UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO
Y
UNIVERSIDAD MICHOACANA DE SAN NICOLÁS DE HIDALGO

**POSGRADO CONJUNTO EN CIENCIAS MATEMÁTICAS
UNAM-UMSNH**

Coarse Geometry on Proper Geodesic Spaces

T E S I N A

que para obtener el grado de

Maestro en Ciencias Matemáticas

presenta

Carlos Adrián Pérez Estrada.

Director: Noé Bárcenas Torres.

Morelia, Michoacán, México.
Junio del 2022.

Acknowledgments

I would like to thank everyone that made possible the preparation of this work and, consequently, the culmination of my master studies.

In the first place, I acknowledge the support I received by the National Autonomous University of Mexico via the UNAM-PAPIIT grant IN100221 “Topología Algebraica y teoría de índice”. This support would have not been possible without the help of my mentor Noé Bárcenas Torres, who I greately thank for his kindness, teachings, for introducing me to Algebraic Topology and Coarse Geometry, and for the many stimulating conversations and remarks that helped shaping the present work.

In regards to my academic formation, thanks are also due to Jesús Hernández Hernández for his patience and guidance in my preparation as a geometric group theorist; for the stimulating conversations regarding this work and all the grammar and mathematical remarks and corrections he made as a referee.

I warmly thank Sandy Aguilar for all the love and emotional support she has provided me since we have been living in Morelia and since we have met. Also I would like to thank her for the times she has helped me economically, for all her LaTeX tips (including the previous formatting of the word LaTeX), and for the helpful conversations we had for some technical parts of this work. Specifically, she helped me formalizing the proof of Lemma 3.25, gave remarkable suggestions for the proof of Theorem 3.33 and helped improving the explanation of the correspondence between the Freudenthal compactification and the spectrum of the algebra of Freudenthal functions contained in Page 13.

I thank my cousin Abigail Estrada for her sweetness and the (very very) funny moments my family and I have had with her. I would like to use this space to express my deepest condolences for the recent loss of her husband, Adrián Lua, occurred during the preparation of this work. He was close to my family, a personal friend of mine and one of the most charismatic people I have ever met. His regretful decease saddens us all and I sincerely admire the strength my cousin has had on her grief.

I greately thank my mother, Laura Estrada, for all the love and emotional support she has always given me and is continuing giving. Besides, I thank the full economic support she provided me in my first year living in Morelia and for helping me to be able to take master courses before being formally enrolled in the master program.

I warmly thank my sister, Karen Nirilse, for all the affection and wisdom she has taught me along my life. I also take advantage of this space to congratulate her husband and her for the recent birth of their first daughter, Olympia, whom I already thank for being a lovely inspiration. My family and I love her since the day we knew she was going to be born and she has gladden us ever since. I am looking forward to the day when she is able to read these acknowledgments and hopefully the mathematical part of this work too.

Abstract

In this work the large-scale geometry of proper geodesic spaces is discussed in the framework of Coarse Geometry, which is briefly introduced. Also, the Higson corona for general coarse spaces, the space of ends for proper geodesic spaces and the visual boundary for proper hyperbolic spaces are studied as large-scale invariants.

Key words: Large-scale geometry, proper geodesic spaces, large-scale invariants, metric compactifications, space of ends, visual boundary.

Resumen

En el presente trabajo se discute la geometría a gran escala de los espacios geodésicos propios en el contexto de la Geometría Gruesa, la cual es brevemente introducida. Además, la corona de Higson para espacios gruesos arbitrarios, el espacio de fines para espacios geodésicos propios y la frontera visual para espacios hiperbólicos propios son estudiados como invariantes a gran escala.

Palabras clave: Geometría a gran escala, espacios geodésicos propios, invariantes a gran escala, compactificaciones métricas, espacio de fines, frontera visual.

Contents

1 Introduction

Coarse Geometry is, roughly speaking, the study of the large-scale structure of geometric objects. In contrast with the study of the "very small-scale structure" of geometric objects carried out by Differential Topology, Coarse Geometry is mainly concerned with the invariants that distinguish an object when it is seen at a "afar". In more concrete terms, if topological spaces are determined by open sets that indicate how said spaces look in a small-scale, then the coarse spaces that will be considered in this work (Definition 2.2) are determined by a certain type of sets that specify the structure of such spaces when seen at a large-scale.

Just like continuous functions preserve the small-scale structure of topological spaces, *coarse maps* (introduced in Definition 2.15) do the same with the large-scale structure of coarse spaces. Inverting such maps, up to natural considerations, give rise to the notion of *coarse equivalence*; which (as expected) suggests when two coarse spaces are equivalent. A celebrated example of such coarse equivalence which is far from being a topological equivalence is the canonical inclusion of the discrete space $\mathbb{Z}^n$ on $\mathbb{R}^n$. On the other hand, every homeomorphism between the euclidean space $\mathbb{R}^n$ and the hyperbolic space $\mathbb{H}^n$ cannot be a coarse equivalence because the former is not hyperbolic while the latter is. This dichotomy is best exemplified when studying the (canonical) *bounded coarse structure* associated to a metric space (Example 2.3) and looking at other metrics which induce the same coarse structure but not the same topology and vice versa.

A great deal of motivation and examples on Coarse Geometry originally arised on Geometric Group Theory, where the notion of *quasi-isometry* (Definition 2.18) is already the preferred notion of large-scale equivalence between metric spaces. Such large-scale equivalence is stronger than the coarse equivalence that will be used in the present text; but it turns out that at least for *geodesic metric spaces* ([BH99, Part I, Definition 1.3]), which will be the main focus on this work, they are equivalent. This makes available the tools of Coarse Geometry in the study of finitely generated groups as they inherit a canonical, up to coarse equivalence, coarse structure arising for any word metric coming from a finite generating set. For instance, a coarse version of the Baum-Connes conjecture can be formulated for coarse spaces, and for some suitable finitely generated groups it implies the injectivity part of the actual Baum-Connes conjecture, and thus the Novikov conjecture for such groups ([Roe93], [Roe03]).

Another use of Coarse Geometry (somehow related to the previous conjectures) is the generalization of the Atiyah-Singer Index Theorem for complete non-compact Riemannian manifolds ([Roe93], [Roe96]). Namely, considering the bounded coarse structure naturally associated to the Riemannian metric, a coarse cohomology theory reflecting the induced coarse structure can be used as a replacement of the ordinary cohomology used on the classical Index Theorem for compact Riemannian manifolds. One of the main applications of this generalized Index Theorem is the fact that a complete simply-connected Riemannian manifold of non-positive curvature cannot carry a Riemannian metric of uniformly positive scalar curvature ([Roe03]).

In the present work a concise and brief introduction to Coarse Geometry with emphasis on proper geodesic metric spaces is given. In particular, Section 2 gives the basic notions of coarse spaces and discusses them in the framework of proper geodesic spaces, and Section 3 introduces some invariants of such spaces in the form of compact Hausdorff spaces. For general coarse spaces the coarse invariant to be considered is the space known as the "Higson corona" (Definition 3.11) and for proper geodesic spaces another invariant to be considered is the "space of ends" (Definition 3.21). Under mild hypothesis, these two invariants come from compactifications respectively called the Higson compactification, which is constructed in Subsection 3.1, and the Freudenthal compactification, which is constructed in Subsection 3.2.

In addition to the above, in Subsection 3.1 it is defined what is a coarse compactification (Definition 3.6), it is given two criteria to decide when an arbitrary map between proper metric spaces is a coarse equivalence (Theorem 3.17 and Theorem 3.18) and it is given two equivalences of the connectedness of the Higson corona: one for arbitrary metric spaces (Theorem 3.19) and the other one for geodesic spaces (Theorem 3.20). This last equivalence is used in Subsection 3.2 to prove that a finitely generated group is one-ended if and only if its Higson corona is non-empty and connected. Also, in Subsection 3.2 it is proved that the Freudenthal compactification is a coarse compactification for proper geodesic spaces (Theorem 3.28) with a technique that is used in Subsection 3.3 as well to show that the visual compactification for proper hyperbolic spaces ([BH99, pages 427-429]) is a coarse compactification too (Corollary 3.42).

2 Coarse Structures

In this section the notion of coarseness, briefly discussed in the introduction, is formalized along with the appropriate objects that allows the discussion of properties of coarse spaces. The information presented can be consulted in John Roe's books [Roe03], [Roe93], [Roe96] and [HR00].

Some notation needed to define a coarse space is established in the following definition.

Definition 2.1. Let X, Y and Z be arbitrary sets.

1. If $E \subseteq X \times Y$, then the relation
$$E^{-1} := \{(y, x) \mid (x, y) \in E\} \subseteq Y \times X$$
is known as the **inverse** of E.

2. If $E \subseteq X \times Y$ and $E' \subseteq Y \times Z$, then $E \circ E' \subseteq X \times Z$ denotes its **composition** defined by
$$\{(x, z) \in X \times Z \mid \text{there exists } y \in Y \text{ such that } (x, y) \in E \text{ and } (y, z) \in E'\}.$$

3. If $E \subseteq X \times Y$ and $K \subseteq Y$, then the **section** of K in E is defined as
$$E[K] = \{x \in X \mid \text{there exists } y \in K \text{ such that } (x, y) \in E\}.$$
If K is a singleton $\{x\}$, $E[\{x\}] =: E_x$ and $E^{-1}[\{x\}] =: E^x$.

Definition 2.2. (Coarse structure on a set) A **coarse space** is an ordered pair $(X, \mathcal{B})$, where X is an arbitrary set and $\mathcal{B} \subseteq \mathcal{P}(X \times X)$ is a family of relations, called the **controlled sets** or the **entourages** of the **coarse structure** $\mathcal{B}$, for which the following conditions hold:

1. $\Delta_X := \{(x, y) \in X \times X \mid x = y\} \in \mathcal{B}$.
2. If $B \in \mathcal{B}$ and $E \subseteq B$, then $E \in \mathcal{B}$.
3. If $E \in \mathcal{B}$, then $E^{-1} \in \mathcal{B}$.
4. If $A, B \in \mathcal{B}$, then $A \cup B, A \circ B \in \mathcal{B}$.

   Moreover, if each point of $X \times X$ belongs to some controlled set, the coarse structure is said to be **coarsely connected**.

The main example of a coarse space that will be concerned in this work is of the bounded metric coarse structure naturally associated to a metric space.

Example 2.3. (Bounded Metric Coarse Structure) Let (X, d) be a metric space. The **bounded metric coarse structure** is the one for which the controlled sets $A \subseteq X \times X$ are those such that
$$\sup\{d(x, x') \mid (x, x') \in A\} < +\infty.$$

Unless otherwise stated, metric spaces will be assumed to carry this coarse structure when regarded as coarse spaces.

Another example of coarse structure on metric spaces, due to Nick Wright ([Wri03, Definition 1.1]), is the C_0-coarse structure.

Example 2.4. (C_0-coarse structure) Let (X, d) be a metric space. The **C_0-coarse structure** on X is the coarse structure for which a set $A \subseteq X \times X$ is controlled if and only if for every $\epsilon > 0$ there exists a compact subset $K \subseteq X$ such that
$$d(x, y) < \epsilon \ \text{ if } (x, y) \in A \setminus (K \times K).$$

Note that the two examples above are coarsely connected since metrics are assumed to take only finite values.

An usual way to regard topological spaces as coarse spaces is by compactifications. The idea behind this is to reflect the behavior at infinity of the boundary of a compactification of a topological space on a coarse structure determined by controlled sets whose growth is continuosly controlled by such compactification.

Example 2.5. (Topological Coarse Structure) Let X be a paracompact and locally compact Hausdorff space equipped with a compactification $\overline{X}$ with boundary ∂X. The **topological coarse structure** associated to the compactification $\overline{X}$ is the one for which the controlled sets $E \subseteq X \times X$ fulfill one of the following equivalent conditions:

1. The closure $\overline{E}$ of E in $\overline{X} \times \overline{X}$ meets the complement of $X \times X$ only in the diagonal $\Delta_{\partial X}$.
2. For every relatively compact subset $K \subseteq X$, the sections $E[K]$ and $E^{-1}[K]$ are relatively compact as well; and for every net $\{x_\alpha, y_\alpha\} \subseteq E$, if $\{x_\alpha\}$ converges to a point $\xi \in \partial X$, then $\{y_\alpha\}$ also converges to ξ.

Remark 2.6. The topological coarse structure of a paracompact and locally compact Hausdorff space is always coarsely connected.

In a coarse structure there is a clear notion of nearness between maps that formalizes the idea of two maps looking the same when seen from a large distance.

Definition 2.7. Let X be a coarse space and let S be any set. Two maps $f_1, f_2\colon S \to X$ are **close** if the set $\{(f_1(x), f_2(x)) \mid x \in S\} \subseteq X \times X$ is controlled.

For metric spaces equipped with the bounded metric coarse structure, the concept of nearness of maps coincides with the one used in Metric Geometry:

Example 2.8. Let (X, d) be a metric space. Two maps $f_1, f_2\colon S \to X$ are close with respect to the bounded metric coarse structure if and only if

$$\sup_{s\in S} d\big(f_1(s), f_2(s)\big) < +\infty.$$

This closeness relation depends only on the large-scale properties of the metric. For example, the metric $\rho\colon X \times X \to \mathbb{R}_{\geq 0}$ given by

$$\rho(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \max\{1, d(x,y)\} & \text{if } x \neq y, \end{cases} \tag{1}$$

defines the same closeness relation as d but the topology it generates is always the discrete one. This shows that the topology of a metric space has little to do with its (bounded metric) coarse structure.

Other notions of metric spaces that have a clear generalization to coarse spaces are the *boundedness* of sets and the *uniform boundedness* of families of subsets.

Definition 2.9. (**Boundedness of a set**) Let X be a coarse space. A subset $B \subseteq X$ is said to be **bounded** if it satisfies one of the following equivalent conditions:

1. $B \times B$ is controlled.
2. The canonical inclusion $B \to X$ is close to a constant map.

Moreover, a family of subsets $\mathcal{U} \subseteq \mathcal{P}(X)$ is **uniformly bounded** if the union $\bigcup_{U\in\mathcal{U}} U \times U$ is controlled.

Example 2.10. In the bounded coarse structure associated to a metric space (X, d), the bounded sets in the sense just defined are just the bounded ones with respect to d and the uniformly bounded families of subsets defined above are just the uniformly bounded ones with respect to d. If d is a proper metric (i.e. if closed balls with respect to d are compact in X), it is also true that the bounded sets of the C_0-coarse structure are the bounded ones with respect to d.

When a topological space is also endowed with a coarse structure, it is natural to require some compatibility between the coarse structure and the topology. With the idea of boundedness, such compatibility can be stated as follows:

Definition 2.11. A coarse structure on a paracompact Hausdorff topological space X is said to be **proper** if

1. X has a uniformly bounded open cover, and
2. every bounded set of X has compact closure.

Theorem 2.12. ([Roe03, Page 25]) If X is a paracompact Hausdorff topological space equipped with a proper coarse structure, then X is locally compact. Moreover, if X is connected as a topological space, then it is coarsely connected as a coarse space and a subset $A \subseteq X$ is bounded if and only if $\overline{A}$ is compact.

Example 2.13. If X is a metric space, the bounded metric coarse structure is proper when X is regarded as a paracompact Hausdorff space if and only if the metric space is proper in the sense of Metric Geometry (i.e. if the closed balls on X are compact); whereas the C_0-coarse structure is always proper.

Example 2.14. The topological coarse structure associated to a compactification of a paracompact and locally compact space X is always proper.

With the notions of boundedness of sets and nearness of maps, the equivalences between coarse spaces can be given as follows.

Definition 2.15. Let X and Y be coarse spaces, and let $f\colon X \to Y$ be an arbitrary map.

1. The map f is **bornologous** if for each controlled subset $E \subseteq X \times X$ the set
$$(f \times f)(E) := \{\big(f(x_1), f(x_2)\big) \mid (x_1, x_2) \in E \times E\}$$
is a controlled subset of $Y \times Y$.
2. The map f is **coarsely proper** if for every bounded subset $B \subseteq Y$, the preimage $f^{-1}(B)$ is bounded in X.
3. The map f is **coarse** if it is coarsely proper and bornologous.
4. Two coarse spaces X and Y are **coarsely equivalent** if there exist coarse maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $f \circ g$ and $g \circ f$ are close to the identity maps on Y and X respectively. The maps f and g are known as **(inverse) coarse equivalences**.

Example 2.16. As was stated in the introduction, the prototypal example of a coarse equivalence between metric spaces is the inclusion $\mathbb{Z}^n \hookrightarrow \mathbb{R}^n$ with coarse inverse the n-fold floor map $\lfloor\cdot\rfloor\colon \mathbb{R}^n \to \mathbb{Z}^n$. Actually, if $\mathbb{R}^n$ is regarded as a discrete space under the metric ρ defined by Expression (1), the inclusion $\mathbb{Z}^n \hookrightarrow \mathbb{R}^n$ is also a coarse map with coarse inverse the floor map too. This implies that the floor map $\lfloor\cdot\rfloor\colon (\mathbb{R}^n, d) \to (\mathbb{R}^n, \rho)$, where d is the standard metric, is its own coarse inverse as a coarse equivalence between the n-euclidean space $(\mathbb{R}^n, d)$ and the discrete space $(\mathbb{R}^n, \rho)$.

Example 2.17. Generalizing Example 2.16, if G is a finitely generated group and $S \subseteq G$ is a symmetric finite generating set which does not contain the identity, the canonical injection of G on the geometric realization of the Cayley graph $\mathrm{Cay}(G, S)$ is a coarse equivalence when G is equipped with the word metric d_S. Moreover, if $S' \subseteq G$ is another finite generating set, the identity function $\mathrm{Id}\colon (G, d_S) \to (G, d_{S'})$ is a bi-Lipschitz homeomorphism, which implies that it is a coarse equivalence.

Indeed, the maps of Example 2.17 turn out to be *quasi-isometries* when they are considered as coarse equivalences and *large-scale Lipschitz* maps when they are considered as bornologous maps. Since for the main coarse spaces treated in this work all coarse equivalences are quasi-isometries and all bornologous maps are large-scale Lipschitz, the definition of such maps is recalled as follows.

Definition 2.18. (**Large-scale Lipschitz Map**) Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f\colon X \to Y$ (not necessarily continuous) is said to be (λ, c)-**large-scale Lipschitz** if there exists constants $\lambda \geq 1$ and $c \geq 0$ such that for all $x_1, x_2 \in X$
$$d_Y\big(f(x_1), f(x_2)\big) \leq \lambda d_X(x_1, x_2) + c.$$
In such case λ and c are referred, respectively, as the **multipliticative** constant and the **additive** constant of f.

If there exists a large-scale Lipschitz map $g\colon Y \to X$ such that $f \circ g$ is close to Id_Y and $g \circ f$ is close to Id_X with respect to the bounded metric coarse structures on X and Y, then f is said to be a **quasi-isometry** with **quasi-inverse** g (which is a quasi-isometry as well). In this case the metric spaces X and Y are said to be **quasi-isometric**.

Remark 2.19. Large-scale Lipschitz maps are also often called coarse Lipschitz maps.

The following characterization of bornology on metric spaces makes it clear why every large-scale Lipschitz map is bornologous. It is based on the observation that bornologous maps can be regarded as large-scale analogs of uniformly continuous maps in the same vein that a coarse structure can be thought as a large-scale analog of a uniform structure.

Theorem 2.20. Let (X, d_X) and (Y, d_Y) be metric spaces equipped with their respective bounded metric coarse structures. A map $f\colon X \to Y$ is bornologous if and only if for every $R > 0$ there exists $S > 0$ such that

$$\forall x, x' \in X,\ d_X(x, x') < R \implies d_Y\big(f(x), f(x')\big) < S. \tag{2}$$

Notice that the Condition (2) is actually the definition of a uniformly continuous function with the usual ϵ and δ the wrong way around. Actually, the numbers R and S have to be thought as being large rather than small because the intuition behind the equivalence above is that bornologous functions between metric spaces are the ones for which there is a uniform control on their expansiveness.

An example exhibiting the impossibility of changing the quantifiers in the previous equivalence is the map $f\colon \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = n^2$ since it is uniformly continuous but not bornologous. Despite this, for the class of "length spaces" ([BH99, Part I, Definition 3.1]) uniform continuity does imply bornology ([Roe93, Proposition 2.2]). As was mentioned before, for such spaces every bornologous map is large-scale Lipschitz.

Theorem 2.21. ([Roe03, Lemma 1.10]) Let X be a length space and Y be any metric space. For every map $f\colon X \to Y$ the following conditions are equivalent:

1. f is large-scale Lipschitz.
2. f is bornologous with respect to the bounded metric coarse structures.
3. There exist $R, S > 0$ such that for all $x, x' \in X$, $d(x, x') < R$ implies $d\big(f(x), f(x')\big) < S$.

It is relatively easy to prove that a large-scale Lipschitz map $f\colon X \to Y$ that turns out to be a quasi-isometry has to be a *quasi-isometric embedding*. That is to say, for the map f there exist a multiplicative constant $\lambda \geq 1$ and an additive constant $c \geq 0$ for which the stronger condition

$$\lambda^{-1} d_X(x_1, x_2) - c \leq d_Y\big(f(x_1), f(x_2)\big) \leq \lambda d_X(x_1, x_2) + c$$

holds for all $x_1, x_2 \in X$. The lower bound seen above implies that a quasi-isometry f is a coarse map for which its quasi-inverse also has to be a coarse map, so f ends up being a coarse equivalence. Combining this with Theorem 2.21 it is concluded that on length metric spaces the notions of quasi-isometry and coarse equivalence coincide:

Corollary 2.22. If $f\colon X \to Y$ is a coarse equivalence between length spaces endowed with their bounded metric coarse structures, then f is a quasi-isometry.

An inmediate consequence of the above result and Example 2.17 is that any coarse equivalence between finitely generated groups is automatically a quasi-isometry. As was said in the introduction, this put in motion the tools of Coarse Geometry in the study of finitely generated groups.

3 Invariant Coarse Compactifications

In the present section topological invariants of coarse spaces are introduced. In Subsection 3.1 the Higson corona is constructed for general coarse spaces as a functor from the category of coarse spaces with coarse maps modulo closeness to the category of compact Hausdorff spaces with continuous maps (Theorem 3.12). In a similar fashion, the space of ends is discussed in Subsection 3.2 as a functor from the category of proper geodesic spaces with Borel quasi-isometric embeddings (modulo closeness) to the category of Stone spaces (Corollary 3.34). Under certain considerations to be explained in Subsection 3.2, the space of ends arises as a natural quotient of the Higson corona in a manner preserved by coarse equivalences as shown in Corollary 3.34. This is the essence of the notion of coarseness (Definition 3.6) for the Freudenthal compactification arising from the space of ends; that also appears in Subsection 3.3 for the visual compactification coming from the visual boundary for proper hyperbolic spaces.

Focusing on the Higson corona of proper metric spaces, in Subsection 3.1 it is shown how said corona determines when an arbitrary map is a coarse equivalence. Additionally, to prove in Subsection 3.2 that a finitely generated group is one-ended exactly when it has a non-empty connected Higson corona, in Subsection 3.1 it is also given a criterion to determine when a geodesic space has a connected Higson corona.

3.1 Higson Compactification

Recall from Example 2.5 that for a paracompact and locally compact Hausdorff space X every (Hausdorff) compactification $\overline{X}$ gives rise to a topological coarse structure on X that induces a proper coarse structure on it (Definition 2.12 and Example 2.14). The Higson compactification serves as a pseudo-converse of such construction. Namely, given a proper coarse space (that is, a paracompact Hausdorff space equipped with a proper coarse structure), the Higson compactification will arise as a natural compactification somehow reflecting the proper coarse structure via the universal property to be discussed in the present subsection. To what extent these passages from proper coarse spaces to compactifications and vice versa are inverses of one another can be checked in [Roe03, Proposition 2.45]. In the present subsection the main objectives are: showing the coarse invariance of the Higson corona (Corollary 3.14), describing how the universal property of the Higson compactification leads to the notion of coarseness in compactifications of proper coarse spaces; and giving the mentioned criteria to determine when a map between proper metric spaces is a coarse equivalence and when the Higson corona of a geodesic space is connected.

Definition 3.1. Let X be a proper coarse space. A bounded complex-valued continuous function $f \in C_b(X)$ is a **(continuous) Higson function** if the gradient

$$\begin{aligned} df\colon X \times X &\longrightarrow \mathbb{C} \\ (x,y) &\longmapsto f(x) - f(y) \end{aligned} \tag{3}$$

vanishes at infinity on every controlled set. That is to say, for every controlled set $E \subseteq X \times X$ and for every $\epsilon > 0$ there exists a compact subset $K \subseteq X$ such that $|f(x) - f(y)| < \epsilon$ whenever $(x,y) \in E \setminus K \times K$. The set of Higson functions on X is denoted by $C_h(X)$.

Theorem 3.2. ([Roe03, Proposition 2.36]) The set of Higson functions on a proper coarse space X is a unital C*-subalgebra of the C*-algebra of bounded continuous functions $C_b(X)$ which contains the algebra $C_0(X)$ as an essential closed *-ideal.

It follows from the Gelfand-Naimark duality that the C*-algebra of Higson functions is the C*-algebra of continuous functions on some compactification of X. Indeed, if $\mathrm{sp}(C_h(X))$ is the space of non-trivial multiplicative complex linear functionals $\omega\colon C_h(X) \to \mathbb{C}$ equipped with the topology of pointwise convergence, then $\mathrm{sp}(C_h(X))$ is a compact Hausdorff space in which the space X is topologically embedded as a dense open subspace by the evaluation map

$$\begin{aligned} X &\longrightarrow \mathrm{sp}(C_h(X)) \\ x &\longmapsto \big(f \mapsto f(x)\big). \end{aligned} \tag{4}$$

In this context, the Gelfand-Naimark duality ensures that the map

$$\begin{aligned} C_h(X) &\longrightarrow C\big(\mathrm{sp}(C_h(X))\big) \\ f &\longmapsto \big(\omega \mapsto \omega(f)\big) \end{aligned}$$

is an *-isomorphism of C*-algebras.

Definition 3.3. (**Higson Compactification**) The **Higson compactification** of a proper coarse space X, denoted by hX, is defined as the space of non-trivial multiplicative complex linear fuctionals $\mathrm{sp}(C_h(X))$ equipped with the topology of pointwise convergence. Its boundary $hX \setminus X$, denoted by $\partial_h(X)$, is known as the **Higson corona** of X.

By Gelfand-Naimark duality again, the Higson corona $\partial_h(X) = hX \setminus X$ is homeomorphic by a map analogous to the evaluation map seen at Expression (4) to the compact Hausdorff space $\mathrm{sp}(C_h(X)/C_0(X))$, which is canonically homeomorphic (by the First Isomorphism Theorem) to the subspace of the Higson compactification hX consisting on the multiplicative linear functionals that vanish on $C_0(X)$.

Remark 3.4. Although the present discussion is based on John Roe's works, the Higson compactification and the Higson corona were originally considered by Nigel Higson from considerations of index theorems for non-compact complete Riemannian manifolds, as mentioned by Roe in [Roe93]. That is why Roe named this space after Higson.

The universal property that characterizes the Higson compactification resembles the universal property of the Stone-Čech compactification among Hausdorff compactifications. To state it, is necessary to compare different coarse structures on a same set in order to establish the *coarse compactifications* for which the Higson compactification is universal.

Definition 3.5. Let $\mathcal{A}$ and $\mathcal{B}$ be coarse structures on the same set X. If $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{A}$ is said to be **finer** than $\mathcal{B}$ and (consequently) $\mathcal{B}$ is said to be **coarser** than $\mathcal{A}$.

Definition 3.6. (**Coarse Compactification**) Let X be a proper coarse space. A **Coarse Compactification** of X is a compactification whose topological coarse structure is coarser than the original coarse structure on X.

Theorem 3.7. ([Roe03, Proposition 2.39]) If X is a proper coarse space, then the Higson compactification hX is a coarse compactification of X. Moreover, the Higson compactification is the universal coarse compactification in the sense that for any other coarse compactification $\overline{X}$ of X, the identity map on X extends uniquely to a continuous map of hX onto $\overline{X}$.

In terms of C*-algebras, the Theorem 3.7 states that a compactification $\overline{X}$ of a proper coarse space X is coarse if and only if every continuous function $f\colon X \to \mathbb{C}$ continuously extendable to $\overline{X}$ is a Higson function. This criterion will be used to show that the visual compactification of a proper hyperbolic space ([BH99, Part III, Definition 3.5]) is a coarse compactification (Corollary 3.42) as well as the Freudenthal compactification for proper geodesic spaces (Theorem 3.28).

While the Higson compactification can be defined only for proper coarse spaces, the Higson corona can be defined for all coarse spaces in a coarse invariant way. That is to say, the Higson corona defined for all coarse spaces turns out to be a functor from the category of coarse spaces with coarse maps modulo closeness to the category of compact Hausdorff spaces with continuous maps. To see this, the Higson corona will be defined without appealing to continuous functions by natural analogs of maps *tending to infinity* in an arbitrary coarse space.

Definition 3.8. Let X be a coarse space and let $f\colon X \to \mathbb{C}$ be an arbitrary function.

1. The function f **tends to zero at infinity** if for every $\epsilon > 0$ there exists a bounded set $B \subseteq X$ (Definition 2.9) such that $|f(x)| < \epsilon$ whenever $x \notin B$.
2. The function f **fulfills the Higson condition** if for every controlled set $E \subseteq X \times X$ and for every $\epsilon > 0$ there exists a bounded set $B \subseteq X$ such that $|f(x) - f(y)| < \epsilon$ whenever $(x, y) \in E \setminus B \times B$.

The set of all bounded functions $X \to \mathbb{C}$ fulfilling the Higson condition is denoted by $B_h(X)$ and the set of all bounded functions $X \to \mathbb{C}$ that tend to zero at infinity is denoted by $B_0(X)$.

It is clear that $B_h(X)$ is a C*-subalgebra of the algebra of bounded complex functions $\ell^\infty(X)$ and that $B_0(X)$ is a closed essential *-ideal of $B_h(X)$. Since for proper coarse spaces the bounded subsets are precisely the relatively compact ones, for such spaces a bounded function $f\colon X \to \mathbb{C}$ belongs to the algebra $B_0(X)$ if and only if for every $\epsilon > 0$ there exists a compact subset $K \subseteq X$ for which $|f(x)| < \epsilon$ whenever $x \notin K$. With this characterization the relation between the previously defined algebras and the algebras of continuous functions used to define the Higson Corona of a proper coarse space can be given as follows.

Lemma 3.9. ([Roe03, Lemma 2.40]) Let X be a proper coarse space. Then

1. $C_0(X) = C_h(X) \cap B_0(X)$,

2. $B_h(X) = C_h(X) + B_0(X)$.

Theorem 3.10. For every proper coarse space X

$$\frac{C_h(X)}{C_0(X)} \cong \frac{B_h(X)}{B_0(X)}.$$

Definition 3.11. (**Higson Corona**) The **Higson corona** of any coarse space X, denoted by $\partial_h(X)$, is defined as the space of non-trivial multiplicative complex linear functionals $\mathrm{sp}(B_h(X)/B_0(X))$ equipped with the topology of pointwise convergence.

By Theorem 3.10, the two definitions of the Higson corona for proper coarse spaces coincide.

As was mentioned before, the Higson corona is functorial in such a way that close coarse maps induce the same continuous map.

Theorem 3.12. ([Roe03, Proposition 2.41]) The Higson corona is a functor from the category of coarse spaces with coarse maps modulo closeness to the category of compact Hausdorff spaces with continuous maps, which assigns to every coarse map $f\colon X \to Y$ the continuous map

$$\begin{aligned} f_*\colon \partial_h(X) &\longrightarrow \partial_h(Y) \\ \omega &\longmapsto \big([g] \longmapsto \omega([g \circ f])\big). \end{aligned} \tag{5}$$

Proof. To see that the Higson corona is functorial, by Gelfand-Naimark duality it suffices to induce for every coarse map $f\colon X \to Y$ a *-homomorphism

$$f^*\colon C\big(\partial_h(Y)\big) \cong B_h(Y)/B_0(Y) \longrightarrow B_h(X)/B_0(X) \cong C\big(\partial_h(X)\big)$$

in a functorial and close-invariant way. To construct said *-homomorphism, first note that for every $g \in B_h(Y)$, $g \circ f \in B_h(X)$. Indeed, for every controlled set $E \subseteq X \times X$, the set $A := (f \times f)(E)$ is controlled in Y by bornology. As a consequence, the Higson condition implies that for every $\epsilon > 0$ there exists a bounded subset $B \subseteq Y$ such that

$$|g(y) - g(y')| < \epsilon \text{ for all } (y, y') \in A \setminus (B \times B). \tag{6}$$

Thus, by properness, the preimage $C := f^{-1}(B) \subseteq X$ is bounded and for every $(x, x') \in E \setminus (C \times C)$

$$|g \circ f(x) - f \circ g(x')| = |g\big(f(x)\big) - g\big(f(x')\big)| < \epsilon$$

because $\big(f(x), f(x')\big) \in A \setminus (B \times B)$. In a similar fashion it can be showed that for every $g \in B_0(Y)$, $g \circ f \in B_0(X)$. Therefore, the coarse map $f\colon X \to Y$ induces the *-homomorphism

$$\begin{aligned} f^*\colon B_h(Y)/B_0(Y) &\longrightarrow B_h(X)/B_0(X) \\ [g] &\longmapsto [g \circ f]. \end{aligned}$$

Since composition of functions is clearly associative, it is straightforward to prove the functoriality of the previous correspondence. It only remains to show that close coarse maps $f, h\colon X \to Y$ induce the same *-homomorphisms f^*, h^*. For such purpose, it is enough to prove that for every $g \in B_h(Y)$, the map $g \circ f - g \circ h \in B_0(X)$. Note that, as f and h are close, the set

$$A := \{\big(f(x), h(x)\big) \mid x \in X\} \subseteq Y$$

is an entourage. Therefore, the Higson condition (6) of g applies for every $x \in X$ for which

$$\big(f(x), h(x)\big) \in A \setminus (B \times B).$$

As a consequence, if $x \in X \setminus \big(f^{-1}(B) \cap h^{-1}(B)\big)$, so that $\big(f(x), h(x)\big) \in A \setminus (B \times B)$, we have that

$$|g \circ f(x) - g \circ h(x)| = |g \circ \big(f(x)\big) - g \circ \big(h(x)\big)| < \epsilon.$$

Since the intersection $f^{-1}(B) \cap h^{-1}(B)$ is bounded, $g \circ f - g \circ h$ ends up being a bounded function on X that tends to zero at infinity. $\square$

Remark 3.13. To prove Theorem 3.17 below, Theorem 3.12 has to be complemented with the fact that for any coarse map $f\colon X \to Y$ between proper coarse spaces, the induced map $\overline{f}\colon hX \to hY$ turns out to be continuous on the points of $\partial_h(X)$.

Corollary 3.14. Coarsely equivalent coarse spaces have homeomorphic Higson coronas. In particular, if $f\colon X \to Y$ is a coarse equivalence between coarse spaces, then $f_*\colon \partial_h(X) \to \partial_h(Y)$ is a homeomorphism.

It is possible to give a pseudo-converse of Corollary 3.14 for proper metric spaces ([ALC11, Theorems 4.16 and 4.17]). To state and prove such results, some facts are recalled following [ALC11] and [Roe03].

Proposition 3.15. ([ALC11, Proposition 4.4]) Let X and Y be paracompact and locally compact Hausdorff spaces equipped with topological coarse structures coming from compactifications $\overline{X}$ and $\overline{Y}$. An arbitrary map $\varphi\colon X \to Y$ is coarse if it has an extension $\overline{\varphi}\colon \overline{X} \to \overline{Y}$ that is continuous on the points of $\partial(X)$ and is such that $\overline{\varphi}\big(\partial(X)\big) \subseteq \partial(Y)$.

Proposition 3.16. ([Roe03, Proposition 2.47]) The bounded metric coarse structure of a proper metric space is the topological coarse structure associated to its Higson compactification.

Theorem 3.17. ([ALC11, Theorem 4.16]) Let X and Y be proper (non-compact) metric spaces. Then, a map $\varphi\colon X \to Y$ is a coarse equivalence if and only if it has an extension $\overline{\varphi}\colon hX \to hY$ such that all the following conditions are satisfied:

1. $\overline{\varphi}\big(\partial_h(X)\big) \subseteq \partial_h(Y)$.
2. The map $\overline{\varphi}$ is continuous on the points of $\partial_h(X)$.
3. The restriction $\varphi^*\colon \partial_h(X) \to \partial_h(Y)$ is a homeomorphism.

Proof. By Theorem 3.12 and Remark 3.13, the coarseness of the map φ implies Points 1 and 2 below. On the other hand, Propositions 3.15 and 3.16 indicate that Points 1 and 2 imply coarseness. Thus, it only remains to see that if $\varphi\colon X \to Y$ is a coarse map, then Point 3 is equivalent to the coarse invertibility of φ. Indeed, if $\psi\colon Y \to X$ is a coarse map coarsely inverse to φ, the induced continuous map $\psi_*\colon \partial_h(Y) \to \partial_h(X)$ is an inverse of φ_* by functoriality.

Conversely, suppose that the map $\varphi_*\colon \partial_h(X) \to \partial_h(Y)$ is a homeomorphism but that φ does not admit a coarse inverse coarse map. As the non-existence of the coarse inverse is equivalent to the non-coarsely-surjectivity, there exists a sequence $\{y_n\} \subseteq Y$ such that the union W of the metric balls $\overline{B_Y}(y_n, n)$ is disjoint from the image $\varphi(X)$ by denying that φ is coarsely surjective. Following [ALC11, Proposition 4.12], without loss of generality it can be supposed that the balls $\overline{B_Y}(y_n, n)$ are mutually disjoint. This implies that the sequence $\{y_n\}$ is necessarily unbounded so it does converge to a point $p \in \partial_h(Y)$. Using the fact that φ^* is a homeomorphism and the density of X in hX, consider a sequence $\{x_n\} \subseteq X$ converging to a $q \in \partial_h(X)$ for which $\varphi_*(q) = p$. By Remark 3.13, the sequence $\{\varphi(x_n)\} \subseteq \varphi(X)$ converges to p in hY. Consequently, by Point 1 of Example 2.5, the sequence $\{(\varphi(x_n), y_n)\} \subseteq Y \times Y$ is an entourage of the bounded metric coarse structure of Y, so there exists an $R \in \mathbb{R}^+$ such that

$$\sup_{n\in\mathbb{N}} d_Y\big(\varphi(x_n), y_n\big) < R.$$

Then, by taking an $n \in \mathbb{N}$ with $n > R$ we arrive to a contradiction because $n < d_Y\big(\varphi(x_n), y_n\big)$. ∎

Theorem 3.18. ([ALC11, Theorem 4.17]) Let X and Y be proper metric spaces. If there exists an algebraic isomorphism $\varphi\colon C_h(X) \to C_h(Y)$ between their respective algebras of Higson functions, then X and Y are coarsely equivalent.

Proof. Since $C_h(X)$ and $C_h(Y)$ are C*-algbras, the algebraic isomorphism φ turns out to be an *-isometric isomorphism. Hence, by Gelfand-Naimark duality, φ induces a homeomorphism of Higson compactifications $\overline{\varphi}\colon hX \to hY$. Such homeomorphism must map X homeomorphically to Y since X is metrizable but no point in $\partial_h(Y)$ is a G_δ-set ([ALC11, Proposition 4.10]). Therefore, the homeomorphism $\overline{\varphi}|_X\colon X \to Y$ has to be a coarse equivalence as a consequence of Theorem 3.17 since the restriction $\overline{\varphi}|_{\partial_h(X)}\colon \partial_h(X) \to \partial_h(Y)$ is a homeomorphism as well. ∎

The present subsection finishes with two results due to [Wei16] concerning the connectedness of the Higson corona for metric and geodesic spaces.

Theorem 3.19. ([Wei16, Theorems 4.1 and 4.6]) The Higson corona of an arbitrary metric space (X, d) is disconnected if and only if there are two unbounded subsets A and B of X such that

1. $X = A \cup B$,
2. for any $R > 0$ there is a bounded subset $C_R \subseteq X$ with the property that

$$a \in A \setminus C_R \text{ and } b \in B \setminus C_R \implies d_X(a, b) \geq R.$$

Proof. Suppose the existence of unbounded subsets A and B fulfilling the conditions above. Note that for every $R > 0$, the intersection $A \cap B \subseteq C_R$ since $d(x, x) = 0 < R$ for every $x \in A \cap B$. Because of that, it can be supposed that $B \setminus A$ is an unbounded subset. Such choice will allow us to see $\partial_h(X)$ as the topological sum of $\partial_h(A)$ with $\partial_h(A \setminus B)$. Since neither A nor $A \setminus B$ are bounded, their Higson coronas are not empty and thus $\partial_h(X)$ ends up being a disconnected space. To achieve such realization, it will be verified that X is coarsely equivalent to the *coarse coproduct* of A with $B \setminus A$ ([Wei16, Definition 3.1]), which turns out to have a Higson corona homeomorphic to the topological sum of $\partial_h(A)$ with $\partial_h(A \setminus B)$.

Consider any pair of points $a_0 \in A$ and $b_0 \in B \setminus A$ and assume, without loss of generality, that each C_R contains such points. With this assumption, the metric d is slightly modified to a metric ρ which restricted to A and $B \setminus A$ coincides with d but for points $a \in A$ and $b \in B \setminus A$ assigns the distance $d(a, a_0) + d(a_0, b_0) + d(b, b_0)$. It can be verified that (X, ρ) is coarsely equivalent to the coproduct of A with $B \setminus A$ constructed in [Wei16, Definition 3.1]. By definition, the identity $\mathrm{Id}\colon (X, \rho) \to (X, d)$ is bornologous (actually 1-Lipschitz), so to see that it is actually a coarse equivalence, it will be corroborated that $\mathrm{Id}\colon (X, d) \to (X, \rho)$ is also bornologous. Indeed, by construction such identity is an isometry when restricted to A and $B \setminus A$, so let $a \in A$ and $b \in B \setminus A$, and choose a bounded set C_R corresponding to any value $R > d(a, b)$. Then one of a or b must be in D_R, let us say b. Thus, if $D(R)$ is the diameter of C_R, then

$$\begin{aligned}
\rho(a, b) &= d(a, a_0) + d(a_0, b_0) + d(b, b_0) \\
&\leq d(a, b) + d(b, a_0) + d(a_0, b_0) + d(b, b_0) \\
&\leq d(a, b) + d(a_0, b_0) + 2D(R) \\
&< R + 2D(R) + d(a_0, b_0).
\end{aligned}$$

As a consequence, if $R > 0$ and $d(x, y) < R$, then $\rho(x, y) < R + 2D(R) + d(a_0, b_0)$.

Now, by [Wei16, Proposition 4.5], the map between C*-algebras

$$\begin{aligned}
C\big(\partial_h(X, d)\big) = C\big(\partial_h(X, \rho)\big) &\longrightarrow C\big(\partial_h(A, d|_A)\big) \times C\big(\partial_h(B \setminus A, d|_{B\setminus A})\big) \\
[f] &\longmapsto ([f|_A], [f|_{B\setminus A}])
\end{aligned}$$

is an *-isometric isomorphism. Then, by Gelfand-Naimark duality, the Higson corona $\partial_h(X)$ is homeomorphic to $\mathrm{sp}(C(\partial_h(A)) \times C(\partial_h(B))) \cong \partial_h(A) + \partial_h(B \setminus A)$, which is a non-trivial topological sum and thus it is disconnected.

Conversely, suppose that $\partial_h(X)$ is a disconnected space. By Gelfand-Naimark duality, the C*-algebra $B_h(X)/B_0(X)$ has a non-trivial idempotent element $[f]$. In particular, as $[f(f-1)] = 0$, the function $f(f-1)$ vanishes at infinity, so there is a bounded subset $C \subseteq X$ such that $|f(x)(1 - f(x))| < 1/16$ for every $x \in X \setminus C$. As a consequence, the image $f(X \setminus C) \subseteq B(0, 1/4) \cup B(1, 1/4)$. Moreover, as neither f nor $1 - f$ vanish at infinity, the preimages $A := f^{-1}\big(B(0, 1/4)\big)$ and $B := X \setminus A$ are unbounded. On the other hand, as f is Higson, the bounded set C can be enlarged to a bounded subset C_R for which

$$x, x' \notin C_R \text{ and } d(x, x') \leq R \implies |f(x) - f(x')| \leq 1/4.$$

In particular, if $d(x, x') \leq R$ and $x, x' \notin C_R$, then x and x' must either both be in A or both be in B. Consquently, the sets A and B fulfill the two conditions of the theorem. □

Theorem 3.20. ([Wei16, Theorem 7.1]) The Higson corona of a geodesic metric space X is disconnected if and only if there exists a bounded subset $X_0 \subseteq X$ such that for any bounded subset $C \subseteq X$ containing X_0 the complement $X \setminus C$ is disconnected.

Proof. Suppose that $\partial_h(X)$ is disconnected, so there exist unbounded subsets A and B of X fulfilling the Conditions 1 and 2 of Theorem 3.19. In particular, there exists a bounded subset $X_0 \subseteq X$ for which $d(a,b) \geq 1$ if $a \in A \setminus X_0$ and $b \in B \setminus X_0$. Taking a subset $C \subseteq X$ containing X_0, the sets $A \setminus C$ and $B \setminus C$ are an open partition of $X \setminus C$ since for every $a \in A \setminus C$

$$B_{X \setminus C}(a, 1) \subseteq A \setminus C$$

and similarly for $B \setminus C$. Therefore, the complement $X \setminus C$ is disconnected.

Conversely, suppose that $X_0 \subseteq X$ fulfills the condition of the theorem. By assumption, every connected component of $X \setminus X_0$ must have an unbounded complement on $X \setminus X_0$ because if it were not the case, then the union of such bounded complement with X_0 would be a bounded subset containing X_0 with connected complement. It follows that the connected components of $X \setminus X_0$ can be arranged into two unbounded subsets A and B' partitioning $X \setminus X_0$. We argue that the sets A and $B = B' \cup X_0$ satisfy the Condition 2 of Theorem 3.19. Indeed, let $R > 0$ and consider the bounded set

$$C_R := B(X_0, R/2) := \{x \in X \mid d(x, X_0) < R/2\}.$$

As $A \setminus C_R \subseteq A$ and $B \setminus C_R \subseteq B'$, every geodesic connecting a point $a \in A \setminus C_R$ with a point $b \in B \setminus C_R$ must pass through a point $p \in X_0$. Hence,

$$d(a,b) = d(a,p) + d(p,b) \geq R/2 + R/2 = R.$$

□

3.2 Freudenthal Compactification

One of the most successful coarse invariants among finitely generated groups is the *space of ends* of the geometric realizations of their Cayley graphs. For instance, the celebrated Stalling's Theorem (proved in [SW79]) characterizes the algebraic structure of finitely generated groups that have two or more ends. In fact, the first formulation of the space of ends was given by Hans Freudenthal in 1931 in his dissertation precisely motivated by topological and group theoretical questions on topological groups. However, an earlier appearance of such space of ends had taken place in the renowed work of Béla von Kerékjártó about the classification of non-compact surfaces. In [KT09] there is a well-resumed and more complete history of the space of ends as well as many equivalent definitions of said space that will be used in this work.

Intuitively speaking, the ends of a topological space X encode the distinct topological ways one can go to infinity inside the space X. One way to see this is to consecutively remove arbitrarily large compact sets of X and examine the "unbounded" connected components of the remainders. This can be thought as the π_0 *at infinity* of the space X ([DK18, Definition 9.8]), in analogy with the well known set $\pi_0(X)$ of connected components of X, and it turns out to be the original formulation devised by Freudenthal.

Although the space of ends can be defined for all locally compact Hausdorff spaces via proximities ([Ü82], [KT09]), the realization of the space of ends as the π_0 at infinity is (a priori) restricted to locally compact, connected, locally path-connected Hausdorff spaces. Since the main focus is on proper geodesic metric spaces, there is no loss of generality in restricting to such topological spaces. Nevertheless, to see the coarse invariance of the space of ends (Theorem 3.28), we will relate it with the algebra of real-valued continuous functions that assume only finitely many values outside some compact subset. As a consequence, many results concerning this relationship can be stated for general locally compact Hausdorff spaces.

Following [DK18] and Freudenthal's definition recalled on [KT09], the space of ends of a locally compact, connected, locally path-connected Hausdorff space is constructed as follows. Given such space X consider the set $\mathcal{K}$ of compact subspaces of X partially ordered by inclusion. For all $K \in \mathcal{K}$ denote by $\pi_0(K^c)$ the set of connected components of the complement $X \setminus K$. If $K_1 \subseteq K_2$ are compact subspaces of X, there exists a map

$$f_{K_1,K_2} \colon \pi_0(K_2^c) \to \pi_0(K_1^c)$$

which assigns to a connected component of $X \setminus K_2$ the unique connected component of $X \setminus K_1$ that contains it. Notice that the collection of maps $\{f_{K_1,K_2}\}_{K_1 \subset K_2 \in \mathcal{K}}$ is an inverse system. That is to say, the collection is such that

$$f_{K_1,K_2} \circ f_{K_2,K_3} = f_{K_1,K_3} \text{ and } f_{K,K} = \mathrm{Id}\,.$$

Based on this, the space of ends of X is defined as the topological inverse limit of such inverse system:

Definition 3.21. (**Space of Ends**) If X is a locally compact, connected, locally path-connected Hausdorff space and $\mathcal{K}$ is its family of compact subsets ordered by inclusion, the **space of ends** of X is defined as the topological inverse limit

$$\begin{aligned}\mathrm{Ends}\,(X) &:= \varprojlim \pi_0(K^c) \\ &= \left\{ e \in \prod_{K \in \mathcal{K}} \pi_0(K^c) \;\middle|\; e(K_1) = f_{K_1,K_2}\big(e(K_2)\big)\ \forall K_1 \subseteq K_2 \right\},\end{aligned}$$

where each $\pi_0(K^c)$ is regarded as a discrete space. The elements of $\mathrm{Ends}\,(X)$ are known as the **ends** of X.

In technical terms, Definition 3.21 considers an end of X as a map $e\colon \mathcal{K} \to \mathcal{P}(X)$ which sends each compact subset $K \subseteq X$ to a connected component of $X \setminus K$ in a containment-preserving fashion. That is,

$$K_1 \subseteq K_2 \implies e(K_2) \subseteq e(K_1).$$

The topologies on $\mathrm{Ends}\, X$ and X extend to a topology on $\overline{X} := X \cup \mathrm{Ends}\,(X)$, which makes this space a compactification of X known as the **Freudenthal compactification**. Namely, a neighborhood basis at an end $e \in \mathrm{Ends}\,(X) \subset \overline{X}$ is the collection of subsets $\{B_{K,e}\}_{K\in\mathcal{K}} \subseteq \overline{X}$ such that

$$B_{K,e} := e(K) \cup \{e' \in \mathrm{Ends}\,(X) \mid e'(K) = e(K)\}.$$

Proposition 3.22. ([DK18, Exercise 9.9, Exercise 9.13 and Corollary 9.15]) If X is a locally compact, connected, locally path-connected Hausdorff space, then the space of ends $\mathrm{Ends}(X)$ is a totally disconnected compact Hausdorff space (i.e. a Stone space). Additionally, if X is second-countable, then $\mathrm{Ends}(X)$ and $\overline{X}$ are compact metrizable spaces and thus the space of ends $\mathrm{Ends}(X)$ is homeomorphic to a closed subspace of the Cantor set.

It is clear from Definition 3.21 that the space of ends is a topological invariant, as noted by Kerékjártó in his classification of non-compact surfaces. At the same time, such definition reflects the idea of going to infinity by leaving arbitrarily large compact sets. This suggests that the space of ends should also be a coarse invariant among metric spaces that at least fulfill the conditions above, and for proper geodesic spaces it is indeed the case (Corollary 3.34). This, for example, allow us to define unambiguously the space of ends of a finitely generated group, and by Stalling's result such space determines the algebraic structure of infinite finitely generated groups that are not one-ended. For one-ended groups, although an algebraic characterization is still not known, [Wei16] characterizes them as the finitely generated groups with non-empty connected Higson corona. At the end of the present subsection said characterization is reviewed.

By [Ü82, Theorem 2.2, Point (d)] and [KT09], for a general locally compact Hausdorff space the Freudenthal compactification is canonically homeomorphic to the *Gelfand spectrum* of the algebra of real-valued continuous functions that assume only finitely many values outside some compact subset. Since these concepts are central in the subsequent discussion, they are collected in the following definitions.

Definition 3.23. (**Gelfand Spectrum**) The **Gelfand spectrum** of a real algebra A, denoted by $\mathrm{sp}(A)$, is the space of non-trivial multiplicative linear functionals $\omega\colon A \to \mathbb{R}$ equipped with the topology of pointwise convergence.

Definition 3.24. (Freudenthal Function) A real-valued continuous function f on a locally compact Hausdorff space X is said to be a **Freudenthal function** of X if there exists a compact subset $K \subseteq X$ for which the image $f(X \setminus K) \subseteq \mathbb{R}$ is finite. The set of all Freudenthal functions on X is denoted by $C_d(X, \mathbb{R})$.

It is clear that $C_d(X, \mathbb{R})$ is a subalgebra of the algebra of real-valued bounded continuous functions $C_b(X, \mathbb{R})$, and by Ünlü's result cited above its Gelfand spectrum is the Freudenthal compactification of X. However, Ünlü's theorem concerns the realization of the Freudenthal compactification via proximities and the definition of the Gelfand spectrum as the set of maximal ideals equipped with the Zarisky topology; but in the context of the present work, such homeomorphism can be seen as follows. Let X be a locally compact, connected, locally path-connected, second-countable Hausdorff space X (for instance, a proper geodesic metric space). By hypothesis, the space X is Tychonoff and has a base of open sets with compact boundaries, so by [Dom03, Theorem 2.2] the algebra $C_d(X,\mathbb{R})$ separates points from closed sets in X. Consequently, the evaluation map

$$\begin{aligned} X &\longrightarrow \operatorname{sp}(C_d(X,\mathbb{R})) \\ x &\longmapsto \big(f \mapsto f(x)\big) \end{aligned}$$

embeds X as a dense open subset of $\operatorname{sp}(C_d(X,\mathbb{R}))$. It remains to see that the complement of X on $\operatorname{sp}(C_d(X,\mathbb{R}))$ is canonically related to the space of ends $\operatorname{Ends}(X)$. For that purpose, consider an arbitrary end $e \in \operatorname{Ends}(X)$ constructed as in Definition 3.21 and let $f \in C_d(X,\mathbb{R})$ be a Freudenthal function over X. By definition, there exists a compact subset $K \subseteq X$ for which $f(X \setminus K)$ is a finite set. By continuity, the function f is constant on every connected component of the complement $X \setminus K$ and particularly, it is constant on $e(K)$. Denote such constant value on $e(K)$ by $f\big(e(K)\big)$. It can be verified that such end e is associated to the multiplicative linear functional

$$\begin{aligned} \widetilde{e} : C_d(X,\mathbb{R}) &\longrightarrow \mathbb{R} \\ f &\longmapsto f\big(e(K)\big), \end{aligned} \tag{7}$$

which does not depend on the choice of the compact subset $K \subseteq X$ for which $f(X \setminus K)$ is a finite set.

On the other hand, given a multiplicative functional $\omega \in \operatorname{sp}(C_d(X,\mathbb{R})) \setminus X$, consider a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ such that $f(x_n) \to \omega(f)$ for every $f \in C_d(X,\mathbb{R})$. It can be verified that for any compact subset $K \subseteq X$ such sequence is eventually contained on a unique connected component of the complement $X \setminus K$, say $\omega(K)$. Thus, associated to the multiplicative functional ω is the end

$$\begin{aligned} \widetilde{\omega} : \mathcal{K} &\longrightarrow \mathcal{P}(X) \\ K &\longmapsto \omega(K). \end{aligned}$$

A natural consequence of the previous construction is that every Freudenthal function $f \in C_d(X,\mathbb{R})$ can be continuously extended to the Freudenthal compactification $\overline{X}$ (hence the name for such functions) by assigning to every end $e \in \operatorname{Ends}(X)$ the number $\widetilde{e}(f)$ defined on Equation (7). In this context Ünlü's theorem implies that the evaluation map

$$\begin{aligned} C_d(X,\mathbb{R}) &\longrightarrow C\big(\overline{X}\big) \\ f &\longmapsto \big(\xi \mapsto f(\xi)\big), \end{aligned}$$

where the codomain is the C*-algebra of complex-valued continuous functions on $\overline{X}$, is an algebraic embedding for which the C*-algebra generated by the image is precisely $C\big(\overline{X}\big)$. As a result, to see that the Freudenthal compactification of a proper coarse space is a coarse compactification, it is enough to check that every Freudenthal function is a Higson function. In the present work it will be done in the context of proper geodesic metric spaces by first exhibiting a technical property of the Freudenthal functions on such spaces.

Lemma 3.25. Let (X,d) be a proper geodesic metric space with basepoint x_0. For every Freudenthal function $f \in C_d(X,\mathbb{R})$ and every $R > 0$ there exists a $K = K(f,R) > 0$ such that for all $x, y \in X$

$$d(x,y) \leq R \text{ and } d(x,x_0) \geq K \text{ imply } f(x) = f(y). \tag{8}$$

Proof. By properness of the space X, a real-valued continuous function $f\colon X \to \mathbb{R}$ is Freudenthal if and only if there exists $M = M(f) > 0$ such that f assumes only finitely many values outside the closed ball $\overline{B}(x_0,M)$. Since X is geodesic as well, every open ball is path-connected (in particular, it is a geodesic star domain with respect to its center) and thus X is connected and locally path-connected. As a result, the connected components of $X \setminus \overline{B}(x_0,M)$ are clopen in $X \setminus \overline{B}(x_0,M)$ and for that f takes a constant value in each of them. Consequently, the lemma will follow provided that the assumptions proposed in Condition (8) imply that x and y are in the same connected component of $X \setminus \overline{B}(x_0,M)$ for a sufficiently large K depending on R. In fact, it is going to be proved that any $K > R + M$ works for the proposed objective as follows: let $x, y \in X$ be as in Condition (8) and consider any geodesic $c\colon [0, d(x,y)] \to X$

connecting x with y. It is claimed that such geodesic is disjoint from $\overline{B}(x_0, M)$. If this were not the case, then for any $t \in [0, d(x, y)]$ for which $p := c(t) \in \overline{B}(x_0, M)$ it would occur that

$$\begin{aligned} R + M < K \leq d(x, x_0) &\leq d(x, p) + d(p, x_0) \\ &\leq t + M, \end{aligned}$$

thus implying $t \leq d(x, y) \leq R < t$, which is a contradiction. Hence the geodesic c connects x with y entirely outside of the closed ball $\overline{B}(x_0, M)$ suggesting that such points are in the same path-connected component of $X \setminus \overline{B}(x_0, M)$. □

Notice that by path-connectedness and properness, the Condition 8 below is independent of the choice of base point.

Lemma 3.26. Let (X, d) be a proper metric space equipped with the bounded metric coarse structure. Every bounded complex-valued continuous function $f \in C_b(X)$ satisfying Condition (8) with respect to a base point x_0 is a Higson function.

Proof. Let $E \subseteq X \times X$ be a controlled set. That is to say,

$$R := \sup\{d(x, y) \mid (x, y) \in E\} < +\infty.$$

Applying Condition (8) on R considering the number K for which said condition holds and the compact set $B = \overline{B}(x_0, K)$, it is concluded that for every $\epsilon > 0$ and $(x, y) \in E \setminus \big(B \times B\big)$, $|f(x) - f(y)| = 0 < \epsilon$ as required by the Higson condition. □

Remark 3.27. By Lemma 3.26, the gradient (Expression $\big(3\big)$) of the Higson functions $f \in C_h(X)$ fulfilling Condition (8) are compactly supported on every controlled set instead of just vanishing at infinity there.

Combining Lemmas 3.25 and 3.26 the expected coarseness of the end space of a proper geodesic metric space arises as an immediate consequence.

Theorem 3.28. For every proper geodesic metric space X the space of ends $\mathrm{Ends}(X)$ is a coarse compactification with respect to the bounded metric coarse structure.

A corollary of Theorem 3.28 is that the space of ends of a proper geodesic space X is a quotient of the Higson corona $\partial_h(X)$ in a natural way. Namely, since $C_d(X, \mathbb{R}) \subseteq C_h(X)$, the equivalence relation $\sim$ on $\partial_h(X)$ for which $\omega_1 \sim \omega_2$ if $\omega_1(f) = \omega_2(f)$ for every Freudental function f, gives rise to the homeomorphism

$$\begin{aligned} \partial_h(X)/\sim &\longrightarrow \mathrm{sp}(C_d(X, \mathbb{R})/C_c(X, \mathbb{R})) \cong \mathrm{Ends}(X) \\ [\omega] &\longmapsto \big([f] \mapsto \omega(f)\big). \end{aligned} \tag{9}$$

On the other hand, if Y is another proper geodesic space, then every **continuous** coarse map $f\colon X \to Y$ induces the continuous map

$$\begin{aligned} \hat{f}\colon \mathrm{Ends}(X) \cong \mathrm{sp}(C_d(X, \mathbb{R})/C_c(X, \mathbb{R})) &\longrightarrow \mathrm{sp}(C_d(Y, \mathbb{R})/C_c(Y, \mathbb{R})) \cong \mathrm{Ends}(Y) \\ \omega &\longmapsto \big([g] \mapsto \omega([g \circ f])\big), \end{aligned}$$

since the preimage of a compact subset of Y under f is a compact subset of X due to coarseness and continuity. Considering such continuous map along with the analogous map $f_*\colon \partial_h(X) \to \partial_h(Y)$ given by Expression (5) on Theorem 3.12 and the projection maps $\pi_X\colon \partial_h(X) \to \mathrm{Ends}(X)$ and $\pi_Y\colon \partial_h(Y) \to \mathrm{Ends}(Y)$ associated to the homeomorphisms given by Expression (9), the next diagram is commutative:

$$\begin{tikzcd} \partial_h(X) \arrow[r, "f_*"] \arrow[d, "\pi_X"'] & \partial_h(Y) \arrow[d, "\pi_Y"] \\ \mathrm{Ends}(X) \arrow[r, "\hat{f}"] & \mathrm{Ends}(Y). \end{tikzcd}$$

Such commutativity suggests that the quotient induced by the homeomorphism in (9) is naturally preserved by continuous coarse maps between proper geodesic spaces. In what follows it will be verified that such quotient is also preserved by Borel quasi-isometric embeddings between said spaces. Since every quasi-isometric embedding between proper spaces is close to a Borel quasi-isometric embedding, the preservation of such quotient and Corollary 2.22 lead to the expected conclusion that the space of ends is a coarse invariant for proper geodesic spaces and that the way it sits as a quotient of the Higson corona is a coarse invariant as well. For these purposes, it is enough to see that Borel quasi-isometric embeddings preserve the space of ends in a functorial way analogous to the Higson corona in Theorem 3.12.

Following the algebraic ideas used to prove the invariance of the Higson corona, the first thing to do is relate in a canonical way the space of ends of a proper geodesic space X with the Gelfand spectrum of the quotient algebra $C_d(X,\mathbb{R})/C_c(X,\mathbb{R})$. Looking at the treatment done for the Higson corona, Gelfand-Naimark duality would seem the way to go, but the totally disconnectedness mentioned in Proposition 3.22 will allow us to use a natural generalization of the more widely known Stone duality ([Dom03, Theorem 1.8]) by using the real algebra of *locally constant functions* rather than complex C*-algebras of continuous functions. For the sake of concreteness, such locally constant functions are recalled in the following definition.

Definition 3.29. (**Locally Constant Function**) A map $f\colon X \to Y$ from a topological space X to a set Y is **locally constant** if for every $x \in X$ there exists a neighborhood $U \subseteq X$ around x for which the restricted function $f|_U$ is constant.

Notice that every locally constant map is continuous irrespective of the topology considered on the codomain. An useful fact for the proof of the Proposition 3.33 is that any map $f\colon X \to Y$ from a locally connected space to an arbitrary topological space is locally constant if and only if f is constant on any connected component of X.

Definition 3.30. Let X be a topological space. The algebra of real-valued locally constant functions on X is denoted by $\mathrm{LC}(X)$.

Putting the Stone duality recalled in [Dom03, Theorem 1.8] in the context of spaces of ends, it can be concluded that for every pair of locally compact Hausdorff spaces X and Y, their corresponding spaces of ends $\mathrm{Ends}(X)$ and $\mathrm{Ends}(Y)$ are homeomorphic if and only if the algebras $\mathrm{LC}(\mathrm{Ends}(X))$ and $\mathrm{LC}(\mathrm{Ends}(Y))$ are isomorphic. On the other hand, by Ünlü's result [Ü82, Theorem 2.2, Point (d)] previously cited, the corresponding Freudenthal compactifications $\overline{X}$ and $\overline{Y}$ are homeomorphic if and only if the algebras $C_d(X,\mathbb{R})$ and $C_d(Y,\mathbb{R})$ are isomorphic as well. The anticipated relation between these dualities, which is also implicity stated by Ünlü in [Ü82, Theorem 2.2, Point (a)], is recorded in the next proposition.

Proposition 3.31. [Dom03, Theorem 3.2] For every locally compact Hausdorff space X the map

$$\begin{aligned} C_d(X,\mathbb{R})/C_c(X,\mathbb{R}) &\longrightarrow \mathrm{LC}(\mathrm{Ends}(X)) \\ [f] &\longmapsto \overline{f}|_{\mathrm{Ends}(X)}, \end{aligned}$$

where $\overline{f}|_{\mathrm{Ends}(X)}$ is the continuous extension of f to $\overline{X}$ restricted to $\mathrm{Ends}(X)$, is a canonical isomorphism of real algebras.

As seen with the Higson corona, the continuity appearing in Proposition 3.31 can be relaxed to measurability in the case of proper geodesic spaces. The next definition introduces the appropriated algebras of Borel functions for that matter.

Definition 3.32. Let X be a proper geodesic metric space and let $f\colon X \to \mathbb{R}$ be an arbitrary function.

1. The function f **has bounded support** if there exists a bounded set $B \subseteq X$ such that $f(x) = 0$ whenever $x \notin B$.

2. The function f is **coarse Freudenthal** if there exists a bounded set $B \subseteq X$ for which the function f is locally constant outside B.

The real algebra of all Borel coarse Freudenthal functions $X \to \mathbb{R}$ is denoted by $B_d(X,\mathbb{R})$ and the real algebra of all Borel functions $X \to \mathbb{R}$ with bounded support is denoted by $B_c(X,\mathbb{R})$.

According to [Roe93, Page 15], for a proper geodesic space X the equalities $B_d(X,\mathbb{R}) = B_c(X,\mathbb{R}) + C_d(X,\mathbb{R})$ and $C_c(X,\mathbb{R}) = B_c(X,\mathbb{R}) \cap C_d(X,\mathbb{R})$ hold, so by the Second Isomorphism Theorem it follows that

$$\frac{C_d(X,\mathbb{R})}{C_c(X,\mathbb{R})} \cong \frac{B_d(X,\mathbb{R})}{B_c(X,\mathbb{R})}$$

in complete analogy with Theorem 3.10. Consequently, the space of ends $\mathrm{Ends}(X)$ is canonically homeomorphic to the Gelfand spectrum $\mathrm{sp}(B_d(X,\mathbb{R})/B_c(X,\mathbb{R}))$.

Theorem 3.33. Let (X, d_X) and (Y, d_Y) be proper geodesic metric spaces. A Borel quasi-isometric embedding $f\colon X \to Y$ induces in a functorial way the homomorphism

$$\begin{aligned} f^*\colon B_d(Y,\mathbb{R}) &\longrightarrow B_d(X,\mathbb{R}) \\ g &\longmapsto g \circ f \end{aligned}$$

which, in turn, maps $B_c(Y,\mathbb{R})$ to $B_c(X,\mathbb{R})$.

Proof. Let $g \in B_d(Y,\mathbb{R})$ be a Borel coarse Freudenthal function. By properness of Y, without loss of generality it can be supposed that g is locally constant outside a closed ball $\overline{B_Y}(f(x),R) \subseteq Y$ centered at the image of an $x \in X$ via f. Consequently, the function g is constant on every path-connected component of $Y \setminus \overline{B_Y}(f(x),R)$. Thus, in order to see that the composition $f \circ g$ is locally constant outside a closed ball $B := \overline{B_X}(x,r)$ of radius r; it suffices to prove that any path-connected component C of $X \setminus \overline{B_X}(x,r)$ is mapped via f into a path-connected component D of $Y \setminus \overline{B_Y}(f(x),R)$, since in that scenario

$$g \circ f(C) = g\big(f(C)\big) \subseteq g(D)$$

and the latter set is a singleton on $\mathbb{R}$. Following the proof of [DK18, Lemma 9.5], using the multiplicative constant $\lambda \geq 1$ and the additive constant $c \geq 0$ of the map f, it will be seen that any number $r > \lambda(R + \lambda + 2c)$ works. Indeed, it will be proved that the $\lambda + c-$neighborhood

$$\mathcal{N}_{\lambda+c}\big(f(C)\big) = \left\{ y \in Y \;\middle|\; \text{there exists } x' \in C \text{ such that } d\big(y, f(x')\big) < \lambda + c \right\}$$

is a path-connected subset of Y which is disjoint from the ball $\overline{B_Y}\big(f(x),R\big)$. As a consequence, such $\lambda+c$-neighborhood, which certainly contains $f(C)$, will be entirely contained in a path-connected component of $Y \setminus \overline{B_Y}\big(f(x),R\big)$.

To corroborate the disjointness, by contradiction suppose there exists $y \in Y$ and $x' \in C \subseteq X \setminus B$ with the property that $d_Y\big(y, f(x')\big) < \lambda + c$ and $d_Y\big(y, f(x)\big) \leq R$. By the triangle inequality

$$\begin{aligned} \lambda^{-1} d_X(x', x) - c &\leq d_Y\big(f(x), f(x')\big) \\ &\leq d_Y\big(y, f(x)\big) + d_Y\big(y, f(x')\big) \\ &< R + \lambda + c, \end{aligned}$$

which implies that

$$d_X(x, x') < \lambda(R + \lambda + 2c).$$

Since $\lambda(R + \lambda + 2c) < r < d_X(x, x')$ by hypothesis on x', the desired inconsistency is reached.

Now, to see that $\mathcal{N}_{\lambda+c}\big(f(C)\big)$ is path-connected let $y, z \in \mathcal{N}_{\lambda+c}\big(f(C)\big)$. If $z = f(x')$ for a $x' \in C$ and $d_Y(y,z) < \lambda + c$, then for every geodesic $c\colon [0, d_Y(y,z)] \to Y$ connecting y with z and $t \in [0, d_Y(y,z)]$,

$$\begin{aligned} d_Y(c(t), z) &= d_Y(y,z) - d_Y\big(y, c(t)\big) \\ &= d_Y(y,z) - t \leq d_Y(y,z) < \lambda + c, \end{aligned}$$

thus implying that c connects y with z entirely inside $\mathcal{N}_{\lambda+c}\big(f(C)\big)$. Instead, if $y, z \in f(C)$ and $x', x'' \in C$ are such that $f(x') = y$ and $f(x'') = z$, then consider a continuous path p connecting x' with x'' inside C and a sequence of points $x' = x_0, x_1, \dots, x_m = x'' \in C$ such that $d_X(x_i, x_{i+1}) \leq 1$ for all $i = 1, \dots, m-1$. Since

$$d_Y\big(f(x_i), f(x_{i+1})\big) \leq \lambda d_X(x_i, x_{i+1}) + c \leq \lambda + c,$$

every geodesic c_i connecting $f(x_i)$ with $f(x_{i+1})$ in Y has to be in $\mathcal{N}_{\lambda+c}\big(f(C)\big)$. Hence, the concatenation of such $c_i's$ is a path joining y with z entirely inside $\mathcal{N}_{\lambda+c}\big(f(C)\big)$. Since the remaining cases can be reduced to these two, it is concluded that $\mathcal{N}_{\lambda+c}\big(f(C)\big)$ is a path-connected subset of Y.

Finally, to corroborate that $f^*\big(B_d(Y,\mathbb{R})\big) \subseteq B_d(X,\mathbb{R})$, just note that for a function $g \in B_d(Y,\mathbb{R})$ vanishing outside a bounded set $B \subseteq Y$, the function $g \circ f$ vanishes outside the bounded set $f^{-1}(B)$ since

$$g \circ f\big(X \setminus f^{-1}(B)\big) \subseteq g(Y \setminus B) = \{0\}.$$

□

As was discussed before in Theorem 3.33, given Stone duality and the fact that coarse equivalences between proper geodesic spaces are quasi-isometries close to Borel quasi-isometries, they give rise to the coarse invariance of the space of ends and the natural transformation $\pi_- \colon \partial_h(-) \to \mathrm{Ends}(-)$. Such discussion is summed up in the next Corollary.

Corollary 3.34. The space of ends is a functor from the category of proper geodesic spaces with Borel quasi-isometric embeddings (modulo closeness) to the category of compact Hausdorff spaces, which assigns to every Borel quasi-isometric embedding $f\colon X \to Y$ the continuous map

$$\begin{aligned} \hat{f}\colon\ \mathrm{Ends}(X) \cong \mathrm{sp}(B_d(X,\mathbb{R})/B_c(X,\mathbb{R})) &\longrightarrow \mathrm{sp}(B_d(Y,\mathbb{R})/B_c(Y,\mathbb{R})) \cong \mathrm{Ends}(Y) \\ \omega &\longmapsto \big([g] \mapsto \omega([g \circ f])\big), \end{aligned}$$

that induces the commutative diagram

$$\begin{tikzcd} \partial_h(X) \arrow[r, "f_*"] \arrow[d, "\pi_X"'] & \partial_h(Y) \arrow[d, "\pi_Y"] \\ \mathrm{Ends}(X) \arrow[r, "\hat{f}"] & \mathrm{Ends}(Y). \end{tikzcd}$$

Consequently, two coarsely equivalent proper geodesic spaces have homeomorphic spaces of ends that are preserved by the functor arising from the Higson corona.

As was promised, the present subsection is finished with the realization of one-ended groups as the ones which have a non-empty connected Higson corona by using Theorem 3.20 of the previous subsection.

Theorem 3.35. ([Wei16, Corollary 7.3]) A finitely generated group G with the word length metric has a connected Higson corona if and only if it has at most one end.

Proof. Consider a finite symmetric generating set $S \subseteq G$ which does not contain the identity 1_G. Let $\mathrm{Cay}(G,S)$ denote the geometric realization of the Cayley graph of G with respect to S. Since $\mathrm{Cay}(G,S)$ is a proper geodesic space, the compact cover

$$\overline{B(1_G,1)} \subseteq \overline{B(1_G,2)} \subseteq \overline{B(1_G,3)} \subseteq \dots$$

is an *exhaustion* ([DK18, Definition 1.21]) of $\mathrm{Cay}(G,S)$, so by [DK18, Page 291] the ends of G are determined by decreasing sequences

$$U_1 \supseteq U_2 \supseteq U_3 \supseteq \dots,$$

where each U_n is an unbounded connected component of $X \setminus \overline{B(1_g,n)}$.

If G has more than one end, then there exists an $n \in \mathbb{N}$ such that the complement of $\mathrm{Cay}(G,S) \setminus \overline{B(1_G,n)}$ has at least two unbounded connected components. Consequently, if $K \subseteq \mathrm{Cay}(G,S)$ is a bounded subset containing $\overline{B(1_G,n)}$, the complement $X \setminus K$ has at least two unbounded components since the difference $K \setminus \overline{B(1_G,n)}$ is bounded. Then, by Theorem 3.20, G must have a disconnected Higson corona.

Conversely, if G has a disconnected Higson Corona, then by Theorem 3.20 again, there is some $n \in \mathbb{N}$ such that $\mathrm{Cay}(G,S) \setminus K$ contains at least two connected components for every bounded set $K \subseteq \mathrm{Cay}(G,S)$ containing $\overline{B(1_G,n)}$. Since the Cayley graph of G is locally finite, the complement $\mathrm{Cay}(G,S) \setminus \overline{B(1_g,n)}$ has finitely many connected components. It follows that at least two of such connected components must be unbounded since if it were not the case, the union of $\overline{B(1_g,n)}$ with the bounded connected components of $\mathrm{Cay}(G,S) \setminus \overline{B(1_g,n)}$ would be a bounded set containing $\overline{B(1_g,n)}$ with connected complement contradicting Theorem 3.20. Hence, G has more than one end. □

3.3 Visual Compactification

The last compactification to be treated in the present text can be said to concern the behavior of geodesic rays at infinity on a proper hyperbolic space X. At an intuitive level, it reflects the distinct ways one can go to infinity inside the space X in a metric way by following the geodesic rays emanating from a fixed point. Following [BH99], one way to achieve this is by considering (equivalence classes of close) geodesic rays as boundary points in what is called the *visual compactification* of X. Such realization responds to the idea that geodesic rays that do not look alike when seen at a large scale, which is to say that are not close for the bounded coarse structure of X (Example 2.8), should give different points at infinity, and thus they must have a different behavior there as suggested by the non-closeness in the bounded coarse structure.

Since the main spaces worked out in the present subsection are proper hyperbolic spaces, the next definition recalls the notion of hyperbolicity.

Definition 3.36. (**Hyperbolic Space**) Let $\delta \geq 0$. A geodesic triangle in a metric space is said to be **δ-thin** if each of its sides is contained in the δ-neighborhood of the union of the other two sides. A geodesic space is said to be **(δ-)hyperbolic** if there exists a $\delta \geq 0$ such that every triangle in X is δ-thin.

Definition 3.37. (**Visual Boundary for Proper Hyperbolic Spaces**) The **Visual Boundary** of a proper hyperbolic space X, denoted by $\partial(X)$, is the set of equivalence classes of geodesic rays $\gamma\colon [0, +\infty) \to X$, where two geodesic rays γ_1 and γ_2 are **equivalent** if

$$\sup_{t \in \mathbb{R}_{\geq 0}} d\big(\gamma_1(t), \gamma_2(t)\big) < +\infty,$$

equipped with the quotient topology coming from the compact-open topology on the space of geodesic rays of X.

As the name of the visual boundary suggests, $\partial(X)$ is the boundary of a compactification of X which is known as the **visual compactification**. Recalling [BH99, Part III, Definition 3.5], in the visual compactification $\overline{X} := X \cup \partial(X)$ a sequence $\{x_n\} \subseteq X$ is defined to converge to a boundary point $\xi \in \partial(X)$ if there exists a basepoint $p \in X$ and a sequence of geodesics $\{c_n\colon [0, d(p, x_n)] \to X\}$ connecting p with x_n such that when each c_n is extended to a continuous ray by defining $c_n(t) = x_n$ for $t > d(p, x_n)$, the sequence $\{c_n\}$ converges to a geodesic ray $\gamma \in \xi$ starting at p in the compact-open topology.

It can be proved that the convergence above does not depend on the choice of basepoint. Since the remaining theorems and definitions also need such basepoint but do not depend on its choice either, the basepoint p is implicity assumed from now on. Moreover, since by [BH99, Part III, Lemma 3.1] every $\xi \in \partial(X)$ has a representative geodesic ray emanating from p, it will be assumed that every geodesic ray on X starts at p when talking about the visual boundary.

In order to see that the visual compactification of a proper hyperbolic space X is a coarse compactification, the complex-valued continuous functions continuously extendable to $\overline{X}$ are characterized via the *Gromov product* (to be recalled next). Because of that, such functions are called *Gromov functions* in Definition 3.39, and they are shown to be Higson functions in Theorem 3.41.

Definition 3.38. (**Gromov Product**) Let X be a proper hyperbolic space with basepoint p. The **Gromov product** of two elements $x, y \in X$ (with respect to the basepoint p) is defined as

$$(x|y) := \frac{1}{2}\big(d(p, x) + d(p, y) - d(x, y)\big).$$

Definition 3.39. (**Gromov Function**) A **Gromov function** on a proper hyperbolic space X is a continuous function $f\colon X \to \mathbb{C}$ with the property that for every $\epsilon > 0$ there exists $K > 0$ such that

$$(x|y) > K \text{ implies } |f(x) - f(y)| < \epsilon. \tag{10}$$

The set of Gromov functions on X is denoted by $C_g(X)$ and the property above is known as the **Gromov condition**.

Theorem 3.40. A complex-valued continuous function $f\colon X \to \mathbb{C}$ on a proper hyperbolic space is continuously extendable to $\overline{X}$ if and only if it is a Gromov function. Besides, the continuous extension of a Gromov function $f \in C_g(X)$ is given by the map

$$\begin{aligned} \overline{f}\colon \overline{X} &\longrightarrow \mathbb{C} \\ [\gamma] &\longmapsto \left(\lim_{t\to+\infty} f\big(\gamma(t)\big)\right) \end{aligned} \tag{11}$$

Proof. First is corroborated that for every $f \in C_g(X)$ the limit $\lim_{t\to+\infty} f(\gamma(t))$ exists for each geodesic ray $\gamma\colon [0,+\infty) \to X$ starting at p and that such limit does not distinguish between equivalent rays. Indeed, note that for every $t, s \in [0,+\infty)$ with $s \geq t$

$$\begin{aligned} \big(\gamma(t)|\gamma(s)\big) &= \frac{1}{2}\big(d(p,\gamma(t)) + d(p,\gamma(s)) - d(\gamma(t),\gamma(s))\big) \\ &= \frac{1}{2}\big(t + s - s + t\big) = t. \end{aligned}$$

By the above and the Gromov condition on f, for every $\epsilon > 0$, $|f(\gamma(t)) - f(\gamma(s))| < \epsilon$ provided that $t, s > K$. By completeness, this implies that the limit $\lim_{t\to+\infty} f(\gamma(t))$ exists. Now, if γ_1 and γ_2 are equivalent geodesic rays on X at distance c, then for every $t \geq 0$

$$\begin{aligned} \big(\gamma_1(t)|\gamma_2(t)\big) &= \frac{1}{2}\big(d(p,\gamma_1(t)) + d(p,\gamma_2(t)) - d(\gamma_1(t),\gamma_2(t))\big) \\ &\geq t - \frac{c}{2}, \end{aligned}$$

so $f(\gamma_1(t)) - f(\gamma_2(t)) \to 0$ since $|f\big(\gamma_1(t)\big) - f\big(\gamma_2(t)\big)| < \epsilon$ as $t > K + c/2$.

To see that the extension $\overline{f}$ given by Expression (11) is continuous on every boundary point of $\partial(X)$, first consider the case where a sequence $\{x_n\} \subseteq X$ converges to $[\gamma] \in \partial(X)$. As the sequence $\{\gamma(n)\}$ converges to $[\gamma]$ too, by [BH99, Part III, Definition 3.12 and Lemma 3.13], $\lim_{n,m\to+\infty}(x_n|\gamma(m)) = +\infty$. In particular, the sequence $\{(x_n|\gamma(n))\} \to +\infty$, so by the Gromov condition on f

$$|f(x_n) - f(\gamma(n))| \to 0.$$

Since $\{f(\gamma(n))\}$ converges to $\overline{f}([\gamma])$ by definition, the sequence $\{f(x_n)\}$ converges to $\overline{f}([\gamma])$ too. For the case where a sequence of boundary points $\{[\gamma_n]\}_{n\in\mathbb{N}}$ converges to $[\gamma]$, consider the double sequence $\{\gamma_n(m)\}_{n,m\in\mathbb{N}}$ for which

$$\lim_{m\to+\infty} \gamma_n(m) = [\gamma_n]$$

for every $n \in \mathbb{N}$. Since $\overline{X}$ is metrizable ([BH99, Page 433]), passing to subsequences if necessary, it is assumed that $\{\gamma_n(n)\}_{n\in\mathbb{N}}$ converges to $[\gamma]$. Consequently, by the case considered above, $\lim_{n\to+\infty} f\big(\gamma_n(n)\big) = \overline{f}([\gamma])$ and also $\lim_{m\to+\infty} f\big(\gamma_n(m)\big) = \overline{f}([\gamma_n])$ accordingly. Thus, after possibly passing to a subsequence, $\lim_{n\to+\infty} \overline{f}([\gamma_n]) = \overline{f}([\gamma])$. Because every situation of convergence on $\overline{X}$ can be reduced to these two, it is concluded that the extension $\overline{f}$ is continuous.

Reciprocally, assume for the sake of contradiction that $f\colon X \to \mathbb{C}$ is a continuous map which is not a Gromov function but can be continuously extended to $\overline{X}$. By continuity, such extension has to be the map $\overline{f}$ given by Expression (11). Because f is not a Gromov function, there exists an $\epsilon > 0$ and two sequences $\{x_n\}, \{y_n\}$ such that for all $n \in \mathbb{N}$

$$(x_n|y_n) \geq n \text{ but } |f(x_n) - f(y_n)| \geq \epsilon.$$

Notice that, modulo subsequence, at least one of the sequences $\{x_n\}$ or $\{y_n\}$ has to converge to a boundary point $[\gamma]$, since if $\{x_n\}$ and $\{y_n\}$ converged to two respective points $x, y \in X$, then

$$+\infty > (x|y) = \lim_{n\to+\infty}(x_n|y_n) \geq \lim_{n\to+\infty} n,$$

which is not true. Hence, assuming that $\{x_n\}$ converges to $[\gamma]$, by [BH99, Part III, Definition 3.12 and Lemma 3.13] again, $\{y_n\}$ also converges to $[\gamma]$ because $\lim_{n,m\to+\infty}(x_m|y_n) = +\infty$. Then, as f extends continuously to $\overline{X}$,

$$0 = \lim_{n\to+\infty} |f(x_n) - f(y_n)| \geq \epsilon,$$

which is a clear contradiction. Thus, every complex-valued continuous function on X continuously extendable to $\overline{X}$ is necessarily a Gromov function. $\square$

Proposition 3.41. The set of Gromov functions on a proper hyperbolic space X is a unital separable C*-subalgebra of the C*-algebra of Higson functions $C_h(X)$ which contains the algebra $C_0(X)$ as an essential closed *-ideal.

Proof. It is clear that $C_g(X)$ is a complex algebra which, by Theorem 3.40, coincides with the C*-algebra of continuous functions $f\colon X \to \mathbb{C}$ continuously extendable to $\overline{X}$. As the latter is unital, it is contained in $C_b(X)$ and it contains $C_0(X)$ as an essential closed *-ideal, the former algebra inherits said conditions. Moreover, since $\overline{X}$ is metrizable, $C_g(X)$ is separable. Accordingly, it only remains to see that $C_g(X)$ is contained in $C_h(X)$. For such purpose, it is convenient to recall that a bounded continuous function $f\colon X \to \mathbb{C}$ is Higson if and only if for every $r > 0$ and $\epsilon > 0$ there is a $C > 0$ such that

$$d(x,p) > C \text{ and } d(x,y) < r \text{ imply } |f(x) - f(y)| < \epsilon. \tag{12}$$

With that in mind, consider a $K > 0$ such that $(x|y) > K$ implies $|f(x)-f(y)| < \epsilon$. Taking a $C > 2K+r$, it is noted that under the hypotheses of Condition (12) above

$$\begin{aligned}(x|y) &= \frac{1}{2}\big(d(x,p) + d(y,p) - d(x,y)\big)\\ &> \frac{1}{2}\big(C - r + d(y,p)\big)\\ &\geq \frac{C-r}{2} > K,\end{aligned}$$

so certainly $|f(x) - f(y)| < \epsilon$ as desired by the Higson condition.

$\square$

Corollary 3.42. For a proper hyperbolic space the Gromov compactification is a coarse compactification.

By Gelfand-Naimark duality, Theorem 3.40 implicity states that for every proper hyperbolic space X the map

$$\begin{aligned}\partial(X) &\longrightarrow \operatorname{sp}(C_g(X)/C_0(X))\\ [\gamma] &\longmapsto \Big([f] \mapsto \lim_{t\to+\infty} f\big(\gamma(t)\big)\Big)\end{aligned}$$

is a homeomorphism between the visual boundary of X and the spectrum of the C*-algebra $C_g(X)/C_0(X)$. Consequently, since $C_g(X) \subseteq C_h(X)$ as seen in Theorem 3.41, the equivalence relation $\sim$ on $\partial_h(X)$ for which $\omega_1 \sim \omega_2$ if $\omega_1(f) = \omega_2(f)$ for every Gromov function f, gives rise to the homeomorphism

$$\begin{aligned}\partial_h(X)/\sim &\longrightarrow \operatorname{sp}(C_g(X)/C_0(X)) \cong \partial(X)\\ [\omega] &\longmapsto \big([f] \mapsto \omega(f)\big).\end{aligned}$$

This homeomorphism reflects the coarseness of $\partial(X)$ recorded in Corollary 3.42 as a natural quotient map from the Higson corona to the visual boundary.

References

[ALC11] Jesús A. Álvarez López and Alberto Candel. Algebraic characterization of quasi-isometric spaces via the Higson compactification. *Topology Appl.*, 158(13):1679–1694, 2011.

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[DK18] Cornelia Druţu and Michael Kapovich. *Geometric group theory*, volume 63 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2018. With an appendix by Bogdan Nica.

[Dom03] Jesús M. Domínguez. A generating family for the freudenthal compactification of a class of rimcompact spaces. *Fundamenta Mathematicae*, 178:203–215, 2003.

[HR00] Nigel Higson and John Roe. *Analytic K-homology*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000. Oxford Science Publications.

[KT09] Bernhard Krön and Elmar Teufl. Ends—Group theoretical and Topological aspects. *Preprint* revised in homepage.univie.ac.at/bernhard.kroen/kroen_teufl_ends.pdf, 2009.

[Roe93] John Roe. Coarse cohomology and index theory on complete Riemannian manifolds. *Mem. Amer. Math. Soc.*, 104(497):x+90, 1993.

[Roe96] John Roe. *Index theory, coarse geometry, and topology of manifolds*, volume 90 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1996.

[Roe03] John Roe. *Lectures on coarse geometry*, volume 31 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2003.

[SW79] Peter Scott and Terry Wall. Topological methods in group theory. In *Homological group theory (Proc. Sympos., Durham, 1977)*, volume 36 of *London Math. Soc. Lecture Note Ser.*, pages 137–203. Cambridge Univ. Press, Cambridge-New York, 1979.

[Ü82] Yusuf Ünlü. Spaces for which the generalized Cantor space 2^J is a remainder. *Proc. Amer. Math. Soc.*, 86(4):673–678, 1982.

[Wei16] Thomas Weighill. On spaces with connected Higson coronas. *Topology Appl.*, 209:301–315, 2016.

[Wri03] Nick Wright. C_0 coarse geometry and scalar curvature. *J. Funct. Anal.*, 197(2):469–488, 2003.