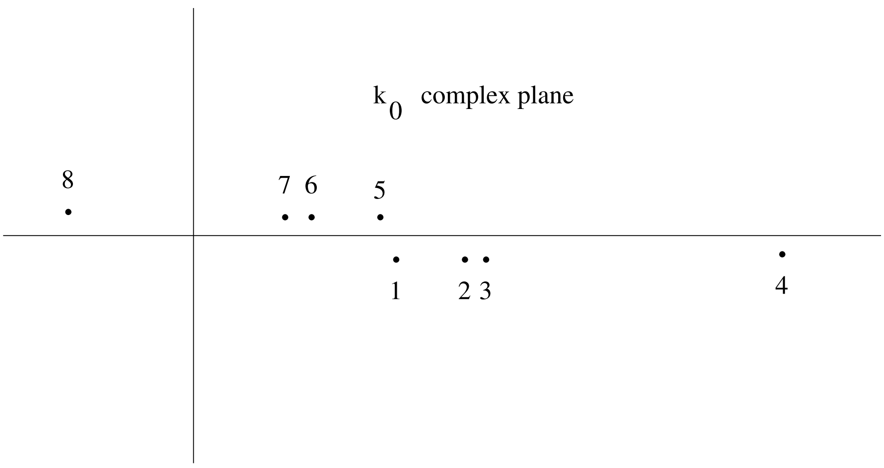

Fig. 2.8: Singularities of the box diagram in the complex k_0 plane when $|\ \mathbf{k}\ |$ is small. As $|\ \mathbf{k}\ |$ increases singularities in the lower half plane move to the right and those in the upper half plane move to the left.

There are eight poles as shown in the figure (2.8) and obtained from the zeros of the denominators in eq. (2.34) as shown below.

$$D_1 = m_1^2 + (P-k)^2 - i\epsilon = \underbrace{(E_1 - W + k_0 - i\epsilon)}_{5}\underbrace{(E_1 + W - k_0 - i\epsilon)}_{4} \tag{2.35}$$

where $W = E_1(p) + E_2(p) = E_1(p') + E_2(p')$ in the center of mass frame.

$$D_2 = m_2^2 + k^2 - i\epsilon = \underbrace{(E_2 + k_0 - i\epsilon)}_{8}\underbrace{(E_2 - k_0 - i\epsilon)}_{1} \tag{2.36}$$

$$D_0 = \mu^2 + (k-p)^2 - i\epsilon = \underbrace{(\omega - E_2(p) + k_0 - i\epsilon)}_{6}\underbrace{(\omega + E_2(p) - k_0 - i\epsilon)}_{2} \tag{2.37}$$

$$D_0^{'} = \mu^2 + (k-p')^2 - i\epsilon = \underbrace{(\omega' - E_2(p') + k_0 - i\epsilon)}_{7}\underbrace{(\omega + E_2(p') - k_0 - i\epsilon)}_{3} \tag{2.38}$$

and

$$E_i = \sqrt{m_i^2 + \mathbf{k}^2} \qquad \omega = \sqrt{\mu^2 + (\mathbf{k} - \mathbf{p})^2} \tag{2.39}$$

$$E_i(p) = \sqrt{m_i^2 + \mathbf{p}^2} \qquad \omega' = \sqrt{\mu^2 + (\mathbf{k} - \mathbf{p'})^2} \tag{2.40}$$

If we evaluate the box diagram in the lower half of the complex p_0 plane, we see that the pole at E_2 dominates as it is close to the singularity $k_0 = W - E_1$ in the upper half plane. Now, retaining this term only, the box diagram reduces to

$$\mathcal{M} \simeq -\lambda_1^2\lambda_2^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_2(E_1^2 - (W-E_2)^2 - i\epsilon)(\omega^2 - (E_2 - E_2(p))^2)(\omega'^2 - (E_2 - E_2(p'))^2)} \tag{2.41}$$

Using the approximation $E_2 \sim m_2$ and $E_1 + W - E_2 \sim 2m_1$ in terms where weak k-dependence is not critical we have

$$\mathcal{M} \simeq -\frac{\lambda_1^2\lambda_2^2}{4m_1m_2} \int \frac{d^3k}{(2\pi)^3} \frac{2m_r}{(E_1 - E_2 - W - i\epsilon)\omega^2\omega'^2} \tag{2.42}$$

and m_r is the reduced mass.

Considering the case when m_1 and m_2 are both large and $\mathbf{p} = \mathbf{p}'$ with meson energies $\omega = \omega'$ and $k =\mid \mathbf{k} \mid\simeq \mu$ we have

$$\mathcal{M} \simeq -\frac{\lambda_1^2\lambda_2^2}{4m_1m_2}\int \frac{d^3k}{(2\pi)^3}\frac{2m_r}{(k^2-p^2-i\epsilon)(\mu^2+(\mathbf{k}-\mathbf{p})^2)^2} \tag{2.43}$$

$$= -\frac{\lambda_1^2\lambda_2^2}{4m_1m_2}\int_0^\infty \frac{k^2dk}{2\pi^2}\frac{2m_r}{(k^2-p^2-i\epsilon)\big[(k^2+p^2+\mu^2)^2+4k^2p^2\big]} \tag{2.44}$$

$$= -\frac{\lambda_1^2\lambda_2^2}{16\pi}\frac{1}{(m_1+m_2)\mu^2}\frac{1}{\big(\mu-2ip\big)} \tag{2.45}$$

Comparing eq. (2.45) with one boson exchange amplitude, we have

$$\frac{\lambda_1\lambda_2}{\mu^2} \simeq -\frac{\lambda_1^2\lambda_2^2}{16\pi}\frac{1}{(m_1+m_2)\mu^2}\frac{1}{\big(\mu-2ip\big)} \tag{2.46}$$

Now, substituting effective dimensionless coupling strength $g_{eff} = \frac{\lambda_1^2\lambda_2^2}{4m_1m_2}$ for the ϕ^3 Yukawa interaction in eq. (2.45) we have

$$\frac{g_{eff}}{4\pi}\frac{m_r}{\big(\mu-2ip\big)} \simeq 1 \tag{2.47}$$

When this condition is satisfied, the fourth order box digram is comparable with second order one boson exchange (OBE) term. Now let $p = i\delta$. With $\epsilon = m_1+m_2-W$ and $p = \sqrt{2m_r\epsilon}$, we have

$$\frac{g_{eff}}{4\pi}\frac{m_r}{\big(\mu+2\delta\big)} \simeq 1 \tag{2.48}$$

Relativistic bound state equation has solutions when equation condition (2.48) is satisfied.

Eq. (2.48) tells us that a potential with a finite range $(\mu \neq 0)$ will have a bound state $(\delta \geq 0)$ only when

$$\frac{g_{eff}}{4\pi}\frac{m_r}{\mu} \gtrsim 1 \tag{2.49}$$

A potential with an infinite range $(\mu = 0)$ will have always have a bound state, and the ground state energy estimated from $-\frac{\delta^2}{2m_r}$, will be of the order of

$$E_0 \simeq -\frac{\delta^2}{2m_r} \simeq -\frac{m_r}{8}\left(\frac{g_e f f^2}{4\pi}\right)^2 \tag{2.50}$$

2.7 Role of crossed ladders

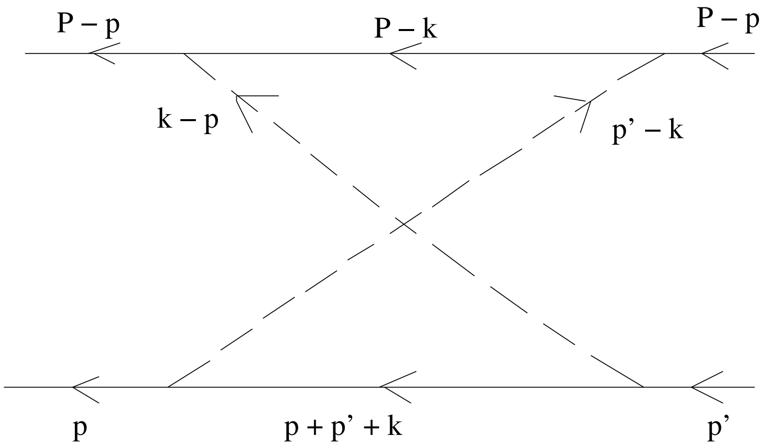

Fig. 2.9: The crossed box diagram.

The crossed box is the only fourth order diagram which describes a long range (two boson exchange) interaction. For the crossed box shown in the figure above,

$$\mathcal{M} = -i\lambda_1^2\lambda_2^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{D_1 D_2^X D_0 D_0'} \tag{2.51}$$

where D_1, D_0 and D_0' are identical to (2.33), but

$$D_2^x = m_2^2 - (p + p' - k)^2 - i\epsilon = \underbrace{(E_1 - W + k_0 - i\epsilon)}_{5}\underbrace{(E_1 + W - k_0 - i\epsilon)}_{4} \tag{2.52}$$

$$= \underbrace{(E_2^x + 2E_2(p) - k_0 - i\epsilon)}_{8_x}\underbrace{(E_2^x - 2E_2(p) + k_0 - i\epsilon)}_{1_x} \tag{2.53}$$

where

$$E_2^x = \sqrt{m_2^2 + (\mathbf{p} + \mathbf{p}' - \mathbf{k})^2} \tag{2.54}$$

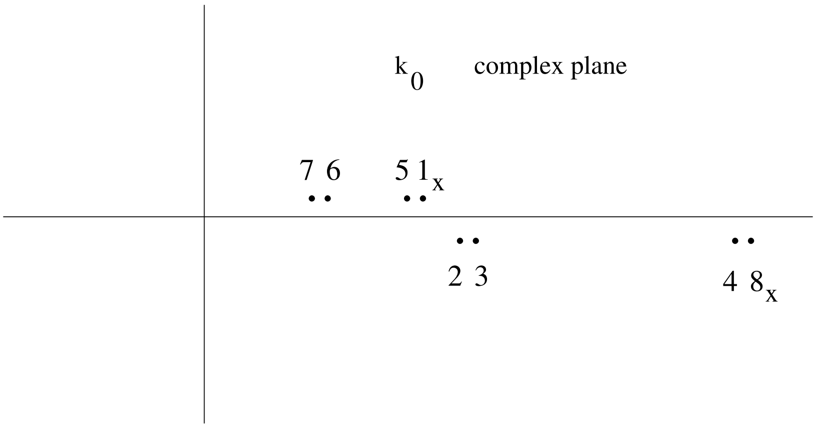

Fig. 2.10: Singularities of the crossed box diagram in the complex k_0 plane when $\mid \mathbf{k} \mid$ is small. Compare with figure (2.8) and note that the only difference is that pole 1 is replaced by 1_x and pole 8 is replaced by 8_x.

There are still eight poles in the complex k_0 plane, but two of the poles 1_x and 8_x are in different locations as shown in the figure (2.11) above. The major difference between the box and crossed box is that pole 1, which dominated the box, has moved from the lower half plane to the upper half plane. These two poles are located at

$$pole1: \; k_0 = E_2 - i\varepsilon \cong m_2 + \frac{k^2}{2m_2} - i\varepsilon \tag{2.55}$$

$$pole1_x: \; k_0 = 2E_2(p) - E_2^x \cong m_2 + \frac{p^2}{m_2} - \frac{(\mathbf{p}+\mathbf{p}^{'}-\mathbf{k})^2}{2m_2} + i\varepsilon \tag{2.56}$$

Evaluating the crossed box by closing the contour in the lower half plane leads to the following observations: The contribution which dominated the box (pole 1) is no longer present in the lower half plane, and hence the leading contribution is missing from the crossed box. The meson poles dominated the crossed box, and the only difference between their contribution to the crossed box and the box is the denominator D_2.

Introducing $k_0 = k_0^{'} + E_2(p)$, two denominators become

$$box: \quad \frac{1}{D_2} \cong \frac{1}{2m_2(\frac{k^2-p^2}{2m_2} - k_0^{'})}$$

$$crossedbox: \quad \frac{1}{D_2^x} \cong \frac{1}{2m_2(\frac{(\mathbf{p}+\mathbf{p}^{'}-\mathbf{k})^2}{2m_2} - \frac{p^2}{2m_2} + k_0^{'})} \tag{2.57}$$

If m_2 is very large, the terms in (2.57) is proportional to m_2^{-1} and may be neglected compared to $k_0^{'}$ (which is equal to ω or $\omega^{'}$ at the meson poles), and we have

$$\frac{1}{D_2} \cong -\frac{1}{D_2^x}$$

Hence, in this approximation the dominant contributions from the crossed box are equal to the meson pole contributions from the box but have the opposite sign, so that their sum (box plus crossed box), cancels. The role of crossed box is to cancel the meson pole contribution from the box.

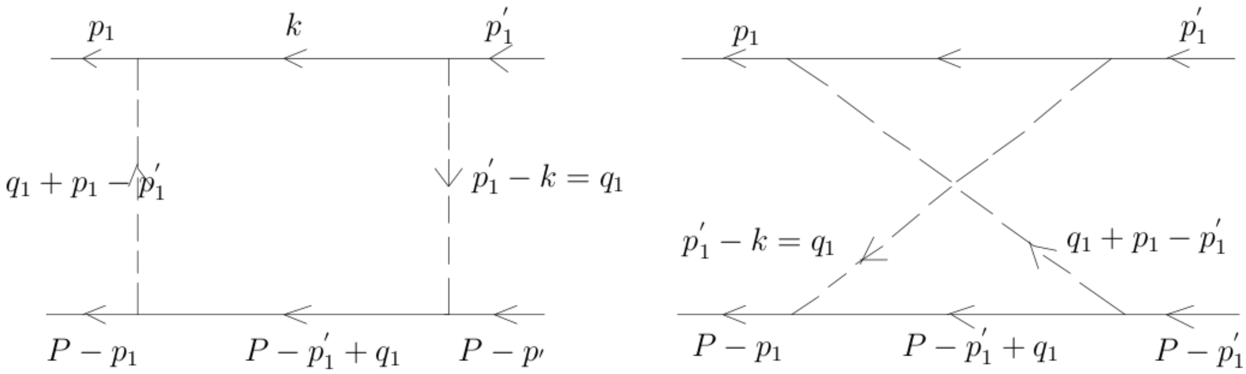

Fig. 2.11: The box and crossed box diagrams with new labeling of momenta to prove the cancelation theorem.

Cancellation theorem: In a theory in which spin zero particle of mass m_1 interacts with a heavy particle of mass m_2 (with no charge states) exchanging a spin zero meson of mass μ, the meson pole contributions from the ladder diagram are cancelled by meson pole contributions from crossed ladder diagrams, and this cancellation is exact in the limit $m_2 \to \infty$.

CHAPTER 3

THE WORLDLINE FORMALISM

A Worldline is a curve in space time that traces out the time history of a particle. The Worldline formalism, also referred to as string inspired formalism, is an alternative to the usual second quantized formalism in Quantum Field Theory, based on a relativistic path integral approach. It allows to calculate amplitudes and effective actions etc in QFT. It was invented by Feynman as an alternative to the formulation of second quantization.

In 1948, Feynman developed the path integral approach to nonrelativistic quantum mechanics (based on earlier work by Wentzel and Dirac). Two years later, he started his famous series of papers that laid the foundations of relativistic quantum field theory (essentially quantum electrodynamics at the time) and introduced Feynman diagrams. However, at the same time, he developed a representation of the QED S-matrix in terms of relativistic path integrals. It appears that he considered this approach less promising, since he relegated the information on it to appendices of [38] and [8]. And indeed, no essential use was made of those path integral representations for many years after; and even today path integrals are used in field theory mainly as integrals over fields, not over particles. Except for an early brilliant application by Affleck et al. [39] in 1984, the position of this particle path integral or "worldline" formalism to improve on standard field theory methods at least for certain types of computations, was recognised in the early nineties through the work of Bern Kosower [20] and Strassler [40] and [10].

3.1 *Free Particle Propagator*

In this section we follow the presentation in [41] closely. The Feynman propagator for a free scalar particle of mass m in D-dimensional space-time is a specific solution of the inhomogeneous Klein-Gordon equation

$$(\Box_x + m^2)\,\triangle_F\,(x-y) = \delta^D(x-y) \tag{3.1}$$

such that positive frequencies propagate forward in time, and negative frequencies backward. An explicit expression for $\triangle_F(x-y)$ in terms of a Fourier integral is

$$\triangle_F(x-y) = \int \frac{d^D p\, e^{ip(x-y)}}{(2\pi)^D(p^2+m^2-i\epsilon)} \tag{3.2}$$

In the limit $\epsilon \to 0^+$ the simple real pole at positive $p^0 = E(\overrightarrow{p}) = \sqrt{\overrightarrow{p}^2 + m^2}$ gives the mass of the particle as $m = E(0)$. The arbitrarily small imaginary part $i\epsilon$ implements the causality condition.

There is a straightforward connection between this propogator and the classical mechanics of a relativistic point−particle through the path integral formalism. To establish this, let us first consider the very general problem of finding the inverse of a non-singular hermitian operator $\hat{H}$. Following Schwinger, we construct the formal solution

$$\hat{H}^{-1} = \lim_{\epsilon\to 0^+} i\int_0^\infty dTe^{-iT(\hat{H}-i\epsilon)} \tag{3.3}$$

Here the exponential operator

$$\hat{K}_\epsilon(T) = e^{-iT(\hat{H}-i\epsilon)} \tag{3.4}$$

is the solution of the Schrodinger equation

$$\frac{\partial\hat{K}_\epsilon(T)}{\partial T} = \left(\hat{H}-i\epsilon\right)\hat{K}_\epsilon(T) \tag{3.5}$$

$$\hat{K}_\epsilon(0) = \hat{I},\ \lim_{\epsilon\to\infty}\hat{K}_\epsilon(T) = 0 \tag{3.6}$$

If the operator $\hat{H}$ acts on a single−particle state−space with a complete coordinate basis$\mid x>$, the matrix elements of the operator in the coordinate basis are

$$\hat{K}_\epsilon(x-y\mid T) =< x\mid \hat{K}_\epsilon(T)\mid y> \tag{3.7}$$

Completeness of basis then implies Huygens composition principle

$$\int d^Dx'K_\epsilon(x-x'\mid T'')K_\epsilon(x'-x''\mid T'') = K_\epsilon(x-x''\mid T'+T'') \tag{3.8}$$

Note that $\epsilon = (\epsilon'T'+\epsilon''T'')\,/(T'+T'')$ stays arbitrarily small if ϵ' and ϵ'' are small enough. Repeated use of eq.(3.8) allows one to write

$$\hat{K}_\epsilon(x-y\mid T) = \int\prod_{n=1}^{N} d^Dx_n\prod_{m=0}^{N}K_\epsilon\left(x_{m+1}-x_m\mid\triangle T\right) \tag{3.9}$$

with $\triangle T = T/(N+1)$, and $x_0 = y, x_{N+1} = x$. Keeping T fixed, the limit $N\to\infty$ becomes an integral over continuous (but generally non−differential) paths in co-ordinate space-time between points y and x. (Observe that $K_\epsilon(x-y\mid\triangle T)$ depends only on the difference $(x-y)$ and converges to $\delta^D(x-y)$ for $\triangle T\to 0$.)

If the operator $\hat{H}$ is an ordered expression in terms of a canonical set of operators $(\hat{x}^\mu,\hat{p}_\mu)$:

$$\hat{H} = \sum_{k,l}\hat{p}_{\mu 1}\ldots\hat{p}_{\mu k}H^{\mu 1\ldots\mu k}_{\nu 1\ldots\nu l}\hat{x}^{\nu 1}\ldots\hat{x}^{\nu l} \tag{3.10}$$

then we expand the co−ordinate path-integral expression (3.9) further to a phase−space path−integral

$$\hat{K}_\epsilon(x-y\mid T) = \int\int\prod_{n=1}^{N}\frac{d^Dx_nd^Dp_n}{(2\pi)^D}\exp\,[i\sum_{k=0}^{N}(p_k.(x_{k+1}-x_k)-\triangle TH(p_k,x_k))]$$

$$\to\int_y^x\mathcal{D}p(\tau)\mathcal{D}x(\tau)\exp^{[i\int_0^T d\tau(p.\dot{x}-H(p,x))]} \tag{3.11}$$

Here $H(p,x)$ is the $c-$number of the ordered operator $\hat{H}$, and we have tacitly assumed that the ordered symbol of the exponential can be replaced by the exponential of the ordered symbol. This is certainly correct for the main applications we consider in this work, as may be checked by explicit calculations. It is now clear that one may interpret the symbol $H(p, x)$ as the hamiltonian of some classical system, and the argument of the exponential as the classical action. Integration over momentum variables $p(\tau)$ then in general leads to the lagrangian form of this action

$$\hat{K}_\epsilon(x - y \mid T) = \int_y^x \mathcal{D}x(\tau) \exp\left[i \int_0^T d\tau L(\dot{x}, x)\right] \tag{3.12}$$

where the precise meaning of the integration measure can be recovered either from the phase-space expression (11), or from requiring the path–integral to satisfy Huygens' composition principle (8). Returning to eq.(1), it states that $\triangle_F(x - y)$ is the inverse of the Klein–Gordon operator (in the space of square–integrable functions). Rescaling it for later convenience by a factor $1/2m$, we consider the evolution operator

$$K_\epsilon(T) = \exp\left[-\frac{iT}{2m}\left(-\Box + m^2 - i\epsilon\right)\right] \tag{3.13}$$

In the co–ordinate representation the explicit expression of the matrix element of this operator is

$$\hat{K}_\epsilon(x - y \mid T) = -i\left(\frac{m}{2\pi T}\right)^{D/2} e^{i\frac{m}{2T}(x-y)^2 - \frac{iT}{2}(m - i\varepsilon)} \tag{3.14}$$

The Feynman propogator can then be written as

$$\triangle_F(x - y) = \frac{i}{2m}\int_0^\infty dT K_\epsilon(x - y), \tag{3.15}$$

Using previous results this can be cast in the form of a path integral eq.(3.14)

$$\triangle_F(x - y) = \frac{i}{2m}\int_0^\infty dT \int_y^x \mathcal{D}x(\tau) \exp\left[\frac{im}{2}\int_0^T d\tau(\dot{x_\mu}^2 - 1)\right]$$

Now, to change to euclidean space we set $T \to T\frac{2m}{i}$ and $\tau \to \tau\frac{2m}{i}$ and we use this transformation in the last part of the eq. (3.16) to get

$$\int_0^T d\tau(\dot{x_\mu}^2 - 1) = \int_0^T \left(\frac{d}{d\tau}x^\mu \frac{d}{d\tau}x_\mu - 1\right) \tag{3.16}$$

Similarly,

$$\exp\left[\frac{im}{2}\int_0^T d\tau(\dot{x_\mu}^2 - 1)\right] = exp\frac{im}{2}\int_0^T d\tau\left(\frac{i}{2m}\dot{x}^2 - \frac{2m}{i}\right) \tag{3.17}$$

$$= exp\int_0^T d\tau\left(-\frac{\dot{x}^2}{4} - m^2\right) \tag{3.18}$$

$$= e^{-m^2 T - \int_0^T d\tau \frac{\dot{x}^2}{4}} \tag{3.19}$$

Finally, eq. (3.16) reduces to

$$\triangle_F(x - y) = \int_0^\infty dT \int_y^x \mathcal{D}x(\tau)\, e^{-m^2 T - \int_0^T d\tau \frac{\dot{x}^2}{4}} \tag{3.20}$$

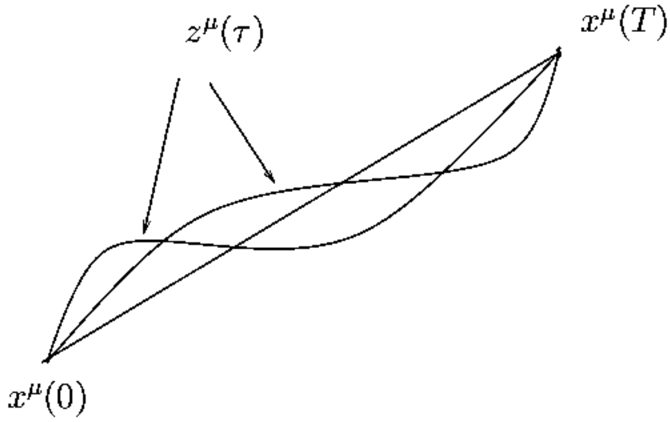

Fig. 3.1: Here we have a straight line representing the background $x^\mu_{bg}(\tau) = \left[y^\mu + (x - y)^\mu \frac{\tau}{T}\right]$ and $z^\mu(\tau)$ representing quantum fluctuations, where $z^\mu(T) = z^\mu(0) = 0$ at the two ends as shown in figure above.

3.2 Sum of tree level propagator corrections

The full propagator in the scalar background field in the worldline formalism is represented as

$$\langle 0|T\phi_q(x)\phi_q(y)|0\rangle = \frac{1}{-\Box + m^2 + \lambda\phi(x)} = \int_0^\infty dT \int_{x(0)=y}^{x(1)=x} \mathcal{D}x \; e^{-\int_0^T d\tau \left[\frac{1}{4}\dot{x}^2 + m^2 + \lambda\phi(x)\right]} \; . \tag{3.21}$$

We will now demonstrate this by comparing with standard Feynman diagrams.

<u>Order λ^0</u>

At zeroeth order the worldline formalism gives a standard representation of the free propagator,

$$\langle 0|T\phi_q(x)\phi_q(y)|0\rangle_{(0)} = \int_0^\infty dT \int_{x(0)=y}^{x(T)=x} \mathcal{D}x \; e^{-\int_0^T d\tau \, (\frac{1}{4}\dot{x}^2 + m^2)} = \int_0^\infty \frac{dT}{(4\pi T)^{\frac{D}{2}}} \, e^{-\frac{(x-y)^2}{4T} - Tm^2} \; . \tag{3.22}$$

The path integral is computed by splitting $x^\mu(\tau)$ into a background part $x^\mu_{bg}(\tau)$, which encodes the boundary conditions, and a quantum part $z^\mu(\tau)$, which has zero Dirichelet boundary conditions at $\tau = 0, T$,

$$x^\mu(\tau) = x^\mu_{bg}(\tau) + z^\mu(\tau) = \left[y^\mu + (x - y)^\mu \frac{\tau}{T}\right] + z^\mu(\tau) \; . \tag{3.23}$$

For future use, we record the propagator for $z^\mu(\tau)$ [42]

$$\begin{aligned} &\langle z^\mu(\tau) z^\nu(\sigma)\rangle = -2\delta^{\mu\nu}\Delta_T(\tau,\sigma) \\ &\Delta_T(\tau,\sigma) = \frac{\tau\sigma}{T} - \tau\theta(\sigma-\tau) - \sigma\theta(\tau-\sigma) = \frac{\tau\sigma}{T} + \frac{|\tau-\sigma|}{2} - \frac{\tau+\sigma}{2} \\ &\Delta_T(\tau,\tau) = \frac{\tau^2}{T} - \tau \; . \end{aligned} \tag{3.24}$$

Fourier transforming to momentum space gives (momenta will always be *ingoing* in the following)

$$\begin{aligned}\langle\tilde\phi(p_1)\tilde\phi(p_2)\rangle_{(0)} &\equiv \iint dxdy\, e^{ip_1x}e^{ip_2y}\,\langle 0|T\phi_q(x)\phi_q(y)|0\rangle_{(0)}\\ &= (2\pi)^D\delta^D(p_1+p_2)\int_0^\infty dT\,\frac{e^{-Tm^2}}{(4\pi T)^{\frac{D}{2}}}\int dx\, e^{-\frac{x^2}{4T}}\,e^{ip_1x}\\ &= (2\pi)^D\delta^D(p_1+p_2)\int_0^\infty dT\, e^{-T(p_1^2+m^2)}\\ &= (2\pi)^D\delta^D(p_1+p_2)\,\frac{1}{p_1^2+m^2}\ .\end{aligned} \tag{3.25}$$

<u>Order λ</u>

We set the background field $\phi(x)=\varepsilon\, e^{ikx}$, pick the linear term in ε from (3.21), and set $\varepsilon=1$

$$\langle 0|T\phi_q(x)\phi_q(y)|0\rangle_{(1)} = \int_0^\infty \frac{dT}{(4\pi T)^{\frac{D}{2}}}\, e^{-\frac{(x-y)^2}{4T}-Tm^2}(-\lambda)\int_0^T d\tau\,\langle e^{ik\cdot x(\tau)}\rangle\ . \tag{3.26}$$

We need the Wick contraction

$$\langle e^{ik\cdot z(\tau)}\rangle = e^{k^2(\frac{\tau^2}{T}-\tau)} \tag{3.27}$$

so that

$$\langle 0|T\phi_q(x)\phi_q(y)|0\rangle_{(1)} = \int_0^\infty \frac{dT}{(4\pi T)^{\frac{D}{2}}}\, e^{-\frac{(x-y)^2}{4T}-Tm^2}(-\lambda)\int_0^T d\tau\, \underbrace{e^{iky}e^{ik\cdot(x-y)\frac{\tau}{T}}}_{classical\ path}\ \underbrace{e^{-k^2(\tau-\frac{\tau^2}{T})}}_{Wick\ con.} \tag{3.28}$$

We Fourier transform, rescale $\tau=Tu$, do the T integral and obtain the product of two propagators in the Feynman parametrization

$$\begin{aligned}\langle\tilde\phi(p_1)\tilde\phi(p_2)\rangle_{(1)} &= (2\pi)^D\delta^D(p_1+p_2+k)(-\lambda)\int_0^1 du\int_0^\infty dT\,T\,e^{-T[p_1^2+m^2+(k^2+2p_1\cdot k)u]}\\ &= (2\pi)^D\delta^D(p_1+p_2+k)\,(-\lambda)\int_0^1 du\,\frac{\Gamma(2)}{[p_1^2+m^2+(k^2+2p_1\cdot k)u]^2}\\ &= (2\pi)^D\delta^D(p_1+p_2+k)\,\frac{1}{p_1^2+m^2}\,(-\lambda)\,\frac{1}{(p_1+k)^2+m^2}\ .\end{aligned} \tag{3.29}$$

The standard Feynman rule coefficient $(-\lambda)$ assigned to the vertex arises this way.

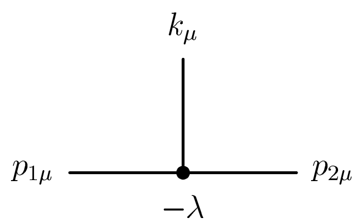

Fig. 3.2: Scalar vertex

Order λ^2

We set the background field $\phi(x) = \varepsilon_1 e^{ik_1 x} + \varepsilon_2 e^{ik_2 x}$, pick the multi-linear terms in ε_1 and ε_2 from (3.21), and then set $\varepsilon_i = 1$. In the process we have to compute

$$\langle e^{ik_1\cdot z(\tau_1)} e^{ik_2\cdot z(\tau_2)}\rangle = e^{-\frac{1}{2}\sum_{ij} k_{i\mu}\langle z^\mu(\tau_i) z^\nu(\tau_j)\rangle k_{j\nu}} = e^{\sum_i k_i^2(\frac{\tau_i^2}{T}-\tau_i)+\sum_{i<j} 2k_i\cdot k_j\, \Delta_T(\tau_i,\tau_j)} \,. \quad (3.30)$$

Again, we Fourier transform, rescale $\tau_i = Tu_i, i = 1, 2$, and perform the T integral. This yields

$$\begin{aligned}\langle\tilde\phi(p_1)\tilde\phi(p_2)\rangle_{(2)} &= (2\pi)^D \delta^D(p_1+p_2+k_1+k_2)(-\lambda)^2 \int_0^1 du_1 \int_0^1 du_2 \\ &\times \frac{2!}{\big[p_1^2+m^2+(k_1^2+2p_1\cdot k_1)u_1+(k_2^2+2p_1\cdot k_2)u_2+2k_1\cdot k_2\big(u_1\theta(u_2-u_1)+u_2\theta(u_1-u_2)\big)\big]^3}\end{aligned} \quad (3.31)$$

Recalling the standard Feynman parametrization to represent products of propagators

$$\frac{1}{A_1 A_2\cdots A_N} = \int_0^1 da_1 \int_0^1 da_2 \cdots \int_0^1 da_N \, \frac{\Gamma(N)\delta(1-\sum a_i)}{[\sum a_i A_i]^N} \quad (3.32)$$

it is now easy to rewrite (3.31) as

$$\begin{aligned}\langle\tilde\phi(p_1)\tilde\phi(p_2)\rangle_{(2)} &= (2\pi)^D \delta^D(p_1+p_2+k_1+k_2) \\ &\times \frac{1}{p_1^2+m^2}(-\lambda)\left[\frac{1}{(p_1+k_1)^2+m^2}+\frac{1}{(p_1+k_2)^2+m^2}\right](-\lambda)\frac{1}{p_2^2+m^2}\,.\end{aligned} \quad (3.33)$$

It thus corresponds to the sum of the two graphs shown in figure 3.3.

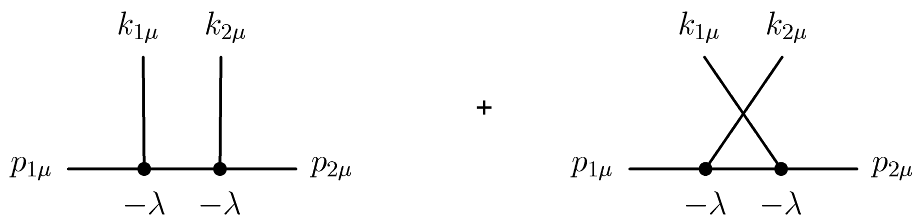

Fig. 3.3: Order λ^2 tree graphs

Order λ^N

The generalization to N external particles is now easy to see. It produces

$$\begin{aligned}\langle\tilde{\phi}(p_1)\tilde{\phi}(p_2)\rangle_{(N)} &= (2\pi)^D\delta^D(p_1+p_2+\sum_i k_i)(-\lambda)^N\int_0^\infty dT\int_0^T d\tau_1\cdot\cdot\int_0^T d\tau_N \\ &\times\exp\Big\{-T\Big(p_1^2+m^2\Big)-\sum_i(k_i^2+2p_1\cdot k_i)\tau_i-\sum_{i<j}2k_i\cdot k_j\big(\tau_i\theta(\tau_j-\tau_i)+\tau_j\theta(\tau_i-\tau_j)\big)\Big\} \\ &= (2\pi)^D\delta^D(p_1+p_2+\sum_i k_i)(-\lambda)^N N!\int_0^1 du_1\cdot\cdot\int_0^1 du_N \\ &\times\Big[p_1^2+m^2+\sum_i(k_i^2+2p_1\cdot k_i)u_i+\sum_{i<j}2k_i\cdot k_j\big(u_i\theta(u_j-u_i)+u_j\theta(u_i-u_j)\big)\Big]^{-N-1}.\end{aligned} \tag{3.34}$$

Each of the $N!$ orderings of the $u_1,..,u_N$ parameters along the worldline region $[0,1]$ identifies a range of integration. Each range of integration produces the product of the $(N+1)$ propagators where the momentum flows according to momentum conservation. This gives the total of $N!$ contributions corresponding to the various exchanges of the external lines carrying momentum k_i^μ. The explicit proof is given in the appendix C.
Fourier transforming (3.34) back to x - space one obtains

$$\begin{aligned}\langle 0|T\phi_q(x)\phi_q(y)|0\rangle_{(N)} &= \int\frac{dp_1}{(2\pi)^D}\int\frac{dp_2}{(2\pi)^D}\,\mathrm{e}^{-ip_1x-ip_2y}\langle\tilde{\phi}(p_1)\tilde{\phi}(p_2)\rangle_{(N)} \\ &= (-\lambda)^N\int_0^\infty\frac{dT}{(4\pi T)^{\frac{D}{2}}}\,e^{-\frac{(x-y)^2}{4T}-m^2T}\int_0^T d\tau_1\cdot\cdot\int_0^T d\tau_N\,\mathrm{e}^{i\sum_i k_i\cdot\left(y+\frac{\tau_i}{T}(x-y)\right)} \\ &\times\exp\Big[\sum_{i,j=1}^N k_i\cdot k_j\Delta_T(\tau_i,\tau_j)\Big]\end{aligned} \tag{3.35}$$

Our main interest is to find representations like (3.35) and (3.34) that unify the Feynman diagrams corresponding to different orderings. However, we wish to mention also that the contribution of any ordered sector to (3.35) can be recast in a form that is a finite-dimensional analogue of the initial path-integral (3.21). First, introducing the inverse of the $N\times N$ matrix $-\Delta_{ij}=-\Delta_1(u_i,u_j)$, as well as its determinant $|-\Delta|$, we can write the final exponential factor in (3.35) in terms of a gaussian integral over auxiliary variables $\xi_1,\ldots\xi_N$ as

$$\begin{aligned}\exp\Big[T\sum_{i,j=1}^N k_i\cdot k_j\Delta_1(u_i,u_j)\Big] &= \int d^D\xi_1\cdots\int d^D\xi_N\Big((4\pi T)^N|-\Delta|\Big)^{-\frac{D}{2}} \\ &\times\exp\Big[-\frac{1}{4T}\sum_{i,j=1}^N(-\Delta^{-1})_{ij}\xi_i\cdot\xi_j+i\sum_{i=1}^N k_i\cdot\xi_i\Big]\end{aligned} \tag{3.36}$$

It is sufficient to consider the standard ordering $1\geq u_1\geq u_2\geq\ldots\geq u_N\geq 0$. For this sector, it is straightforward to show inductively that $|-\Delta|$ and $(-\Delta^{-1})$ are given by

$$|-\Delta|=(1-u_1)(u_1-u_2)(u_2-u_3)\cdots(u_{N-1}-u_N)u_N \tag{3.37}$$

and

$$-\Delta^{-1} = \begin{pmatrix} \frac{1}{1-u_1}+\frac{1}{u_1-u_2} & -\frac{1}{u_1-u_2} & 0 & 0 & 0 \\ -\frac{1}{u_1-u_2} & \frac{1}{u_1-u_2}+\frac{1}{u_2-u_3} & -\frac{1}{u_2-u_3} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -\frac{1}{u_{N-2}-u_{N-1}} & \frac{1}{u_{N-2}-u_{N-1}}+\frac{1}{u_{N-1}-u_N} & -\frac{1}{u_{N-1}-u_N} \\ 0 & 0 & 0 & -\frac{1}{u_{N-1}-u_N} & \frac{1}{u_{N-1}-u_N}+\frac{1}{u_N} \end{pmatrix} \tag{3.38}$$

Thus in the first term in the exponent in (3.36) we can rewrite

$$\sum_{i,j=1}^{N} (-\Delta^{-1})_{ij}\xi_i \cdot \xi_j = \frac{\xi_1^2}{1-u_1} + \sum_{i=1}^{N-1} \frac{(\xi_i - \xi_{i+1})^2}{u_i - u_{i+1}} + \frac{\xi_N^2}{u_N} \tag{3.39}$$

Using (3.39) in (3.35) and performing a linear shift

$$\xi_i \to \xi_i - y - u_i(x-y) \tag{3.40}$$

we get

$$\begin{aligned} \langle 0|T\phi(x)\phi(y)|0\rangle^{(12\dots N)}_{(N)} &= (-\lambda)^N \int_0^\infty \frac{dT}{(4\pi T)^{\frac{D}{2}}}\, e^{-m^2T} T^N \int_0^1 du_1 \int_0^{u_1} du_2 \cdots \int_0^{u_{N-1}} du_N \\ &\times \int d^D\xi_1 \cdots \int d^D\xi_N \Big((4\pi T)^N | - \Delta| \Big)^{-\frac{D}{2}} \\ &\times \exp\Bigg\{ -\frac{1}{4T}\Bigg[\frac{(x-\xi_1)^2}{1-u_1} + \sum_{i=1}^{N-1}\frac{(\xi_i-\xi_{i+1})^2}{u_i-u_{i+1}} + \frac{(\xi_N - y)^2}{u_N}\Bigg] + i\sum_{i=1}^{N} k_i\cdot\xi_i \Bigg\} \end{aligned} \tag{3.41}$$

Here on the lhs the superscript $(12\dots N)$ indicates the restriction to the standard ordering. Comparing with the original path integral (3.21) it will be observed that (3.41) can be viewed as a restriction of this path integral to the finite-dimensional set of *polygonal* paths leading from x to y, corresponding to the propagation of a particle that is free in between absorbing (or emitting), at proper-time $\tau_i = u_i T$ and the space-time point ξ_i, the momentum k_i. Despite of its simplicity we have not been able to find this formula in the literature.

Chapter 4

Work of Tjon and Nieuwenhuis

In the work of Tjon and Nieuwenhuis [7], the inadequacy of the Bethe-Salpeter equation to explain the relativistic two body bound state problem beyond the ladder approximation which was already discussed earlier, has been addressed. We recall the main objections to the ladder approximation: The one body limit does not lead to the Klein-Gordon equation and gauge invariance is not satisfied. Hence, it becomes imperative to include all cross ladders diagrams. Now, since the study of two-body Green's function beyond the ladder approximation is not viable, this led to consider several quasipotential equations as possible explanation for an effective theory (to study systems like mesons, small nuclei, few electron atom and positronium).

Tjon and Nieuwenhuis then go on to use the worldline formalism to obtain the four-point Green's function in the $\varphi^2\chi$ theory.

4.1 Worldline Formalism

Neglecting the contributions from self energy and vertex corrections, we have an expression for the four point Greens function in the ladder approximation (i.e. after neglecting the possible occurrence of vacuum fluctuation $\varphi\varphi$ loops) in terms of path integrals over particle trajectories z and $\bar{z}$ of two φ particles as below.

$$G = \int_0^\infty ds \int_0^\infty d\bar{s} \int (\mathcal{D}_z)_{xy} (\mathcal{D}_z)_{\bar{x}\bar{y}} \times \exp(-K[z,s] - K[\bar{z},\bar{s}] + V[z,\bar{z},s,\bar{s}]) \tag{4.1}$$

where K and V are given by

$$K[z,s] = m^2 s + \frac{1}{4s}\int_0^1 d\tau \dot{z}_\lambda^2(\tau), \tag{4.2}$$

$$V[z,\bar{z},s,\bar{s}] = g^2 s\bar{s}\int_0^1 d\tau \int_0^1 d\bar{\tau}\Delta(z(\tau) - \bar{z}(\tau)). \tag{4.3}$$

After the functional integration over all possible paths in eq. (4.1), subject to the boundary conditions $z(0) = x, z(1) = y$, and similarly for $\bar{z}$. With the free two point

$\Delta(x)$ function in $3+1$ dimensions, given by

$$\Delta(x) = \frac{\mu}{4\pi^2 \mid x \mid} k_1(\mu \mid x \mid) \tag{4.4}$$

The authors obtain an expression for G in large timelike separations $T = \frac{1}{2}(y_4 + \bar{y}_4 - x_4 - \bar{x}_4)$ and the bound state spectrum can be determined by studying the behaviour of G with respect to variations of its initial points $(x, \bar{x})$ and final points $(y, \bar{y})$

$$G = \sum_{n=0}^{\infty} c_n \exp(-m_n T), \overset{T\to\infty}{\simeq} c_0 \exp(-m_0 T) \tag{4.5}$$

This implies that asymptotically the Green function is dominated by the ground state contribution.

Since the path integrals in (4.1) are quantum mechanical ones, not field path integrals, this amounts to a considerable reduction in the number of degrees of freedom.

The solutions to Feynmann-Schwinger representation could be obtained by discretizing the functional integral according to

$$(\mathcal{D}_z)_{xy} \to \left(\frac{N}{4\pi s}\right)^{2N} \prod_{i=1}^{N-1} \int d^4 z_i \tag{4.6}$$

The normalization in Eq.(4.6) has been chosen such that, when expanded in the coupling g^2, the Green function correctly reproduces the Feynman perturbation series. In terms of the discretized variables the functionals k and V assume the following form:

$$K[z, s] \to m^2 s + \frac{N}{4s} \sum_{i=1}^{N} (z_i - z_{i-1}), \tag{4.7}$$

$$V[z, \bar{z}, s, \bar{s}] \to \frac{g^2 n\bar{s}}{N^2} \sum_{i,j=1}^{N} \Delta(\frac{1}{2}(z_i + z_{i-1} - \bar{z}_j + \bar{z}_{j-1})). \tag{4.8}$$

The boundary conditions are $z_0 = x$ and $z_N = y$ and similarly for $\bar{z}$.

The ground state mass is given by (see eq.(4.1) for the definition of G).

$$L(T) = -\frac{d}{dT}[\ln G(T)] \overset{T\to\infty}{\to} m_0 \tag{4.9}$$

Substituting Z for the full set of degrees of freedom we have $S(Z) = K[z,s] - V[z,\bar{z},s,\bar{s}]$, therefore $L(t)$

$$L(T) = \frac{\int \mathcal{D}ZS'[Z]\mathrm{e}^{-\mathrm{S[Z]}}}{\int \mathcal{D}Z\mathrm{e}^{-\mathrm{S[Z]}}} \tag{4.10}$$

where prime denotes differentiation with respect to T. Therefore the ground state mass is the average of $S'[Z]$ over an ensemble generated by an action $S[Z]$ for sufficiciently large T. The FSR ground state wave function Ψ can be obtained by an additional integration of G in eq.(4.1) over the spatial relative components $r \equiv \bar{y} - y$ of the final point and incorporating this coordinate in the set Z. Keeping track of the distribution of $\mid r \mid$'s while computing $L(T)$ the r dependence of Ψ can be determined. The dependence on N was studied and the bound state mass was found to become independent of N at values of $N = 35 - 40$.

4.2 *Results and Comparison*

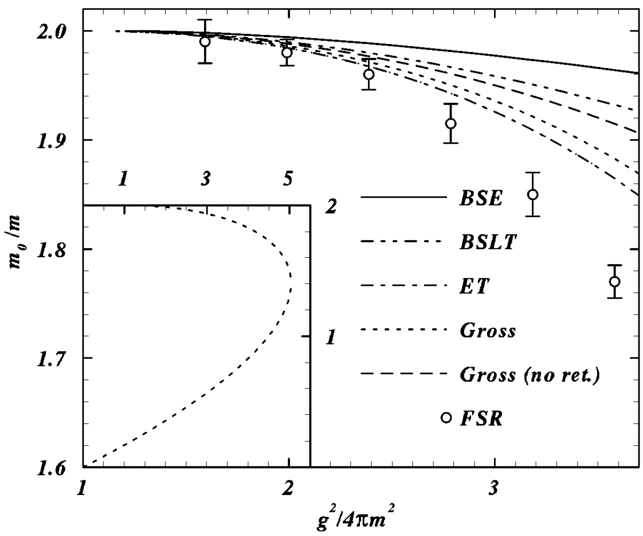

Fig. 4.1: The ground state mass as a function of the dimensionless coupling constant $\frac{g^2}{4\pi m^2}$ for $\frac{\mu}{m} = 0.15$, taken from Tjon and Nieuwenhuis ([7]).

Fig.1 shows the ground state mass as a function of the dimensionless coupling constant $\frac{g^2}{4\pi m^2}$ for $\frac{\mu}{m} = 0.15$. Since the self-energy has been neglected in the FSR calculations, predictions could directly be compared to those of Ladder BSE and the various QPEs. The range of validity of the ladder theory is restricted to small coupling. For large couplings all approximations tend to underbind the system as compared to FSR results. All QPEs generate more binding energy than the ladder BSE, and the results are closer to FSR ones. The equal-time (ET) approximation provides the best correspondence with the FSR ones.

Fig. 2 compares FSR and BSE ground state wave function for relative time $t = 0$

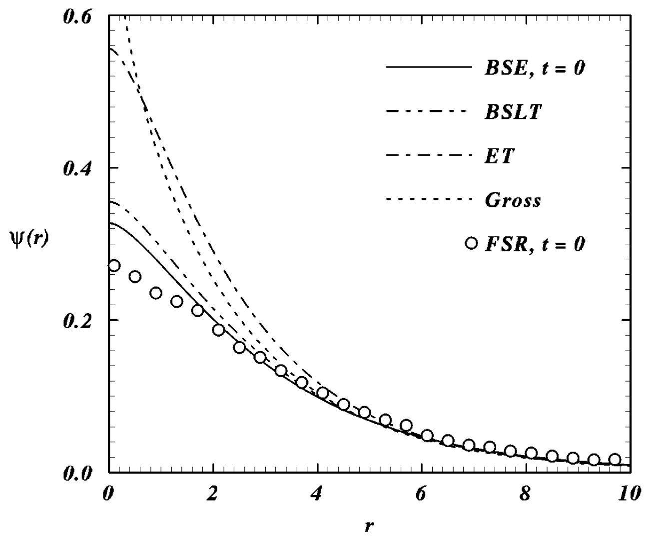

Fig. 4.2: The ground state wave function $\Psi(r)$ as a function of relative distance r taken from Tjon and Nieuwenhuis ([7]).

with various QPE wave functions is made. The coupling constant g is adjusted such that the same value of the ground state mass $m_0 = 1.882m$ is found. At large separations wave function behaviour is essentially determined by the binding energy of the composite system. At short distances, the main difference between the QPE predictions is due to the asymptotic behaviour of their two-particle free propagator $S_{QEP}(q)$ for large values of q.

An important role is played by the relative time t dependence of the wave function, especially at small spatial separations between the constituents.

The authors present for the scalar case the first calculations of bound state properties beyond the ladder approximation using the Feynman-Schwinger representation. When compared with results from Bethe-Salpeter equation in the ladder approximation, it is observed that cross ladders contribute significantly to the binding energy.

Chapter 5

Ladder Graphs in Scalar Field Theory

We use an euclidean signature. The action for $\lambda\varphi^3$ theory is

$$S[\phi] = \int d^D x \left(\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{3!}\phi^3 \right) . \tag{5.1}$$

In the background field formalism ($\phi \to \phi_q + \phi$) the action quadratic in quantum fields ϕ_q is

$$S^{(2)}[\phi_q, \phi] = \int d^D x \; \frac{1}{2}\phi_q \Big(-\Box + m^2 + \lambda\phi \Big) \phi_q + ... \tag{5.2}$$

It is now simple to use equation number (3.34) to sum all the propagator graphs (simply called "N - propagators“ in the following) for constructing the sum of all ladder and crossed-ladder graphs with N rungs (simply called "N-ladders" in the following) in ϕ^3 theory, and more generally, putting a different mass μ for the rungs, for scalar Yukawa theory. Let us start with the graphs in momentum space. Starting with the product of two copies of (3.34), identifying k_i of one N - propagator with $-k_i$ of the second one, and inserting the connecting propagator integrals $\int \frac{dk_1}{(2\pi)^D} \frac{1}{k_1^2+\mu^2} \cdots \int \frac{dk_N}{(2\pi)^D} \frac{1}{k_N^2+\mu^2}$ produces precisely $N!$ times the N-ladder graphs (see fig. 1.4):

$$\begin{aligned}
\langle \tilde\phi(q_1)\tilde\phi(q_2)\tilde\phi(p_1)\tilde\phi(p_2)\rangle_{(N)} \;=\;& (2\pi)^D \delta^D(p_1+p_2+q_1+q_2) \frac{\lambda^{2N}}{N!} \int_0^\infty dS \int_0^\infty dT \, \mathrm{e}^{-m^2(S+T)} \\
&\times \int_0^S d\sigma_1 \cdots \int_0^S d\sigma_N \int_0^T d\tau_1 \cdots \int_0^T d\tau_N \\
&\times \int \frac{dk_1}{(2\pi)^D} \frac{1}{k_1^2+\mu^2} \cdots \int \frac{dk_N}{(2\pi)^D} \frac{1}{k_N^2+\mu^2} (2\pi)^D \delta^D(p_1+p_2+\sum_i k_i) \\
&\times \exp\Big\{ -S p_1^2 - \sum_i (k_i^2 + 2 p_1 \cdot k_i)\sigma_i - \sum_{i<j} 2 k_i \cdot k_j D(\sigma_i, \sigma_j) \Big\} \\
&\times \exp\Big\{ -T q_1^2 - \sum_i (k_i^2 - 2 q_1 \cdot k_i)\tau_i - \sum_{i<j} 2 k_i \cdot k_j D(\tau_i, \tau_j) \Big\}
\end{aligned} \tag{5.3}$$

where we have now also defined

$$D(\tau_i,\tau_j) \quad := \quad \tau_i\theta(\tau_j-\tau_i)+\tau_j\theta(\tau_i-\tau_j) \tag{5.4}$$

Next, we introduce Schwinger parameters $\alpha_1,\dots,\alpha_N$ to exponentiate the connecting propagators,

$$\frac{1}{k_i^2+\mu^2} \quad = \quad \int_0^\infty d\alpha_i\,\mathrm{e}^{-\alpha_i(k_i^2+\mu^2)} \tag{5.5}$$

and we also (re-)exponentiate the second δ - function factor,

$$(2\pi)^D\delta^D\bigl(p_1+p_2+\sum_i k_i\bigr) \quad = \quad \int dv\,\mathrm{e}^{iv\cdot\left(p_1+p_2+\sum_i k_i\right)} \tag{5.6}$$

The k_i - integrals are now Gaussian, and performing them involves only the inverse and the determinant of the symmetric $N\times N$ - matrix A_N with entries

$$\begin{aligned} A_{Nii} &= \sigma_i+\tau_i+\alpha_i \\ A_{Nij} &= D(\sigma_i,\sigma_j)+D(\tau_i,\tau_j) \quad (i\neq j) \end{aligned} \tag{5.7}$$

The v - integral then also becomes Gaussian. Doing it one is left with the following integral representation for the N - ladder (henceforth we will omit the global δ function factor $(2\pi)^D\delta^D(p_1+p_2+q_1+q_2)$):

$$\begin{aligned} \langle\tilde\phi(q_1)\tilde\phi(q_2)\tilde\phi(p_1)\tilde\phi(p_2)\rangle_{(N)} &= \frac{1}{(4\pi)^{(N-1)\frac{D}{2}}}\frac{\lambda^{2N}}{N!}\int_0^\infty dS\int_0^\infty dT\,\mathrm{e}^{-m^2(S+T)} \\ &\times\int_0^S d\sigma_1\cdots\int_0^T d\tau_N\int_0^\infty d\alpha_1\cdots\int_0^\infty d\alpha_N\,\frac{\mathrm{e}^{-\mu^2\sum_i\alpha_i}}{(a_N\,\mathrm{det}A_N)^{\frac{D}{2}}} \\ &\times\exp\biggl\{-Sp_1^2-Tq_1^2-\frac{b_N^2}{a_N}+(p_1\vec\sigma-q_1\vec\tau)\cdot A_N^{-1}\cdot(p_1\vec\sigma-q_1\vec\tau)\biggr\} \end{aligned} \tag{5.8}$$

Here we have further defined

$$\begin{aligned} a_N &:= \vec 1\cdot A^{-1}\cdot\vec 1 \\ b_N &:= p_1+p_2-\vec 1\cdot A_N^{-1}\cdot\vec\sigma\,p_1+\vec 1\cdot A_N^{-1}\cdot\vec\tau\,q_1 \end{aligned} \tag{5.9}$$

with $\vec 1:=(1,\dots,1)$, $\vec\sigma:=(\sigma_1,\dots,\sigma_N)$ etc. It is understood that the matrix A_N acts trivially on Lorentz indices.

Note that (5.8) is still valid in D dimensions, and that it could be modified in an obvious way to give different masses to the two N - propagators and/or to the internal propagators.

Fourier transforming (5.8) we obtain the corresponding amplitude in x - space in the form

$$\langle\phi_q(x)\phi_q(\bar{x})\phi_q(y)\phi_q(\bar{y})\rangle_{(N)} = \frac{1}{(4\pi)^{(N+2)\frac{D}{2}}}\frac{\lambda^{2N}}{N!}\int_0^\infty dS\int_0^\infty dT\,\mathrm{e}^{-m^2(S+T)} \times\int_0^S d\sigma_1\cdot\cdot\int_0^T d\tau_N\int_0^\infty d\alpha_1\cdot\cdot\int_0^\infty d\alpha_N\,\frac{\mathrm{e}^{-\mu^2\sum_i\alpha_i}}{(\mathrm{det}L\mathrm{det}A_N)^{\frac{D}{2}}} \times\exp\biggl\{-\frac{1}{4}\Bigl[a_N(y-\bar{y})^2+(w,\bar{w})L^{-1}(w,\bar{w})\Bigr]\biggr\} \tag{5.10}$$

where

$$\begin{aligned} L &= \begin{pmatrix} S-\vec{\sigma}A_N^{-1}\vec{\sigma} & \vec{\sigma}A_N^{-1}\vec{\tau} \\ \vec{\sigma}A_N^{-1}\vec{\tau} & T-\vec{\tau}A_N^{-1}\vec{\tau}\end{pmatrix} \\ \mathrm{det}L &= (S-\vec{\sigma}A_N^{-1}\vec{\sigma})(T-\vec{\tau}A_N^{-1}\vec{\tau})-(\vec{\sigma}A_N^{-1}\vec{\tau})^2 \\ w &= x-y+\vec{1}A_N^{-1}\vec{\sigma}\,(y-\bar{y}) \\ \bar{w} &= \bar{x}-\bar{y}-\vec{1}A_N^{-1}\vec{\tau}\,(y-\bar{y}) \end{aligned} \tag{5.11}$$

Starting from equation (3.35) we obtain

$$\langle\phi_q(x)\phi_q(\bar{x})\phi_q(y)\phi_q(\bar{y})\rangle_{(N)} = \frac{1}{(4\pi)^{(N+2)\frac{D}{2}}}\frac{\lambda^{2N}}{N!}\int_0^\infty dS\int_0^\infty dT\,\mathrm{e}^{-m^2(S+T)-\frac{(x-y)^2}{4S}-\frac{(\bar{x}-\bar{y})^2}{4T}} \times\int_0^S d\sigma_1\cdot\cdot\int_0^T d\tau_N\int_0^\infty d\alpha_1\cdot\cdot\int_0^\infty d\alpha_N\,\frac{\mathrm{e}^{-\mu^2\sum_i\alpha_i-\frac{1}{4}\vec{r}M_N^{-1}\vec{r}}}{(ST\mathrm{det}M_N)^{\frac{D}{2}}} \tag{5.12}$$

where M_N is the symmetric $N\times N$ matrix

$$M_{Nij} = \delta_{ij}\alpha_i-\Delta_S(\sigma_i,\sigma_j)-\Delta_T(\tau_i,\tau_j) \tag{5.13}$$

and

$$\vec{r} = (y-\bar{y})\vec{1}+\frac{x-y}{S}\vec{\sigma}-\frac{(\bar{x}-\bar{y})}{T}\vec{\tau} \tag{5.14}$$

We note that the two x-space representations (5.10),(5.12) can be related by

$$\begin{aligned} M_N &= A_N-\frac{\vec{\sigma}\otimes\vec{\sigma}}{S}-\frac{\vec{\tau}\otimes\vec{\tau}}{T} \\ M_N^{-1} &= A_N^{-1}+L_{11}^{-1}A_N^{-1}\cdot\vec{\sigma}\,\vec{\sigma}\cdot A_N^{-1}-L_{12}^{-1}A_N^{-1}\cdot\vec{\sigma}\,\vec{\tau}\cdot A_N^{-1}-L_{21}^{-1}A_N^{-1}\cdot\vec{\tau}\,\vec{\sigma}\cdot A_N^{-1}+L_{22}^{-1}A_N^{-1}\cdot\vec{\tau}\,\vec{\tau}\cdot A_N^{-1} \end{aligned} \tag{5.15}$$

which also implies that

$$ST\mathrm{det}M_N = \mathrm{det}L\mathrm{det}A_N \tag{5.16}$$

Chapter 6

Projection on the lowest bound state

We proceed to the simplest possible application of our formulas for the ladder graphs, which is the extraction of the lowest bound state. Following [7], this can be done by considering the limit of large timelike separation, i.e., $t \to \infty$, where

$$t = \frac{1}{2}(y_4 + \bar{y}_4 - x_4 - \bar{x}_4) \tag{6.1}$$

namely, defining our four – point Green's function in the ladder approximation by G, so that

$$G = \sum_{N=0}^{\infty} G_N = \sum_{N=0}^{\infty} \langle \phi_q(x)\phi_q(\bar{x})\phi_q(y)\phi_q(\bar{y})\rangle_{(N)} \tag{6.2}$$

one has

$$G \overset{t\to\infty}{\simeq} c_0\, \mathrm{e}^{-m_0 t} \tag{6.3}$$

with m_0 the lowest bound state mass. We can further simplify by setting

$$y = \bar{y}, \quad x = \bar{x} \tag{6.4}$$

so that $t = y_4 - x_4 = \bar{y}_4 - \bar{x}_4$. Further, since the limit $t \to \infty$ is taken at fixed spatial separation, in this limit we can effectively set

$$t^2 = (y-x)^2 = (\bar{y}-\bar{x})^2 \tag{6.5}$$

Using this approximation in eqs. (5.10), (5.11) one finds

$$w = \bar{w} = -t\vec{e} \tag{6.6}$$

where $\vec{e} := (1,0,0,0)$, and

$$
\begin{aligned}
G_N \;=\;& \frac{1}{(4\pi)^{(N+2)\frac{D}{2}}}\frac{\lambda^{2N}}{N!}\int_0^\infty dS\int_0^\infty dT\,\mathrm{e}^{-m^2(S+T)} \\
&\times\int_0^S d\sigma_1\cdot\cdot\int_0^T d\tau_N\int_0^\infty d\alpha_1\cdot\cdot\int_0^\infty d\alpha_N\,\frac{\mathrm{e}^{-\mu^2\sum_i\alpha_i}}{(\mathrm{det}L\mathrm{det}A_N)^{\frac{D}{2}}} \\
&\times\exp\Bigl\{-\frac{t^2}{4\mathrm{det}L}\Bigl[S+T-(\vec\sigma+\vec\tau)A_N^{-1}(\vec\sigma+\vec\tau)\Bigr]\Bigr\}
\end{aligned}
\tag{6.7}
$$

Rescaling all integration variables $S,T,\sigma_i,\tau_i,\alpha_i$ by $1/m^2$, one arrives at (setting now also $D=4$)

$$
\begin{aligned}
G_N \;=\;& \frac{m^4}{(4\pi)^4}\frac{g^N}{N!}\int_0^\infty dS\int_0^\infty dT\,\mathrm{e}^{-(S+T)} \\
&\times\int_0^S d\sigma_1\cdot\cdot\int_0^T d\tau_N\int_0^\infty d\alpha_1\cdot\cdot\int_0^\infty d\alpha_N\,\frac{\mathrm{e}^{-\frac{\mu^2}{m^2}\sum_i\alpha_i}}{(\mathrm{det}L\mathrm{det}A_N)^2} \\
&\times\exp\Bigl\{-\hat t^2\frac{1}{\mathrm{det}L}\Bigl[S+T-(\vec\sigma+\vec\tau)A_N^{-1}(\vec\sigma+\vec\tau)\Bigr]\Bigr\}
\end{aligned}
\tag{6.8}
$$

where we have introduced the dimensionless parameters

$$
g \;:=\; \Bigl(\frac{\lambda}{4\pi m}\Bigr)^2, \quad \hat t := tm/2\,. \tag{6.9}
$$

We now specialize to the ϕ^3 - case, $\mu=m$. Here it is useful to introduce the total proper-time Λ,

$$
\Lambda \;:=\; S+T+\sum_{i=1}^N\alpha_i \tag{6.10}
$$

As usual, this variable can be integrated out in closed form: After another rescaling of $S,T,\sigma_i,\tau_i,\alpha_i$ by Λ we have

$$
\begin{aligned}
G_N \;=\;& \frac{m^4}{(4\pi)^4}\frac{g^N}{N!}\int_0^\infty d\Lambda\Lambda^{N-3}\,\mathrm{e}^{-\Lambda}\int_0^1 dS\int_0^1 dT \\
&\times\int_0^S d\sigma_1\cdot\cdot\int_0^T d\tau_N\int_0^1 d\alpha_1\cdot\cdot\int_0^1 d\alpha_N\,\delta\Bigl(1-S-T-\sum_{i=1}^N\alpha_i\Bigr)\,\frac{1}{(\mathrm{det}L\mathrm{det}A_N)^2} \\
&\times\exp\Bigl\{-\frac{1}{\Lambda}\hat t^2\frac{1}{\mathrm{det}L}\Bigl[S+T-(\vec\sigma+\vec\tau)A_N^{-1}(\vec\sigma+\vec\tau)\Bigr]\Bigr\}
\end{aligned}
\tag{6.11}
$$

The Λ - integral can be done using

$$
\int_0^\infty dx\,x^n\,\mathrm{e}^{-x-\frac{z}{4}\frac{1}{x}} \;=\; 2^{-n}z^{\frac{n+1}{2}}K_{-n-1}(\sqrt{z}) \tag{6.12}
$$

where K_n is the modified Bessel function of the second kind. Thus we have

$$\begin{aligned} G_N &= 2\frac{m^4}{(4\pi)^4}\frac{g^N}{N!}\hat{t}^{N-2}\int_0^1 dS\int_0^1 dT\int_0^1 d\alpha_1\cdot\cdot\int_0^1 d\alpha_N\,\delta\Big(1-S-T-\sum_{i=1}^N\alpha_i\Big) \\ &\times\int_0^S d\sigma_1\cdot\cdot\int_0^T d\tau_N\;\frac{\gamma^{N-2}K_{-N+2}(2\gamma\hat{t})}{(\det L\det A_N)^2} \end{aligned} \tag{6.13}$$

with

$$\gamma \;:=\; \sqrt{\frac{S+T-(\vec{\sigma}+\vec{\tau})A_N^{-1}(\vec{\sigma}+\vec{\tau})}{\det L}} \tag{6.14}$$

In the limit $t\to\infty$, we can use the asymptotic expansion of $K_n(z)$,

$$K_n(z)\;\sim\;\mathrm{e}^{-z}\sqrt{\frac{\pi}{2}}\sqrt{\frac{1}{z}}\Big(1+O(1/z)\Big) \tag{6.15}$$

leading to

$$\begin{aligned} G_N &\sim \frac{m^4}{(4\pi)^4}\frac{g^N}{N!}\sqrt{\pi}\,\hat{t}^{N-5/2}\int_0^1 dS\int_0^1 dT\int_0^1 d\alpha_1\cdot\cdot\int_0^1 d\alpha_N\,\delta\Big(1-S-T-\sum_{i=1}^N\alpha_i\Big) \\ &\times\int_0^S d\sigma_1\cdot\cdot\int_0^T d\tau_N\;\frac{\gamma^{N-5/2}\,\mathrm{e}^{-2\gamma\hat{t}}}{(\det L\det A_N)^2} \end{aligned} \tag{6.16}$$

One should now analyze the function γ; presumably it is non-negative and the leading contributions at large $\hat{t}$ come from regions in the integrand where it tends to zero. We return to the general case of two arbitrary different masses. If one uses (5.12) instead of (5.10), then one obtains the same (6.8) rewritten as

$$\begin{aligned} G_N &= \frac{m^4}{(4\pi)^4}\frac{g^N}{N!}\int_0^\infty dS\int_0^\infty dT\,\mathrm{e}^{-(S+T)} \\ &\times\int_0^S d\sigma_1\cdot\cdot\int_0^T d\tau_N\int_0^\infty d\alpha_1\cdot\cdot\int_0^\infty d\alpha_N\;\frac{\mathrm{e}^{-\frac{\mu^2}{m^2}\sum_i\alpha_i}}{(ST\det M_N)^2} \\ &\times\exp\Big\{-\hat{t}^2\Big[\frac{1}{S}+\frac{1}{T}+\Big(\frac{\vec{\sigma}}{S}-\frac{\vec{\tau}}{T}\Big)M_N^{-1}\Big(\frac{\vec{\sigma}}{S}-\frac{\vec{\tau}}{T}\Big)\Big]\Big\} \end{aligned} \tag{6.17}$$

Here it seems preferable to also rescale $\sigma_i = Su_i, \tau_i = Tv_i$, leading to

$$\begin{aligned} G_N &= \frac{m^4}{(4\pi)^4}\frac{g^N}{N!}\int_0^\infty dS\,S^{N-2}\int_0^\infty dT\,T^{N-2}\,\mathrm{e}^{-(S+T)} \\ &\times\int_0^1 du_1\cdot\cdot\int_0^1 dv_N\int_0^\infty d\alpha_1\cdot\cdot\int_0^\infty d\alpha_N\;\frac{\mathrm{e}^{-\frac{\mu^2}{m^2}\sum_i\alpha_i}}{(\det\hat{M}_N)^2} \\ &\times\exp\Big\{-\hat{t}^2\Big[\frac{1}{S}+\frac{1}{T}+(\vec{u}-\vec{v})\hat{M}_N^{-1}(\vec{u}-\vec{v})\Big]\Big\} \end{aligned} \tag{6.18}$$

where now

$$\hat{M}_{Nij} \quad = \quad \delta_{ij}\alpha_i - S\Delta_1(u_i,u_j) - T\Delta_1(v_i,v_j) \tag{6.19}$$

Eq. (6.13) now takes the form

$$\begin{aligned} G_N \quad = \quad & 2\frac{m^4}{(4\pi)^4}\frac{g^N}{N!}\hat{t}^{N-2}\int_0^1 dS\, S^{N-2}\int_0^1 dT\, T^{N-2}\int_0^1 d\alpha_1 \cdot\cdot \int_0^1 d\alpha_N\, \delta\Big(1 - S - T - \sum_{i=1}^N \alpha_i\Big) \\ & \times \int_0^1 du_1 \cdot\cdot \int_0^1 dv_N\, \frac{\hat{\gamma}^{N-2} K_{-N+2}(2\hat{\gamma}\hat{t})}{(\mathrm{det}\hat{M}_N)^2} \end{aligned} \tag{6.20}$$

with

$$\hat{\gamma} \quad = \quad \sqrt{\frac{1}{S} + \frac{1}{T} + (\vec{u}-\vec{v})\hat{M}_N^{-1}(\vec{u}-\vec{v})} \tag{6.21}$$

We return to (6.18), and perform the following further rescalings

$$S \to \hat{t}S, \quad T \to \hat{t}T, \quad \alpha_i \to \hat{t}\alpha_i \tag{6.22}$$

This yields

$$\begin{aligned} G_N \quad = \quad & \frac{m^4}{(4\pi)^4\hat{t}^2}\frac{(\hat{t}g)^N}{N!}\int_0^\infty dS\, S^{N-2}\int_0^\infty dT\, T^{N-2} \\ & \times \int_0^1 du_1 \cdot\cdot \int_0^1 dv_N \int_0^\infty d\alpha_1 \cdot\cdot \int_0^\infty d\alpha_N\, \frac{1}{(\mathrm{det}\hat{M}_N)^2} \\ & \times \exp\Big\{-\hat{t}\Big[S + T + \frac{1}{S} + \frac{1}{T} + \frac{\mu^2}{m^2}\sum_i \alpha_i + (\vec{u}-\vec{v})\hat{M}_N^{-1}(\vec{u}-\vec{v})\Big]\Big\} \end{aligned} \tag{6.23}$$

CHAPTER 7

SADDLE POINT APPROXIMATION

Further, we will now use the large $\hat{t}$ limit to eliminate, at fixed S, T, α_i, the v_i integrals by a gaussian approximation around the point $\vec{v} = \vec{u}$. Around the point $\vec{v} = \vec{u}$ the term containning $\vec{v} - \vec{u}$ in the exponential in equ. (6.23) reduces to unity. The result we have.

$$
\begin{aligned}
G_N &= \frac{m^4}{(4\pi)^4 \hat{t}^2} \frac{(\pi \hat{t} g^2)^{N/2}}{N!} \int_0^\infty dS\, S^{N-2} \int_0^\infty dT\, T^{N-2} \\
&\times \int_0^1 du_1 \cdot\cdot \int_0^1 du_N \int_0^\infty d\alpha_1 \cdot\cdot \int_0^\infty d\alpha_N \frac{1}{(\det \bar{M}_N)^{3/2}} \\
&\times \exp\left\{ -\hat{t} \left[S + T + \frac{1}{S} + \frac{1}{T} + \frac{\mu^2}{m^2} \sum_i \alpha_i \right] \right\}
\end{aligned}
\tag{7.1}
$$

where now

$$
\bar{M}_{Nij} = \delta_{ij} \alpha_i - (S+T) \Delta_1(u_i, u_j) \tag{7.2}
$$

After a further rescaling

$$
\alpha_i \to (S+T)\alpha_i \tag{7.3}
$$

and summation over N, we obtain the following representation for the full Green's function:

$$
\begin{aligned}
G &= \frac{m^4}{(4\pi)^4 \hat{t}^2} \int_0^\infty \frac{dS}{S^2} \int_0^\infty \frac{dT}{T^2} \exp\left\{ -\hat{t} \left[S + T + \frac{1}{S} + \frac{1}{T} \right] \right\} \\
&\times \sum_{N=1}^\infty \frac{(\pi \hat{t} g^2)^{N/2}}{N!} \left[\frac{ST}{(S+T)^{1/2}} \right]^N c_N\left(\hat{t}(S+T)\mu^2/m^2 \right)
\end{aligned}
\tag{7.4}
$$

where

$$
c_N(x) = \int_0^1 du_1 \cdots \int_0^1 du_N \int_0^\infty d\alpha_1 \cdot\cdot \int_0^\infty d\alpha_N \frac{\mathrm{e}^{-x \sum_i \alpha_i}}{(\det \tilde{M}_N)^{3/2}} \tag{7.5}
$$

$$\tilde{M}_{Nij} = \delta_{ij}\alpha_i - \Delta_1(u_i, u_j) \tag{7.6}$$

The integrals in (60) are convergent, however this is not very transparent the way they are written. This motivates the following transformations. First, let us rewrite the matrix $\hat{M}_N$ as

$$\tilde{M}_N = D_N(\mathbb{1} - R_N) \tag{7.7}$$

where D_N is the diagonal part of $\tilde{M}_N$

$$D_{Nij} := \delta_{ij}(\alpha_{ij} - \Delta_1(u_i, u_j)) = \delta_{ij}(\alpha_i + u_i(1 - u_i)) \tag{7.8}$$

and

$$R_N := \Delta_1^{'} D^{-1} \tag{7.9}$$

where Δ_1' denotes the matrix Δ_{1ij} with its diagonal terms deleted. Then, we perform a change of variables from α_i to β_i

$$\beta_i := \sqrt{\frac{-\Delta_{1ii}}{\alpha_i - \Delta_{1ii}}} \tag{7.10}$$

The integrals (60) then turn into

$$c_N(x) = 2^N \int_0^1 \frac{du_1}{\sqrt{u_1(1-u_1)}} \cdots \int_0^1 \frac{du_N}{\sqrt{u_N(1-u_N)}} \int_0^1 d\beta_1 \cdots \int_0^1 d\beta_N \frac{\exp^{-x\sum_i(-\Delta_{ii})(\frac{1}{\beta_i^2}-1)}}{\det^{\frac{3}{2}}(\mathbb{1} - R_N)} \tag{7.11}$$

Note that now $D_{Nij}^{-1} = \delta_{ij}\beta_i^2/(-\Delta_{1ii})$.
Further, since the integrand is permutation symmetric the full u_i integrals can be replaced by $N!$ times the integral over the ordered sector $u_1 \geq u_2 \geq u_3 \cdots \geq u_N$. Thus we define

$$\begin{aligned}\bar{c}_N(x) := \frac{c_N(x)}{2^N N!} = \int_0^1 \frac{du_1}{\sqrt{u_1(1-u_1)}} \int_0^{u_1} \frac{du_2}{\sqrt{u_2(1-u_2)}} \cdots \int_0^{u_{N-1}} \frac{du_N}{\sqrt{u_N(1-u_N)}} \\ \times \int_0^1 d\beta_1 \cdots \int_0^1 d\beta_N \frac{\exp^{-x\sum_i(-\Delta_{ii})(\frac{1}{\beta_i^2}-1)}}{\det^{\frac{3}{2}}(\mathbb{1} - \mathrm{R_N})}\end{aligned} \tag{7.12}$$

For $\mu = 0$, case functions $\bar{c}_N(x)$ reduce to numbers i.e.

$$\bar{c}_N(0) =: \bar{c}_N \tag{7.13}$$

The first coefficient is

$$\bar{c}_1 = \int_0^1 \frac{du_1}{\sqrt{u_1(1-u_1)}} = \pi \tag{7.14}$$

For $N > 1$, inspection of the determinant $\det(\mathbb{1} - \mathrm{R_N})$ shows that it simplifies considerably if, instead of $u_1, \ldots, u_N$ one writes it in terms of new variables $z_2, \ldots, z_N$ defined by

$$z_i := \sqrt{\frac{u_i(1-u_{i-1})}{u_{i-1}(1-u_i)}} \tag{7.15}$$

Changing variables from u_i to z_i for $i = 2,\ldots,N$, we obtain

$$\bar{c}_N = 2^{N-1} \int_0^1 dz_2 \int_0^1 dz_3 \cdots \int dz_N \mathcal{M}_N \int_0^1 d\beta_1 \cdots \int_0^1 d\beta_N \frac{1}{\det^{\frac{3}{2}}(\mathbb{1} - \mathrm{R_N})} \tag{7.16}$$

where R is now written as a function of $\beta_1, \ldots, \beta_N, z_2, \ldots, z_N$ and $\mathcal{M}_N$ is a function of $z_2 \ldots, z_N$ defined as

$$\mathcal{M}_N := \frac{1}{z_2 z_3 \cdots z_N} \int_0^1 du_1 \sqrt{\frac{u_2(1-u_2)u_3(1-u_3)\cdots u_N(1-u_N)}{u_1(1-u_1)}} \tag{7.17}$$

It is understood that here first $u_2, \ldots, u_N$ are, backwards starting from u_N, transformed to $z_2, \ldots, z_N$ via

$$u_i = \frac{u_{i-1} z_i^2}{1 - u_{i-1}(1 - z_i^2)} \tag{7.18}$$

$(i \geq 2)$ and then one performs the u_1 integral. For $N = 2, 3$, one finds

$$\mathcal{M}_2 = \frac{2 \log z_2}{z_2^2 - 1} \tag{7.19}$$

$$\mathcal{M}_3 = \frac{\pi}{(z_2+1)(z_3+1)(z_2 z_3 + 1)} \tag{7.20}$$

Now we write an expression for R_2

$$\mathcal{R}_2 = \begin{bmatrix} 0 & (u_1 u_2 + \frac{|u_1 - u_2|}{2} - \frac{u_1+u_2}{2})\frac{\lambda_2}{u_2(1-u_2)} \\ (u_1 u_2 + \frac{|u_1 - u_2|}{2} - \frac{u_1+u_2}{2})\frac{\lambda_1}{u_1(1-u_1)} & 0 \end{bmatrix} \tag{7.21}$$

A few main terms for $\mathcal{M_N}$ and $\mathcal{R_N}$ are given in appendix B.

After this transformation, the integral for the second coefficient can be done in closed form:

$$\bar{c}_2 = 2 \int_0^1 dz_2 \mathcal{M}_2 \int_0^1 d\beta_1 \int_0^1 d\beta_2 \frac{1}{(1 - \beta_1^2 \beta_2^2 z_2^2)^{\frac{3}{2}}} = \frac{\pi^3}{6} \tag{7.22}$$

Some more coefficients we have obtained by numerical integration:

Tab. 7.1: The coefficients $\bar{c}_n$.

n	1	2	3	4	5	6	7	8	9	10	11
$\bar{c}_n$	π	$\frac{\pi^3}{6}$	5.9319	5.3402	4.0192	2.6243	1.5349	0.8044	0.378	0.175	0.0761

CHAPTER 8

EXPANDING AND MATCHING

Let us now ask what the asymptotic behavior of the coefficients $\bar{c}_n$ *should* be to get the expected correction to the ground state mass. For $\mu = 0$ in the nonrelativistic limit, the exact bound state energy would be, (8.1), (8.2)

$$E_b = \frac{1}{4} m\alpha^2 \tag{8.1}$$

where

$$\alpha = \frac{\lambda^2}{16\pi m^2} = \pi g \tag{8.2}$$

This corresponds to a large $\hat{t}$ exponential factor

$$\mathrm{e}^{-Et} = \mathrm{e}^{-(2m-E_b)t} = \mathrm{e}^{-(2m-E_b)2\hat{t}/m} = \mathrm{e}^{(-4+\frac{1}{2}\pi^2 g^2)\hat{t}} \tag{8.3}$$

This should become the exact answer for small g. Now, we know that the trivial exponent $-4\hat{t}$ corresponds to a saddle point at $S = T = 1$; thus at least for small g it should be a good approximation to set $S = T = 1$ also in the prefactor $\left[\frac{ST}{(S+T)^{1/2}}\right]^N$. This leaves us with the series

$$\sum_N \frac{c_N}{N!}\left(\frac{\pi\hat{t}}{2}\right)^{N/2} g^N = \sum_N \bar{c}_N \left(2\pi\hat{t}\right)^{N/2} g^N \quad \stackrel{!}{=} \quad \mathrm{e}^{\frac{1}{2}\pi^2 g^2 \hat{t}} \tag{8.4}$$

From the Taylor series

$$\sum_{n=0}^{\infty} \frac{x^n}{\Gamma(1+n/2)} = (1+\mathrm{Erf}(x))\,\mathrm{e}^{x^2} \stackrel{x\to\infty}{\sim} 2\,\mathrm{e}^{x^2} \tag{8.5}$$

we then conclude that the $\bar{c}_N$ should have the asymptotic behavior

$$\bar{c}_N \stackrel{N\to\infty}{\sim} \frac{c_\infty \beta^N}{\Gamma(1+N/2)} \tag{8.6}$$

which would lead to an exponential

$$e^{2\pi\beta^2 g^2 \hat{t}} \tag{8.7}$$

Comparison with (8.4) yields

$$\beta \overset{!}{=} \frac{\sqrt{\pi}}{2} = 0.886 \tag{8.8}$$

To compare with our numerical results for the $\bar{c}_N$, we note that from (8.6) it follows that the sequence

$$\beta_N := \frac{\bar{c}_{N+1}\Gamma\left(1+\frac{N+1}{2}\right)}{\bar{c}_N\Gamma\left(1+\frac{N}{2}\right)} \tag{8.9}$$

should converge to β for $N \to \infty$. From table 7.1 we find

Tab. 8.1: The coefficients β_n.

n	1	2	3	4	5	6	7	8	9	10
β_n	1.856	1.525	1.355	1.251	1.179	1.134	1.081	1.035	1.061	1.043

In fig. 8.1 we graph the coefficients β_n together with the supposed asymptotic limit β.

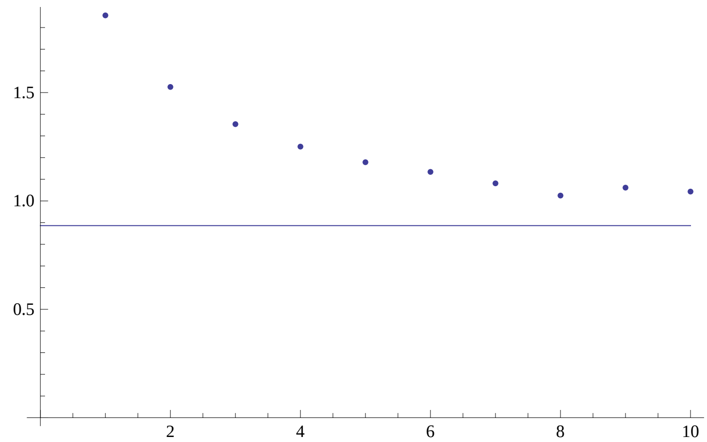

Fig. 8.1: The coefficients β_n

The graph clearly suggests that, if there is convergence at all, it will be to a higher value than β.

To understand what this means, let us now return to the coefficients $\bar{c}_n$ of table 7.1, and plot the combination

$$\tilde{c}_N := \Gamma\left(1+\frac{N}{2}\right)\frac{\bar{c}_N}{\beta^N} \tag{8.10}$$

If (8.6) were true, the coefficients would converge to the constant c_∞; instead we find (see fig. 8.2) a curve which looks parabolic.

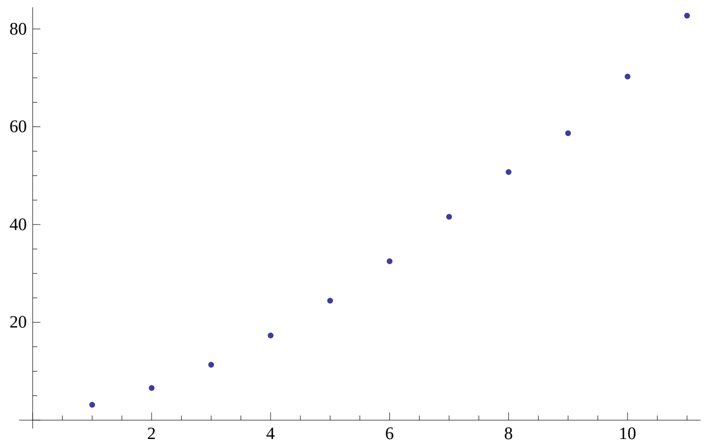

Fig. 8.2: The coefficients $\tilde{c}_n$

Therefore, let us look at yet another set of coefficients c'_N,

$$c'_N := \frac{\tilde{c}_N}{N^2} \tag{8.11}$$

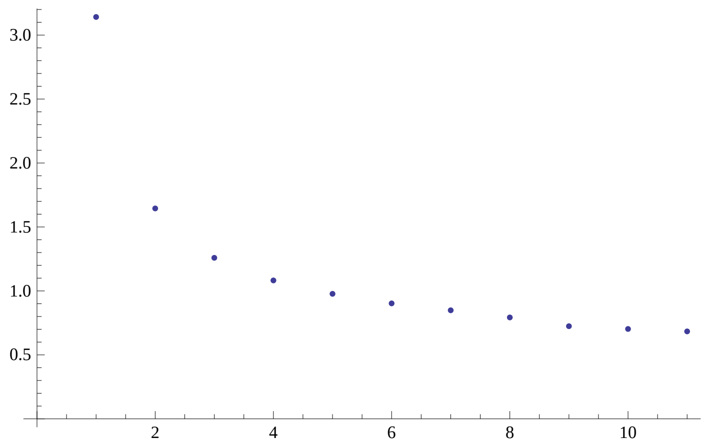

Fig. 8.3: The coefficients c'_n

These modified coefficients indeed seem to converge to a constant (see fig. 8.3); let us call this constant c'_∞. Thus we now have, instead of (8.6), the asymptotic behaviour

$$\bar{c}_N \overset{N\to\infty}{\sim} \frac{c'_\infty N^2 \beta^N}{\Gamma(1+N/2)} \tag{8.12}$$

Fortunately, this does not change anything essential: instead of (8.5) we now have

$$\sum_{n=0}^{\infty} n^2 \frac{x^n}{\Gamma(1+n/2)} \overset{x\to\infty}{\sim} 8x^4\, \mathrm{e}^{x^2} \tag{8.13}$$

So, there is no modification of the exponent, only of the prefactor, which does not interest us right now[1]

[1](however, it is curious to note that this change of the prefactor *precisely removes the* $1/\hat{t}^2$ *in the master formula* (7.4))

Using eq. (8.12) , we obtain, in analogy to eq.(8.9), an expression

$$\beta'_N = \frac{(N+1)^2}{N^2}\beta_N = \frac{\bar{c}_{N+1}N^2\Gamma\big(1+\frac{N+1}{2}\big)}{\bar{c}_N(N+1)^2\Gamma\big(1+\frac{N}{2}\big)} \tag{8.14}$$

Should converge to β for $N \to \infty$, where $\beta = 0.886$ from eq. (8.8)

Tab. 8.2: The coefficients β'_n.

n	1	2	3	4	5	6	7	8	9	10
β'_n	0.464	0.678	0.762	0.801	0.818	0.833	0.828	0.818	0.859	0.861

This table forβ'_N is generated by substituting values for β_N in eq. (8.14).

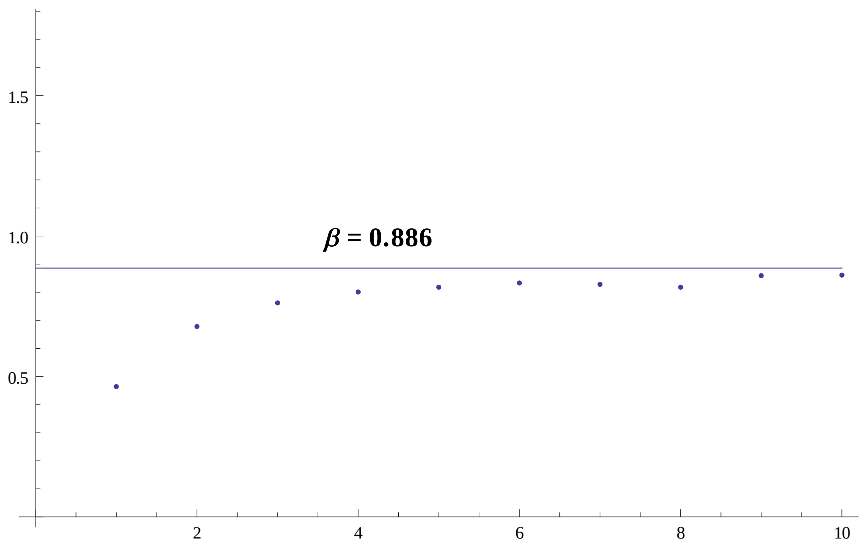

Fig. 8.4: The coefficients β'_N.

Fig. 8.4 the coefficient β'_n is plotted for various values of N. The graph shows β'_N converging to value β for $N \to \infty$.

Assuming that (8.6) with (8.8) are true, let us now undo the assumptions of small g and of the saddle point at $S = T = 1$ and return to (7.4). The asymptotic summation formula (8.5) with (8.8) now leads to a total exponential factor

$$\exp\left[-\hat{t}\Big(S + T + \frac{1}{S} + \frac{1}{T} - \pi^2 g^2 \frac{S^2T^2}{S+T}\Big)\right] \tag{8.15}$$

As long as $g^2 < 1/3\pi^2$, one finds a saddle point (local maximum) of the exponent at

$$S = T = \sqrt{\frac{2}{3}} \frac{1}{\pi g} \sqrt{1 - \sqrt{1 - 3\pi g}} \tag{8.16}$$

with saddle point value

$$\exp\left\{ -\hat{t} \frac{4\sqrt{2}}{3} \left[\left(1 + \sqrt{1 - 3\pi^2 g^2} \right)^{-1/2} + \left(1 + \sqrt{1 - 3\pi^2 g^2} \right)^{1/2} \right] \right\} \tag{8.17}$$

From (6.3), (6.9) this gives for the lowest bound state mass m_0

$$\frac{m_0}{m} = \frac{2\sqrt{2}}{3} \left[\left(1 + \sqrt{1 - 3\pi^2 g^2} \right)^{-1/2} + \left(1 + \sqrt{1 - 3\pi^2 g^2} \right)^{1/2} \right] \tag{8.18}$$

As g^2 increases from zero to its maximal value $1/3\pi^2$, the result (8.18) for this mass m_0 decreases monotonically from $2m$ to $\frac{4\sqrt{2}}{3} m = 1.886m$. An expansion of (8.18) in g yields

$$\exp\left[-\hat{t} \left(4 - \frac{\pi^2 g^2}{2} - \frac{9}{32} (\pi^2 g^2)^2 - \frac{81}{256} (\pi^2 g^2)^3 - \dots \right) \right] \tag{8.19}$$

Conjectured formula for the lowest bound state mass,

$$\frac{m_0}{m} = 2 - \frac{\pi^2 g^2}{4} - \frac{9}{64} (\pi^2 g^2)^2 - \frac{81}{512} (\pi^2 g^2)^3 - \dots \tag{8.20}$$

In the second term of the expansion we find again, of course, the nonrelativistic limit (8.1) of the binding energy, which we have already used as an input for our matching procedure; but the order g^4 term is already new. It is remarkable, that in the expansion (8.20) of the bound state mass in powers of g no term of the order $g^3 \ln g$ appears, as it would be the case for the corresponding result in the Wick-Cutkosky model, *i.e.*, for the ladder approximation of the Bethe-Salpeter equation in the same model theory [43]. As we have mentioned before in the introduction, such a contribution is generally considered to be unphysical.

Our result for the mass of the lowest bound state may be compared to the result of the relativistic eikonal approximation or Todorov's equation [33],[34], in our notation

$$\frac{m_0}{m} = \sqrt{2} \left(1 + \sqrt{1 - \pi^2 g^2} \right)^{1/2}$$

$$\frac{m_0}{m} = 2 - \frac{\pi^2 g^2}{4} - \frac{5}{64} (\pi^2 g^2)^2 - \dots \tag{8.21}$$

In terms of diagrams, the eikonal approximation sums up all ladder and crossed ladder diagrams, but neglects any self-energy contributions and vertex corrections just as in our approach. To reproduce the contributions of the ladder and crossed ladder diagrams correctly up to the order g^4 ([35]). The coefficients of the g^4- term in the expansion of (8.21) of the bound state mass in powers of the coupling constant is somewhat smaller (in absolute value) than in our approximation, but it has the same sign.

Finally, we compare the maximal value of the coupling constant, $g^2 = \frac{1}{3\pi^2}$, to the critical value of the variational worldline approximation ([35]). The latter value is (approximately) $\alpha = 0.814$ (without self- energy and vertex corrections, for a massless exchanged particle), somewhat larger than our value $\alpha = \pi g = \frac{1}{\sqrt{3}} = 0.577$. The existence of a critical coupling constant isattributed to the instability of the vacuum in a scalar field theory ([35]).

Chapter 9

N half ladder in x-space

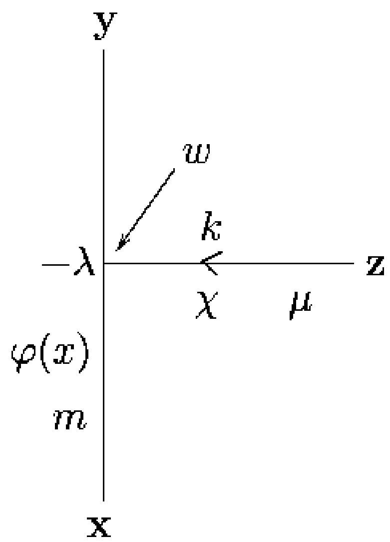

Fig. 9.1: Three point function.

Here we study the N - ladder in x - space, that is, a scalar propagator from y to x with N propagators attached, connecting it to points $z_1, \ldots, z_N$. In fig. 9.1 we show the case $N = 1$, where x, y and z are the three points in x-space, λ the order of interaction, φ is the field associated with the propagator between x and y with mass m and χ is the field associated with the connecting propagator between the vertex w and z with mass μ.
The starting point is the worldline representation of the dressed propagator,

$$\begin{aligned} \langle 0|T\phi_q(x)\phi_q(y)|0\rangle &= \frac{1}{-\Box + m^2 + \lambda\phi(x)} \\ &= \int_0^\infty dT \int_{x(0)=y}^{x(T)=x} \mathcal{D}x \; e^{-\int_0^T d\tau\,\left[\frac{1}{4}\dot{x}^2 + m^2 + \lambda\phi(x)\right]} \end{aligned} \tag{9.1}$$

As in chapter 3, we will generally use the shift

$$x^\mu(\tau) = \left[y^\mu + (x-y)^\mu \frac{\tau}{T}\right] + q^\mu(\tau) \tag{9.2}$$

where the first term on the rhs is the equation of a straight line and $q^\mu(\tau)$ represents the quantum fluctuation. The path integral over $q^\mu(\tau)$ will then be calculated using the worldline propagator Δ_T,

$$\begin{aligned}
&\langle z^\mu(\tau) z^\nu(\sigma)\rangle = -2\delta^{\mu\nu}\Delta_T(\tau,\sigma) \\
&\Delta_T(\tau,\sigma) = \frac{\tau\sigma}{T} - \tau\theta(\sigma-\tau) - \sigma\theta(\tau-\sigma) = \frac{\tau\sigma}{T} + \frac{|\tau-\sigma|}{2} - \frac{\tau+\sigma}{2} \\
&\Delta_T(\tau,\tau) = \frac{\tau^2}{T} - \tau\ .
\end{aligned} \tag{9.3}$$

From the propagator with one interaction 3.26 we can get the three-point function replacing $e^{ik\cdot x(\tau)}$ by

$$\int d^4k \frac{e^{ik\cdot(x(\tau)-z)}}{k^2+\mu^2} \tag{9.4}$$

where $x(\tau)$ is the interaction point and z is a fixed point, as shown in the 9.1 Now, making use of the Schwinger parameter α to exponentiate the connecting propagator,

$$\frac{1}{k^2+\mu^2} \;=\; \int_0^\infty d\alpha\, \mathrm{e}^{-\alpha(k^2+\mu^2)} \tag{9.5}$$

we have

$$\int d^4k \frac{e^{-ik\cdot z}}{k^2+\mu^2} \;=\; \int d^4k \int_0^\infty d\alpha\, \mathrm{e}^{-(ik\cdot z+\alpha(k^2+\mu^2))} \tag{9.6}$$

This leads to

$$\begin{aligned}
\Gamma(x,y,z,m,\mu) \;=\; & -\lambda \int_0^\infty \frac{dT}{(4\pi T)^{\frac{D}{2}}}\, e^{-\frac{(x-y)^2}{4T} - m^2 T} \int \frac{d^D k}{(2\pi)^D} \int_0^\infty d\alpha\, \mathrm{e}^{-(ik\cdot z+\alpha(k^2+\mu^2))} \\
& \times \int_0^T d\tau\, e^{ik\cdot y} e^{ik\cdot(x-y)\frac{\tau}{T}}\, e^{-k^2(\tau-\frac{\tau^2}{T})}
\end{aligned} \tag{9.7}$$

Performing the gaussian k-integral and rescaling $\tau = Tu$ as well as $\alpha = T\hat{\alpha}$, we obtain

$$\begin{aligned}
\Gamma(x,y,z,m,\mu) \;=\; & -\frac{\lambda}{(4\pi)^D} \int_0^\infty \frac{dT}{T^{D-2}}\, e^{-\frac{(x-y)^2}{4T} - m^2 T} \int_0^\infty d\hat{\alpha}\, e^{-\hat{\alpha}\mu^2 T} \\
& \times \int_0^1 du \frac{1}{[\hat{\alpha}+u(1-u)]^{\frac{D}{2}}}\, e^{\frac{-(y-z+(x-y)u)^2}{4T(\hat{\alpha}+u(1-u))}}
\end{aligned} \tag{9.8}$$

Now, we specialize to the massless case, $m = \mu = 0$. The T - integral then becomes elementary, and one gets

$$\Gamma(x,y,z,0,0) = -\frac{\lambda}{(4\pi)^D}\Gamma(D-3)\int_0^1 du \int_0^\infty d\hat{\alpha} \times \frac{1}{[\hat{\alpha}+u(1-u)]^{\frac{D}{2}}} \frac{4^{D-3}}{\Big[(x-y)^2+\frac{[y-z+(x-y)u]^2}{\hat{\alpha}+u(1-u)}\Big]^{D-3}}$$

Further simplification is possible if we now also assume $D = 4$. This makes the $\hat{\alpha}$ - integral elementary, and results in

$$\Gamma(x,y,z,0,0) = -\frac{\lambda}{64\pi^4}\int_0^1 du \frac{1}{uc+(1-u)b-u(1-u)a}\log\Big[\frac{uc+(1-u)b}{u(1-u)a}\Big] \tag{9.9}$$

where we have now abbreviated

$$(x-y)^2 = a \qquad (y-z)^2 = b \qquad (x-z)^2 = c \tag{9.10}$$

The u - integral can be reduced to the standard integral

$$\int du \frac{\ln(Au+B)}{u-C} = \ln(Au+B)\ln\Big(1-\frac{Au+B}{AC+B}\Big) + \mathrm{Li}_2\Big(\frac{Au+B}{AC+B}\Big) \tag{9.11}$$

The final result is then easy to identify with the well-known representation of the massless triangle function due to Ussyukina and Davydychev [44],

$$\Gamma(x,y,z,0,0) = 4\pi^4\lambda\frac{1}{a}\Phi^{(1)}\Big(\frac{b}{a},\frac{c}{a}\Big) \tag{9.12}$$

where

$$\Phi^{(1)}(x,y) \equiv \frac{1}{\Lambda}\Big\{2\Big(\mathrm{Li}_2(-\rho x)+\mathrm{Li}_2(-\rho y)\Big) + \ln\frac{y}{x}\ln\frac{1+\rho y}{1+\rho x} + \ln(\rho x)\ln(\rho y) + \frac{\pi^2}{3}\Big\} \tag{9.13}$$

with

$$\begin{aligned} \Lambda &\equiv \sqrt{(1-x-y)^2-4xy}, \\ \rho &\equiv 2(1-x-y+\Lambda)^{-1}. \end{aligned} \tag{9.14}$$

After this warm-up, we proceed to the much more challenging $N = 2$ case. Eq. (9.7) generalizes straightforwardly to

We proceed to the two-rung case. We now have to replace

$$e^{ik_i x_i(\tau_i)} \longrightarrow \frac{e^{ik_i(x(\tau_i)-z_i)}}{k_i^2+\mu^2} \tag{9.15}$$

for $i = 1, 2$. Using again

$$\frac{1}{k_i^2+\mu^2} = \int_0^\infty d\alpha_i e^{-\alpha_i k_i^2+\mu^2} \tag{9.16}$$

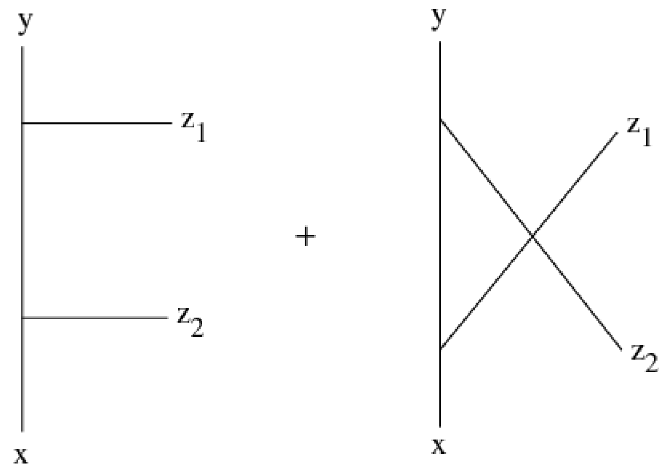

Fig. 9.2: Four-point function

.

$$\begin{aligned}\Gamma(x,y,z_1,z_2,m,\mu) &= (-\lambda)^2 \int_0^\infty \frac{dT\,T^2}{(4\pi T)^{D/2}} e^{-\frac{(x-y)^2}{4T}-m^2T} \int \frac{d^Dk_1}{(2\pi)^D} \int \frac{d^Dk_2}{(2\pi)^D}\, e^{-i(k_1\cdot z_1+k_2\cdot z_2)} \\ &\quad \times \int_0^\infty d\alpha_1 e^{-\alpha_1(k_1^2+\mu^2)} \int_0^\infty d\alpha_2\, e^{-\alpha_2(k_2^2+\mu^2)} \int_0^1 du_1 \int_0^1 du_2 \\ &\quad \times e^{ik_1\cdot[y+(x-y)u_1]} e^{ik_2\cdot[y+(x-y)u_2]} e^{T\left[k_1^2\triangle_1(u_1,u_1)+k_2^2\triangle_1(u_2,u_2)+2k_1\cdot k_2\triangle_1(u_1,u_2)\right]}\end{aligned} \tag{9.17}$$

Here we have already rescaled $\tau_i = Tu_i$, $i = 1, 2$. The ordered sector $u_1 < u_2$ of this integral corresponds to the first diagram shown in 9.1 (for $N = 2$), the sector $u_1 > u_2$ to the second one.

As before, we first do the gaussian $k_{1,2}$ - integrals, and obtain (in the following we abbreviate $\triangle_1(u_i, u_j)$ by $\triangle_{ij}$)

$$\Gamma(x,y,z_1,z_2,m,\mu) = \frac{\lambda^2}{(4\pi)^D}\int_0^\infty \frac{dT\,T^2}{(4\pi T)^{D/2}} e^{-\frac{(x-y)^2}{4T}-m^2T}\int_0^1 du_1 du_2 \int_0^\infty d\alpha_1 d\alpha_2 e^{-(\alpha_1+\alpha_2)\mu^2}$$
$$\times \frac{\exp\Bigl\{-\frac{(\alpha_1-T\triangle_{11})\beta_2^2+(\alpha_2-T\triangle_{22})\beta_1^2+2T\triangle_{12}\beta_1\cdot\beta_2}{4\bigl[(\alpha_1-T\triangle_{11})(\alpha_2-T\triangle_{22})-T^2\triangle_{12}^2\bigr]}\Bigr\}}{\Bigl[(\alpha_1-T\triangle_{11})(\alpha_2-T\triangle_{22})-T^2\triangle_{12}^2\Bigr]^{\frac{D}{2}}} \tag{9.18}$$

where we have defined

$$\beta_i := y - z_i + u_i(x-y) \tag{9.19}$$

Specializing to the massless case $m=\mu=0$, and changing from α_i to $\hat{\alpha}_i$ via

$$\alpha_i = T(\hat{\alpha}_i + \triangle_{ii}), \quad i=1,2, \tag{9.20}$$

we can do the T - integral. This leads to

$$\Gamma(x,y,z_1,z_2,0,0) = \frac{\lambda^2}{(4\pi)^{\frac{3}{2}D}}\Gamma\Bigl(1+\frac{3}{2}(D-4)\Bigr)\int_0^1 du_1 du_2\int_{-\triangle_{11}}^\infty d\hat{\alpha}_1\int_{-\triangle_{22}}^\infty d\hat{\alpha}_2$$
$$\times\frac{1}{\Bigl[\hat{\alpha}_1\hat{\alpha}_2-\Delta_{12}^2\Bigr]^{\frac{D}{2}}}\left[\frac{4}{(x-y)^2+\frac{\hat{\alpha}_1\beta_2^2+\hat{\alpha}_2\beta_1^2+2\Delta_{12}\beta_1\cdot\beta_2}{\hat{\alpha}_1\hat{\alpha}_2-\Delta_{12}^2}}\right]^{1+\frac{3}{2}(D-4)} \tag{9.21}$$

Setting $D=4$, this becomes

$$\Gamma(x,y,z_1,z_2,0,0) = \frac{4\lambda^2}{(4\pi)^6}\int_0^1 du_1 du_2\int_{-\triangle_{11}}^\infty d\hat{\alpha}_1\int_{-\triangle_{22}}^\infty d\hat{\alpha}_2$$
$$\times\frac{1}{\Bigl[\hat{\alpha}_1\hat{\alpha}_2-\Delta_{12}^2\Bigr]\Bigl[(x-y)^2(\hat{\alpha}_1\hat{\alpha}_2-\Delta_{12}^2)+\hat{\alpha}_1\beta_2^2+\hat{\alpha}_2\beta_1^2+2\Delta_{12}\beta_1\cdot\beta_2\Bigr]} \tag{9.22}$$

Performing the $\hat{\alpha}_1$ - integral, which is elementary, we find

$$\Gamma(x,y,z_1,z_2,0,0) = \frac{4\lambda^2}{(4\pi)^6}\int_0^1 du_1 du_2\int_{-\triangle_{22}}^\infty d\hat{\alpha}_2$$
$$\times\frac{\ln\Bigl\{\frac{\hat{\alpha}_2\bigl[\hat{\alpha}_2(\beta_1^2-\Delta_{11}(x-y)^2)+2\Delta_{12}\beta_1\cdot\beta_2-\Delta_{11}\beta_2^2-\Delta_{12}^2(x-y)^2\bigr]}{(\hat{\alpha}_2(-\Delta_{11})-\Delta_{12}^2)(\hat{\alpha}_2(x-y)^2+\beta_2^2)}\Bigr\}}{(\hat{\alpha}_2\beta_1+\Delta_{12}\beta_2)^2} \tag{9.23}$$

The $\hat{\alpha}_2$ - integral is still a straightforward one. Introducing the zeroes $\hat{\alpha}_\pm$ of the quadratic form in the denominator,

$$\hat{\alpha}_\pm := -\frac{\Delta_{12}}{\beta_1^2}\left[\beta_1\cdot\beta_2 \pm i\sqrt{\beta_1^2\beta_2^2-(\beta_1\cdot\beta_2)^2}\right] \tag{9.24}$$

we can write the result as

$$\begin{aligned}\Gamma(x,y,z_1,z_2,0,0) &= \frac{4\lambda^2}{(4\pi)^6}\int_0^1 du_1du_2\frac{1}{(\hat{\alpha}_+-\hat{\alpha}_-)}\left[\ln\left(\frac{-\Delta_{11}a+\beta_1^2}{-\Delta_{11}a}\right)\ln\left(\frac{-\Delta_{22}-\hat{\alpha}_-}{-\Delta_{22}-\hat{\alpha}_+}\right)\right.\\ &\left.+I(0)+I\left(\frac{2\Delta_{12}\beta_1\cdot\beta_2-\Delta_{11}\beta_2^2-\Delta_{12}^2a}{\beta_1^2-\Delta_{11}a}\right)-I\left(\frac{\Delta_{12}^2}{\Delta_{11}}\right)-I\left(\frac{\beta_2^2}{a}\right)\right]\end{aligned} \tag{9.25}$$

where

$$\begin{aligned}I(A) &:= (\hat{\alpha}_+-\hat{\alpha}_-)\int_{-\triangle_{22}}^{\infty}d\hat{\alpha}_2\frac{\ln(\hat{\alpha}_2+A)}{(\hat{\alpha}_2-\hat{\alpha}_+)(\hat{\alpha}_2-\hat{\alpha}_-)}\\ &= \left\{\mathrm{Li}_2\left(\frac{A-\Delta_{22}}{A+\hat{\alpha}_-}\right)+\ln(A-\Delta_{22})\ln\left(\frac{\hat{\alpha}_-+\Delta_{22}}{\hat{\alpha}_-+A}\right)\right.\\ &\left.+\frac{1}{2}\ln^2\left(-\frac{1}{A+\hat{\alpha}_-}\right)\right\}-(\hat{\alpha}_-\to\hat{\alpha}_+)\end{aligned} \tag{9.26}$$

and we have abbreviated $a \equiv (x-y)^2$ as before. To rewrite the new integrand completely in terms of the external Lorentz invariants, we further introduce

$$\begin{aligned}b_i &:= (x-z_i)^2\\ c_i &:= (y-z_i)^2\\ d &:= (z_1-z_2)^2\end{aligned} \tag{9.27}$$

In terms of those variables,

$$\begin{aligned}\beta_i^2 &= u_ib_i+(1-u_i)c_i-u_i(1-u_i)a\\ 2\beta_1\cdot\beta_2 &= (2u_1u_2-u_1-u_2)a+u_2b_1+u_1b_2+(1-u_2)c_1+(1-u_1)c_2-d\end{aligned} \tag{9.28}$$

Although we are not able to perform the remaining two integrals analytically, the representation (9.25) is the most explicit representation available for this integral which plays an important role in SYM theory [45, 46, 47].

For the general N-rung case, the formulas (9.7), (9.17) generalize immediately to

$$
\begin{aligned}
\Gamma(x,y,z_1,z_2,\ldots,z_N) &= (-\lambda)^N \int_0^\infty \frac{dT\,T^N}{(4\pi T)^{D/2}} e^{-\frac{(x-y)^2}{4T}-m^2T} \int \frac{d^Dk_1}{(2\pi)^D}\cdots\frac{d^Dk_N}{(2\pi)^D}\, e^{-i\sum_{i=1}k_i\cdot z_i} \\
&\times \int d\alpha_1\cdots d\alpha_N e^{-\sum_{i=1}^N \alpha_i(k_i^2+\mu^2)} \int du_1\ldots du_N e^{i\sum_{i=1}^N k_i\cdot(y+(x-y)u_i)} \\
&\times \exp\Bigl[T\sum_{i,j=1}^N \Delta_{ij} k_i\cdot k_j\Bigr] \qquad (9.29)
\end{aligned}
$$

Chapter 10

Conclusions

To summarize, in this thesis we have used the worldline formalism to derive integral expression for three classes of amplitudes - the $N-$ propagators, $N-$ half ladders - in scalar field theory involving an exchange of N momenta, and in each case have given a compact expression combining the $N!$ Feynman diagrams contributing to the amplitude. For the $N-$ propagators and $N-$ ladders we have given these representations in both x and (off-shell) momentum space, for the $N-$ half-ladders in $x-$ space only. These amplitudes are not only of interest in their own right, but being off-shell, can also be used as building blocks for many more complex amplitudes.

For the N half ladders in $x-$space, eq. (9.9) is a new integral representation for the famous massless triangle function ([44]). The four-point integral corresponding to $N = 2$ eq. (9.25) figures prominently in $N = 4$ SYM theory [45, 46, 47, 48] but is presently still not known in closed form. We have achieved a new two-parameter integral representation for this integral, which is not only more explicit than other known representations, but also promising as a starting point for a closed form calculation.

We have derived a compact expression for the sum of all ladder graphs with N rungs, including all possible crossings of the rungs, and we have used this to extract an approximate formula for the mass of the lowest-lying bound state, explicitly for the case of a massless particle exchange between the constituents. Technically, we apply a saddle point approximation to our formula for the N-rung ladders, after summing over all N.

In our approach the truncation to the non-crossed ladder graphs is induced naturally by Gaussian approximation $\overrightarrow{v} = \overleftarrow{u}$, rather than done ad hoc from the beginning. Our final result (8.18) for the mass of the lowest bound state does not display any obvious inconsistencies.

In particular, in the expansion (8.20) of the bound state mass in powers of g no term of the order $g^3 \ln g$ appears, as in the case for the corresponding result in the Wick-Cutkosky model, *i.e.*, for the ladder approximation of the Bethe-Salpeter equation in the same model theory [43]. Such contribution is considered unphysical.

Equation (8.18) is similar to the result of the relativistic eikonal approximation ([33],[34]). In terms of diagrams, the eikonal approximation sums up all ladder and crossed ladder diagrams, but neglects any self-energy contributions and vertex corrections just as in our approach. The resulting coefficients of the g^4- term in the expansion of (8.21) of the bound state mass in powers of the coupling constant is somewhat smaller (in absolute value) than in our approximation, but it has the same sign.

Finally, we compare the maximal value of the coupling constant, $g^2 = \frac{1}{3\pi^2}$, to the critical value of the variational worldline approximation ([35]). The latter value is (approximately) $\alpha = 0.814$ (without self- energy and vertex corrections, for a massless exchanged particle), somewhat larger than our value $\alpha = \pi g = \frac{1}{\sqrt{3}} = 0.577$. The existence of a critical coupling constant is attributed to the instability of the vacuum in a scalar field theory ([35]).

It would be straightforward to extend our various master formulas to the case of scalar QED (i.e. scalar lines and photon exchanges). In the spinor QED case (fermion lines and photon exchanges) closed-form expression for general N could still be achieved using the worldline super-formalism ([9]), however at the cost of introducing additional multiple Grassman integrals. For an eventual extension to the nonabelian case it may turn out essential to work with a path integral representation of the color degrees of freedom, such as the one given in ([49]), rather than with explicit color factors. Finally even a closed-form treatment of ladder graphs involving the exchange of gravitons between the scalars or spinors - a completely hopless task in the Feynman diagram approach due to the existence of vertices involving an arbitrary number of gravitons - may be feasible in the worldline formalism along the lines of ([28, 29]).

APPENDIX A

THE VARIABLES c_N, $\mathcal{M}_\mathcal{N}$ AND $\mathcal{R}_\mathcal{N}$

In the representation for the full Green's function coefficient c_N is given by

$$c_N(x) = \int_0^1 du_1 \cdots \int_0^1 du_N \int_0^\infty d\alpha_1 \cdot\cdot \int_0^\infty d\alpha_N \, \frac{\mathrm{e}^{-x\sum_i \alpha_i}}{(\det \tilde{M}_N)^{3/2}} \tag{A.1}$$

$$\tilde{M}_{Nij} \;\; = \;\; \delta_{ij}\alpha_i - \Delta_1(u_i, u_j) \tag{A.2}$$

$$\bar{c}_N = 2^{N-1} \int_0^1 dz_2 \int_0^1 dz_3 \cdots \int dz_N \mathcal{M}_N \int_0^1 d\beta_1 \cdots \int_0^1 d\beta_N \frac{1}{\det^{\frac{3}{2}}(\mathbb{1} - \mathrm{R_N})} \tag{A.3}$$

Here R_N given by eq.(7.7), (7.8) and (7.9) see chapter 7. $\mathcal{M}_\mathcal{N}$ is a function of $z_2, \ldots, z_N$ defined as

$$\mathcal{M}_N := \frac{1}{z_2 z_3 \cdots z_N} \int_0^1 du_1 \sqrt{\frac{u_2(1-u_2)u_3(1-u_3)\cdots u_N(1-u_N)}{u_1(1-u_1)}} \tag{A.4}$$

where $z_2, , z_N$ are defined in terms of $u_1, \ldots, u_N$ as

$$z_i := \sqrt{\frac{u_i(1-u_{i-1})}{u_{i-1}(1-u_i)}} \tag{A.5}$$

$u_2, \ldots, u_N$ are, backwards starting from u_N, transformed to $z_2, \ldots, z_N$ via

$$u_i = \frac{u_{i-1} z_i^2}{1 - u_{i-1}(1 - z_i^2)} \tag{A.6}$$

for $(i \geq 2)$ one performs the u_1 integral. One finds $\mathcal{M}_\mathcal{N}$ for $N = 2, 3,$

$$\mathcal{M}_2 = \frac{2 \log z_2}{z_2^2 - 1} \tag{A.7}$$

$$\mathcal{M}_3 = \frac{\pi}{(z_2+1)(z_3+1)(z_2 z_3 + 1)} \tag{A.8}$$

$$\mathcal{M}_4 = \frac{(2z_3(-z_2^2(-1+z_3^2)(-1+z_4^2)(-1+z_3^2z_4^2)\log[z_2]}{((-1+z_2^2)(-1+z_3^2)(-1+z_2^2z_3^2)(-1+z_4^2)(-1+z_3^2z_4^2)(-1+z_2^2z_3^2z_4^2))} + \frac{(-1+z_2^2)((-1+z_4^2)(-1+z_2^2z_3^4z_4^2)\log[z_3]-(-1+z_3^2)(-1+z_2^2z_3^2)z_4^2\log[z_4])))}{((-1+z_2^2)(-1+z_3^2)(-1+z_2^2z_3^2)(-1+z_4^2)(-1+z_3^2z_4^2)(-1+z_2^2z_3^2z_4^2))} \tag{A.9}$$

Similarly the expression for $\mathcal{R}_2$, $\mathcal{R}_3$ and $\mathcal{R}_4$ can be written as

$$\mathcal{R}_2 = \begin{bmatrix} 0 & (u_1u_2 + \frac{|u_1-u_2|}{2} - \frac{u_1+u_2}{2})\frac{\lambda_2}{u_2(1-u_2)} \\ (u_1u_2 + \frac{|u_1-u_2|}{2} - \frac{u_1+u_2}{2})\frac{\lambda_1}{u_1(1-u_1)} & 0 \end{bmatrix} \tag{A.10}$$

$$\mathcal{R}_3 = \begin{bmatrix} 0 & (u_1u_2 + \frac{|u_1-u_2|}{2} - \frac{u_1+u_2}{2})\frac{\lambda_2}{u_2(1-u_2)} & (u_1u_3 + \frac{|u_1-u_3|}{2} - \frac{u_1+u_3}{2})\frac{\lambda_3}{u_3(1-u_3)} \\ (u_2u_1 + \frac{|u_2-u_1|}{2} - \frac{u_2+u_1}{2})\frac{\lambda_1}{u_1(1-u_1)} & 0 & (u_2u_3 + \frac{|u_2-u_3|}{2} - \frac{u_2+u_3}{2})\frac{\lambda_3}{u_3(1-u_3)} \\ (u_3u_1 + \frac{|u_3-u_1|}{2} - \frac{u_3+u_1}{2})\frac{\lambda_1}{u_1(1-u_1)} & (u_3u_2 + \frac{|u_3-u_2|}{2} - \frac{u_3+u_2}{2})\frac{\lambda_2}{u_2(1-u_2)} & 0 \end{bmatrix} \tag{A.11}$$

$$\mathcal{R}_4 = \begin{bmatrix} 0 & (u_1u_2 + \frac{|u_1-u_2|}{2} - \frac{u_1+u_2}{2})\frac{\lambda_2}{u_2(1-u_2)} & (u_1u_3 + \frac{|u_1-u_3|}{2} - \frac{u_1+u_3}{2})\frac{\lambda_3}{u_3(1-u_3)} & (u_1u_4 + \frac{|u_1-u_4|}{2} - \frac{u_1+u_4}{2})\frac{\lambda_4}{u_4(1-u_4)} \\ (u_2u_1 + \frac{|u_2-u_1|}{2} - \frac{u_2+u_1}{2})\frac{\lambda_1}{u_1(1-u_1)} & 0 & (u_2u_3 + \frac{|u_2-u_3|}{2} - \frac{u_2+u_3}{2})\frac{\lambda_3}{u_3(1-u_3)} & (u_2u_4 + \frac{|u_2-u_4|}{2} - \frac{u_2+u_4}{2})\frac{\lambda_4}{u_4(1-u_4)} \\ (u_3u_1 + \frac{|u_3-u_1|}{2} - \frac{u_3+u_1}{2})\frac{\lambda_1}{u_1(1-u_1)} & (u_3u_2 + \frac{|u_3-u_2|}{2} - \frac{u_3+u_2}{2})\frac{\lambda_2}{u_2(1-u_2)} & 0 & (u_3u_4 + \frac{|u_3-u_4|}{2} - \frac{u_3+u_4}{2})\frac{\lambda_4}{u_4(1-u_4)} \\ (u_4u_1 + \frac{|u_4-u_1|}{2} - \frac{u_4+u_1}{2})\frac{\lambda_1}{u_1(1-u_1)} & (u_4u_2 + \frac{|u_4-u_2|}{2} - \frac{u_4+u_2}{2})\frac{\lambda_2}{u_2(1-u_2)} & (u_4u_3 + \frac{|u_4-u_3|}{2} - \frac{u_4+u_3}{2})\frac{\lambda_3}{u_3(1-u_3)} & 0 \end{bmatrix} \tag{A.12}$$

Appendix B

Comparison with Feynman diagrams

Let's consider the term appearing in (3.35)

$$N! \int_0^1 du_1 \cdots \int_0^1 du_N \left[p_1^2 + m^2 + \sum_i (k_i^2 + 2p_1 \cdot k_i) u_i + \sum_{i<j} 2k_i \cdot k_j \big(u_i \theta(u_j - u_i) + u_j \theta(u_i - u_j) \big) \right]^{-N-1} \tag{B.1}$$

The integration region can be split into $N!$ subregions specified by a unique ordering $\sigma(i)$ of the indices $i = 1, 2, .., N$ so that $t_i = u_{\sigma(i)}$ are ordered as $1 \geq t_1 \geq t_2 \geq ... \geq t_N \geq 0$. Then each integration subregion contributes

$$\begin{aligned}
& N! \int_0^1 dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \cdots \int_0^{t_{N-1}} dt_N \Big[p_1^2 + m^2 + \sum_i (k_{\sigma(i)}^2 + 2k_{\sigma(i)} \cdot p_1) t_i \\
& \qquad + \sum_{i<j} 2k_{\sigma(i)} \cdot k_{\sigma(j)} \big(t_i \underbrace{\theta(t_j - t_i)}_{=0} + t_j \underbrace{\theta(t_i - t_j)}_{=1} \big) \Big]^{-N-1} \\
&= N! \int_0^1 dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \cdots \int_0^{t_{N-1}} dt_N \left[p_1^2 + m^2 + \sum_i (k_{\sigma(i)}^2 + 2k_{\sigma(i)} \cdot p_1) t_i + \sum_{i<j} 2k_{\sigma(i)} \cdot k_{\sigma(j)} t_j \right]^{-N-1} \\
&= N! \int_0^1 dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \cdots \int_0^{t_{N-1}} dt_N \left[p_1^2 + m^2 + \sum_i \Big[(k_{\sigma(i)}^2 + 2k_{\sigma(i)} \cdot (p_1 + \sum_{j=1}^{i-1} k_{\sigma(j)}) \Big] t_i \right]^{-N-1} \\
&= \frac{1}{p_1^2 + m^2} \, \frac{1}{(p_1 + k_{\sigma(1)})^2 + m^2} \, \frac{1}{(p_1 + k_{\sigma(1)} + k_{\sigma(2)})^2 + m^2} \cdots \frac{1}{(p_1 + \sum_{i=1}^N k_{\sigma(i)})^2 + m^2}
\end{aligned} \tag{B.2}$$

This shows that in each internal propagator flows the momentum as implied by momentum conservation at each vertex. The last integration above has been carried out by using the formula

$$\begin{aligned}
\frac{1}{A_0 A_1 A_2 \cdots A_N} &= N! \int_0^1 dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \cdots \int_0^{t_{N-1}} dt_N \\
&\times \frac{1}{[A_0 + (A_1 - A_0)t_1 + (A_2 - A_1)t_2 + \cdots + (A_N - A_{N-1})t_N]^{N+1}}
\end{aligned} \tag{B.3}$$

which can be derived from (3.32) by enforcing the delta function and suitably changing integration variables.

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