UNAM-UMSNH
POSGRADO CONJUNTO EN CIENCIAS MATEMÁTICAS

Centro de Ciencias Matemáticas
Instituto de Fisica y Matemáticas

Infinitary combinatorics and its applications

Tesis

que para obtener el grado de

doctor en ciencias matemáticas

presenta

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Morelia, Michoacán   Marzo Del 2022

Agradecimientos

Antes que nada me gustaría agradecer a mi asesor, Michael Hrušák, por todo el conocimiento que me ha transmitido durante estos años y por enseñarme a ver las Matemáticas de una forma muy distinta. Todo el trabajo presente y futuro va a estar fuertemente influenciado por estas enseñanzas. Del mismo modo, me gustaría agradecer a Paul Szeptycki, que a pesar de no aparecer así formalmente, ha sido, de cierta manera, un co-asesor en mi proceso de posgrado.

Para completar el comité que evaluó mi tesis, tuve la suerte de tener a David Meza, que fue mi asesor de licenciatura, a Fernando Hernández, que fue mi asesor en la maestría, y a Osvaldo Guzmán, que aunque nunca fue mi asesor, siempre aprendí mucho de él, incluso en la época en que ambos eramos estudiantes. A todos ellos les agradezco por el tiempo que se tomaron en revisar este trabajo y que le dedicaron a mi formación.

A mi esposa, Thelma, y a mi hijo, Matias. Las personas más importantes en mi vida y mi motivación para todo lo que hago, incluyendo este doctorado.

A mis padres, Ismael y Ana, que se que siempre me van a apoyar y que les debo al menos una pequeña alegría después de los corajes que les he hecho pasar. En general, a toda mi familia: mis hermanas, mis cuñados y todos mis sobrinos (si, también te cuento a ti, Vale). Sin duda hacen mi vida más amena y divertida.

Al CONACyT, que financió todos estos años de trabajo.

A la UNAM y a la Universidad Michoacana. Principalmente a todos los que forman parte del CCM: mis amigos y compañeros que hicieron estos años muy amenos, y todos los que aquí trabajan y nos ayudan en lo posible.

¡GRACIAS!

ABSTRACT

The aim of this thesis is to study some topological problems from a set theoretic point of view. These two areas of mathematics are very related and it is almost impossible to study one of them without reach out the other. Hence, we will focus on set-theoretic topology in order to show the interplay between these two areas.

On the one hand, several topological problems have a combinatorial translation which make them good candidates to be solved with set theoretical tools. We will see examples of this situation using almost disjoint families in chapter 2 and chapter 3 and then, we will see a similar situation using ladder systems in chapter 4.

On the other hand, several new tools and ideas in set theory arose motivated by topological problems. One example are the models of the form $\mathsf{PFA}(S)[S]$ that came up to solve Katětov problem [38]. These kind of models will be studied in chapter 4. Of course, the interplay between set theory and topology is not exclusive. Uniformization properties on ladder systems, for example, were defined by Shelah in his study of Whitehead groups [50].

These coaction of set theory also shows up with other areas like Algebra, Real Analysis, Functional Analysis, Dynamics, Geometry and Algebraic Topology, but we will focus on general and set theoretic topology. Even so, having this in mind we can use similar strategies for solving problems in all other mentioned areas. We now turn to a general description of this work:

In the first chapter we will introduce basic notions which will be used along this work, as well as fix some notation.

In chapter 2 we will study convergence properties on almost disjoint fam-

ilies.[1] Mainly, we will study strong Fréchet properties (like bisequentiality and the concept of absolutely Fréchet) and the α_i properties introduced by Arhangel'skii. We will construct several examples of Fréchet almost disjoint families which satisfy some of the α_i properties whilst fail to be bisequential. We will do this under several assumptions like CH, $\diamondsuit(\mathfrak{b})$ and several cardinal invariant (in)equalities. We will also show that there are absolutely Fréchet non-bisequential almost disjoint families in ZFC.

In chapter 3 we will continue studying almost disjoint families, this time focusing in normality-like properties. It is well known that diverse properties on almost disjoint families imply that the space naturally associated to the family is not normal. Hence, the study of weakenings of normality in the realm of almost disjoint families becomes more interesting. We will show that there are almost disjoint families which are almost-normal but fails to be normal under CH, and no MAD family is almost-normal under PFA. We will also construct more almost disjoint families satisfying specific normality-like properties in ZFC and under CH.

Finally, in chapter 4, we will study uniformization and anti-uniformization properties of ladder systems. We will begin by showing that after forcing with a Suslin tree, every ladder system fails to satisfy most of the uniformization properties considered. Then, we will show that if we first force an intermediate model of the form $\mathsf{PFA}(S)$ (i.e., we have PFA for those posets that preserve a fixed Suslin tree S), then after forcing with the Suslin tree, all ladder systems satisfy some uniformization properties while fail to satisfy any anti-uniformization property. Therefore, we will characterize completely which uniformization and anti-uniformization properties satisfies each ladder system in models of the form $\mathsf{PFA}(S)[S]$.

The main contributions of this thesis are the following:

1. There are α_3 and Fréchet almost disjoint families which are not bisequential under several assumptions ($\mathsf{non}(\mathcal{M}) = \mathfrak{c}$, $\mathfrak{s} \leq \mathfrak{b}$ and in consequence $\mathfrak{c} \leq \aleph_2$, and under $\diamondsuit(\mathfrak{b})$, the latter of size ω_1) [14]. This solves a problem of Gary Gruenhage [28] and additionally, some questions of Peter Nyikos [45].

[1] Every time we talk about a topological property in an almost disjoint family, we are referring to the Ψ-space naturally associated to it.

2. Under CH, there is a countable, α_1, absolutely Fréchet space which is not bisequential [14]. This partially solves two questions of Arhangel'skii [3]. We give an alternative construction with a space of size ω_1 in the last chapter (see [11]).

3. In ZFC, there is a countable absolutely Fréchet space which is not bisequential [11]. This solves an old question of Arhangel'skii [3].

4. Under CH There is an almost-normal almost disjoint family which is not normal and under PFA no MAD family is almost-normal [13]. This consistently solves questions of García-Balán and Szeptycki [26].

5. There is a quasi normal almost disjoint family which fails to be partly normal [13]. This solves a question of García-Balán and Szeptycki [26].

6. Under CH, there is a strongly $\aleph_0$-separated almost disjoint family that is not almost-normal [13]. This solves a question of Oliveira-Rodrigues and Santos-Ronchim [46].

7. We have completely determined which uniformization and anti uniformization properties satisfy ladder systems in models of the form $\mathsf{PFA}(S)[S]$ [15].

Key words: Almost disjoint family, MAD family, ladder system, Fréchet, α_i-property, bisequential, almost normal, uniformization, CH, PFA.

RESUMEN

El propósito de este trabajo es presentar un estudio de diversos problemas topológicos atacados desde un punto de vista de la teoría de conjuntos. Estas dos áreas de la matemática están entrelazadas a tal grado que estudiar una, sin interactuar con la otra, se ha convertido en algo casi imposible de conseguir. A esta interacción suele referirse como la Topología de Conjuntos.

En este texto atacaremos algunos problemas que forman parte de la Topología de Conjuntos, para mostrar la interacción entre estas dos áreas. Por un lado, muchos problemas topológicos tienen una traducción puramente combinatoria que los hace accesibles para ser atacados con herramientas conjuntistas, al menos, dentro de alguna clase especial de espacios que conserva la escencia del problema en general. Ejemplos de este fenómeno son los presentados en los capítulos 2 y 3 usando familias casi ajenas sobre ω, y en el capítulo 4 usando sistemas de escaleras en ω_1.
Por otro lado, muchos problemas topológicos hacen emerger problemas combinatorios que son interesantes por si mismos, y en ocasiones, llevan a desarrollar nuevas herramientas en la teoría de conjuntos. Un ejemplo de este suceso son los modelos de la forma $PFA(S)[S]$ (ver capítulo 4) introducidos por Todorčević y Larson para resolver el problema de Katětov [38]. Por supuesto, la interacción entre estas dos áreas no es exclusiva. Por ejemplo, los sistemas de escaleras y algunas de sus propiedades subyacentes, fueron introducidas por Shelah en su trabajo sobre los grupos de Whitehead [50].

Esta interacción nata de la teoría de conjuntos con la topología, también se ha dado con otras áreas como el Álgebra, la Análisis Real, y más recientemente, con el Análisis Funcional, Dinámica, Geometría y Topología Algebraica. Con esta versatilidad en mente, las estrategías usadas en los

siguientes capítulos bien podrían adaptarse para atacar problemas relacionados a áreas distintas de la Topología de Conjuntos. Ahora daremos una descripción general del trabajo:

En el primer capítulo daremos las definiciones básicas de los conceptos usados a lo largo del texto, a la vez que fijamos cierta notación.

En el capítilo 2, estudiaremos propiedades de convergencia en familias casi ajenas. En particular, estudiaremos fortalecimientos de ser Fréchet (como ser bisecuencial o absolutamente Fréchet), las propiedades α_i introducidas por Arhangelskii, y las relaciones que hay entre estos dos tipos de propiedades. Daremos varios ejemplos de espacios Fréchet satisfaciendo algún α_i y que no son bisecuenciales bajo varios axiomas, como CH, $\diamondsuit(\mathfrak{b})$ y algunas desigualdades entre invariantes cardinales. También construiremos en ZFC una familia casi ajena que es absolutamente Fréchet pero no bisecuencial.

En el capítulo 3, continuaremos con el estudio de familias casi ajenas, esta vez centrándonos en propiedades de tipo normalidad. Es bien sabido que varias propiedades sobre familias casi ajenas implican que su espacio topológico asociado no es normal. Consecuentemente, el estudio de varios debilitamientos de normalidad gana importancia en el contexto de familias casi ajenas. Usando CH y PFA, probaremos que hay familias casi ajenas cuyo espacio no es normal y que pueden, o no, satisfacer la propiedad de casi-normalidad (ver sección 3.1). Construiremos también algunas familias casi ajenas con propiedades específicas de normalidad en ZFC y bajo la prescencia de CH.

Finalmente, en el capítulo 4, estudiaremos propiedades de uniformización y antiuniformización en sistemas de escaleras. Comenzaremos por ver que propiedades cumplen los sistemas de escaleras después de forzar con un árbol de Suslin. Posteriormente, analizaremos que pasa si forzamos primero un modelo intermedio que preserve el árbol de Suslin y después forzamos con el propio árbol de Suslin. El tipo de modelos considerados, son los llamados modelos de $\mathsf{PFA}(S)[S]$. En este caso particular, caracterizaremos el comportamiento de todos los sistemas de escaleras respecto a las propiedades de uniformización y antiuniformización consideradas.

Las principales contribuciones de este trabajo son las siguientes:

1. La construcción de una familia casi ajena que es α_3, Fréchet y no bisecuencial (bajo $\mathsf{non}(\mathcal{M}) = \mathfrak{c}$, $\mathfrak{s} \leq \mathfrak{b}$ y en consecuencia $\mathfrak{c} \leq \aleph_2$ y bajo $\diamondsuit(\mathfrak{b})$, este último de tamaño ω_1) [14]. Respondiendo una pregunta de Gary Gruenhage [28] y adicionalmente algunas preguntas de Peter Nyikos [45].

2. Bajo CH, existe un espacio numerable, α_1, absolutamente Fréchet que no es bisecuencial [14]. Esto resuelve parcialmente dos problemas de Arhangel'skii [3]. Esto lo hacemos en el capítulo 2. Alternativamente, usando forcing damos una versión alternativa con un espacio de tamaño ω_1 en el capítulo 4 (ver [11]).

3. En ZFC, existe un espacio numerable y absolutamente Fréchet no bisecuencial [11]. Esto resuelve un antigua pregunta de Arhangel'skii [3].

4. Bajo CH existe una familia casi ajena, casi normal, que no es normal, mientras que bajo PFA ninguna familia MAD es casi normal [13]. Esto resuelve consistentemente preguntas de García-Balán y Szeptycki [26].

5. Existe una familia casi ajena que es quasi normal pero no es parcialmente normal [13]. Esto resuelve una pregunta de García-Balán y Szeptycki [26].

6. Bajo CH, existe una familia casi ajena que es fuertemente $\aleph_0$-separada y que no es casi normal [13]. Esto resuelve una pregunta de Oliveira-Rodrigues y Santos-Ronchim [46].

7. La determinación de las propiedades de uniformización y antiuniformización que cumplen los sistemas de escaleras en modelos de la forma $\mathsf{PFA}(S)[S]$ [15].

Palabras clave: Almost disjoint family, MAD family, ladder system, Fréchet, α_i-property, bisequential, almost normal, uniformization, CH, PFA.

Palabras clave: Familia casi disjunta, familia MAD, sistema de escalera, Fréchet, αi-property, bisecuencial, casi normal, uniformización, CH, PFA.

Contents

Chapter 1

Introduction and preliminaries

In this short chapter we will fix some notation and terminology as well as define some of the basic notions which will be used along this work. Our set theoretic notation is mainly standard and follows [37]. By $[X]^\kappa$ we denote the set of all subsets of X of size κ and $[X]^{<\kappa} = \bigcup_{\lambda<\kappa}[X]^\lambda$. Similarly, we will use κ^λ to denote the set of all functions $f : \lambda \to \kappa$. A partial function $f; A \to B$ is a function such that $\mathrm{dom}(f)$ is a (possibly proper) subset of A. We will denote by $\mathcal{P}(X)$ the power set of X. By $A \subseteq^* B$ we mean $|A \setminus B| < \omega$ and we will say that A is *almost contained* in B. Given two functions $f, g \in \omega^\omega$, we will say that $f \leq^* g$ if $\{n \in \omega : f(n) > g(n)\}$ is finite, and correspondingly, we will use $f <^* g$ if $\{n \in \omega : f(n) \geq g(n)\}$ is finite. We will also need symbols for quantify for all but finitely many elements of a given set. Then, $\exists^\infty$ and $\forall^\infty$ stand for "there exists infinitely many" and "for all but finitely many", respectively.

Throughout this work, we will name a statement as "Problem" to refer that the statement will be solved along the corresponding chapter. On the other hand, a statement will be called as "Question" if it still remains open. This is done in order to easily distinguish which statements are still the target of future work.

Filters and ideals

Given $\mathcal{F} \subseteq \mathcal{P}(X)$, we will say that $\mathcal{F}$ is a *filter* if

1. $X \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$,
2. $(A \in \mathcal{F}) \wedge (A \subseteq B) \Rightarrow \ B \in \mathcal{F}$ and
3. $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$.

A subset $\mathcal{G} \subseteq \mathcal{P}(X)$ is a *filter base* if $\emptyset \notin \mathcal{G}$ and for every $A, B \in \mathcal{G}$ there exists $C \in \mathcal{G}$ such that $C \subseteq A \cap B$. Then, the filter generated by $\mathcal{G}$ is

$$\langle \mathcal{G} \rangle = \{A \subseteq X : \exists B \in \mathcal{G}(B \subseteq A)\}.$$

The dual notion of a filter on X is called an *ideal* on X. We say that $\mathcal{I} \subseteq X$ is an ideal if $\mathcal{I}^* = \{X \setminus I : I \in \mathcal{I}\}$ is a filter. In the same way, if $\mathcal{F}$ is a filter, $\mathcal{F}^*$ stands for the dual ideal. It follows from the definition that an ideal contains $\emptyset$, does not contain X as an element and it is closed under finite unions and subsets.
We only consider filters $\mathcal{F}$ containing all cofinite sets (i.e., $|X \setminus A| < \omega \Rightarrow A \in \mathcal{F}$), thus every ideal contains all finite subsets of X. Given a family $\mathcal{A} \subseteq \mathcal{P}(X)$, the ideal generated by $\mathcal{A}$ is defined as

$$\mathcal{I}(\mathcal{A}) = \Big\{Y \subseteq X : \exists \mathcal{H} \in [\mathcal{A}]^{<\omega}\Big(Y \subseteq^* \bigcup \mathcal{H}\Big)\Big\}.$$

The family of positive sets $\mathcal{I}^+$, with respect to an ideal $\mathcal{I}$ is $\mathcal{P}(X) \setminus \mathcal{I}$. We will also use $\mathcal{F}^+ = (\mathcal{F}^*)^+$. It is easy to see that $\mathcal{I}^+$ is the family of all subsets of X which intersect every element in the dual filter $\mathcal{I}^*$ in an infinite set.

Almost disjoint families

We will mainly use filters and ideals defined on ω and other countable structures. Of particular interest will be those generated by almost disjoint families. A family $\mathcal{A} \subseteq [\omega]^\omega$ is *almost disjoint* (*AD*), if $A \cap B$ is finite for every $A, B \in \mathcal{A}$. $\mathcal{A}$ is a *maximal almost disjoint* (MAD*)*-family if it is an AD family and it is maximal with respect to this property (i.e., for every

$X \in \omega$, there exists $A \in \mathcal{A}$ such that $A \cap X$ is infinite). Given a family $\mathcal{X} \subseteq \mathcal{P}(\omega)$, and a set $Y \subseteq \omega$, we will say that Y is almost disjoint (AD) with $\mathcal{X}$ if $Y \cap X$ is finite for every $X \in \mathcal{X}$ and

$$\mathcal{X}^{\perp} = \{Y \subseteq \omega : \ \forall X \in \mathcal{X} \ (|X \cap Y| < \omega)\}$$

will denote the sets of all subsets of ω which are AD with $\mathcal{X}$. Finally, for $A, B \subseteq \omega$, we will say that A meets B if $|A \cap B| = \omega$.

Cardinal invariants

The cardinality of $\mathbb{R}$ will be denoted by $\mathfrak{c}$ and will be called continuum. The least size of a MAD family is denoted by $\mathfrak{a}$. For two infinite subsets X, Y of ω, we will say that X *splits* Y if $|X \cap Y| = \omega = |Y \setminus X|$. A family $\mathcal{S} \subseteq [\omega]^\omega$ is called a *splitting family* if for every $X \in [\omega]^\omega$ there is $S \in \mathcal{S}$ such that S splits X. We denote by $\mathfrak{s}$, the minimum size of a splitting family.
For a family of functions $\mathcal{B} \subseteq \omega^\omega$, we will say that $f \in \omega^\omega$ *dominates* $\mathcal{B}$ if $f \geq^* b$ for every $b \in \mathcal{B}$. Then, $\mathcal{B}$ is an *unbounded family* if no single $f \in \omega^\omega$ dominates all functions from $\mathcal{B}$. We will say that $\mathcal{D} \subseteq \omega^\omega$ is a *dominating family* if for every $f \in \omega^\omega$ there is $d \in \mathcal{D}$ such that $d \geq^* f$. The minimum cardinality of an unbounded family is denoted by $\mathfrak{b}$ and the minimum cardinality of a dominating family is denoted by $\mathfrak{d}$. It is easy to show that there are no countable unbounded families by a diagonalization argument and it is also easy to check that every dominating family is also unbounded, hence $\omega_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$. It is also known that $\mathfrak{b} \leq \mathfrak{a} \leq \mathfrak{c}$ and $\omega_1 \leq \mathfrak{s} \leq \mathfrak{d}$. Also, CH is the statement $\mathfrak{c} = \omega_1$, and under CH it is clear that all these cardinal invariants equal ω_1, on the other hand, all possible inequalities stated above can be consistently strict.

Another class of cardinal invariants is defined from an ideal. We will say that $N \subseteq \mathbb{R}$ is *nowhere dense* if $U \setminus \overline{N} \neq \emptyset$ for every $U \subseteq \mathbb{R}$ open. A set $M \subseteq \mathbb{R}$ is *meager* if it is a countable union of closed nowhere dense sets. Thus $\mathcal{M}$ is the ideal on $\mathbb{R}$ generated by the set of meager subsets of $\mathbb{R}$. We define the following cardinal invariants:

1. $\mathsf{add}(\mathcal{M}) = \min\{|\mathcal{X}| : \mathcal{X} \subseteq \mathcal{M} \ \wedge \ \bigcup \mathcal{X} \notin \mathcal{M}\}$,
2. $\mathsf{non}(\mathcal{M}) = \min\{|A| : A \subseteq \mathbb{R} \ \wedge \ A \notin \mathcal{M}\}$,

3. $\mathsf{cov}(\mathcal{M}) = \min\{|\mathcal{X}| : \mathcal{X} \subseteq \mathcal{M} \quad \wedge \ \bigcup \mathcal{X} = \mathbb{R}\}$,

4. $\mathsf{cof}(\mathcal{M}) = \min\{|\mathcal{X}| : \mathcal{X} \subseteq \mathcal{M} \quad \wedge \ \forall M \in \mathcal{M} \ \exists X \in \mathcal{X} \ (M \subseteq X)\}$.

It is easy to see that $\mathsf{add}(\mathcal{M}) \leq \mathsf{non}(\mathcal{M}), \mathsf{cov}(\mathcal{M}) \leq \mathsf{cof}(\mathcal{M})$.

Topology

For a space X we refer to a completely regular topological space. Given $A \subseteq X$, the closure of A will be denoted by $\overline{A}$. Countable unions of closed sets are called F_σ and countable intersections of open sets are called G_δ. We will denote by βX the Stone-Cěch compactification of X.

Given an AD family $\mathcal{A}$ the *Mrówka-Isbell* space $\Psi(\mathcal{A})$ is the space $\omega \cup \mathcal{A}$, where ω is discrete and the basic open neighborhoods of $A \in \mathcal{A}$ are of the form $\{A\} \cup A \setminus n$, i.e., the set $\{n \in \omega : n \in A\}$ converges to A for every $A \in \mathcal{A}$. This space is locally compact, then $\Psi(\mathcal{A})^* = \Psi(\mathcal{A}) \cup \{\infty\}$ will denote its one-point compactification. Following [28], we will call the subspace $\omega \cup \{\infty\}$ of $\Psi(\mathcal{A})^*$ the *AD space generated by* $\mathcal{A}$. Notice that a basic neighborhood of ∞ in the AD space generated by $\mathcal{A}$ is of the form $(\{\infty\} \cup \omega) \setminus (F \cup \bigcup \mathcal{B})$, where F is a finite subset of ω and $\mathcal{B}$ is a finite subset of $\mathcal{A}$. Hence, it is easy to see that a sequence $S \subseteq \omega$ converges to ∞ if and only if $S \cap A$ is finite for every $A \in \mathcal{A}$. Similarly, $\infty \in \overline{S}$, if and only if $S \setminus \bigcup \mathcal{B} \neq \emptyset$ for every finite subset $\mathcal{B} \subseteq \mathcal{A}$.

We will say that an AD family $\mathcal{A}$ *satisfies a topological property* $\mathcal{P}$ if the AD space associated $\omega \cup \{\infty\}$ does. For more on AD families and Mrówka-Isbell spaces see [35, 34]. As a general reference for topology we refer the reader to [22].

Clubs, stationary sets and guessing principles in ω_1

We will denote the set of limit ordinals of ω_1 by $\lim(\omega_1)$. We say that $X \subseteq \omega_1$ is *unbounded* if for every $\alpha \in \omega_1$ there exists $\beta \in X \setminus \alpha$. A set $C \subseteq \omega_1$ is a *club* if it is **cl**osed (with respect to the usual topology on ω_1) and **un**bounded. Since the intersection of countable many clubs is a club, we can consider the filter $\mathrm{Club}(\omega_1)$ generated by all club subsets of ω_1. Of particular interests are the positive sets with respect to this filter: A subset

$S \subseteq \omega_1$ is *stationary* if $S \cap C \neq \emptyset$ for every club C. In particular, the next result about stationary sets will be useful.

Lemma 1.0.1. *Fodor Let $S \subseteq \omega_1$ be a stationary set and $f : S \to \omega$ be a regressive function (i.e., $f(\alpha) < \alpha$ for every $\alpha \in S$), then there exists $\beta \in \omega_1$ such that $f^{-1}(\beta)$ is stationary.* $\square$

With the definition of stationary sets at hand, we can define the following guessing principle: *Jensen's diamond principle* ($\diamondsuit$)

$$\diamondsuit \equiv \text{There is a sequence } \{A_\alpha : \alpha < \omega_1\} \text{ such that } A_\alpha \subseteq \alpha \text{ and for all}$$

$$X \subseteq \omega_1 \text{ the set } \{\alpha \in \omega_1 : X_\alpha \cap \alpha = A_\alpha\} \text{ is stationary.}$$

It is easy to see that $\diamondsuit$ implies CH. In the forthcoming chapters, we will use some weak versions of $\diamondsuit$, namely, the parametrized diamond principle $\diamondsuit(\mathfrak{b})$ (see [44]) and Ostaszewski's principle $\clubsuit$ (see [47]).

Trees

Recall that a tree T is a partially ordered set, such that for all $t \in T$, the set of predecessors of t is well ordered. For every $t \in T$, $l_T(t)$ is the order type of $\{s \in T : s < t\}$ and is called the length of t. We will omit the subindex T when no confusion arises. The level α of a tree T is $\mathrm{Lev}_\alpha(T) = \{t \in T : l(t) = \alpha\}$ and $T_\alpha = \bigcup_{\beta<\alpha} \mathrm{Lev}_\beta(T)$. We will say that a tree T has height α if $\alpha = \min\{\beta : \mathrm{Lev}_\beta(T) = \emptyset\}$ and we will denote it by $h(T) = \alpha$. Elements of trees will be called *nodes*. Two nodes $s, t \in T$ are *incomparable* if $s \nleq t$ and $t \nleq s$. An antichain of a tree T is a subset consisting of pairwise incomparable nodes. A branch of T is a maximal chain in T, i.e., a maximal set of pairwise comparable nodes. We will say that a branch b is *cofinal* in T if $\mathrm{Lev}_\alpha(T) \cap b \neq \emptyset$ for every $\alpha < h(T)$.
An *Aronszajn tree* is a tree T of height ω_1 with countable levels and such that there is no cofinal branch. Similarly, a *Souslin tree* is a tree of height ω_1 with no uncountable branches and no uncountable antichains. Aronszajn trees do exist in ZFC whilst Souslin trees exist only under some assumptions beyond ZFC (like $\diamondsuit$).

We will use trees as forcing notions with the reverse order. Some previous knowledge of forcing and elementary submodels is assumed in the last chapter. The rest of this work can be read without it, with some minor exceptions.

Chapter 2

Fréchet-like properties in almost disjoint families

Recall that a point x in a topological space X is a *Fréchet point* if whenever $x \in \overline{A} \subseteq X$, there is a sequence $\{x_n : n \in \omega\} \subseteq A$ such that $x_n \to x$. A space X is *Fréchet* if every point $x \in X$ is a Fréchet point.

Recall also ([3]) that a point $x \in X$ is an *α_i-point* $(i = 1, 2, 3, 4)$ if given a family $\{S_n : n \in \omega\}$ of sequences converging to x, there is a sequence $S \to x$ (we identify a convergent sequence with its range) such that

(α_1) $S \setminus S_n$ is finite for all $n \in \omega$,

(α_2) $S \cap S_n \neq \emptyset$ for all $n \in \omega$,

(α_3) $|S \cap S_n| = \omega$ for infinitely many $n \in \omega$,

(α_4) $S \cap S_n \neq \emptyset$ for infinitely many $n \in \omega$.

Notice that for an α_2-point, it is equivalent that $|S \cap S_n| = \omega$ for every $n \in \omega$. With this in mind, it should be obvious that the properties get progressively weaker. A space X is an *α_i-space* if every point $x \in X$ is an α_i-point. We say that a space X is *α_i-FU* if X is both, Fréchet and α_i.

Definition 2.0.1. [3] A space X is *absolutely Fréchet* if every $x \in X$ is a Fréchet point in every compactification bX of X.

We now show that absolutely Fréchetness only needs to be checked for a fixed arbitrary compactification bX. This result is mentioned in Arhangel'skii's paper [3] and it is probably folklore, however, we were not able to find a good reference for this result with its proof. We need the following preliminary lemma:

Lemma 2.0.2. *Let $\varphi : K \to C$ be a continuous function between compact spaces and let $M \subseteq K$. Then $\varphi[\overline{M}] = \overline{\varphi[M]}$.*

Lemma 2.0.3. *Let X be a space and let bX be a compactification of X. If $x \in X$ is a Fréchet point in bX, then x is a Fréchet point in every compactification of X*

Proof. Assume $x \in X$ is a Fréchet point in bX. We begin by showing that $x \in X$ is Fréchet in βX.
Let $\varphi : \beta X \to bX$ be the continuous extension of the identity map on X. Let $M \subseteq \beta X$ such that $x \in \overline{M}$. Then $x \in \varphi[\overline{M}] = \overline{\varphi[M]}$ since φ is a continuous function between compact spaces. Let $\{y_n : n \in \omega\} \subseteq \varphi[M]$ such that $y_n \to x$. Take $x_n \in \varphi^{-1}(y_n) \cap M$ for every $n \in \omega$. Thus

$$\varphi\left[\overline{\{x_n : n \in \omega\}}\right] = \overline{\{y_n : n \in \omega\}} \ni x.$$

Then x is in the closure of $\{x_n : n \in \omega\}$. To see this, remember that $\varphi[\beta X \setminus X] = bX \setminus X$, and since $\varphi \restriction X$ is the identity, $\varphi^{-1}(x) = \{x\}$.

The same argument shows that x is in the closure of $\{x_n : n \in A\}$ for every $A \in [\omega]^\omega$, which implies that $x_n \to x$.

Now let γX be a compactification of X. Let $N \subseteq \gamma X$ and $x \in X$ such that $x \in \overline{N}$. Let $f : \beta X \to \gamma X$ be the continuous extension of the identity map and consider $B = f^{-1}[N]$. It follows that $x \in \overline{B}$ since

$$f[\overline{B}] = \overline{f[B]} = \overline{N} \ni x,$$

(we have used that f is onto and the previous lemma). Then we can take a sequence $\{x_n : n \in \omega\} \subseteq B$ such that $x_n \to x$ and in consequence $y_n \to x$ where $y_n = f(x_n) \in N$. □

Given a family $\mathcal{A} \subseteq \mathcal{P}(X)$ we will say that $x \in \overline{\mathcal{A}}$ if $x \in \overline{A}$ for every $A \in \mathcal{A}$. A filter base $\mathcal{G}$ converges to a point $x \in X$ if for every neighborhood U of x, there is a $G \in \mathcal{G}$ such that $G \subseteq U$. We then write $\mathcal{G} \to x$. Given a filter $\mathcal{F}$, recall that $\mathcal{F}^+$ denotes the family of all sets which intersects every element of $\mathcal{F}$ in an infinite set.

Definition 2.0.4. [42] X is *bisequential* at $x \in X$ if for every filter $\mathcal{F}$ in X such that $x \in \overline{\mathcal{F}}$, there is a decreasing sequence $\{G_n : n \in \omega\} \subseteq \mathcal{F}^+$ such that $\{G_n\}_{n\in\omega} \to x$. A space X is bisequential if it is bisequential at every point.

Bisequentiality was introduced by E. Michael in his study of general types of mappings [42] and the concept of Absolutely Fréchet (as well as the α_i-properties), were introduced by A. Arhangel'skii in [3], where he also studied bisequentiality and the effects of all these properties in the product of Fréchet spaces.

All these concepts are related; every bisequential space is absolutely Fréchet and every absolutely Fréchet space is, of course, Fréchet [3]. Concerning the α_i-properties, every absolutely Fréchet space is α_4, and every bisequential space is α_3.

Most of the properties defined so far, impose certain conditions in the product of Fréchet spaces. For instance, if X is bisequential and Y is α_4-FU, then $X \times Y$ is Fréchet [3].

Notice that the study of α_i-spaces could be restricted to countable spaces, since a space X is α_i if and only if every countable subset of X is.

We will deal with G. Gruenhage's question of whether the properties of α_3-FU and bisequentiality are equivalent for AD spaces [28]. As a by-product we also solve some questions of Nyikos [45], and the construction gives new consistent examples of absolutely Fréchet spaces with strong α_i-properties which are not bisequential.

We will say that an AD family $\mathcal{A}$ is *hereditarily* α_3 if for every $\mathcal{B} \subseteq \mathcal{A}$, $\mathcal{B}$ is α_3. Since $\mathcal{B} \subseteq \mathcal{A}$ is Fréchet for every Fréchet AD family $\mathcal{A}$, hereditarily α_3-FU is the same as Fréchet and hereditarily α_3.

Problem. [28] *Is every α_3-FU (hereditarily α_3-FU) AD family $\mathcal{A}$ bisequential?*

Recall that if $\mathcal{A}$ is bisequential then it is hereditarily α_3-FU [28] and clearly, every hereditarily α_3-FU is α_3-FU.

Figure 2.1 shows ZFC implications between these properties (of course, hereditarily α_3 only makes sense for almost disjoint families).

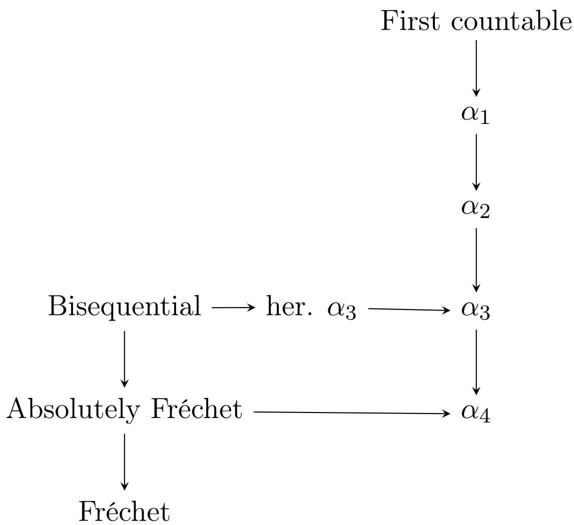

Figure 2.1: Fréchet-like properties.

2.1 AD spaces and bisequentiality

A large class of AD families is bisequential, namely, those that are $\mathbb{R}$-embeddable. An AD family $\mathcal{A}$ is $\mathbb{R}$-embeddable [33] if there is a one-to-one function $f : \omega \to \mathbb{Q}$ which extends to a continuous one-to-one $\widehat{f} : \psi(\mathcal{A}) \to \mathbb{R}$. However, there are ZFC examples of bisequential $\mathcal{A}$ that are not $\mathbb{R}$-embeddable. On the other hand, under $\mathfrak{b} = \mathfrak{c}$, there is an AD family which is not even α_3 [45]. We will prove that under the same assumption, there is an α_3-FU AD family which is not bisequential, but before, we are going to give combinatorial characterizations of these properties for AD families. Since ω is a discrete subspace of the AD space of $\mathcal{A}$, the only point of interest is ∞. A sequence $X \subseteq \omega$ converges to ∞ iff $X \in \mathcal{A}^\perp$ (we are identifying a sequence with its range since the space is Hausdorff, see the introduction for the definition of $\mathcal{A}^\perp$). Also, $\infty \in \overline{X}$ iff $X \in \mathcal{I}(\mathcal{A})^+$. Then, an AD family $\mathcal{A}$ is Fréchet iff it is nowhere MAD, i.e., for every $X \in \mathcal{I}(\mathcal{A})^+$, there exists $Y \in \mathcal{A}^\perp$ such that $|Y \cap X| = \omega$. The family $\mathcal{A}$ is α_3 iff for every sequence $\{X_n : n \in \omega\} \subseteq \mathcal{A}^\perp$ there is an $X \in \mathcal{A}^\perp$ which intersects infinitely many X_n in an infinite set. We will need the following fact:

Theorem 2.1.1. [8] *The cardinal* $\mathsf{non}(\mathcal{M})$ *is the smallest size of a family* $\mathcal{F} \subseteq \omega^\omega$ *such that*

$$\forall g \in \omega^\omega\ \exists f \in \mathcal{F}\ \exists^\infty n \in \omega\ (f(n) = g(n)).$$

□

Then for every family $\mathcal{F} \subseteq \omega^\omega$ of size less than $\mathsf{non}(\mathcal{M})$, there is a function $g \in \omega^\omega$ which is eventually different from $\mathcal{F}$, i.e., for every $f \in \mathcal{F}$ and all but finitely many $n \in \omega$, $g(n) \neq f(n)$. Moreover, a slight modification to the proof shows that for every of these families $\mathcal{F}$, the set of functions which are eventually different from $\mathcal{F}$ is not meager. Therefore we get the next corollary:

Corollary 2.1.2. *For every* $\mathcal{F} \subseteq \omega^\omega$ *of size less than* $\mathsf{non}(\mathcal{M})$ *and every* G_δ *dense subset* $G \subseteq \omega^\omega$*, there exists a function* $g \in G$ *which is eventually different from* $\mathcal{F}$*.* □

Lemma 2.1.3. *Let* $\mathcal{A}$ *be an* α_3 *AD family that is not hereditarily* α_3*. Then, there is* $\mathcal{B} = \{A_n : n \in \omega\} \subseteq \mathcal{A}$ *such that* $\mathcal{A} \setminus \mathcal{B}$ *is not* α_3*. Moreover, the family* $\{A_n : n \in \omega\}$

Proof. Let $\mathcal{C} \subseteq \mathcal{A}$ that is not α_3. Pick a sequence $\{D_n : n \in \omega\}$ witnessing this. Hence each D_n is AD with $\mathcal{C}$ and since $\mathcal{A}$ is α_3, we can assume (taking a subsequence of $\{D_n : n \in \omega\}$ if necessary) that for every $n \in \omega$, there exists $A(n) \in \mathcal{A}$ such that $|A(n) \cap D_n| = \omega$. By shrinking each D_n we can further assume that $D_n \subseteq A(n)$. Then if we define $\mathcal{B} = \{A(n) : n \in \omega\}$, it is clear that the same sequence witnesses that $\mathcal{A} \setminus \mathcal{C}$ is not α_3. □

Lemma 2.1.4. *An almost disjoint family* $\mathcal{A} \subseteq [\omega]^\omega$ *is* α_3*-FU and non-hereditarily* α_3 *if*

1. $\mathcal{A}$ *is nowhere MAD.*

2. $\forall\, (D_n : n \in \omega) \subseteq \mathcal{A}^\perp\ \exists Y \in \mathcal{A}^\perp\ (|\{n \in \omega : |Y \cap D_n| = \omega\}| = \omega)$.

3. $\exists\{A_n : n \in \omega\} \subseteq \mathcal{A} \ \ \forall X \in [\omega]^\omega$

$$\left(\left|\left\{n \in \omega : \left|X \cap A_n\right| = \omega\right\}\right| = \omega\right) \implies \left(\exists A \in \mathcal{A}' \ \left|A \cap X\right| = \omega\right),$$

where $\mathcal{A}' = \mathcal{A} \setminus \{A_n : n \in \omega\}$.

Proof. It is well known that an AD family is Fréchet if and only if it is nowhere MAD and the equivalence of being α_3 with point 2 follows easily from the observation that a sequence $S \subseteq \omega$ converges to ∞ if and only if $S \in \mathcal{A}^\perp$. Hence, we only need to proof that $\mathcal{A}$ is non-hereditarily α_3 if it satisfies item 3. If item 3 holds, then $\{A_n : n \in \omega\}$ is a family of convergent sequences to ∞, such that no $X \subseteq \omega$ that meets infinitely many A_n converges to ∞. That is, $\mathcal{A}'$ is not α_3. $\square$

Notation 2.1.5. A column in $\omega \times \omega$ will be a set of the form $\{n\} \times \omega$ for some $n \in \omega$. For an indexed set $\mathcal{H} = \{H_\alpha : \alpha < \kappa\}$ and $\eta < \kappa$ we denote the restriction of $\mathcal{H}$ to η by $\mathcal{H}_\eta = \{H_\alpha : \alpha < \eta\}$.

Theorem 2.1.6. ($\mathsf{non}(\mathcal{M}) = \mathfrak{c}$) *There is an α_3-FU AD family $\mathcal{A}$ such that $\mathcal{A}$ is not hereditarily α_3-FU.*

Proof. We will build recursively three families $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$, $\mathcal{Y} = \{Y_\alpha : \omega \leq \alpha < \mathfrak{c}\}$ and $\mathcal{Z} = \{Z_\alpha : \omega \leq \alpha < \mathfrak{c}\}$ such that:

1. $A_n = \{n\} \times \omega$ for every $n \in \omega$,
2. A_α, Y_α and Z_α are graphs of functions for every infinite ordinal $\alpha < \mathfrak{c}$,
3. Y_α and Z_α are AD with $\mathcal{A}_\alpha$ for every infinite $\alpha < \mathfrak{c}$ and
4. A_α is AD with $\mathcal{A}_\alpha \cup \mathcal{Y}_{\alpha+1} \cup \mathcal{Z}_{\alpha+1}$ for every infinite $\alpha < \mathfrak{c}$.

For every function $f \in \omega^\omega$ we will use f for both, the function and its graph as a subset of $\omega \times \omega$. Enumerate $([\omega \times \omega]^\omega)^\omega = \{\overrightarrow{D_\alpha} : \omega \leq \alpha < \mathfrak{c}\}$ and $[\omega \times \omega]^\omega = \{X_\alpha : \omega \leq \alpha < \mathfrak{c}\}$. Then, each $\overrightarrow{D_\alpha} = (D_{\alpha,n} : n \in \omega)$. Assume that we have defined A_β for $\beta < \alpha$ and Y_β, Z_β for $\omega \leq \beta < \alpha$.

If $X_\alpha \in \mathcal{I}(\mathcal{A}_\alpha)^+ \subseteq \mathcal{I}(\mathcal{A}_\omega)^+$, the set

$$G = \{f \in \omega^\omega : |f \cap X_\alpha| = \omega\}$$

is G_δ and dense in ω^ω. Hence, by Corollary 2.1.2, we can find a function $Y_\alpha \in \omega^\omega$ such that Y_α meets X_α and Y_α is eventually different from A_β for every $\beta < \alpha$. This in particular implies that Y_α is AD with $\mathcal{A}_\alpha$ as graphs of functions. If $X_\alpha \notin \mathcal{I}(\mathcal{A}_\alpha)^+$, define Y_α AD with $\mathcal{A}_\alpha$ arbitrarily using Corollary 2.1.2.

Whenever $D_{\alpha,n}$ is AD with $\mathcal{A}_\alpha$, it must intersects infinitely many columns. Thus, if $D_{\alpha,n}$ is AD with $\mathcal{A}_\alpha$ for every $n \in \omega$, the set

$$G = \{f \in \omega^\omega : \forall n \in \omega \ (|f \cap D_{\alpha,n}| = \omega)\}$$

is a dense G_δ subset of ω^ω. Applying Corollary 2.1.2, we can find a function $Z_\alpha \in \omega^\omega$ such that Z_α is AD with $\mathcal{A}_\alpha$ and Z meets $D_{\alpha,n}$ for all $n \in \omega$. If $D_{\alpha,n} \cap A$ is finite for some $n \in \omega$ and $A \in \mathcal{A}_\alpha$, define $Z_\alpha \in \omega^\omega$ AD with $\mathcal{A}_\alpha$ arbitrarily.

Finally, if there are infinitely many $n \in \omega$ such that X_α meets A_n (remember that X meets A stands for $|X \cap A| = \omega$), then $X_\alpha \in \mathcal{I}(\mathcal{A}_\alpha)^+$ and we already know that the set

$$G = \{f \in \omega^\omega : |f \cap X_\alpha| = \omega\}$$

is G_δ and dense. Then we can find A_α AD with $\mathcal{A}_\alpha \cup \mathcal{Y}_{\alpha+1} \cup \mathcal{Z}_{\alpha+1}$ such that A_α meets X_α. Otherwise, chose A_α AD with $\mathcal{A}_\alpha \cup \mathcal{Y}_{\alpha+1} \cup \mathcal{Z}_{\alpha+1}$ arbitrarily. This finishes the construction.

We will show that $\mathcal{A} = \{A_\beta : \beta < \mathfrak{c}\}$ is the desired family. From the definition it is clear that $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ is almost disjoint. Given $X \in \mathcal{I}(\mathcal{A})^+$ there exists $\alpha < \mathfrak{c}$ such that $X = X_\alpha$ and then Y_α meets X since $\mathcal{I}(\mathcal{A})^+ \subseteq \mathcal{I}(\mathcal{A}_\alpha)^+$. Moreover, since A_β is AD with $\mathcal{Y}_{\beta+1}$ for every $\beta \geq \alpha$, the set Y_α is AD with $\mathcal{A}$. Hence $\mathcal{A}$ is Fréchet. The same idea shows that $\mathcal{A}$ is α_3 (even α_2) using Z_α as a witness for the sequence of convergent sequences $\overrightarrow{\mathcal{D}_\alpha}$.

Now define $\mathcal{B} = \mathcal{A} \setminus \mathcal{A}_\omega$. Then A_n is AD with $\mathcal{B}$ for every $n \in \omega$, but for every possible witness $X \subseteq \omega \times \omega$ for the property α_3, i.e., for every X such that X meets A_n for infinitely many $n \in \omega$, there exists $\omega \leq \alpha < \mathfrak{c}$ such that $X = X_\alpha$ and then $A_\alpha \in \mathcal{B}$ satisfies that A_α meets X. In consequence X is not AD with $\mathcal{B}$, which shows that $\mathcal{A}$ is not hereditarily α_3. □

2.2 The splitting and unbounding numbers

A MAD family is said to be *completely separable* if for every $X \in \mathcal{I}(\mathcal{A})^+$ there is an $A \in \mathcal{A}$ such that $A \subseteq X$. It was shown by Balcar and Simon (see [5]) that completely separable MAD families exists under one of the following axioms: $\mathfrak{a} = \mathfrak{c}$, $\mathfrak{b} = \mathfrak{d}$, $\mathfrak{d} \leq \mathfrak{a}$ and $\mathfrak{s} = \omega_1$. A more general theorem was proved by Shelah, who proved that completely separable MAD families exists if either $\mathfrak{s} < \mathfrak{a}$ or if $\mathfrak{s} = \mathfrak{a}$ and a certain PCF-hypothesis holds or if $\mathfrak{s} > \mathfrak{a}$ and a stronger PCF-hypothesis holds. The method of Shelah is a powerful tool to construct almost disjoint families and it was improved by Mildenberg, Raghavan and Steprāns in [43], eliminating the PCF-hypothesis in the case $\mathfrak{s} = \mathfrak{a}$. This improvement was the result of the introduction of a new cardinal invariant $\mathfrak{s}_{\omega,\omega}$ which turned out to be equal to $\mathfrak{s}$. Recall that a family $\mathcal{S} \subseteq [\omega]^\omega$ is *splitting* if for every $X \in [\omega]^\omega$ there is $S \in \mathcal{S}$ such that $|X \cap S| = |X \setminus S| = \omega$ and we will say that it is (ω, ω)-*splitting* if for every sequence $\langle X_n : n \in \omega\rangle \subseteq [\omega]^\omega$, there is $S \in \mathcal{S}$ such that the sets $\{n \in \omega : |X_n \cap S| = \omega\}$ and $\{n \in \omega : |X_n \setminus S| = \omega\}$ are both infinite. Thus, $\mathfrak{s}$ is the least size of a splitting family and $\mathfrak{s}_{\omega,\omega}$ is the least size of a (ω, ω)-splitting family. In [43], it is proved that $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$. The importance of (ω, ω)-splitting families is due to the next result:

Lemma 2.2.1. [49] *let $\mathcal{S}$ be an (ω, ω)-splitting family and let $\mathcal{A}$ be an AD family. For every $X \in \mathcal{I}(\mathcal{A})^+$ there exists $S \in \mathcal{S}$ such that $S \cap X \in \mathcal{I}(\mathcal{A})^+$ and $(\omega \setminus S) \cap X \in \mathcal{I}(\mathcal{A})^+$.*

Notation 2.2.2. For a set $X \subseteq \omega$ we will denote $X^0 = X$ and $X^1 = \omega \setminus X$.

We will also need the following fact about ideals defined from an AD family:

Lemma 2.2.3. [4] *Given an AD family $\mathcal{A}$, for every sequence of decreasing positive sets $\{X_n : n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$, there exists $Y \in \mathcal{I}(\mathcal{A})^+$ such that $Y \subseteq^* X_n$ for every $n \in \omega$.*

Theorem 2.2.4. ([52],[43]) *Assume $\mathfrak{s} \leq \mathfrak{a}$. Then there is a completely separable MAD family.*

Proof. Let $\{S_\alpha : \alpha < \mathfrak{s}\}$ be an (ω, ω)-splitting family and let $[\omega]^\omega = \{X_\alpha : \alpha < \mathfrak{c}\}$. We will recursively construct $\{A_\alpha : \alpha < \mathfrak{c}\}$ and $\{\sigma_\alpha : \alpha < \mathfrak{c}\} \subseteq 2^{<\mathfrak{s}}$ such that:

1. $\{A_\alpha : \alpha < \mathfrak{c}\}$ is a MAD family.

2. If $X_\alpha \in \mathcal{I}(\mathcal{A}_\alpha)^+$ then $A_\alpha \subseteq X_\alpha$.

3. $A_\alpha \subseteq^* S_\eta^{\sigma_\alpha(\eta)}$ for all $\alpha < \mathfrak{c}$ and all $\eta \in dom(\sigma_\alpha)$.

4. If $\alpha < \beta$ then $\sigma_\beta \nsubseteq \sigma_\alpha$.

Assume we have constructed $\mathcal{A}_\alpha = \{A_\beta : \beta < \alpha\}$ and $\{\sigma_\beta : \beta < \alpha\}$. Let $X = X_\alpha$ if $X_\alpha \in \mathcal{I}(\mathcal{A}_\alpha)^+$ and let X be any element in $\mathcal{I}(\mathcal{A}_\alpha)^+$ otherwise.

For each $s \in 2^{<\omega}$ define recursively $X_s \in \mathcal{I}(\mathcal{A}_\alpha)^+$ and $\tau_s \in 2^{<\mathfrak{s}}$ such that $\tau_s \subseteq \tau_t$ and $X_s \supseteq X_t$ whenever $s \subseteq t$ and such that $X_s \cap S_\xi^{1-\tau_s(\xi)} \in \mathcal{I}(\mathcal{A}_\alpha)$ for every $\xi \in dom(\tau_s)$ while $X_s \cap S_{|\tau_s|} \in \mathcal{I}(\mathcal{A}_\alpha)^+$ and $X_s \setminus S_{|\tau_s|} \in \mathcal{I}(\mathcal{A}_\alpha)^+$. Start with $X_\emptyset = X$ and $\tau_\emptyset = \emptyset$. Given $s \in 2^{<\omega}$, define $X_{s^\frown 0} = X_s \setminus S_{|\tau_s|}$ and $X_{s^\frown 1} = X_s \cap S_{|\tau_s|}$. There exist $\delta_i < \mathfrak{s}$ for $i < 2$ such that $X_{s^\frown i} \cap S_{\delta_i} \in \mathcal{I}(\mathcal{A}_\alpha)^+$ and $X_{s^\frown i} \setminus S_{\delta_i} \in \mathcal{I}(\mathcal{A}_\alpha)^+$ and since $X_{s^\frown i} \subseteq X_s$ it follows that $\delta_i > |\tau_s|$. Hence let $\tau_{s^\frown i}$ such that $dom(\tau_{s^\frown i}) = \delta_i$ and $\tau_{s^\frown i}(\xi) = j$ iff $X_{s^\frown i} \cap S_\xi^j \in \mathcal{I}(\mathcal{A}_\alpha)^+$ for every $\xi \in dom(\tau_{s^\frown i})$.

It is easy to see that if s and t are incompatible, then τ_s and τ_t are incompatible as well. Thus if we define $\tau_f = \bigcup_{n\in\omega} \tau_{f\restriction n}$ for every $f \in 2^\omega$, they are a family of incompatible nodes. Moreover, every $\tau_f \in 2^{<\mathfrak{s}}$ since $\mathfrak{s}$ has uncountable cofinality. Pick $f \in 2^\omega$ such that $\tau_f \nsubseteq \sigma_\beta$ for every $\beta < \alpha < \mathfrak{c}$ and define $\sigma_\alpha = \tau_f$. Notice that $\{X_{f\restriction n} : n \in \omega\}$ is a decreasing sequence of positive sets. Then we can find a positive pseudointersection Y, i.e., $Y \subseteq^* X_{f\restriction n}$ for every $n \in \omega$ and $Y \in \mathcal{I}(\mathcal{A}_\alpha)^+$. In consequence $Y \cap S_\xi^{1-\sigma_\alpha(\xi)} \in \mathcal{I}(\mathcal{A}_\alpha)$ for every $\xi \in dom(\sigma_\alpha)$.

For every $\xi \in dom(\sigma_\alpha)$ let $F_\xi \in [\mathcal{A}_\alpha]^{<\omega}$ such that $Y \cap S_\xi^{1-\sigma_\alpha(\xi)} \subseteq \bigcup F_\xi$. Define $W = \{A_\beta : \sigma_\beta \subseteq \sigma_\alpha\} \cup \bigcup_{\xi < |\sigma_\alpha|} F_\xi$. Note that $|W| < \mathfrak{s} \leq \mathfrak{a}$, hence there is $A_\alpha \in [Y]^\omega$ which is almost disjoint with every element of W. Since $A_\alpha \subseteq Y$ it follows that $A_\alpha \subseteq X$. Since F_ξ is a finite subset of W for every $\xi \in dom(\sigma_\alpha)$ and $Y \setminus S_\xi \subseteq \bigcup F_\xi$, it also follows that $A_\alpha \subseteq^* S_\eta^{\sigma_\alpha(\eta)}$ for all $\eta \in dom(\sigma_\alpha)$. It remains to prove that A_α is indeed almost disjoint with $\mathcal{A}_\alpha$. Let $\beta < \alpha$ such that $A_\beta \notin W$. Let $\xi = \min\{\eta : \sigma_\beta(\eta) \neq \sigma_\alpha(\eta)\}$. Then $A_\beta \subseteq^* S_\eta^{1-\sigma_\alpha(\eta)}$ and $A_\alpha \subseteq^* S_\eta^{\sigma_\alpha(\eta)}$ implies that $A_\beta \cap A_\alpha =^* \emptyset$. Moreover, the final family $\mathcal{A}$ is MAD, since for every infinite $X = X_\alpha$, if X is AD with $\mathcal{A}_\alpha$ then $|A_\alpha \cap X| = \omega$. $\square$

The cardinal $\mathfrak{s}_{\omega,\omega}$ was introduced in [49] in order to construct a weakly tight MAD family using the method of Shelah mentioned before. A MAD family $\mathcal{A}$ is *tight* if for every family $\{X_n : n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ there is $A \in \mathcal{A}$ such that $|A \cap X_n| = \omega$ for every $n \in \omega$. It is shown in [27] that the existence of a tight MAD family is equivalent to the existence of a Cohen-indestructible MAD family and the notion of weakly tight MAD family is introduced: A MAD family $\mathcal{A}$ is *weakly tight* if for every collection $\{X_n : n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ there is $A \in \mathcal{A}$ such that $|A \cap X_n| = \omega$ for infinitely many $n \in \omega$.

It is an open problem whether weakly tight MAD families exist in ZFC. Raghavan and Steprāns showed that they exist assuming $\mathfrak{s} \leq \mathfrak{b}$:

Theorem 2.2.5. [49] ($\mathfrak{s} \leq \mathfrak{b}$) *There is a weakly tight mad family.*

Proof. Let $\{s_\alpha : \alpha < \mathfrak{s}\}$ be an (ω, ω)-splitting family and let $\leq_{lex}$ denote the lexicographic order on $\mathfrak{c} \times (\omega + 1)$. We will say that $\overrightarrow{\mathcal{D}} = \{D(n) : n \in \omega\} \subseteq [\omega]^\omega$ is a *disjoint sequence* if $D(n) \cap D(m) = \emptyset$, for every $n \neq m$. Given two disjoint sequences $\overrightarrow{C}$ and $\overrightarrow{D}$, we will say that $\overrightarrow{C}$ refines $\overrightarrow{D}$, and write $\overrightarrow{C} \prec \overrightarrow{D}$, if there is an increasing sequence $\{k_n : n \in \omega\} \subseteq \omega$ such that $C(n) \subseteq D(k_n)$ for every $n \in \omega$.

Fix an enumeration $\{b_\alpha : \alpha < \mathfrak{c}\} = ([\omega]^\omega)^\omega$. We will recursively construct $\{a_\alpha^\xi : (\alpha < \mathfrak{c}) \wedge (\xi \leq \omega)\} \subseteq [\omega]^\omega$ and $\{\tau_\alpha^\xi : (\alpha < \mathfrak{c}) \wedge (\xi \leq \omega)\} \subseteq 2^{<\mathfrak{s}}$ such that:

(1) $\mathcal{A} = \{a_\alpha^\omega : \alpha < \mathfrak{c}\}$ is almost disjoint.

(2) If $b_\alpha(n) \in \mathcal{I}(\mathcal{A}_\alpha)^+$, then $|a_\alpha^n \cap b_\alpha(n)| = \omega$.

(3) $a_\alpha^n \subseteq^* s_\eta^{\tau_\alpha^n(\eta)}$ for all $\alpha < \mathfrak{c}$, $n \in \omega$ and $\eta < dom(\tau_\alpha^n)$.

(4) $\overrightarrow{C_\alpha} := \{a_\alpha^n : n \in \omega\}$ is a disjoint sequence.

(5) $\exists \overrightarrow{D_\alpha} \prec \overrightarrow{C_\alpha} \left(a_\alpha^\omega = \bigcup_{n\in\omega} D_\alpha(n)\right)$.

(6) $\forall \xi < dom(\tau_\alpha^\omega)\ \forall^\infty n \in \omega \ \left(D_\alpha(n) \subseteq s_\xi^{\tau_\xi^\omega(\xi)}\right)$.

If we manage to do this then the resulting MAD family $\mathcal{A}$ is weakly tight by properties (2), (4) and (5). Assume we have defined $\{a_\alpha^\xi : (\alpha <$

$\delta) \wedge (\xi \leq \omega)\}$ and $\{\tau_\alpha^\xi : (\alpha < \delta) \wedge (\xi \leq \omega)\}$ for some $\delta < \mathfrak{c}$. We shall recursively define a_δ^n and τ_δ^n for $n \in \omega$. So, we can also assume that a_δ^i and τ_δ^i have been defined for $i < \omega$. Let $b = b_\alpha(n)$ if $b_\alpha(n) \in \mathcal{I}(\mathcal{A}_\delta)^+$ and $b = b'$ for some $b' \in \mathcal{I}(\mathcal{A}_\delta)^+$ otherwise.

For each $t \in 2^{<\omega}$ define recursively $b_t \in \mathcal{I}(\mathcal{A}_\delta)^+$ and $\tau_t \in 2^{<\mathfrak{s}}$ such that $\tau_t \subseteq \tau_r$ and $b_t \supseteq b_r$ whenever $t \subseteq r$. Moreover, $b_t \cap s_\xi^{1-\tau_t(\xi)} \in \mathcal{I}(\mathcal{A}_\delta)$ for every $\xi \in dom(\tau_t)$ while $b_t \cap s_{|\tau_t|} \in \mathcal{I}(\mathcal{A}_\delta)^+$ and $b_t \setminus s_{|\tau_t|} \in \mathcal{I}(\mathcal{A}_\delta)^+$. Start with $b_\emptyset = b$ and $\tau_\emptyset = \emptyset$. Given $t \in 2^{<\omega}$, define $b_{t^\frown 0} = b_t \setminus s_{|\tau_t|}$ and $b_{t^\frown 1} = b_t \cap s_{|\tau_t|}$. There exist $\eta_i < \mathfrak{s}$ for $i < 2$ such that $b_{t^\frown i} \cap s_{\eta_i} \in \mathcal{I}(\mathcal{A}_\delta)^+$ and $b_{t^\frown i} \setminus s_{\eta_i} \in \mathcal{I}(\mathcal{A}_\delta)^+$ and since $b_{t^\frown i} \subseteq b_t$, it follows that $\eta_i > |\tau_t|$. Hence let $\tau_{t^\frown i}$ such that $dom(\tau_{t^\frown i}) = \eta_i$ and $\tau_{t^\frown i}(\xi) = j$ iff $b_{t^\frown i} \cap s_\xi^j \in \mathcal{I}(\mathcal{A}_\delta)^+$ for every $\xi \in dom(\tau_{t^\frown i})$.

Notice that if we define $\tau_f = \bigcup_{n\in\omega} \tau_{f\restriction n} \in 2^{<\mathfrak{s}}$ for every $f \in 2^\omega$, which is well defined since $\mathfrak{s}$ has uncountable cofinality, we will get a family of incompatible nodes $\{\tau_t : t \in 2^\omega\} \subseteq 2^{<\mathfrak{c}}$ and then there exists $f \in 2^\omega$ such that $\tau_f \not\subseteq \tau_\alpha^k$ for every $(\alpha, k) \leq_{lex} (\delta, n)$. Define $\tau_\delta^n = \tau_f$. Since $\{b_{f\restriction n} : n \in \omega\}$ is a decreasing sequence in $\mathcal{I}(\mathcal{A}_\delta)^+$, we can find $b_f \in \mathcal{I}(\mathcal{A}_\delta)^+$ such that $b_f \subseteq^* b_{f\restriction n}$ for all $n \in \omega$. Then, $b_f \cap s_\xi^{1-\tau_\delta^n(\xi)} \in \mathcal{I}(\mathcal{A}_\delta)$ for all $\xi < dom(\tau_\delta^n)$ and there exists a finite subset $\mathcal{F}_\xi \in [\mathcal{A}_\delta]^{<\omega}$ such that $b_f \cap s_\xi^{1-\tau_\delta^n(\xi)} \subseteq \bigcup \mathcal{F}_\xi$. Define

$$W = \{a_\alpha^i : ((\alpha, i) \leq_{lex} (\delta, n)) \wedge \left(\tau_\alpha^i \subseteq \tau_\delta^n\right)\}$$

and

$$W' = \{a_\alpha^\omega : (\alpha < \delta) \wedge \left(\exists i < \omega(\tau_\alpha^i \subseteq \tau_\delta^n)\right)\}.$$

Let $\mathcal{W} = W \cup W' \cup \left(\bigcup_{\xi < |\tau_\delta^n|} \mathcal{F}_\xi\right)$ Since $|\mathcal{W}| < \mathfrak{s} \leq \mathfrak{b} \leq \mathfrak{a}$, we can find a set $x \subseteq b_f \subseteq b$ such that $|x \cap a| < \omega$ for every $a \in \mathcal{W}$. Define $a_\delta^n = x$. We claim that $|a_\delta^n \cap a_\alpha^i| < \omega$ for every $(\alpha, i) \leq_{lex} (\delta, n)$. First notice that for every $\xi < dom(\tau_\delta^n)$, a_δ^n satisfies property 3 since $a_\delta^n \setminus s_\xi^{\tau_\delta^n(\xi)} \subseteq \bigcup \mathcal{F}_\xi$ and $|a_\delta^n \cap \mathcal{F}_\xi| < \omega$ by the choice of a_δ^n.

Suppose $a_\alpha^i \notin \mathcal{W}$. Then there exists $\eta < \min\{dom(\tau_\alpha^i), dom(\tau_\delta^n)\}$ such that $\tau_\alpha^i(\eta) = 1 - \tau_\delta^n(\eta)$. If $i < \omega$ then $|a_\alpha^i \cap a_\delta^n| < \omega$ since $a_\alpha^i \subseteq^* s_\eta^{\tau_\alpha^i(\eta)}$ and $a_\delta^n \subseteq^* s_\eta^{1-\tau_\alpha^i(\eta)}$. On the other hand, if $i = \omega$, again $a_\alpha^i \subseteq^* s_\eta^{\tau_\alpha^i(\eta)}$ and $a_\alpha^\omega = \bigcup_{m\in\omega} C(m)$ where $\{C(m) : m \in \omega\} \prec \{a_\alpha^m : m \in \omega\}$. Thus there exists $l \in \omega$ such that $C(m) \subseteq s_\eta^{\tau_\alpha^\omega(\eta)}$ for all $m > l$, but this implies that $a_\delta^n \cap a_\alpha^\omega \subseteq^* \bigcup_{m \leq l} C(m)$. Therefore $|a_\delta^n \cap a_\alpha^\omega| < \omega$ since $a_\delta^n \cap C(m) \subseteq a_\delta^n \cap a_\alpha^{k(m)}$ for some $k(m) \in \omega$.

We can already assume that $\{a_\delta^n : n \in \omega\}$ is a disjoint sequence by replacing a_δ^n with $a_\delta^n \setminus \bigcup_{i<n} a_\delta^i$ if necessary. It remains to define a_δ^ω in order to finish the proof.

We will recursively define $C_t \in ([\omega]^\omega)^\omega$ and $\sigma_t \in 2^{<\mathfrak{s}}$ for $t \in 2^{<\omega}$ such that:

(i) C_t is a disjoint sequence,

(ii) $C_{t^\frown i} \prec C_t$ for $i \in 2$,

(iii) $C_t(n) \subseteq^* s_\eta^{\sigma_t(\eta)}$ for all $\eta \in dom(\sigma_t))$ and for all but finitely many $n \in \omega$,

(iv) $C_{t^\frown i}(n) \subseteq s_\eta^{\sigma_t(\eta)}$ for all $\eta \in dom(\sigma_t)$, all but finitely many $n \in \omega$ and $i \in 2$,

(v) $\{n \in \omega : |C_t(n) \cap s^i_{|\sigma_t|}| = \omega\}$ is infinite for $i \in 2$ and

(vi) t_0 and t_1 are incompatible and $t^\frown i \supsetneq t$ for $i \in 2$.

Define $C_\emptyset = \{a_\delta^n : n \in \omega\}$. There exists $\xi < \mathfrak{s}$ such that s_η (ω,ω)-splits $C_\emptyset$, i.e., $\{n \in \omega : |C_\emptyset(n) \cap s_\xi^i| = \omega\}$ is infinite for $i \in 2$. Let η be the minimum of these ξ and define $\sigma_\emptyset \in 2^{<\mathfrak{s}}$ such that $dom(\sigma_\emptyset) = \eta$ and for every $\xi < \eta$, $\sigma_\emptyset(\xi) = j$ iff $\{n \in \omega : |C_\emptyset(n) \cap s_\xi^j| = \omega\}$.

Assume we have constructed C_t and σ_t. Then, for every $\xi < dom(\sigma_t)$ there exist $n_\xi \in \omega$ such that $C_t(m) \subseteq^* s_\xi^{\sigma_t(\xi)}$ for all $m > n_\xi$. Define $f_\xi \in \omega^\omega$ such that $f(m) = 0$ if $m \leq n_\xi$ and $C_t(m) \cap s_\xi^{1-\sigma_t(\xi)} \subseteq f_\xi$ if $m > n_\xi$. Since $|\sigma_t| < \mathfrak{s} \leq \mathfrak{b}$, there exists a function $f \in \omega^\omega$ which dominates $\{f_\xi : \xi \in dom(\sigma_t)\}$. Define $E(n) = C_t(n) \setminus f(n)$. Notice that $\{E(n) : n \in \omega\}$ also satisfies that $\{n \in \omega : |E(n) \cap s^i_{|\sigma_t|}| = \omega\}$ is infinite for $i \in 2$. Moreover, if $\eta < dom(\sigma_t)$ and $n \in \omega$ is such that $n > n_\eta$ and $f(m) > f_\eta(m)$ for every $m > n$, then $E(m) \subseteq s_\eta^{\sigma_t(\eta)}$ for all $m > n$.

Then define $C_{t^\frown i} = \{E(n) : |E(n) \cap s^i_{|\sigma_t|}| = \omega\}$ with the obvious enumeration. Having in mind the last observation about $\{E(n) : n \in \omega\}$, it is clear that $C_{t^\frown i}$ satisfies conditions (i), (ii) and (iv). For every $i \in 2$, let η_i be the minimum ξ such that $\{n \in \omega : |C_{t^\frown i}(n) \cap s^0_\xi| = \omega\}$ and $\{n \in \omega : |C_{t^\frown i}(n) \cap s^1_\xi| = \omega\}$ are both infinite. Define $\sigma_{t^\frown i}$ such that $dom(\sigma_{t^\frown i}) = \eta_i$ and for every $\xi < \eta_i$, $\sigma_{t^\frown i}(\xi) = j$ iff $\{n \in \omega : |C_{t^\frown i}(n) \cap s^j_\xi| = \omega\}$. It is clear that $\{C_t : t \in 2^\omega\}$ and $\{\sigma_t : t \in 2^\omega\}$ are as desired.

For every $g \in 2^\omega$ define $\sigma_g = \bigcup_{n\in\omega} \sigma_{g\restriction n} \in 2^{\mathfrak{s}}$. Then there exists $h \in 2^\omega$ such that $\sigma_h \not\subseteq \tau^i_\alpha$ for every $(\alpha, i) \leq_{lex} (\delta, \omega)$. Define $\tau^\omega_\delta = \sigma_h$. Also define $D'(n) = C_{h\restriction n}(n)$. For every $\beta < \delta$ such that $\tau^\omega_\beta \subseteq \tau^\omega_\delta$ let $F_\beta \in \omega^\omega$ such that $D'(m) \cap a^\omega_\beta \subseteq F_\beta(m)$. This is possible since $D'(m) \subseteq a^n_\delta$ and $|a^n_\delta \cap a^\omega_\beta| < \omega$. Let $F \in \omega^\omega$ which dominates all F_β with $\beta < \delta$ and $\tau^\omega_\beta \subseteq \tau^\omega_\delta$. Thus, define $D(m) = D'(m) \setminus F(m)$. Of course $\{D(m) : m \in \omega\} \prec \{D'(m) : m \in \omega\}$. It follows from the construction of the C_t that

$$\overrightarrow{D} := \{D(n) : n \in \omega\} \prec C_\emptyset = \{a^n_\delta : n \in \omega\},$$

and then a^ω_δ satisfies (2) and (5). Since also $\overrightarrow{D} \prec C_{h\restriction n}$ for every $n \in \omega$, a^ω_δ satisfies (6).

Let $\beta < \delta$. We will finish if we show that $|a^\omega_\delta \cap a^\omega_\beta| < \omega$. Suppose $\tau^\omega_\beta \not\subseteq \tau^\omega_\delta$. Let η be the minimum ordinal such that $\tau^\omega_\beta(\eta) \neq \tau^\omega_\delta$. Then there are $n_\beta, n_\delta \in \omega$ such that $D_\beta(m) \subseteq s^{\tau^\omega_\beta(\eta)}_\eta$ for every $m > n_\beta$ and $D_\delta(m) \subseteq s^{1-\tau^\omega_\beta(\eta)}_\eta$ for every $m > n_\delta$. Then

$$a^\omega_\beta \cap a^\omega_\delta \subseteq \left(\bigcup_{m<n_\beta} D_\beta(m)\right) \cap \left(\bigcup_{m<n_\delta} D_\delta(m)\right).$$

But the right hand term is finite since $D_\beta \prec \{a^n_\beta : n \in \omega\}$ and $D_\delta \prec \{a^n_\delta : n \in \omega\}$.

On the other hand, if $\tau^\omega_\beta \subseteq \tau^\omega_\delta$, the function F_β was considered at step δ. Hence,

$$a^\omega_\beta \cap a^\omega_\delta \subseteq a^\omega_\beta \cap \left(\bigcup_{m<k_\beta} D_\delta(m)\right),$$

where k_β is the minimum $k \in \omega$ such that $F(n) > F_\beta(n)$ for every $n > k_\beta$. Therefore $a^\omega_\beta \cap a^\omega_\delta$ is finite. $\square$

The proof of their theorem actually shows that under $\mathfrak{s} \leq \mathfrak{b}$, there is a weakly tight MAD family $\mathcal{A}$ such that for every countable collection $\{X_n : n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ there are $\mathfrak{c}$-many $A \in \mathcal{A}$ such that $|A \cap X_n| = \omega$ for infinitely many $n \in \omega$. We will take advantage of this fact in the next theorem.

Theorem 2.2.6. ($\mathfrak{s} \leq \mathfrak{b}$) *There is an α_3-FU AD family $\mathcal{A}$ which is not hereditarily α_3. In particular it is not bisequential.*

Proof. Let $\mathcal{E} = \{e_\alpha : \alpha < \mathfrak{c}\} \subseteq [\omega]^\omega$ be a weakly tight MAD family such that for every $\{X_n : n \in \omega\} \subseteq \mathcal{I}(\mathcal{E})^+$ there are $\mathfrak{c}$-many $e \in \mathcal{E}$ such that $|e \cap X_n| = \omega$ for infinitely many $n \in \omega$. We can assume that $\{e_n : n \in \omega\}$ forms a partition of ω. Enumerate $[\omega]^\omega = \{X_\alpha : \omega \leq \alpha < \mathfrak{c}\}$ and $([\omega]^\omega)^\omega = \{D_\alpha : \omega \leq \alpha < \mathfrak{c}\}$. Define recursively $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathcal{E}$ and $\{Y_{\alpha,i} : \omega \leq \alpha < \mathfrak{c} \wedge i \in 2\} \subseteq \mathcal{E}$ such that $A_\beta \neq A_\alpha \neq Y_{\eta,i}$ for all $\alpha, \beta \in \mathfrak{c}$ with $\alpha \neq \beta$, $\omega \leq \eta < \mathfrak{c}$ and $i \in 2$.

For $n \in \omega$ we start by defining $A_n = e_n$. Let $\omega \leq \alpha < \mathfrak{c}$. If $X_\alpha \in \mathcal{I}(\mathcal{A}_\alpha)^+$ choose $Y_{\alpha,0} \in \mathcal{E} \setminus (\mathcal{A}_\alpha \cup \mathcal{Y}_\alpha)$ where $\mathcal{Y}_\alpha = \{Y_{\beta,i} : \beta < \alpha \wedge i \in 2\}$ such that $|Y_{\alpha,0} \cap X_\alpha| = \omega$. Similarly if $\{D_\alpha(n) : n \in \omega\} \subseteq \mathcal{A}^\perp \subseteq \mathcal{A}_\omega^\perp \subseteq \mathcal{I}(\mathcal{A}_\alpha)^+$, choose $Y_{\alpha,1} \in \mathcal{E} \setminus (\mathcal{A}_\alpha \cup \mathcal{Y}_\alpha)$ such that $|Y_{\alpha,1} \cap D_\alpha(n)| = \omega$ for infinitely many $n \in \omega$. Finally, if $|X_\alpha \cap A_n| = \omega$ for infinitely many $n \in \omega$, then $X \in \mathcal{I}(\mathcal{A}_\alpha)^+$ and choose $A_\alpha \in \mathcal{E} \setminus (\mathcal{A}_\alpha \cup \mathcal{Y}_{\alpha+1})$ such that $|A_\alpha \cap X_\alpha| = \omega$. Therefore, $\mathcal{A}$ is α_3-FU but not hereditarily α_3 by lemma 2.1.4. □

Combining theorems 2.1.6 and 2.2.6 and since $\mathfrak{s} \leq \mathsf{non}(\mathcal{M})$ we get the following corollary.

Corollary 2.2.7. ($\mathfrak{c} \leq \aleph_2$) $\Rightarrow$ *There is an α_3-FU AD family which is not bisequential.* □

2.3 Weak ◇ principles

The almost disjoint families defined so far, have in common that all of them are of size $\mathfrak{c}$. We will use the parametrized diamond $\diamondsuit(\mathfrak{b})$ (see [44]) to construct counterexamples of size ω_1 to Gruenhage's questions. Remember that this principle is defined as follows:

$$\diamondsuit(\mathfrak{b}) \equiv \forall F : 2^{<\omega_1} \to \omega^\omega \text{ Borel } \exists g : \omega_1 \to \omega^\omega \ \forall f \in 2^{\omega_1}$$

$$\{\alpha \in \omega_1 : g(\alpha) \not\leq^* F(f \restriction \alpha)\} \text{ is stationary.}$$

Here, we say that $F : 2^{<\omega_1} \to \omega^\omega$ is Borel if for every $\delta < \omega_1$ the restriction of F to 2^δ is a Borel map.

Theorem 2.3.1. $\diamondsuit(\mathfrak{b})$ *implies the existence of an α_3-FU non-hereditarily α_3 AD family.*

Proof. Let $\{A_n : n \in \omega\}$ be a partition of ω into infinite sets. For every infinite ordinal $\delta < \omega_1$ fix a bijection $e_\delta : \omega \to \delta$. We will define a Borel function F into the set ω^ω and such that its domain is the set of tuples $(\mathcal{A}_\alpha, \mathcal{Y}_\delta, \overrightarrow{X})$ where:

1. $\alpha \in \{\delta, \delta + 1\}$.
2. δ is an infinite countable ordinal.
3. $\mathcal{A}_\alpha = \langle A_\beta : \beta < \alpha \rangle$ is an almost disjoint family.
4. $\mathcal{Y}_\delta = \langle Y_\alpha : \omega \leq \alpha < \delta \rangle \subseteq \mathcal{A}_\alpha^\perp$.
5. $\overrightarrow{X} \in ([\omega]^\omega \times 2) \cup ([\omega]^\omega)^\omega$.
6. If $\overrightarrow{X} \in ([\omega]^\omega)^\omega$, then $\alpha = \delta + 1$ and $X_n := X(n)$ is AD with $\mathcal{A}_{\delta+1}$ for every $n \in \omega$.
7. If $\overrightarrow{X} = (X, i) \in ([\omega]^\omega \times 2)$ then $X \in \mathcal{I}(\mathcal{A}_\alpha)^+$. Moreover, if $i = 0$, then X meets A_n for infinitely many $n \in \omega$.
8. If $\overrightarrow{X} \in ([\omega]^\omega \times 2)$, then $i = 1$ if and only if $\alpha = \delta + 1$.

If $\overrightarrow{X} = (X, 0)$, there are infinitely many $n \in \omega$ such that X meets $A_{e_\delta(n)}$. Let $\{n_k : k \in \omega\}$ be the increasing enumeration of

$$\left\{n \in \omega : \left|X \cap A_{e_\delta(n)}\right| = \omega\right\}$$

and define

$$F(\mathcal{A}_\delta, \mathcal{Y}_\delta, (X, 0))(k) = \min\left(X \cap A_{e_\delta(n_k)} \setminus \bigcup_{i<n_k} \left[A_{e_\delta(i)} \cup Y_{e_\delta(i)}\right]\right).$$

Analogously, if $\overrightarrow{X} = (X, 1)$, there are infinitely many $n \in \omega$ such that the set $X \cap A_{e_{\delta+1}(n)}$ is nonempty. Redefine $\{n_k : n \in \omega\}$ as the increasing enumeration of this set and define

$$F(\mathcal{A}_{\delta+1}, \mathcal{Y}_\delta, (X, 1))(k) = \min \left(X \cap A_{e_{\delta+1}(n_k)} \setminus \bigcup_{i<n_k} A_{e_{\delta+1}(i)}) \right).$$

On the other hand, if $\overrightarrow{X} \in ([\omega]^\omega)^\omega$, then $X_n \in \mathcal{I}(\mathcal{A}_{\delta+1})^+$ for every $n \in \omega$. Define $f_n = F(\mathcal{A}_{\delta+1}, \mathcal{Y}_\delta, (X_n, 1))$. Take $h \in \omega^\omega$ such that $f_n \leq^* h$ for all $n \in \omega$ and define $F(\mathcal{A}_{\delta+1}, \mathcal{Y}_\delta, \overrightarrow{X}) = h$.

Now suppose that $g : \omega_1 \to \omega^\omega$ is a $\diamondsuit(\mathfrak{b})$-sequence for F and assume that the entries of g form a $<^*$-strictly increasing sequence of increasing functions by making them larger if necessary.

We now construct our almost disjoint family $\mathcal{A} = \langle A_\alpha : \alpha < \omega_1 \rangle$ together with a sequence $\mathcal{Y} = \langle Y_\alpha : \omega \leq \alpha < \omega_1 \rangle \subseteq \mathcal{A}^\perp$. If $\langle A_\alpha : \alpha < \delta \rangle$ and $\langle Y_\alpha : \omega \leq \alpha < \delta \rangle$ have been defined for an infinite countable ordinal δ, set

$$A_\delta = \bigcup_{n\in\omega} \left(g(\delta)(n) \cap A_{e_\delta(n)} \setminus \bigcup_{i<n} \left[A_{e_\delta(i)} \cup Y_{e_\delta(i)} \right] \right)$$

and

$$Y_\delta = \bigcup_{n\in\omega} \left(g(\delta)(n) \cap A_{e_{\delta+1}(n)} \setminus \bigcup_{i<n} A_{e_{\delta+1}(i)} \right).$$

It is clear from the definition that Y_δ is AD with $\mathcal{A}_{\delta+1}$ and that A_δ is AD with $\mathcal{A}_\delta \cup \mathcal{Y}_\delta$. Then $\mathcal{A}$ is almost disjoint and $\mathcal{Y} \subseteq \mathcal{A}^\perp$. Let us prove that $\mathcal{A}$ satisfies the properties listed in lemma 2.1.4.

Let us prove first that $\mathcal{A}$ is nowhere MAD. Given $X \in \mathcal{I}(\mathcal{A})^+$ we have that $(\mathcal{A}_{\delta+1}, \mathcal{Y}_\delta, (X, 1))$ is in the domain of F for every $\omega \leq \delta < \omega_1$. Then suppose that g guesses $F(\mathcal{A}, \mathcal{Y}, (X, 1))$ at δ, i.e., $g(\delta) \not\leq^* F(\mathcal{A}, \mathcal{Y}, (X, 1))$. Let $l \in \omega$, we shall find $m > l$ such that $m \in Y_\delta \cap X$, thus $X \cap Y_\delta$ is infinite and $\mathcal{A}$ is nowhere MAD since $Y_\delta \in \mathcal{A}^\perp$. Recall that in this case $\{n_k : k \in \omega\}$ is the increasing enumeration of the natural numbers n such that $A_{e_{\delta+1}(n)}$ has nonempty intersection with X. Find $k \in \omega$ such that $[0, l] \subseteq \bigcup_{i<n_k} A_{e_{\delta+1}(i)}$ and $g(\delta)(k) > F(\mathcal{A}_{\delta+1}, \mathcal{Y}_\delta, (X, 1))(k)$. This is possible since

$$\{A_{e_{\delta+1}(n)} \setminus \bigcup_{i<n} A_{e_{\delta+1}(i)} : n \in \omega\}$$

forms a partition of ω and $g(\delta) \not\leq^* F(\mathcal{A}_{\delta+1}, \mathcal{Y}_\delta, (X, 1))$. Then

$$m = F(\mathcal{A}_{\delta+1}, \mathcal{Y}_\delta, (X, 1)) > l$$

and since $m < g(\delta)(k) \leq g(\delta)(n_k)$ it follows that $m \in Y_\delta \cap X$.

A similar argument shows that if X is like in point 3 of lemma 2.1.4 (where $\{A_n : n \in \omega\}$ there, is exactly the first ω-many elements in this construction), then $(\mathcal{A}_\delta, \mathcal{Y}_\delta, (X, 0))$ is an element of the domain of F for every $\omega \leq \delta < \omega_1$. Hence, if g guesses $F(\mathcal{A}, \mathcal{Y}, (X, 0))$ at δ we have that A_δ meets X.

Finally suppose that $\overrightarrow{X} \in ([\omega]^\omega)^\omega$ and let $\delta \in \omega_1$ such that g guesses $F(\mathcal{A}, \mathcal{Y}, \overrightarrow{X})$ at δ. Since $g(\delta) \not\leq^* F(\mathcal{A}_{\delta+1}, \mathcal{Y}_\delta, \overrightarrow{X})$ and $F(\mathcal{A}_{\delta+1}, \mathcal{Y}_\delta, \overrightarrow{X}) \geq^* f_n$ for every of the associated functions f_n, it follows that $g(\delta) \not\leq^* f_n$ for every $n \in \omega$. Using the same reasoning as before with f_n instead of F, we can prove that Y_δ meets X_n for every $n \in \omega$. Therefore we have proved not only that $\mathcal{A}$ is α_3 but α_2. □

2.4 Further results

As we said, almost disjoint families are of special interest when looking for counterexamples of properties related to convergence. We have used different axioms for building the spaces studied so far, namely, $\mathsf{non}(\mathcal{M}) = \mathfrak{c}$, $\mathfrak{s} \leq \mathfrak{b}$, $\diamondsuit(\mathfrak{b})$ and in consequence, the results follow from $\mathfrak{c} \leq \aleph_2$ since $\mathfrak{s}, \mathfrak{b} \leq \mathsf{non}(\mathcal{M})$.

Notation 2.4.1. For the remainder of this section let Φ be any of the following axioms:

- $\mathsf{non}(\mathcal{M}) = \mathfrak{c}$
- $\mathfrak{s} \leq \mathfrak{b}$
- $\diamondsuit(\mathfrak{b})$
- $\mathfrak{c} \leq \aleph_2$

In [45], Nyikos built under $\mathfrak{b} = \mathfrak{c}$, an AD family $\mathcal{A} \subseteq [\omega \times \omega]^\omega$ consisting of functions which fails to be α_3 and asked whether it is possible to construct an α_3 non-bisequential AD family of this kind under the same assumption.

Theorem 2.1.6 provides a positive answer to this question. He also asked the following:

- Is every compact α_3-FU space $\aleph_0$-bisequential?
- Is there a ZFC example of a compact space X that has Fréchet product with every regular countably compact Fréchet space, but is not $\aleph_0$-bisequential?

In [3], Arhangel'skii proved that a separable space is $\aleph_0$-bisequential iff it is bisequential. Then since $\Psi(\mathcal{A})^*$ is compact and separable, we get the following:

Corollary 2.4.2. (Φ) *There exists a compact α_3-FU space which is not $\aleph_0$-bisequential.*

As Nyikos pointed out, a (consistent) negative answer to the first problem gives an (consistent) affirmative one to the second question in view of the next theorem.

Theorem 2.4.3. [3] *If X is an α_3-FU space, then $X \times Y$ is Fréchet for every regular countably compact Fréchet space.*

Given a Fréchet AD family $\mathcal{A}$, the space $\Psi(\mathcal{A})^*$ is compact and Fréchet since every infinite subset of $\mathcal{A}$ converges to ∞. Then an AD family is Fréchet iff it is absolutely Fréchet. In [3], Arhangel'skii asked the following questions:

1. Is there an absolutely Fréchet space which is not bisequential?
2. Is there a (countable) α_1-Fréchet space which is not bisequential?

The results of the previous sections provide new consistent examples to question 1.

Corollary 2.4.4. (Φ) *There exists an absolutely Fréchet non-bisequential space.*

Malyhin has constructed a consistent example for the second question under $2^{\aleph_0} < 2^{\aleph_1}$ [40]. Here we will construct under CH a countable absolutely Fréchet example by strengthening our previous results in order to get α_1.

Theorem 2.4.5. (CH) *There is a countable α_1 and absolutely Fréchet space which is not bisequential.*

Proof. We will prove the theorem using an AD family $\mathcal{A}$. For this purpose we will recursively define $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ and

$$\mathcal{B} = \{B_{\alpha,i} : \omega \leq \alpha < \mathfrak{c} \ \wedge \ i \in 2\}$$

such that $\mathcal{A}$ is almost disjoint and $\mathcal{B} \subseteq \mathcal{A}^\perp$. We start by defining $\{A_n : n \in \omega\}$ being any partition of ω consisting of infinite sets. Enumerate $[\omega]^\omega = \{X_\alpha : \omega \leq \alpha < \omega_1\}$ and $([\omega]^\omega)^\omega = \{\overrightarrow{D_\alpha} : \omega \leq \alpha < \omega_1\}$ where $\overrightarrow{D}_\alpha = (D_{\alpha,n} : n \in \omega)$. For every $\omega \leq \delta < \omega_1$ let $e_\delta : \omega \to \delta$ be a bijection. Suppose we have constructed $\mathcal{A}_\delta$ and $\mathcal{B}_\delta := \{B_{\alpha,i} : \omega \leq \alpha < \delta \wedge i \in 2\}$.

Define $X = X_\delta$ if $X_\delta \in \mathcal{I}(\mathcal{A}_\delta)^+$ and $X = X'$ for some $X' \in \mathcal{I}(\mathcal{A}_\delta)^+$ otherwise. Pick $x_n \in X \setminus (\{x_i : i < n\} \cup \bigcup_{i<n} A_{e_\delta(i)})$ and define $B_{\delta,0} = \{x_n : n \in \omega\}$.

Similarly define $\overrightarrow{D} = \overrightarrow{D}_\delta$ if $D_{\delta,n}$ is AD with A_δ for every $n \in \omega$ and $\overrightarrow{D} = \overrightarrow{D}'$ for some $\overrightarrow{D}'$ satisfying that each D'_n is AD with $\mathcal{A}_\delta$ otherwise. Define

$$B_{\delta,1} := \bigcup_{n\in\omega} \left[\left(\bigcup_{j \leq n} D_j \right) \cap \left(A_{e_\delta(n)} \setminus \bigcup_{i<n} A_{e_\delta(i)} \right) \right].$$

Notice that $D_j \setminus (\bigcup_{i<j} A_{e_\delta(i)}) \subseteq B_{\delta,1}$ and $\bigcup_{j\leq n} D_j$ has finite intersection with each $A_{e_\delta(n)}$. Hence $B_{\delta,1}$ is almost disjoint with $\mathcal{A}_\delta$ and almost contains each D_j.

Finally, define $\mathcal{G} = \overrightarrow{D}_\delta$ if D_δ is a decreasing family (i.e., $D_{n+1} \subseteq D_n$ for every $n \in \omega$) and each $D_{\delta,n}$ meets A_n for infinitely many $n \in \omega$ and $\mathcal{G} = \overrightarrow{D}'$ for some $\overrightarrow{D}'$ satisfying this property otherwise. Let $\{k_n : n \in \omega\} \subseteq \omega$ be an increasing sequence such that $D_{\delta,n}$ meets A_{k_n} for every $n \in \omega$. Pick $z_n \in D_{\delta,n} \cap A_{k_n} \setminus (\bigcup_{i<k_n} A_i$ and define $A_\delta := \{z_n : n \in \omega\}$.

From the construction it is clear that $\mathcal{A}$ is α_1 because for every sequence $\overrightarrow{D} = (D_n : n \in \omega)$ such that every D_n is AD with $\mathcal{A}$ there exists $\alpha < \omega_1$ such that $\overrightarrow{D} = \overrightarrow{D}_\alpha$ and $B_{\alpha,1}$ almost contains each D_n. With a similar argument we conclude that $\mathcal{A}$ is also Fréchet (hence absolutely Fréchet) using $B_{\alpha,0}$.

To see that it is not bisequential, consider $\mathcal{F}$ as the dual filter of the ideal

$$\mathcal{J} = \{A \subseteq \omega : \exists n \in \omega \; \forall m > n \; (|Z \cap A_m| < \omega)\},$$

(i.e., $\mathcal{F} = \{X \subseteq \omega : \omega \setminus X \in \mathcal{J}\}$). The previous ideal is often defined in $\omega \times \omega$ using $\{n\} \times \omega$ instead of A_n and it is called $\mathbf{fin} \times \mathbf{fin}$. Since every element $A \in \mathcal{A}$ is disjoint from one member of $\mathcal{F}$, it follows that $\infty \in \overline{\mathcal{F}}$. Notice that $\mathcal{F} \cup \{G\}$ generates a filter iff $G \in \mathcal{F}^+ := \mathcal{J}^+$ iff $|\{n \in \omega : |G \cap A_n| = \omega\}| = \omega$.

Let $\mathcal{G} = \{G_n : n \in \omega\} \subseteq \mathcal{F}^+$ be a decreasing sequence, thus there exists an $\alpha < \omega_1$ such that $\mathcal{G} = \overrightarrow{D}_\alpha$, and in consequence, A_α meets each G_n. This shows that no G_n is contained in the open set $(\omega \cup \{\infty\}) \setminus A_\alpha$. □

Lastly, we will get a ZFC result for question 1 using a completely separable (not maximal!) AD family. From the definition of a completely separable MAD family, it is easy to see that maximality already follows from the condition that every positive set contains an element of the family. For an AD family $\mathcal{A}$, define

$$\mathcal{I}(\mathcal{A})^{\oplus} := \{X \subseteq \omega : |\{A \in \mathcal{A} : |X \cap A| = \omega\}| \geq \omega\}.$$

An AD family $\mathcal{A}$ is completely separable if for every $X \in \mathcal{I}^{\oplus}(\mathcal{A})$ there is $A \in \mathcal{A}$ such that $A \subseteq X$. Note that $\mathcal{A}$ is a completely separable MAD family (in the previous sense) iff it is a completely separable AD family (in the new sense) and maximal. Which of the two definitions we are referring to will be understood by the use of the terms AD family or MAD family.

While the existence of a completely separable MAD family in ZFC remains open, the existence of a completely separable AD family is a theorem of ZFC:

Theorem 2.4.6. [4] *There is a completely separable AD family in ZFC.*

The following lemma will be useful for proving the main theorem of this section.

Lemma 2.4.7. *If $\mathcal{A}$ is a completely separable AD family and $X \in \mathcal{I}^{\oplus}(\mathcal{A})$, then $|\{A \in \mathcal{A} : A \subseteq X\}| = \mathfrak{c}$.*

Proof. Let $X \in \mathcal{I}(\mathcal{A})^{\oplus}$ and let $\{A_n : n \in \omega\} \subseteq \mathcal{A}$ such that $A_n \neq A_m$ and $|X \cap A_n| = \omega$ for every $n, m \in \omega$ with $n \neq m$. For every $n \in \omega$ define $B_n \subseteq X \cap A'_n$ infinite and such that $A_n \setminus B_n$ is infinite. Let $\{a_\alpha : \alpha < \mathfrak{c}\}$ be a MAD family of size $\mathfrak{c}$, and for every $\alpha < \mathfrak{c}$ let $X_\alpha = \bigcup_{n \in a_\alpha} B_n \subseteq X$. Notice that $X_\alpha \in \mathcal{I}(\mathcal{A})^{\oplus}$ for every $\alpha < \mathfrak{c}$. Then there exists $A_\alpha \in \mathcal{A}$ such that $A_\alpha \subseteq X_\alpha$ and $A_\alpha \neq A_\beta$ since $X_\alpha \cap X_\beta \subseteq \bigcup_{n \in a_\alpha \cap a_\beta} B_n$. $\square$

Theorem 2.4.8. *There exists an absolutely Fréchet AD family $\mathcal{A}$ which is not bisequential.*

Proof. Let $\mathcal{E} = \{a_\alpha : \alpha < \mathfrak{c}\} \subseteq [\omega]^\omega$ be a completely separable AD family. We can assume that $\{a_n : n \in \omega\}$ forms a partition of ω into infinite sets by replacing a_n with $a'_n = (a_n \cup \{n\}) \setminus \bigcup_{i<n} a'_i$ if necessary. Enumerate $[\omega]^\omega = \{X_\alpha : \omega \leq \alpha < \mathfrak{c}\}$. We will construct recursively two families $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ and $\mathcal{B} = \{B_\alpha : \omega \leq \alpha < \mathfrak{c}\}$ such that

1. $\mathcal{A} \subseteq \mathcal{E}$ (hence, it is almost disjoint).
2. For every $B \in \mathcal{B}$, either $B \in \mathcal{E} \setminus \mathcal{A}$ or $B \in \mathcal{E}^\perp$. In particular $B \in \mathcal{A}^\perp$.
3. If $X_\alpha \in \mathcal{I}^+(\mathcal{A}_\alpha)$, then $|B_\alpha \cap X_\alpha| = \omega$.
4. If $\Big|\{n \in \omega : |X_\alpha \cap A_n| = \omega\}\Big| = \omega$, then $A_\alpha \subseteq X_\alpha$.

For $n < \omega$, define $A_n = a_n$. Assume we have constructed two families $\mathcal{A}_\delta = \{A_\alpha : \alpha < \delta\}$ and $\mathcal{B}_\delta = \{B_\alpha : \omega \leq \alpha < \delta\}$ with the desired properties for an infinite ordinal $\delta < \mathfrak{c}$. Define $\mathcal{B}'_\delta = \mathcal{B}_\delta \cap \mathcal{E}$.

If $X_\delta \notin \mathcal{I}(\mathcal{A}_\delta)^+$ define $B_\delta \in \mathcal{E} \setminus \mathcal{A}$ arbitrarily. Suppose $X_\delta \in \mathcal{I}(\mathcal{A}_\delta)^+$. If $X_\delta \in \mathcal{I}(\mathcal{E})^{\oplus}$, there is $a \in \mathcal{E} \setminus \mathcal{A}_\delta$ such that $a \subseteq X_\delta$ by lemma 2.4.7. Define $B_\delta = a$. On the other hand, if $X_\delta \in \mathcal{I}(\mathcal{A}_\delta)^+ \setminus \mathcal{I}(\mathcal{E})^{\oplus}$, there are two cases:

Case 1: There exists $a \in \mathcal{E} \setminus \mathcal{A}_\delta$ such that a meets X_δ. In this case define $B_\delta = a$.

Case 2: $\{a \in \mathcal{E} : |a \cap X_\delta| = \omega\} \subseteq \mathcal{A}_\delta$. In this case it is easy to find an infinite set $X' \in [X_\delta]^\omega$ such that $X' \in \mathcal{E}^\perp$. Define $B_\delta = X'$.

Now assume that X_δ is as in 4. Then using lemma 2.4.7 again, there is $a \in \mathcal{E} \setminus (\mathcal{A}_\delta \cup (\mathcal{B}_{\delta+1} \cap \mathcal{E}))$ such that $a \subseteq X_\delta$. Define $A_\delta = a$. Otherwise define $A_\delta \in \mathcal{E} \setminus (\mathcal{A}_\delta \cup (\mathcal{B}_{\delta+1} \cap \mathcal{E}))$ arbitrarily. This finishes the construction.

We shall prove that $\mathcal{A}$ is absolutely Fréchet but not bisequential. Recall that in the AD space generated by $\mathcal{A}$, a subset $X \subseteq \omega$ converges to ∞ iff $X \in \mathcal{A}^\perp$ and $\infty \in \overline{X}$ iff $X \in \mathcal{I}(\mathcal{A})^+$. Let $X \in [\omega]^\omega$ such that $X \in \mathcal{I}(\mathcal{A})^+$. There is $\alpha \in [\omega, \mathfrak{c})$ such that $X = X_\alpha$. Hence, since $\mathcal{I}(\mathcal{A})^+ \subseteq \mathcal{I}(\mathcal{A}_\alpha)^+$, it follows that B_α meets X and considering that either, $B_\alpha \in \mathcal{E} \setminus \mathcal{A}$ or $B_\alpha \in \mathcal{E}^\perp$, we conclude that $B_\alpha \in \mathcal{A}^\perp$. Thus $Y = B_\alpha \cap X$ is an infinite set disjoint from $\mathcal{A}$. In view of every infinite subset of $\mathcal{A}$ converges to ∞, $\mathcal{A}$ is nowhere MAD and ω is discrete in $\Psi(\mathcal{A})^*$, it follows that $\Psi(\mathcal{A})^*$ witnesses that $\mathcal{A}$ is absolutely Fréchet.

We use again the following ideal

$$\mathcal{J} = \{X \subseteq \omega : \exists n \in \omega \ \forall m > n \ (|X \cap A_m| < \omega)\}$$

and define $\mathcal{F}$ as the dual filter. Let $\{G_n : n \in \omega\} \subseteq \mathcal{F}^+$ and assume without loss of generality that it is a decreasing sequence of sets. Find an increasing sequence $\{k_n : n \in \omega\}$ such that $|G_n \cap A_{k_n}| = \omega$ for every $n \in \omega$ and define $X = \bigcup_{n\in\omega}(G_n \cap A_{k_n})$. There is $\alpha \in [\omega, \mathfrak{c})$ such that $X = X_\delta$. Notice that X satisfies point 4, then $A_\alpha \subseteq X_\alpha$ and since A_α is almost disjoint from every A_{k_n} and $\{G_n : n \in \omega\}$ is decreasing, $A_\alpha \cap G_n \neq \emptyset$ for every $n \in \omega$. This shows that $\mathcal{A}$ is not bisequential. □

The existence of completely separable AD families implies the existence of an absolutely Fréchet non-bisequential space in ZFC in the same way that the existence of a weakly tight MAD family implies the existence of an α_3-FU non-bisequential AD family under $\mathfrak{s} \leq \mathfrak{b}$. One could expect that the same works using a weakly tight AD family (not maximal!) in ZFC. We will say that $\mathcal{A}$ is a weakly tight AD family if for every sequence of sets $\{b_n : n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^\oplus$, there exists $A \in \mathcal{A}$ such that $\{n \in \omega : |A \cap b_n| = \omega\}$ is infinite.

Question 1. *Is there a weakly tight AD family in ZFC?*

Of course, a positive answer to this question would encourage us to repeat the last construction in ZFC and try to solve the next question:

Question 2. *Is there an α_3-FU AD family in ZFC which is not bisequential?*

Chapter 3

Madness and normality

In this chapter we consider weakenings of normality in Ψ-spaces and prove that the existence of an AD family whose Ψ-space is almost-normal but is not normal follows from CH. On the other hand, under PFA no MAD family is almost normal. We also construct a partly-normal not quasi-normal AD family, answering questions of García-Balan and Szeptycki [26]. We finish by showing that the concepts of almost-normal and strongly $\aleph_0$-separated AD families are different, even under CH, answering a question of Oliveira-Rodrigues and Santos-Ronchim [46].

Mrówka-Isbell spaces provide a wide and numerous source of examples and counterexamples in many areas of topology. Many examples of the use of AD families and their Ψ-spaces can be found in [35]. Normality is no exception. If X is a normal space, then it is pseudocompact if and only if it is countably compact. A MAD family is never normal, since maximality implies that the associated Ψ-space is pseudocompact and it contains a discrete uncountable subspace, hence it is not countably compact. AD families of size $\mathfrak{c}$ are not normal by Jones' lemma, since $\mathcal{A}$ is a discrete subspace of size continuum of a separable space. One of the first examples of an AD family with special combinatorial properties, was a Luzin family [39]. An AD family $\mathcal{A}$ is a *Luzin family*, if it can be enumerated as $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ in such a way that $\{\beta < \alpha : A_\alpha \cap A_\beta \subseteq n\}$ is finite for every $\alpha < \omega_1$ and every $n \in \omega$. The key property of Luzin families is that if $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$ are two uncountable subfamilies, they can not be separated, in consequence, Luzin families are not normal. This suggest that normality is not easily fulfilled for an AD family. One of the first applications of AD families to problems related to normality, was the equivalence of the existence of a normal, separable, non-metrizable Moore space and the existence of an uncountable AD family which is normal. The later was proved to be independent of ZFC [54].

In [26], weak normality properties on Ψ-spaces were considered. Recall that a space X is *normal* if every two disjoint closed sets $C, D \subseteq X$ can be separated by two disjoint open sets $U, V \subseteq X$ (that is $C \subseteq U$, $D \subseteq V$ and $U \cap V = \emptyset$). A subset $C \subseteq X$ of a topological space is *regular closed* if $C = \overline{int(C)}$. Thus, the definition of normality becomes weaker if we require one, or both of the closed sets to be regular closed or a finite intersection of regular closed sets (which is called π-*closed*). Ranging over these possibilities, several weakenings of normality arise, and so do some implications between them (see, [2], [1], [53] and [26]). We summarize these implications in the next diagram without defining all the concepts involved simply to organize them and have a visual support. We will define each term that we will focus on when necessary.

(∗) normal $\Longrightarrow$ almost-normal $\Longrightarrow$ quasi-normal $\Longrightarrow$

partly-normal $\Longrightarrow$ mildly-normal.

Counterexamples of some of these reverse implications were given in [26]: A mildly-normal which is not partly-normal and a quasi-normal which is not almost-normal AD families were constructed, whilst counterexamples of the remaining two implications were left open. In particular, the existence of an almost-normal MAD family, was left open (Questions 4.1, 4.2 and 4.3 in[26]). In Section 3.1, we provide an example of an almost-normal AD family which is not normal under CH. In Section 3.2 we show that under PFA, no MAD family can be almost-normal. In Section 3.3, we build a partly-normal AD family which is not quasi-normal, hence, completing all the counterexamples in (∗), at least, consistently. Finally, in Section 3.4, we will construct a strongly $\aleph_0$-separated AD family which is not almost-normal under CH, answering a question from Oliveira-Rodrigues and Santos-Ronchim [46]. Each undefined weakening of normality can be found in [26].

3.1 An almost-normal AD family that is not normal

As we mentioned above, in [26], several counterexamples for the reverse implications in (∗) were given, however, some questions were left open, among them the following two:

- Is there an almost-normal not normal AD family?
- Is there an almost-normal MAD family?

A space X is *almost-normal* ([53]) if each pair of closed sets $C, D \subseteq X$, where one of them is regular closed, can be separated.

Of course, a positive answer for the second question provides a positive answer for the first one. In [46], it is shown that it is consistent that the first question has a positive answer for. For a subset $X \subseteq 2^\omega$, the AD family $\mathcal{A}_X \subseteq \mathcal{P}(2^{<\omega})$ is defined as the family of all sets of the form $\{f \restriction n : n \in \omega\}$ with $f \in X$. The result in [46] is obtained by defining a special class of subsets of 2^ω, called almost Q-sets, such that $\mathcal{A}_X$ is the desired family whenever X is an almost Q-set and then forcing the set X. This result cannot be improved to get MAD since AD families of the form $\mathcal{A}_X$ are never MAD.

The construction is also showed to be independent of CH. We will show in this section, that the existence of an almost normal AD family which is not normal, does follow from CH.

Definition 3.1.1. Let $\mathcal{A}$ be an almost disjoint family. A set $D \in [\omega]^\omega$ is a *partitioner* for $\mathcal{A}$, if for every $A \in \mathcal{A}$ either, $A \subseteq^* D$ or $A \cap D$ is finite.
We will say that two disjoint subfamilies $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$ *can be separated*, if there is a partinioner D for $\mathcal{A}$, such that $B \subseteq^* D$ for every $B \in \mathcal{B}$ and $|C \cap D| < \omega$ for every $C \in \mathcal{C}$. In this case, we will say that D is a *separator* for $(\mathcal{B}, \mathcal{C})$. In particular, each separator is a partitioner for $\mathcal{A}$.

Notice that if D is a partitioner, $\omega \setminus D$ is a partitioner as well, where the properties of "almost contained" and "is almost disjoint" have been exchanged. Thus, we can always decide which part of our family is almost contained in the partinioner.

It is known that an AD family $\mathcal{A}$ is normal, if and only if for each $\mathcal{B} \subseteq \mathcal{A}$, $\mathcal{B}$ and $\mathcal{A} \setminus \mathcal{B}$ can be separated [54]. The respective result for almost normality also holds (see Proposition 3.1.3 below) . We will need the following easy observation.

Lemma 3.1.2. *Let $\mathcal{A}$ be an AD family and let $K \subseteq \Psi(\mathcal{A})$. The following are equivalent:*

1. *K is regular closed.*
2. *$K = \overline{\omega \cap K}$.*
3. *$K = (\omega \cap K) \cup \{A \in \mathcal{A} : |A \cap K| = \omega\}$.*

Proof. (1) $\Rightarrow$ (3). Let $A \in K \setminus \omega$. If $A \in int(K)$ then there exists $n \in \omega$ such that $A \setminus n \subseteq K$, in particular $A \cap K$ is infinite. Otherwise, $A \in \overline{\omega \cap K}$ since $\mathcal{A}$ is a discrete subspace of $\Psi(\mathcal{A})$, which implies $A \cap (\omega \cap K) = A \cap K$ is infinite. We have proved that the left-side is included in the right-side. The other inclusion follows easily by noting that $\omega \cap K \subseteq int(K)$.
(2) $\Leftrightarrow$ (3). Let $A \in \overline{\omega \cap K} \setminus (\omega \cap K)$, since ω is discrete, $A \in \mathcal{A}$. Hence $A \cap \omega \cap K \subseteq A \cap K$ is infinite. Conversely, if $A \in \mathcal{A}$ and A meets K, it is clear that $A \in \overline{\omega \cap K}$.
(2) $\Rightarrow$ (1). Since $\omega \cap K \subseteq int(K)$ we have $K = \overline{\omega \cap K} \subseteq \overline{int(K)}$. Thus $K = \overline{int(K)}$. □

Proposition 3.1.3. *An AD family $\mathcal{A}$ is almost-normal if and only if, for every $C \in [\omega]^\omega$, there exists a separator for $(\mathcal{B}, \mathcal{A} \setminus \mathcal{B})$, where*

$$\mathcal{B} = \{A \in \mathcal{A} : |A \cap C| = \omega\}.$$

Proof. Assume $\mathcal{A}$ is almost-normal and let $C \subseteq \omega$. Let

$$\mathcal{B} = \{A \in \mathcal{A} : |A \cap C| = \omega\}.$$

Then $K = \mathcal{B} \cup C$ is regular closed and $\mathcal{A} \setminus \mathcal{B}$ is closed in $\Psi(\mathcal{A})$. Let us check that D is a separator for $(\mathcal{B}, \mathcal{A} \setminus \mathcal{B})$. Since $\mathcal{A}$ is almost-normal, we can find disjoint open subsets $U, V \subseteq \Psi(\mathcal{A})$ such that $K \subseteq U$ and $\mathcal{A} \setminus \mathcal{B} \subseteq V$. Define $D = U \cap \omega$ and let $B \in \mathcal{B}$. Since U contains a basic neighborhood of B, it follows that $B \subseteq^* D$. On the other hand, if $A \in \mathcal{A} \setminus \mathcal{B}$, there exists a basic neighborhood of A contained in V, thus $A \subseteq^* V \cap \omega$ and therefore $|A \cap D| < \omega$.

Now suppose that each pair $\mathcal{B}$, $\mathcal{A} \setminus \mathcal{B}$ as in the proposition can be separated. Let $F, K \subseteq \Psi(\mathcal{A})$ be two disjoint closed sets with K regular closed. There exist $C \subseteq \omega$ such that $K = C \cup \mathcal{B}$ with $\mathcal{B} = \{A \in \mathcal{A} : |A \cap C| = \omega\}$. Let D be a separator for $\mathcal{B}$ and $\mathcal{A} \setminus \mathcal{B}$ and let $E = F \cap \omega$. Define $U = (\mathcal{B} \cup C \cup D) \setminus E$.

Claim: U is clopen.

Given that ω is discrete, we only care about the points in $\mathcal{A}$. Let $A \in \mathcal{A} \setminus \mathcal{B}$. Since $A \notin \mathcal{B}$ and $|A \cap D| < \omega$, it follows that $\{A\} \cup (A \setminus (C \cup D))$ is a basic neighborhood of A disjoint from U. Then U is closed. If $B \in \mathcal{B}$, $|B \cap E| < \omega$ (otherwise $B \in F$) and $B \subseteq^* D$. Then $\{B\} \cup (D \setminus E) \subseteq U$ contains a basic neighborhood of B showing that U is open.

Finally note that U is a clopen subset disjoint from F and $K \subseteq U$. Thus $\mathcal{A}$ is almost normal. □

A very related notion on AD families called weakly separated was considered in [10] and [18]. Given $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$, we say that $D \in [\omega]^\omega$ *weakly separates* $\mathcal{B}$ and $\mathcal{C}$, if D meets B for every $B \in \mathcal{B}$ and $D \cap C$ is finite for every $C \in \mathcal{C}$. An AD family is *weakly separated* if for any two disjoint subfamilies $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$, there is a set $D \in [\omega]^\omega$ that weakly separates $\mathcal{B}$ and $\mathcal{C}$. It follows easily that an AD family is normal iff it is almost-normal and weakly separated. As we said before, AD families of size $\mathfrak{c}$ are never normal by Jones' lemma. We state this result since we are going to use it below.

Lemma 3.1.4. *Let X be a separable and normal space. Then for every closed and discrete set $D \subseteq X$ we have $2^{|D|} \leq 2^\omega$.*

Recall that $\mathfrak{b}$ is the least size of an unbounded family and $\mathfrak{s}$ is the least size of a splitting family. We know that both cardinal invariants are uncountable. Hence we can see that for every countable family of functions $\mathcal{F} \subseteq \omega^\omega$, there is a single function $g \in \omega^\omega$ that dominates all $f \in \mathcal{F}$ and for every countable family $\mathcal{S} \subseteq [\omega]^\omega$ there is a single set $R \in [\omega]^\omega$ such that either, $R \subseteq^* S$ or $R \cap S$ is finite for every $S \in \mathcal{S}$.

Lemma 3.1.5. *Let $\mathcal{A}$ be a countable AD family and $C \subseteq \omega$. There is a separator $D \subseteq \omega$ for $\mathcal{B} = \{A \in \mathcal{A} : |A \cap C| = \omega\}$ and $\mathcal{A} \setminus \mathcal{B}$ such that $C \subseteq D$.*

Proof. If $|\mathcal{B}| < \omega$, let $D' = \bigcup \mathcal{B}$ (or $D' = \emptyset$ in case $\mathcal{B}$ is empty). Conversely, if $\mathcal{A} \setminus \mathcal{B}$ is finite or empty, we can define $D' = \omega \setminus \bigcup(\mathcal{A} \setminus \mathcal{B})$. Otherwise, enumerate $\mathcal{A} \setminus \mathcal{B} = \{A_n : n \in \omega\}$, $\mathcal{B} = \{B_n : n \in \omega\}$ and define

$$D' = \bigcup_{n\in\omega} \Big(B_n \setminus \bigcup_{i<n} A_i \Big).$$

In any case, D' is a separator for $\mathcal{B}$ and $\mathcal{A} \setminus \mathcal{B}$ such that $A \subseteq^* D'$ iff $A \in \mathcal{B}$ for every $A \in \mathcal{A}$. Thus, define $D = D' \cup C$. Clearly each B_n is almost contained in D since $D' \subseteq D$. Now let $A \in \mathcal{A} \setminus \mathcal{B}$. Then both $A \cap C$ and $A \cap D'$ are finite and in consequence $|A \cap D| < \omega$. □

Lemma 3.1.6. *Let $\mathcal{A} \subseteq [\omega]^\omega$ be a countable AD family and let $\mathcal{D} = \{D_n : n \in \omega\}$ be a family of partitioners for $\mathcal{A}$. Assume that for each $n \in \omega$ there exists $C_n \in [\omega]^\omega$ such that $C_n \subseteq D_n$ and D_n is a separator for $\mathcal{B} = \{A \in \mathcal{A} : |A \cap C_n| = \omega\}$ and $\mathcal{A} \setminus \mathcal{B}$. Then there exists $A \in [\omega]^\omega$ such that, $\mathcal{A} \cup \{A\}$ is AD, each D_n is a partitioner for $\mathcal{A} \cup \{A\}$ and $A \subseteq^* D_n$ iff $|A \cap C_n| = \omega$ for every $n \in \omega$. Moreover, if $\mathcal{A}$ contains an infinite partition $\{A_n : n \in \omega\}$ of ω, then we can ensure that $|A \cap A_n| \leq 1$ for every $n \in \omega$.*

Proof. Suppose that $\mathcal{A} = \{A_\alpha : \alpha < \delta\}$ for some countable ordinal $\delta \in \omega_1$. Furthermore, assume that $\{A_n : n \in \omega\}$ forms a partition of ω. Define

$$E_n = \{k \in \omega : |A_k \cap C_n| = \omega\}.$$

Notice that $k \in E_n$ iff $A_k \subseteq^* D_n$, otherwise $A_k \cap D_n$ is finite. Since $\{E_n : n \in \omega\}$ is countable, we can find $R \in [\omega]^\omega$ such that either $R \subseteq^* E_n$ or $R \cap E_n$ is finite for every $n \in \omega$.

Now, for every $n \in \omega$ define $f_n \in \omega^\omega$ as follows:

$$f_n(k) = \begin{cases} \max(A_k \setminus D_n) + 1 & \text{if } \ k \in E_n \\ \max(A_k \cap D_n) + 1 & \text{otherwise} \end{cases}$$

Similarly, we define $f_\alpha \in \omega^\omega$ for every $\omega \leq \alpha < \delta$ by $f_\alpha(k) = \max(A_\alpha \cap A_k) + 1$. In order to avoid confusions, we consider $\max(\emptyset) = 0$. Again, since the family $\{f_\alpha : \alpha < \delta\}$ is countable, we can find $f \in \omega^\omega$ that dominates every f_α.

Let $Z = \{n \in \omega : R \subseteq^* E_n\}$. We can find a function $g : R \to Z$ such that

1. $g^{-1}(n)$ is infinite for every $n \in Z$ and

2. $g(r) = n$ implies $r \in E_n$.

To see this, notice that we can assume that $R \subseteq E_{\min(Z)}$ by throwing away a finite set from R. Since for every $n \in Z$ there is $k \in \omega$ such that $(R \setminus k) \subseteq E_n$, then we can define $g(r) = n$ for every $r \in R \setminus k$. Using a bookkeeping argument we can ensure that every $n \in Z$ is chosen infinitely many times.

We proceed to the definition of A. For each $r \in R$, $r \in E_n$ with $n = g(r)$. This implies that $A_r \cap C_n$ is infinite. Let $m_r = \min((A_r \cap C_n) \setminus f(r))$. Then we define $A = \{m_r : r \in R\}$. Since R is infinite, so does A. It is clear from the definition that $|A \cap A_n| \leq 1$ for every $n \in \omega$.

Let $\omega \leq \alpha < \delta$ and let $N \in \omega$ such that $f(n) > f_\alpha(n)$ for every $n > N$. Then, for every $r \in R \setminus N$, $m_r \geq f(r) > f_\alpha(r)$ and $A_\alpha \cap A_r \subseteq f_\alpha(r)$, which implies $m_r \notin A_\alpha$ and hence $A \cap A_\alpha$ is finite.

Let $n \in \omega \setminus Z$. Then $R \cap E_n$ is finite. Let N such that $f(k) > f_n(k)$ for every $k > N$ and such that $R \cap E_n \subseteq N$. Thus for $r \in R \setminus N$, we have that $r \notin E_n$ and the definition of $f_n(r)$ follows the second case. In particular, $f_n(r) > \max(A_r \cap D_n)$ and since $m_r \in A_r$ and $m_r \geq f(r) > f_n(r)$, we conclude that $m_r \notin D_n$. Therefore $A \cap D_n$ is finite (and consequently, $A \cap C_n$ since $C_n \subseteq D_n$).

Finally let $n \in Z$. Find $N \in \omega$ such that $R \setminus N \subseteq E_n$ and $f(k) > f_n(k)$ for every $k > N$. For every $r \in R \setminus N$, we have that $A_r \subseteq^* D_n$ and then $f_n(r)$ was defined by the first case. Hence $m_r \geq f(r) > f_n(r) > \max(A_k \setminus D_n)$. This implies that $A \subseteq^* D_n$. Moreover, since $g^{-1}(n)$ is infinite, we chose infinitely many m_r in C_n, which implies that $A \cap C_n$ is infinite. This finishes the proof. □

Theorem 3.1.7. (CH) *There is an almost-normal AD family which is not normal.*

Proof. Enumerate $[\omega]^\omega = \{C_\alpha : \alpha < \omega_1\}$ with $C_n = \omega$ for every $n \in \omega$. We will recursively construct an AD family $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ and a family of partitioners $\mathcal{D} = \{D_\alpha : \alpha < \omega_1\}$ such that if $\mathcal{A}_\alpha = \{A_\beta : \beta < \alpha\}$ and $\mathcal{D}_\alpha = \{D_\beta : \beta < \alpha\}$ then:

1. $\mathcal{A}_\alpha$ is AD.
2. $|A_\alpha \cap A_n| \leq 1$ for every $n \in \omega$ and $\omega \leq \alpha \leq \omega_1$
3. D_α is a separator for $\mathcal{B} = \{A \in \mathcal{A}_\alpha : |A \cap C_\alpha| = \omega\}$ and $\mathcal{A}_\alpha \setminus \mathcal{B}$.
4. Either, $A_\alpha \subseteq^* D_\beta$ or $|A_\alpha \cap D_\beta| < \omega$ for every $\beta \leq \alpha$.
5. $A_\alpha \subseteq^* D_\beta$ iff $|A_\alpha \cap C_\beta| = \omega$.
6. $C_\alpha \subseteq D_\alpha$

Let $\{A_n : n \in \omega\} \subseteq [\omega]^\omega$ be a partition of ω into infinite pieces and define $D_n = \omega$ for every $n \in \omega$. This family clearly satisfies the above conditions. Assume we have constructed $\mathcal{A}_\alpha$ and $\mathcal{D}_\alpha$ as above. We can apply lemma 3.1.5 to the pair $(\mathcal{A}_\alpha, C_\alpha)$ to obtain D_α.

For the construction of A_α, apply lemma 3.1.6 to the family $\mathcal{A}_\alpha$ and $\{D_\beta : \beta \leq \alpha\}$ with their respective C_β.

It is clear from point (2) that $\mathcal{A}$ is AD. Also, if $C \in [\omega]^\omega$, there exists $\alpha < \omega_1$ such that $C = X_\alpha$. Hence D_α is a partitioner for $\mathcal{A}_\alpha$ as in proposition 3.1.3. Moreover, point (4) and point (5) ensure that D_α is preserved for $\beta \geq \alpha$. Thus D_α is a partitioner for $\mathcal{A}$ as required in proposition 3.1.3 with $C = C_\alpha$. We can conclude that $\mathcal{A}$ is almost-normal. On the other hand, it is not normal by Jones' lemma. □

Due to the extra property that $|A_n \cap A_\alpha| \leq 1$, if we use a bijection of ω with $\omega \times \omega$ that sends A_n to the set $\{n\} \times \omega$, we can assume that the family consists of bars (sets of the form $\{n\} \times \omega$) and graphs of partial functions.

Corollary 3.1.8. (CH) *There is an almost-normal AD family $\mathcal{A} \subseteq [\omega \times \omega]^\omega$, consisting of bars and graphs of functions that is not normal.*

It was mentioned before that in [26], a quasi-normal Luzin MAD family was constructed, then it is natural to ask the following question:

Question 3. (CH) *Is there a Luzin and/or MAD family which is almost-normal?*

3.2 There may be no almost-normal MAD families

There are many reasons for which one could think that it is not possible to obtain Theorem 3.1.7 without assuming CH. The most obvious reason is that after ω_1-many steps, we have already constructed a Luzin family $\mathcal{A}$. Then, we can not get a partitioner as in Proposition 3.1.3 for a given set $C \subseteq [\omega]^\omega$, whenever it meets uncountable many elements of $\mathcal{A}$ and it is almost disjoint from uncountable many elements of $\mathcal{A}$ as well. Indeed, this situation could be unavoidable for MAD families as we will see below.

Recall that the Proper Forcing Axiom (PFA) is the assertion that for every proper forcing $\mathbb{P}$ and every family $\mathcal{D}$ of ω_1-many open dense subsets of $\mathbb{P}$ there exists a $\mathcal{D}$-generic filter for $\mathbb{P}$. If we replace "proper" by "ccc" and "ω_1" by "$< \mathfrak{c}$" we get the definition of Martin's Axiom (MA). It is well known that PFA implies $\mathsf{MA}+\mathfrak{c} = \omega_2$. Under PFA we can not avoid the existence of Luzin subfamilies of MAD families due to the following result.

Theorem 3.2.1. *[19] Each MAD family contains a Luzin subfamily.*

The existence of a set C as above, that "wants to separate" the Luzin subfamily is also insured by the next theorem.

Theorem 3.2.2. *[41] (MA) For every pair of families $\mathcal{A}, \mathcal{B} \subseteq [\omega]^\omega$ of size $< \mathfrak{c}$ such that for every $K \in [\mathcal{A}]^{<\omega}$ and $B \in \mathcal{B}$, $B \setminus \bigcup K$ is infinite, there exists $C \in [\omega]^\omega$ such that $C \cap A$ is finite for every $A \in \mathcal{A}$ and C meets B for every $B \in \mathcal{B}$.*

Now it follows easily that there are no almost-normal MAD families in the presence of PFA.

Corollary 3.2.3. *PFA implies that there are no almost-normal MAD families.*

Proof. Let $\mathcal{A}$ be a MAD family and let $\mathcal{A}' \subseteq \mathcal{A}$ be a Luzin subfamily. In particular, $|\mathcal{A}'| = \omega_1$. We can split $\mathcal{A}'$ into two uncountable disjoint subfamilies $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}'$. By Theorem 3.2.2, and since PFA implies MA and $\mathfrak{c} = \omega_2$, we can find a set $X \subseteq \omega$ that weakly separates $\mathcal{B}$ and $\mathcal{C}$, as they have size $\omega_1 < \mathfrak{c}$. That is, $X \cap C$ is finite for every $C \in \mathcal{C}$ and X meets B for every $B \in \mathcal{B}$. Thus, $\mathcal{K} = \{A \in \mathcal{A} : |X \cap A| = \omega\}$ and $\mathcal{A} \setminus \mathcal{K}$ cannot be separated since $\mathcal{B} \subseteq \mathcal{K}$, $\mathcal{C} \subseteq \mathcal{A} \setminus \mathcal{K}$ and $\mathcal{B}, \mathcal{C}$ are uncountable subfamilies of a Luzin family. Therefore $\mathcal{A}$ is not almost-normal. □

Property	**Separation**
Normal	Closed and Closed
Almost-normal	Regular closed and Closed
Partly-normal	Regular closed and π-closed
Quasi-normal	π-closed and π-closed

Figure 3.1: Weak normality properties

In [21], it is shown that it is consistent with MA that there is a MAD family which contains no Luzin subfamilies. It could be possible that the only thing that blocks the existence of almost-normal MAD families is the existence of Luzin subfamilies, so we ask the following:

Question 4. *Is it consistent with MA that there are almost-normal MAD families?*

3.3 Partly-normal not quasi-normal AD families

In this section, we will consider the next question stated in [26] and will provide a positive answer.

- Is there a partly-normal not quasi-normal AD family?

We will say that a space X is *partly-normal* if any pair of disjoint closed sets $A, B \subseteq X$, where A is regular closed and B is π-closed (a finite intersection of regular closed sets), can be separated [2]. A space X is *quasi-normal* if any two disjoint π-closed sets can be separated [57]. Figure 3.1 summarizes the weak normality properties considered in this section.

Most of the examples in [26] were constructed using AD families of true cardinality $\mathfrak{c}$. For an AD family $\mathcal{A}$ and $W \subseteq \omega$, we will denote by $\mathcal{A} \restriction W$ the set of $A \in \mathcal{A}$ such that A meets W. An AD family is of *true cardinality* $\mathfrak{c}$, if for every $W \subseteq \omega$, either, $\mathcal{A} \restriction W$ is finite or has size $\mathfrak{c}$. It is well known that the existence of (M)AD families of true cardinality $\mathfrak{c}$ is equivalent to the existence of completely separable (M)AD families ([25], see also section 2.2). The original definition is due to Hechler in [32]. Hechler's definition implies that the AD family is maximal and both definitions coincide for MAD families. While completely separable AD families do exist in ZFC, the existence of completely separable MAD families in ZFC, asked first by Erdös and Shelah [24], is one of the more interesting and central questions concerning almost disjoint families. See the discussion in section 2.2 for more about completely separable AD families.

The existence of AD families of true cardinality $\mathfrak{c}$ is particularly useful for constructions of AD families with strong combinatorial properties, since they usually need recursive constructions of length continuum (see, for example, [48]). We will use an AD family of true cardinality $\mathfrak{c}$ to construct a partly-normal not quasi-normal AD family. First, observe that we can always assume that an infinite AD family $\mathcal{A}$, contains an infinite partition of ω into infinite pieces, since we can take $\{A_n : n \in \omega\} \subseteq \mathcal{A}$ and substitute A_n by $A'_n = (A_n \cup \{n\}) \setminus \bigcup_{i<n} A'_i$. We are now ready to prove the following result.

Theorem 3.3.1. *There is a partly-normal AD family which is not quasi-normal.*

Proof. Partition ω in four infinite sets W_0, W_1, V_0 and V_1. Further, partition both, W_0 and W_1 into infinitely many infinite sets, that is,

$$W_0 = \bigcup_{n \in \omega} P_n$$

and

$$W_1 = \bigcup_{n \in \omega} Q_n.$$

Let $\mathcal{A}_{W_0}, \mathcal{A}_{W_1}, \mathcal{A}_{V_0}$ and $\mathcal{A}_{V_1}$ be AD families of true cardinality $\mathfrak{c}$ in each of the four sets W_0, W_1, V_0 and V_1. We can assume that $\{P_n : n \in \omega\} \subseteq \mathcal{A}_{W_0}$ and $\{Q_n : n \in \omega\} \subseteq \mathcal{A}_{W_1}$. We will recursively construct our family putting together some elements of these AD families of true cardinality $\mathfrak{c}$. For ease of notation, let $\mathcal{E} = \mathcal{A}_{W_0} \cup \mathcal{A}_{W_1} \cup \mathcal{A}_{V_0} \cup \mathcal{A}_{V_1}$.

For every $n \in \omega$, define $A_n = P_n \cup Q_n$. Let $\{f_\alpha : \alpha < \mathfrak{c}\}$ be a dominating family of functions in ω^ω. List all pairs $(C, \mathcal{D})$ where $\mathcal{D} \in [[\omega]^\omega]^{<\omega}$ and $C \in [\omega]^\omega$ as $\{(C_\alpha, \mathcal{D}_\alpha) : \alpha < \mathfrak{c}\}$. We will build finite sets $\mathcal{F}^0_\alpha, \mathcal{F}^1_\alpha \in [\mathcal{E}]^{<\omega}$ recursively, so that each $\mathcal{F}^0_\alpha$ will contain exactly one element of each family $\mathcal{A}_{W0}, \mathcal{A}_{V_0}$ and $\mathcal{A}_{V_1}$, and each $\mathcal{F}^1_\alpha$ will intersect exactly two of the families $\mathcal{A}_{W_0}, \mathcal{A}_{W_1}, \mathcal{A}_{V_0}, \mathcal{A}_{V_1}$. In particular, no element of the form A_n or $\cup\mathcal{F}^i_\alpha$ will meet the four sets W_0, W_1, V_0 and V_1

Assume we have constructed $\mathcal{F}^0_\beta$ and $\mathcal{F}^1_\beta$ for $\beta < \alpha$. For $\mathcal{F}^0_\alpha$, consider the set $X = \bigcup_{n\in\omega}(P_n \setminus f_\alpha(n))$. Since X meets infinitely many elements of $\mathcal{A}_{W_0}$, it follows that X meets $\mathfrak{c}$-many elements of $\mathcal{A}_{W_0}$. Choose

$$A \in \mathcal{A}_{W_0} \setminus \left(\bigcup_{\beta<\alpha} (\mathcal{F}^0_\beta \cup \mathcal{F}^1_\beta) \cup \{P_n : n \in \omega\}\right)$$

such that A meets X. Also pick $B \in \mathcal{A}_{V_0} \setminus \bigcup_{\beta<\alpha}(\mathcal{F}^0_\beta \cup \mathcal{F}^1_\beta)$ and $C \in \mathcal{A}_{V_1} \setminus \bigcup_{\beta<\alpha}(\mathcal{F}^0_\beta \cup \mathcal{F}^1_\beta)$ arbitrary and define $\mathcal{F}^0_\alpha = \{A, B, C\}$.

For $\mathcal{F}^1_\alpha$ consider the pair $(C_\alpha, \mathcal{D}_\alpha)$ and let $\mathcal{D}_\alpha = \{D^j_\alpha : j < n\}$. Define

$$\mathcal{C} = \{A \in \mathcal{E} : A \text{ meets } C_\alpha\}$$

and

$$\mathcal{B} = \{A \in \mathcal{E} : \forall\, j < n \;(A \text{ meets } D^j_\alpha)\}.$$

If either $\mathcal{B}$ or $\mathcal{C}$ are finite, simply define $\mathcal{F}^1_\alpha = \emptyset$. Otherwise, we have some cases. Since $\mathcal{B}$ and $\mathcal{C}$ are infinite, there are $Y, Z \in \{W_0, W_1, V_0, V_1\}$ such that $\mathcal{B} \cap \mathcal{A}_Y$ and $\mathcal{C} \cap \mathcal{A}_Z$ are infinite. Since there are infinitely many elements of $\mathcal{A}_Y$ which meet D^j_α for every $j < n$, there are $\mathfrak{c}$-many of these elements. Pick, for every $j < n$, an element $B_j \in \mathcal{A}_Y$ such that B_j meets D^j_α and

$$B_j \in \mathcal{A}_Y \setminus \left(\mathcal{F}^0_\alpha \cup \bigcup_{\beta<\alpha} (\mathcal{F}^0_\beta \cup \mathcal{F}^1_\beta) \right).$$

Notice that $\bigcup_{j<n} B_j \subseteq Y$. Similarly, since $\mathcal{C}$ is infinite, we can find a set $C' \in \mathcal{A}_Z$ such that C' meets C_α and

$$C' \in \mathcal{A}_Z \setminus \left(\mathcal{F}^0_\alpha \cup \bigcup_{\beta<\alpha} (\mathcal{F}^0_\beta \cup \mathcal{F}^1_\beta) \right).$$

Define $\mathcal{F}^1_\alpha = \{C'\} \cup \{B_j : j < n\}$. This finishes the construction of the $\mathcal{F}^i_\alpha$'s.

Now, we can describe our AD family. Let

$$\mathcal{A} = \{A_n : n \in \omega\} \bigcup \{\cup \mathcal{F}^i_\alpha : \alpha < \mathfrak{c} \;\wedge\; i < 2\}.$$

It is clear that it is AD since each of its elements is a finite union of elements of $\mathcal{E}$.

To see that it is partly-normal, let K_0 and K_1 be two disjoint closed subsets of $\Psi(\mathcal{A})$ such that K_0 is regular closed and K_1 is π-closed. Hence, by lemma 3.1.2, there are $C \in [\omega]^\omega$ and $\{D_j : j < n\} \subseteq [\omega]^\omega$ such that $K_0 = \overline{C}$ and $K_1 = \bigcap_{j<n} \overline{D_j}$ in $\Psi(\mathcal{A})$. We consider first the case when one of the K_i has finite intersection with the AD family. Suppose that $K_0 \cap \mathcal{A}$ is finite, thus

$$U = C \cup \bigcup\{\{A\} \cup (A \setminus \cap_{j<n} D_j) : A \text{ meets } C\}$$

is a clopen subset which separates K_0 and K_1. To see this, notice that for every $A \in K_0$, the set $\{A\} \cup (A \setminus \cap_{j<n} D_j)$ is a basic clopen neighborhood of A, as $A \notin K_1$ and this implies $A \cap (\cap_{j<n} D_j)$ is finite. Given that $K_0 \cap \mathcal{A}$ is finite, the union at the right in the definition of U is a finite union of clopen sets. On the other hand, C is open and $\overline{C} \subseteq C \cup \{A \in \mathcal{A} : A \text{ meets } C\} \subseteq U$. It follows that U is clopen and contains K_0. Now observe that K_0 and K_1 are disjoint, which implies that $|A \cap C| < \omega$ for every $A \in \mathcal{A} \cap K_1$ and also $K_1 \cap \omega = \cap_{j<n} D_i$ which is disjoint from U. Henceforth $K_1 \subseteq \Psi(\mathcal{A}) \setminus U$. A similar argument shows that if $K_1 \cap \mathcal{A}$ is finite, we can separate K_0 and K_1.

We can then assume that both $K_0 \cap \mathcal{A}$ and $K_1 \cap \mathcal{A}$ are infinite. Let $\alpha < \mathfrak{c}$ such that $(C, \{D_j : j < n\}) = (C_\alpha, \mathcal{D}_\alpha)$. By the previous assumption, $\mathcal{F}^1_\alpha$ is not empty. So, $\mathcal{F}^1_\alpha = \{C'\} \cup \{B_j : j < n\}$, where C' meets C and B_j meets D_j for every $j < n$, which implies that $\cup \mathcal{F}^1_\alpha \in K_0 \cap K_1$, a contradiction. Hence the case where $K_0 \cap \mathcal{A}$ and $K_1 \cap \mathcal{A}$ are infinite is not possible.

To see that $\mathcal{A}$ is not quasi-normal, consider $W = \overline{W_0} \cap \overline{W_1}$ and $V = \overline{V_0} \cap \overline{V_1}$. These two closed sets are disjoint since no element of $\mathcal{A}$ intersects the four sets W_0, W_1, V_0, V_1, which are a partition of ω. Let U be an open set containing W. We have that A_n meets both W_0 and W_1 for every $n \in \omega$, then each $A_n \in W$. In particular, $P_n \subseteq A_n \subseteq^* U$. Let $f \in \omega^\omega$ such that $P_n \setminus f(n) \subseteq U$. We can find an $\alpha < \mathfrak{c}$ such that $f <^* f_\alpha$. Then, at step α, we defined $\mathcal{F}^0_\alpha = \{A, B, C\}$ in such a way that A meets $\bigcup_{n\in\omega}(P_n \setminus f_\alpha(n))$ (and consequently $\bigcup_{n\in\omega}(P_n \setminus f(n)) \subseteq U$), B meets V_0 and C meets V_1. Therefore $\cup \mathcal{F}^0_\alpha \in V$ but no open set containing it can be disjoint from U, which makes us impossible to separate V from W. □

All known counterexamples of the normality-like properties considered here, with exception of an almost-normal not normal AD family, can be constructed in ZFC alone. Hence, it is natural to ask if such a space can also exist in ZFC. We already know that no counterexample can be MAD by Corollary 3.2.3. In [46], the cardinal $\mathfrak{an}$ is defined as the least cardinality of an almost-normal not normal AD family, and it is noted that $\mathfrak{ap} \leq \mathfrak{an}$, whenever $\mathfrak{an}$ is well defined, i.e., whenever there is an almost-normal not normal AD family. Here $\mathfrak{ap}$ is defined as the least cardinality of an AD family which is not weakly separated [10]. Since it is consistent that $\mathfrak{ap} = \mathfrak{c}$, the unresolved portion of the question of whether there are almost-normal not normal AD families, can be stated as follows:

Question 5. *Does there exist (in ZFC) an almost-normal AD family which is not normal? (an almost-normal AD family of size $\mathfrak{c}$?)*

On the other hand, it was proved in [46], that there is, consistently, an almost-normal not normal AD family of size $\omega_1 < \mathfrak{c}$. Hence, even though the first part of the above question might have a positive answer, the proof may go by cases (in some models all such families have size $< \mathfrak{c}$ while in others, all such families have size $\mathfrak{c}$) and then the second part of the question could have a negative answer.

In [30], a study on the relation between normality and the existence of Luzin-type subfamilies was developed. We call a pair $\mathcal{B} = \{B_\alpha : \alpha < \omega_1\}$ and $\mathcal{C} = \{C_\alpha : \alpha < \omega_1\}$ of subfamilies of $[\omega]^\omega$ a *Luzin gap* if there is an $m \in \omega$ such that:

1. $A_\alpha \cap B_\alpha \subseteq m$ for every $\alpha < \omega_1$ and
2. $(A_\alpha \cap B_\beta) \cup (A_\beta \cap B_\alpha) \nsubseteq m$ but $A_\alpha \cap B_\beta$ is finite for every $\alpha \neq \beta < \omega_1$.

It is known that every Luzin family contains many Luzin gaps and if $\mathcal{B}$ and $\mathcal{C}$ forms a Luzin gap, they can not be separated. Thus, AD families which contain Luzin gaps are not normal. Moreover, Luzin gaps are indestructible by forcing notions which preserve ω_1, thus, Luzin gaps can not be normal in any of these forcing extensions. A generalization of Luzin gaps is the following:

Definition 3.3.2. [30] Let $n \in \omega$ and $B_i = \{B^i_\alpha : \alpha < \omega_1\}$ be disjoint subfamilies of an AD family $\mathcal{A}$ for $i < n$. We say that $\langle B_i : i < n \rangle$ forms an n-Luzin gap if there is an $m \in \omega$ such that:

1. $B^i_\alpha \cap B^j_\alpha \subseteq m$ for all $i \neq j$, $\alpha < \omega_1$ and
2. $\bigcup_{i \neq j} (B^i_\alpha \cap B^j_\beta) \nsubseteq m$ for every $\alpha \neq \beta < \omega_1$.

Let P be any property of AD families. An AD family is said to be *potentially* P [30], if there is a forcing notion $\mathbb{P}$, such that $\Vdash_{\mathbb{P}}$"$\mathcal{A}$ is P". Hence, an AD family fails to be potentially normal if it contains Luzin gaps. An interesting result arises when n-Luzin gaps are considered under MA.

Theorem 3.3.3. *[30] Assume MA and let $\mathcal{A}$ be an AD family. Then $\mathcal{A}$ is normal if and only if $|\mathcal{A}| < \mathfrak{c}$ and $\mathcal{A}$ does not contain n-Luzin gaps for any $n \in \omega$.*

A result in ZFC that could be useful to the study of normality-like properties is the following:

Theorem 3.3.4. *[30] The following are equivalent for an AD family $\mathcal{A}$:*

1. *$\mathcal{A}$ does not contain n-Luzin gaps for any $n \in \omega$,*
2. *$\mathcal{A}$ is potentially normal,*
3. *$\mathcal{A}$ is potentially $\mathbb{R}$-embeddable.*

Recall that $\mathcal{A}$ is $\mathbb{R}$-embeddable if there is an injective and continuous function $\varphi : \Psi(\mathcal{A}) \to \mathbb{R}$. Hence, one could ask the relation between these concepts and the weakenings of normality.

Question 6. *Are almost-normal AD families potentially normal?*

Since it is consistent that there are quasi-normal AD families which contain Luzin families, we can not ask the above question for weaker normality-like properties in ZFC.

Question 7. *Is it consistent that quasi-normal (partly-normal, mildly-normal) AD families are potentially normal?*

3.4 On strongly $\aleph_0$-separated AD families

The concept of strongly $\aleph_0$-separated AD families was introduced in [26] by the authors. An AD family $\mathcal{A}$ is *strongly* $\aleph_0$*-separated*, if for every two disjoint countable subfamilies $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$, there is a separator for $(\mathcal{B}, \mathcal{C})$. There, it was shown that almost-normal AD families are strongly $\aleph_0$-separated and that there is a strongly $\aleph_0$-separated MAD family under CH.

The requirement of one of the subfamilies being countable was modified in [46] in order to define a stronger concept: An AD family is *strongly* $(\aleph_0, < \mathfrak{c})$*-separated*, if for every two disjoint subfamilies $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$, where $\mathcal{B}$ is countable and $|\mathcal{C}| < \mathfrak{c}$, there is a separator for $(\mathcal{B}, \mathcal{C})$. The relation of these two concepts and almost-normality was studied in [26] and [46], however, the next question remained unanswered [46]:

- Does CH imply that strongly $\aleph_0$-separated AD families are almost-normal?

We will answer this question in the negative. For this purpose, recall that $\mathfrak{s}$ is the least size of a splitting family. The splitting number $\mathfrak{s}$ is a cardinal invariant of the continuum, hence $\omega < \mathfrak{s} \leq \mathfrak{c}$. In particular, for every countable family $\mathcal{H} \subseteq [\omega]^\omega$, there exists $X \in [\omega]^\omega$ which is not split by any element of $\mathcal{H}$, i.e., either, $X \cap H$ is finite or $X \subseteq^* H$ for every $H \in \mathcal{H}$ (see [9]).

Theorem 3.4.1. *(CH) There is a strongly $\aleph_0$-separated AD family which is not almost-normal.*

Proof. Let $\{d_\alpha : \alpha < \omega_1\}$ be a dominating family of functions and enumerate all pairs $(a, b) \in [\omega_1]^{\leq\omega} \times [\omega_1]^{\leq\omega}$, such that $a \cap b = \emptyset$ as $\{(a_\alpha, b_\alpha) : \omega \leq \alpha < \omega_1\}$. We can assume without loss of generality that $a_\alpha \cup b_\alpha \subseteq \alpha$ for every $\alpha < \omega_1$. Partition $\omega = V \cup W$ into two infinite sets and let $\varphi : V \to W$ be a bijection. Moreover, partition $V = \bigcup_{n\in\omega} A_n$, into infinitely many infinite sets.

We will recursively construct A_α and D_α for $\omega \leq \alpha < \omega_1$ such that if $\mathcal{A}_\alpha = \{A_\beta : \beta < \alpha\}$, then the following holds:

1. $\mathcal{A}_\alpha$ is an almost disjoint family.

2. $|A_\alpha \cap A_n| \leq 1$ for every $\omega \leq \alpha < \omega_1$ and every $n \in \omega$.

3. A_α meets $\bigcup_{n\in\omega} (A_n \setminus d_\alpha(n))$.

4. D_α is a partitioner for $\mathcal{A}_\alpha$ such that $A_\beta \subseteq^* D_\alpha$ for every $\beta \in a_\alpha$ and $A_\gamma \cap D_\alpha$ is finite for every $\gamma \in b_\alpha$.

5. Either, $A_\alpha \subseteq^* D_\beta$ or $A_\alpha \cap D_\beta$ is finite for every $\beta < \alpha$.

Suppose we have defined $\mathcal{A}_\alpha = \{A_\beta : \beta < \alpha\}$ and $\mathcal{D}_\alpha = \{D_\beta : \omega \leq \beta < \alpha\}$ with the above properties. We shall define D_α and A_α.

Consider the pair (a_α, b_α). Let $\mathcal{B} = \{A_\beta : \beta \in a_\alpha\}$ and $\mathcal{C} = \{A_\beta : \beta < \alpha \ \wedge \ \beta \notin a_\alpha\}$. Since α is countable we can enumerate both sets as $\mathcal{B} = \{B_n : n \in \omega\}$ and $\mathcal{C} = \{C_n : n \in \omega\}$. Define

$$D_\alpha = \bigcup_{n\in\omega} \left(B_n \setminus \bigcup_{i<n} C_i \right).$$

Since $\mathcal{A}_\alpha = \mathcal{B} \cup \mathcal{C}$ is AD, it is easy to see that D_α is a partitioner for $\mathcal{A}_\alpha$ and it follows from the definition that satisfies property (4).

Now we turn to the construction of A_α. For every infinite ordinal $\beta < \alpha$, there is a function f_β such that $A_\beta \cap A_n \subseteq f_\beta(n)$ for every $n \in \omega$. Define for every infinite ordinal $\beta \leq \alpha$, $H_\beta = \{n \in \omega : D_\beta \text{ meets } A_n\}$. Notice that since D_β is a partitioner, $A_n \subseteq^* D_\beta$ whenever D_α meets A_n. Thus we can also define a function $g_\beta \in \omega^\omega$ such that $A_n \setminus g_\beta(n) \subseteq D_\beta$ if $n \in H_\beta$ and $D_\beta \cap A_n \subseteq g_\beta(n)$ otherwise. Let $r \in \omega^\omega$ such that r dominates the family $\{d_\beta : \beta \leq \alpha\} \cup \{f_\beta : \omega \leq \beta < \alpha\} \cup \{g_\beta : \omega \leq \beta \leq \alpha\}$.

Since the family $\{H_\beta : \omega \leq \beta \leq \alpha\}$ is countable, we can also find a set $H \in [\omega]^\omega$ such that for every infinite ordinal $\beta \leq \alpha$ either, $H \cap H_\beta$ is finite or $H \subseteq^* H_\beta$. For every $n \in H$, let $x_n \in A_n \setminus r(n)$. Define $A_\alpha = \{x_n : n \in \omega\}$.

It is clear that A_α satisfies (2), given that the family A_n is a partition of V. To see that A_α satisfies (3), simply note that $r >^* d_\alpha$. We check property 1. Let $\beta < \alpha$ an infinite ordinal and let $k \in \omega$ such that $r(n) > f_\beta(n)$ for every $n > k$. Then

$$x_n \in A_n \setminus r(n) \subseteq A_n \setminus f_\beta(n) \subseteq A_n \setminus A_\beta$$

for every $n > k$, showing that $A_\alpha \cap A_\beta$ is finite. For property (5), consider the set H_β. If $H \cap H_\beta$ is finite, we can find $k \in \omega$ such that $H \cap H_\beta \subseteq k$ and $r(n) > g_\beta(n)$ for every $n > k$. Hence, for every $n \in H \setminus k$, $n \notin H_\beta$ implies that $D_\beta \cap A_n \subseteq g_\beta(n)$ and

$$x_n \in A_n \setminus r(n) \subseteq A_n \setminus g_\beta(n) \subseteq A_n \setminus D_\beta,$$

whence $A_\alpha \cap D_\beta$ is finite. On the other hand, if $H \subseteq^* H_\beta$, we can find $k \in \omega$ such that $H \setminus k \subseteq H_\beta$ and $r(n) > g_\beta(n)$ for every $n > k$. Then, for every $n \in H \setminus k \subseteq H_\beta$, $g_\beta(n)$ was defined so that $A_n \setminus g_\beta(n) \subseteq D_\beta$ and

$$x_n \in A_n \setminus r(n) \subseteq A_n \setminus g_\beta(n) \subseteq D_\beta,$$

proving that $A_\alpha \subseteq^* D_\beta$. This finishes the recursive construction.

We are going to make a last modification to $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ in order to get the desired family. For every $\alpha < \omega_1$ define $\widetilde{A}_\alpha$ as follows:

$$\widetilde{A}_\alpha = \begin{cases} A_\alpha & \text{if } \alpha < \omega \\ A_\alpha \cup \varphi[A_\alpha] & \text{if } \alpha \geq \omega \end{cases}$$

Similarly define $\widetilde{D}_\alpha = D_\alpha \cup \varphi[D_\alpha]$ for $\omega \leq \alpha < \omega_1$. Since φ is a bijection between two disjoint sets V and W, if $\widetilde{\mathcal{A}} = \{\widetilde{A}_\alpha : \alpha < \omega_1\}$ and $\widetilde{\mathcal{A}}_\alpha = \{\widetilde{A}_\beta : \beta < \alpha\}$, properties (1)-(5) also hold replacing A_β by $\widetilde{A}_\beta$ and D_β by $\widetilde{D}_\beta$.

Claim: $\widetilde{\mathcal{A}}$ is strongly $\aleph_0$-separated.
Let $\mathcal{A}', \mathcal{A}'' \in [\widetilde{\mathcal{A}}]^{\leq\omega}$ be disjoint subfamilies. Define $a = \{\delta : \widetilde{A}_\delta \in \mathcal{A}'\}$ and $b = \{\beta : \widetilde{A}_\beta \in \mathcal{A}''\}$. There exists $\alpha < \omega_1$ such that $(a, b) = (a_\alpha, b_\alpha)$. Thus $\widetilde{D}_\alpha$ is a partitioner for $\widetilde{\mathcal{A}}_\alpha$ and was chosen so that $\widetilde{D}_\alpha$ separates $\mathcal{A}'$ and $\mathcal{A}''$ by property (4). Moreover, since $\widetilde{A}_\gamma$ is either, almost disjoint or almost contained in $\widetilde{D}_\alpha$ for every $\gamma \geq \alpha$, $\widetilde{D}_\alpha$ is indeed, a separator for $(\mathcal{A}', \mathcal{A}'')$.

Claim: $\widetilde{\mathcal{A}}$ is not almost-normal.
For every $n \in \omega$, $\widetilde{A}_n = A_n \subseteq V$ which is disjoint from W. In addition, $\widetilde{A}_\alpha \cap W = \varphi[A_\alpha]$ is an infinite set for $\alpha \geq \omega$. It suffices now to prove that $\mathcal{A}_\omega$ and $\mathcal{B} = \{\widetilde{A}_\alpha : \omega \leq \alpha < \omega_1\}$ can not be separated (recall Proposition 3.1.3). Let D be such that $A_n \subseteq^* D$ for every $n \in \omega$. There exists $f \in \omega^\omega$

such that $A_n \setminus f(n) \subseteq D$. Choose $\alpha < \omega_1$ such that $d_\alpha >^* f$. Then $\widetilde{A}_\alpha$ meets $\bigcup_{n\in\omega}(A_n \setminus d_\alpha(n)) \subseteq^* D$. Hence, D is not a separator for $(\mathcal{A}_\omega, \mathcal{B})$. $\square$

We have answered Question 7.3 from [46] in the negative, in particular, under CH, there is a strongly $(\aleph_0, < \mathfrak{c})$-separated AD family which is not almost-normal. This result also follows from PFA, actually, something stronger is true. Let P be a given property. We will say that MAD families with property P *exists generically* if all AD families of size less than $\mathfrak{c}$ can be extended to a MAD family with property P. Generic existence of MAD families was introduced in [31] and it was proved in [46] that under $\mathfrak{b} = \mathfrak{c} = \mathfrak{s}$, completely separable MAD families which are strongly $(\aleph_0, < \mathfrak{c})$-separated exist generically. Since the hypothesis hold under PFA and we have proved that PFA implies no MAD family is almost-normal, we get the following:

Corollary 3.4.2. *(PFA) Completely separable, strongly* $(\aleph_0, < \mathfrak{c})$*-separated MAD families which are not almost-normal exist generically.*

In particular, strongly $(\aleph_0, < \mathfrak{c})$-separated AD families which contain Luzin families (and hence are not potentially normal) exist generically. We do not know if this is always the case, or at least, it follows from MA.

Question 8. *Is it consistent that strongly* $\aleph_0$*-separated (or strongly* $(\aleph_0, < \mathfrak{c})$*-separated) AD families are potentially normal? Is it consistent with MA?*

Chapter 4

Uniformization properties of ladder systems after forcing with a Souslin tree

In this chapter, we will study uniformization and anti-uniformization properties of ladder systems on ω_1 after forcing with a Souslin tree. In particular, we will determine exactly which of these properties are satisfied when we force over a model of $MA_{\omega_1}(\mathcal{K})$. We will end this chapter by studying spaces defined from walks on ladder systems, giving an alternative proof of some results in Chapter 2.

4.1 Basic notions

A ladder system over a stationary subset of limit ordinals $E \subseteq \omega_1$ is a sequence $L = \langle L_\alpha : \alpha \in E \rangle$ such that each L_α is a cofinal subset of α with order type ω. Shelah introduced the notion of a ladder system being uniformizable in relation to his work on Whitehead groups [50].

Definition 4.1.1. A ladder system $L = \langle L_\alpha : \alpha \in E \rangle$ is *uniformizable* if

$$\forall \langle s_\alpha : L_\alpha \to \omega | \alpha \in E \rangle \; \exists f : \omega_1 \to \omega \; \forall \alpha \in E \;\; (f \restriction L_\alpha =^* s_\alpha)$$

Then a ladder system is uniformizable if given any sequence of colorings of the ladders, we can define a function which almost agrees with all of them.

Proposition 4.1.2. (Devlin, Shelah) $MA(\omega_1)$ *implies that all ladder systems are uniformizable.*

Proof. Let $L = \langle L_\alpha : \alpha \in E \rangle$ be a ladder system and let $\langle s_\alpha : \alpha \in E \rangle$ be a sequence of functions $s_\alpha : L_\alpha \to \omega$. Define $\mathbb{P}$ to be the set of all pairs (f, F) such that:

- $f; \omega_1 \to \omega$,
- $|dom(f)| < \omega$,
- $F \in [E]^{<\omega}$,
- $\forall \alpha, \beta \in F(L_\alpha \cap L_\beta \subseteq dom(f))$

and set $(f, F) \leq (g, G) \Leftrightarrow\ f \supseteq g,\ F \supseteq G$ and $f(\beta) = s_\alpha(\beta)$ for all $\alpha \in G$ and all $\beta \in L_\alpha \cap dom(f) \setminus dom(g)$.

First note that $D_\alpha = \{(f, F) : \alpha \in dom(f)\}$ and $H_\eta = \{(f, F) : \eta \in F\}$ are dense subsets of $\mathbb{P}$ for every $\alpha \in \omega_1$ and $\eta \in E$. Hence if G is generic for these ω_1-many sets,

$$f_G = \bigcup \{f : \exists F \in [E]^{<\omega}((f, F) \in G)\}$$

uniformizes the sequence $\langle s_\alpha : \alpha \in E \rangle$.

To finish the proof we shall prove that $\mathbb{P}$ is ccc. Let $\{(f_\alpha, F_\alpha) : \alpha < \omega_1\} \subseteq \mathbb{P}$. Let $a_\alpha = dom(f_\alpha)$. We can assume that $\{a_\alpha : \alpha < \omega_1\}$ and $\{F_\alpha : \alpha \in \omega_1\}$ form delta systems with roots r and R respectively and define $A_\alpha = F_\alpha \setminus R$. Moreover we can assume that there are $n, N \in \omega$ such that $|a_\alpha| = n$ and $|A_\alpha| = N$ for all $\alpha \in \omega_1$ and further, assume that $\min(A_\alpha \cup (a_\alpha \setminus r)) \geq \alpha$ for every α. Let M be a countable elementary submodel of $H(\theta)$ with θ large enough and such that $\mathbb{P}, \{(f_\alpha, F_\alpha) : \alpha \in \omega_1\}, \langle s_\alpha : \alpha \in E \rangle \in M$. Let $\delta = \omega_1 \cap M$. Then $(f_\delta, F_\delta) \notin M$ and $((a_\delta \setminus r) \cup A_\delta) \cap \delta = \emptyset$. For every $\alpha \in \omega_1$ enumerate $A_\alpha = \{\alpha_0, \alpha_1, \ldots, \alpha_{N-1}\}$. Pick $\eta < \delta$ such that $L_{\alpha_i} \cap \delta \subseteq \eta$ for all $i < N$. Finally define $\tau^i_\alpha = s_{\alpha_i} \restriction L_{\alpha_i} \cap \eta$. Then, by elementarity, there exist $\eta < \xi_i < \delta$ such that $L_{\xi_i} \cap \eta = L_{\alpha_i} \cap \eta$ and $s_{\xi_i} \restriction L_{\xi_i} \cap \eta = \tau^i_\delta$ for every $i < N$. Now it is straightforward to see that (f_ξ, F_ξ) and (f_δ, F_δ) are compatible. □

In [7], weak versions of uniformization and natural related *anti uniformization* properties were introduced. These properties arise in relation to constructions of topological spaces as counterexamples concerning the relationship between covering and separation properties. The simplest such construction from a ladder system L is the space $X_L = E \times \{1\} \cup \omega_1 \times \{0\}$ where the points of $\omega_1 \times \{0\}$ are isolated, and a neighborhood base at $(\alpha, 1) \in E \times \{1\}$ is of the form $\{(\alpha, 1)\} \cup ((L_\alpha \times \{0\}) \setminus F)$ where F is finite. Notice the similarity with the Mrowka-Isbell spaces defined from an AD family. The space X_L is always first countable and locally compact. Recall that a space X is *collectionwise Hausdorff* if for every closed and discrete subset D, there exists a family $\{U_d : d \in D\}$ of pairwise disjoint open sets such that $d \in U_d$.

Lemma 4.1.3. *X_L is not collectionwise Hausdorff.*

Proof. First note that if $L = \langle L_\alpha : \alpha \in E\rangle$, then $E \times \{1\}$ is closed and discrete in X_L. Pick an open neighborhood U_α for every $\alpha \in E$. Let $g : E \to \omega_1$ defined by $g(\alpha) = \min\{\beta \in \omega_1 : (\beta, 0) \in U_\alpha\}$. Then by the pressing down lemma there exist $E' \subseteq E$ stationary and $\eta \in \omega_1$ such that $g(\alpha) = \eta$ for all $\alpha \in E'$. In particular, $U_\alpha \cap U_\beta \neq \emptyset$ for every $\alpha, \beta \in E'$. □

We will introduce now weak uniformization properties and some topological equivalences in the associated space X_L introduced in [7].

Definition 4.1.4. A ladder system $L = \langle L_\alpha : \alpha \in E\rangle$ satisfies $\mathcal{M}_n$ ($\mathcal{M}_{<\omega}$ respectively) if for every function $f : E \to \omega$ there exists $F : \omega_1 \to [\omega]^{n+1}$ ($F : \omega_1 \to [\omega]^{<\omega}$ respectively) such that

$$\forall \alpha \in E\ \forall^\infty \beta \in L_\alpha\ \ (f(\alpha) \in F(\beta))\,.$$

If in addition $F \restriction L_\alpha$ is eventually constant for every $\alpha \in E$ then the ladder system satisfies $\mathcal{P}_n$ ($\mathcal{P}_{<\omega}$ respectively).

It is clear that $\mathcal{P}_n \Rightarrow \mathcal{P}_{n+1} \Rightarrow \mathcal{M}_{n+1} \Rightarrow \mathcal{M}_{n+2} \Rightarrow \mathcal{M}_{<\omega}$, $\mathcal{P}_n \Rightarrow \mathcal{P}_{<\omega} \Rightarrow \mathcal{M}_{<\omega}$ and $\mathcal{P}_0$ and $\mathcal{M}_0$ are both equivalent to being uniformizable with respect to constant functions.

Since $MA(\omega_1)$ implies that all ladder systems are uniformizable, it is consistent that all uniformization properties are equivalent. On the other

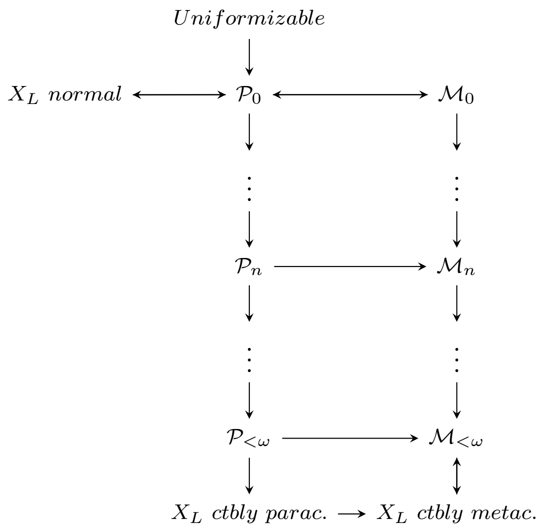

Figure 4.1: Uniformization properties

hand, $2^{\aleph_0} < 2^{\aleph_1}$ implies that no ladder system is uniformizable [17]. Nevertheless, there exists a ladder system which satisfies $\mathcal{M}_{<\omega}$ in ZFC and it is shown in [7] that there are no other ZFC implications between the properties $\mathcal{M}_n$ and $\mathcal{P}_m$ than those drawn in diagram 4.1. Regarding topological properties, we will prove that the property $\mathcal{P}_0$ corresponds to X_L being normal while $\mathcal{M}_{<\omega}$ corresponds to X_L being countably metacompact.

Theorem 4.1.5. (Folklore) *A ladder system L satisfies $\mathcal{P}_0$ iff X_L is normal.*

Proof. Assume $L = \langle L_\alpha : \alpha \in E \rangle$ satisfies $\mathcal{P}_0$. Let $C, D \subseteq X_L$ be two closed subsets. Since $\omega_1 \times \{0\}$ consist of isolated points, we can assume that $C, D \subseteq E \times \{1\}$. Let $f : E \to 2$ such that $f(\alpha) = 0$ if and only if $\alpha \in C$. Since L satisfies $\mathcal{P}_0$ there is a function $g : \omega_1 \to \omega$ which uniformizes f. For every $\alpha \in E$ define $U_\alpha = \{(\alpha, 1)\} \cup \{(\beta, 0) \in L_\alpha \times \{0\} : g(\beta) = f(\alpha)\}$. Finally define $U = \bigcup_{\alpha \in C} U_\alpha$ and $V = \bigcup_{\alpha \in D} U_\alpha$. Then $C \subseteq U$, $D \subseteq V$ and $U \cap V = \emptyset$.

Conversely let $f : E \to \omega$ and define $E_n = \{(\alpha, 1) : f(\alpha) = 1\}$ for every $n \in \omega$. It is easy to see that E_n and $D_n = \bigcup_{m \geq n} E_m$ are closed for every $n \in \omega$. Take U_0 and V_1 which separate E_0 and D_1 with $E_0 \subseteq U_0$. In general, define $U_n \subseteq V_n \subseteq \ldots \subseteq V_1$ and $V_{n+1} \subseteq V_n \subseteq \ldots \subseteq V_1$ such that $E_n \subseteq U_n$, $D_{n+1} \subseteq V_{n+1}$ and $U_n \cap V_{n+1} = \emptyset$. Thus define $g : \omega_1 \to \omega$ such that $g(\alpha) = n$ if and only if $(\alpha, 0) \in U_n$ and $g(\alpha) = 0$ if $\alpha \notin \bigcup_{n\in\omega} U_n$. The constructed g uniformizes f. □

It should be remarked that we have actually proved that uniformization with respect to the constant functions (i.e. $\mathcal{P}_0$) is equivalent when we consider functions with range 2 instead of ω. Recall that given a space X and a cover $\mathcal{U}$ of X, $\mathcal{V}$ refines $\mathcal{U}$ if $\mathcal{V}$ is a cover of X and for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$. A family $\mathcal{U} \subseteq \mathcal{P}(X)$ is point-finite if for every $x \in X$ the set $\{U \in \mathcal{U} : x \in U\}$ is finite. It is locally finite if there exists an open set W with $x \in W$ such that the set $\{U \in \mathcal{U} : W \cap U \neq \emptyset\}$ is finite. Then a space X is (countably) metacompact if for every (countable) open cover $\mathcal{U}$ there exists a point-finite refinement $\mathcal{V}$. The space X is (countably) paracompact if every (countable) open cover $\mathcal{U}$ has a locally finite refinement $\mathcal{V}$.

Theorem 4.1.6. [7] *A ladder system L satisfies $\mathcal{M}_{<\omega}$ iff X_L is countably metacompact.*

Proof. Suppose L is $\mathcal{M}_{<\omega}$ and let $X_L = \bigcup_{n\in\omega} U_n$ where each U_n is open. Define $V_n = U_n \cap (E \times \{1\})$ for every $n \in \omega$. Since $\omega_1 \times \{0\}$ is discrete and $E \times \{1\}$ is a discrete subspace, we can assume that $V_n \cap V_m = \emptyset$ for every $m \neq n$. Then let $f : E \to \omega$ given by $f(\alpha) = n$ if and only if $(\alpha, 1) \in U_n$ and let $F : \omega_1 \to [\omega]^{<\omega}$ be a function which uniformizes f in the sense of $\mathcal{M}_{<\omega}$. Let

$$U'_n := V_n \cup \{(\beta, 0) \in U_n : n \in F(\beta)\}$$

for every $n \in \omega$. Thus, $\{U'_n : n \in \omega\} \cup \{\{(\alpha, 0) : (\alpha, 0) \notin \bigcup_{n\in\omega} U'_n\}\}$ is a point finite refinement of $\{U_n : n \in \omega\}$.

Now assume X_L is countably metacompact, let $f : E \to \omega$ and define

$$U_n = \left[f^{-1}(n) \times \{1\}\right] \cup \left[\left(\bigcup\{L_\alpha : \alpha \in f^{-1}(n)\}\right) \times \{0\}\right]$$

for every $n \in \omega$. Also define $U_\omega = X_L \setminus \left(\bigcup_{n\in\omega} U_n\right)$. Notice that $U_\omega \subseteq \omega_1 \times \{0\}$, and thus it is open. Let $\mathcal{W}$ be a point-finite refinement of $\{U_\alpha : \alpha \leq \omega\}$

and define $V_n = \bigcup\{W \in \mathcal{W} : V \subseteq U_n\}$. Notice that $\mathcal{V} = \{V_n : n \in \omega\}$ is also a point-finite cover of X_L. Define $F : \omega_1 \to [\omega]^{<\omega}$ by

$$F(\beta) = \{n \in \omega : (\beta, 0) \in V_n\} \in [\omega]^{<\omega}.$$

This function is well defined because $\mathcal{V}$ is point finite and it is easy to see that F uniformizes f in the sense of $\mathcal{M}_{<\omega}$. □

A similar argument shows that if a ladder system satisfies $\mathcal{P}_{<\omega}$ then the space X_L is countably paracompact but it is still not known if these properties are equivalent. We will not prove this fact since in lemma 4.2.1 we will give a combinatorial characterization of countably paracompactness. This lemma also provide a better approach in order to try to prove the equivalence of these properties. From ladder systems with specific uniformization and anti uniformization properties (which will be defined later), it is possible to construct topological spaces with interesting properties solving some open questions in topology. We will talk about these spaces after defining anti uniformization properties.

From a ladder system L with specific uniformization and anti uniformization properties, the space X_L gives an example of a normal, first countable, locally countable space, which is not collectionwise Hausdorff and the witness for the last property is a closed discrete non-G_δ set (see [51]). Also, from a ladder system L' called thin and countably metacompact in [7], the space $X_{L'}$ is used to construct a countably paracompact, locally compact, screenable space which is not paracompact, answering a question in [6], however, the existence of such ladder system is left open.

Recall that a forcing notion $\mathbb{P}$ is Knaster if every uncountable subset of $\mathbb{P}$ contains an uncountable subset of pairwise compatible conditions. Then if $\mathcal{K}$ is the class of Knaster forcings, $MA_{\omega_1}(\mathcal{K})$ is the assertion that for every forcing notion $\mathbb{P}$ which is Knaster and for every family $\mathcal{D} = \{D_\alpha : \alpha \in \omega_1\}$ of dense subsets of $\mathbb{P}$, there is a $\mathcal{D}$-generic filter $G \subseteq \mathbb{P}$.

Also, $MA(S)$ is the assertion that there exists a coherent Souslin tree S such that for every poset $\mathbb{P}$ which satisfies that $\mathbb{P} \times S$ is ccc and for every sequence $\mathcal{D} = \{D_\alpha : \alpha \in \omega_1\}$ of dense subsets of $\mathbb{P}$, there exists a $\mathcal{D}$-generic filter $G \subseteq \mathbb{P}$. Models obtained by a forcing extension with the Souslin tree

S over models of $MA(S)$ were introduced by Larson and Todorčević in [38]. We will call this kind of models as models of $MA(S)[S]$.

The motivation of this chapter was to try to find a ladder system which satisfies $\mathcal{M}_{<\omega}$ and some anti-uniformization properties in models of $PFA(S)[S]$ (which is defined in a similar way to $MA(S)[S]$) in order to give consistent answers to some topological questions which were proved to be implied by the existence of this kind of ladders in [7]. Since every Knaster forcing preserves the Souslin tree and since a forcing notion $\mathbb{P}$ preserves the Souslin tree iff $\mathbb{P} \times S$ is ccc, it follows that $MA(S)$ implies $MA_{\omega_1}(\mathcal{K})$. Therefore, since forcings of theorems 4.3.3 and 4.4.2 are Knaster, we will state these results under $MA_{\omega_1}(\mathcal{K})$ instead of $PFA(S)$. On the other hand due to theorem 4.4.2, there are no ladder systems satisfying anti uniformization properties in models of $PFA(S)[S]$, but instead, we have determined exactly which uniformization and anti uniformization properties are satisfied for each ladder system in models obtained by a forcing extension with a Souslin tree S over models of $MA_{\omega_1}(\mathcal{K})$. For more on $PFA(S)$ and its applications see [56] and [20].

In Section 4.2 we shall prove that in models obtained by a forcing extension with a Souslin tree, no ladder system satisfies the property $\mathcal{M}_n$ for every $n \in \omega$ and that the space X_L is never countably paracompact. In Section 4.3 we show that after forcing with the Souslin tree over a model of $MA_{\omega_1}(\mathcal{K})$, every ladder system satisfies $\mathcal{M}_{<\omega}$. In section 4.4 we make some observations concluding that in the forcing extension with a Souslin tree S over a model of $MA_{\omega_1}(\mathcal{K})$, no ladder system satisfies any of the anti-uniformization properties introduced in [7]. Finally, in Section 4.5 we study a space defined from the ladder system using the theory of walks on ordinals developed by Todorčević [55]. With this space, we will find a similar result to theorem 2.4.5, by forcing a ladder system. Hence, this time the counterexample to Arhangelskii's question will have size ω_1.

4.2 Properties after forcing with a Souslin tree.

In [7], it is showed that a ladder system satisfying $\mathcal{P}_{<\omega}$ defines a countably paracompact space X_L. It is not known if these properties are indeed equivalent. We will begin this section by characterizing the property of X_L being countably paracompact under combinatorial properties of the ladder

L. This also provide a more suitable statement for either proving or deny the equivalence between $\mathcal{P}_{<\omega}$ and X_L being countably paracompact.

Lemma 4.2.1. *Let $L = \{L_\alpha : \alpha \in E\}$ be a ladder system. The space X_L is countably paracompact iff for all $f : E \to \omega$, there exist $F : \omega_1 \to [\omega]^{<\omega}$ and $g : E \mapsto [\omega]^{<\omega}$ such that*

$$f(\alpha) \in F(\beta) \subseteq g(\alpha)$$

for all $\alpha \in E$ and for all but finitely many $\beta \in L_\alpha$.

Proof. Notice that X_L is countably paracompact if and only if, for every partition $E \times \{1\} = \bigcup_{n\in\omega} E_n$ into countably many pieces, there exists an open expansion $\{U_n : n \in \omega\}$ (i.e., each $U_n \subseteq X_L$ is open and $U_n \cap (E \times \{1\}) = E_n$) which is locally finite. Identify functions from E to ω with partitions of E. Then, if X_L is countably paracompact and $\{U_n : n \in \omega\}$ is a locally compact open expansion, define $F : \omega_1 \to [\omega]^{<\omega}$ by letting $F(\beta) = \{n \in \omega : (\beta, 0) \in U_n\}$ for all $\beta \in \omega_1$. Also define $g : E \to [\omega]^{<\omega}$ by letting $g(\alpha) = \{n \in \omega : U_n \cap L_\alpha \neq \emptyset\}$.

Conversely, given F, g and a partition $E \times \{1\} = \bigcup_{n\in\omega} E_n$, for every $n \in \omega$ define

$$U_n = E_n \cup \{(\eta, 0) : n \in F(\eta)\}.$$

It is straightforward to see that these constructions prove the lemma. □

We repeat the question mentioned above, first asked in [7] which is still open:

Question 9. *If X_L is countably paracompact, does L satisfies $\mathcal{P}_{<\omega}$?*

We turn now to the main result of this section.

Theorem 4.2.2. *After forcing with a Souslin tree S, the following hold:*

1. *X_L is not countably paracompact for every ladder system L*
2. *For every $n \in \omega$, no ladder system satisfies $\mathcal{M}_n$*

Proof. (1)We will use Lemma 4.2.1. We can assume that $S \subseteq \omega^{<\omega_1}$. Let $b \subseteq S$ be a generic branch. Let $\dot{E}$ be an S-name for a stationary subset of $\lim(\omega_1)$ and $\dot{L} = \{\dot{L}_\alpha : \alpha \in \dot{E}\}$ be an S-name for a ladder system.

We can find a club $C \subseteq \omega_1$ such that for every $\alpha \in \lim(\omega_1)$ and every $s \in S$ with $l(s) = \alpha^+$, where α^+ is the minimum element greater than α living in C, we have that s decides "$\alpha \in \dot{E}$" and if it is the case that $s \Vdash$ "$\alpha \in \dot{E}$", then s also decides $\dot{L}_\alpha$

In the extension $V[b]$, define $f : S \to \omega$ by letting $f(\alpha) = b(\alpha^+)$. Given $t \in S$, since $E_t = \{\beta > l(t) : t \nVdash$ "$\beta \notin \dot{S}$"$\}$ is stationary, we can find $s \geq t$ and $\delta \in E_t \cap C$ such that $l(s) = \delta^+$, $s \Vdash$"$\delta \in E$" and s decides L_δ. Moreover, given $\dot{F}$ an S-name for a function from ω_1 to $[\omega]^{<\omega}$, we can get an elementary submodel $M \prec H((2^{\omega_1})^+)$ such that $S, \dot{F}, t \in M$ and this δ is equal to $M \cap \omega_1$. In this way, s also decides $\dot{F} \restriction \delta$ and in consequence s decides $\dot{F} \restriction L_\delta$.

Now, we can take a generic branch $r \subseteq S$ such that $s \subseteq r$. Define in V[r]

$$H = \bigcap_{n \in \omega} \bigcup_{m \geq n} F(L_\delta(m))$$

where $L_\delta(m)$ is the m-th element of L_δ. Note that if H is infinite, then it can not exist $g(\delta) \in [\omega]^{<\omega}$ such that $F(\beta) \subseteq g(\delta)$ for all but finitely many $\beta \in L_\delta$. So, we can assume that H is finite. Take $m \in \omega \setminus H$, thus $m \notin F(\eta)$ for infinitely many $\eta \in L_\delta$. Then, $s^\frown m \geq s \geq t$ and $s^\frown m \Vdash$ "$f(\delta) = b(\delta^+) = m$". Therefore, for the function f defined above, we have that $f(\delta) \notin F(\beta)$ for infinitely many $\beta \in L_\delta$. By density and as $\dot{F}$ was chosen arbitrarily, we have that X_L is never countably paracompact.

(2) Proceeding as in (1), for every $t \in S$ and every $\dot{F}$ an S-name for a function from ω_1 to $[\omega]^{n+1}$, we can find a $\delta \in \omega_1$, and an $s \in S$ such that $l(s) = \delta^+$, $s \geq t$, $s \Vdash$ "$\delta \in \dot{E}$" and s decides $\dot{F} \restriction L_\delta$. Working in $V[r]$ where r is a generic branch which extends s, define $B \in [\omega]^\omega$ and $A \subseteq \omega$ such that

$$A = \bigcap_{m \in B} F(L_\delta(m))$$

and for all $k \in \omega \setminus A$, we have that $k \in L_\delta(m)$ only for finitely many $m \in B$. Note that this can be carried out recursively in n steps.

Since F has codomain $[\omega]^{n+1}$, it follows that $|A| \leq n+1$. Take $m \in \omega \setminus A$. Then $s^\frown m \geq t$, $s \Vdash$ "$f(\delta) = m$" and B witnesses that $s \Vdash$ "$f(\delta) \notin F(\beta)$" for infinitely many $\beta \in L_\delta$. Again by density and since $\dot{F}$ was chosen arbitrarily, we are done. □

4.3 Forcing over models of $MA_{\omega_1}(\mathcal{K})$

In this section we will consider total ladder systems. A total ladder system is a ladder system $L = \{L_\alpha : \alpha \in E\}$ where $E = \lim(\omega_1)$. Note that the property $\mathcal{M}_{<\omega}$ is hereditary with respect to the stationary sets and for this reason we only need to prove theorem 4.3.3 for total ladder systems. First, remember the next theorem:

Theorem 4.3.1 (Dushnik-Miller [23]). *If κ is a regular cardinal such that $\kappa \geq \omega$, then*

$$\kappa \to (\kappa, \omega + 1)^2$$

□

The previous theorem is interpreted as follows: Given a colouring of the pairs of κ into two colors, either, there is a 0-homogeneous set of size κ or there is a 1-homogeneous set of order type $\omega + 1$.

Lemma 4.3.2. *Let $\mathbb{P}$ be a partial order such that for every uncountable $X \subseteq \mathbb{P}$ there exists a subset $Y = \{p_\alpha : \alpha \in \omega_1\} \subseteq X$ such that for every $A \subseteq \omega_1$ of order type $\omega + 1$, $Y \restriction A = \{p_\alpha : \alpha \in A\}$ is not an antichain. Then $\mathbb{P}$ is Knaster.*

Proof. Let $X \subseteq \mathbb{P}$ be uncountable and $Y = \{p_\alpha : \alpha \in \omega_1\} \subseteq X$. Define $c : [\omega_1]^2 \to 2$, such that $c(\alpha, \beta) = 0$ if and only if p_α and p_β are compatible. By the Erdös-Dushnik-Miller theorem, either:

1. There is $Z \in [\omega_1]^{\omega_1}$ such that $[Z]^2 \subseteq c^{-1}(\{0\})$ or else
2. There is $Z \subseteq \omega_1$ of order type $\omega + 1$ such that $[Z]^2 \subseteq c^{-1}(\{1\})$.

Since by assumption the second possibility is impossible, we get an uncountable set of compatible conditions. □

Theorem 4.3.3. ($MA_{\omega_1}(\mathcal{K})$). *The Souslin tree S forces that all (total) ladder system satisfy $\mathcal{M}_{<\omega}$.*

Proof. Le V be a model for $MA_{\omega_1}(\mathcal{K})$. Given $\dot{L} = \langle \dot{L}_\alpha : \alpha \in \lim(\omega_1)\rangle$ an S-name for a total ladder system and $\dot{f}$ an S-name for a function from $\lim(\omega_1)$ to ω, we can find a club $C \subseteq \omega_1$ such that for every $\alpha \in \lim(\omega_1)$

and every node $s \in S$ such that $l_S(s) = \alpha^+$ (where α^+ is the least element greater than α living in C), s decides $\dot{f}(\alpha)$ and $\dot{L}_\alpha$. For $\alpha \in \omega_1$ define $S_\alpha = \{s \in S : l_S(s) = \alpha\}$. Define the tree $T = \bigcup_{\alpha \in C} S_\alpha$ with the inherited order. Recall that for any I, J, $Fn(I, J)$ is the set of all finite partial functions from I to J. We will define the forcing $\mathbb{P} = \mathbb{P}(\dot{f}, \dot{L})$ as follows:

$$\mathbb{P} = \{(p, F) : p \in Fn(T, [\omega]^{<\omega}) \wedge F \in [\lim(\omega_1)]^{<\omega}\}$$

and $(p, F) \leq (q, G)$ iff $p \supseteq q$, $F \supseteq G$ and $\forall \alpha \in G\ \forall s \in dom(p) \setminus dom(q)\ \forall t \in A(p)$

$$\Big[\Big((s \subseteq t) \wedge (l_T(t) \geq \alpha) \wedge (t \Vdash \text{“}l_T(s) \in \dot{L}_\alpha \wedge \dot{f}(\alpha) = n\text{”})\Big) \implies (n \in p(s))\Big]$$

where $A(p)$ is the set of maximal elements of the domain of p and $t \Vdash \varphi$ is with t considered as an element of S.

Note that a generic filter G over $\mathbb{P}$ gives us a total function

$$h_G = \bigcup \{p : \exists F((p, F) \in G)\} : T \to [\omega]^{<\omega}$$

such that for every generic branch $b \subseteq S$ the function $H = H(b) : \omega_1 \to [\omega]^{<\omega}$ defined by $H(\alpha) = h_G(b \restriction (\alpha^+ \cap C))$ is a function which uniformizes f in the sense of $\mathcal{M}_{<\omega}$ (remember that h_G is defined only in nodes $s \in S$ such that $l_S(s) \in C$). To see this, note that for every $s \in T$, the set $D_s = \{(p, F) : s \in dom(p)\}$ is a dense subset of $\mathbb{P}$ because if $(p, F) \in \mathbb{P}$ is such that $s \notin dom(p)$, we can define $a_s = \{n_t^\alpha : t \in A(p) \wedge \alpha \in F\}$ where n_t^α is defined as follows:

- $n_t^\alpha = n$ if: $t \Vdash$ “$l_T(s) \in \dot{L}_\alpha \wedge \dot{f}(\alpha) = n$”, $(s \subseteq t)$ and $(l_T(t) \geq \alpha)$
- $n_t^\alpha = 0$ otherwise.

In this way $(p \cup (s, a_s), F) \leq (p, F)$, and h_G is actually a total function.

Also, for every $\alpha \in \lim(\omega_1)$, the set $D_\alpha = \{(p, F) : \alpha \in F\}$ is dense in $\mathbb{P}$ because $(p, F \cup \{\alpha\}) \leq (p, F)$ always holds.

Then let $(p_0, F_0) \in G$ such that $\alpha \in F_0$ and take some $\beta \in \omega_1$ such that

$$V[b] \models \beta \in L_\alpha \wedge b \restriction (\beta^+ \cap C) \notin dom(p_0).$$

We can find two conditions $(p_1, F_1) \in G$ and $(p_2, F_2) \in G$ such that $b \upharpoonright (\beta^+ \cap C) \in dom(p_1)$ and $b \upharpoonright (\alpha^+ \cap C) \in dom(p_2)$. Take a common extension $(p, F) \in G$ of (p_i, F_i) $(i \in 3)$. Without loss of generality, we can assume that $b \upharpoonright (\alpha^+ \cap C) \in A(p)$. Hence the following properties hold:

- $\alpha \in F_0$
- $b \upharpoonright (\beta^+ \cap C) \in dom(p) \setminus dom(p_0)$
- $b \upharpoonright (\alpha^+ \cap C) \in A(p)$
- $b \upharpoonright (\beta^+ \cap C) \subseteq b \upharpoonright (\alpha^+ \cap C)$
- $l_T(b \upharpoonright (\alpha^+ \cap C)) \geq \alpha$
- $b \upharpoonright \alpha^+ \Vdash$ "$l_T(b \upharpoonright (\beta^+ \cap C)) \in \dot{L}_\alpha \wedge \dot{f}(\alpha) = n$" for some $n \in \omega$.

By the definition of the forcing, this implies that $f(\alpha) \in H(\beta)$, for all $\alpha \in \lim(\omega_1)$ and all but finitely many $\beta \in L_\alpha$ (in $V[b]$).

It remains to prove that $\mathbb{P}$ is Knaster in order to get the such generic filter G. Let $\langle (p_\alpha, F_\alpha) : \alpha \in \omega_1 \rangle \subseteq \mathbb{P}$. We can assume that $\{dom(p_\alpha) : \alpha \in \omega_1\}$ and $\{F_\alpha : \alpha \in \omega_1\}$ form Δ-systems with roots r and R respectively. Also we can assume that $(p_\alpha \upharpoonright r = p_\beta \upharpoonright r)$ for all $\alpha, \beta \in \omega_1$, and that there exists an increasing function $h : \omega_1 \to \omega_1$ such that $\{l_T(s) : s \in r\} \cup R \subseteq h(0)$ and $\{l_T(s) : s \in dom(p_\alpha) \setminus r\} \cup (F_\alpha \setminus R) \subseteq (h(\alpha), h(\alpha + 1))$ for all $\alpha \in \omega_1$.

It is now sufficed by lemma 4.3.2 to prove that for every $X \subseteq \omega_1$ of order type $\omega + 1$, there are $\alpha, \beta \in X$ such that (p_α, F_α) and (p_β, F_β) are compatible. Let $\{x_\alpha : \alpha \in \omega + 1\}$ be the increasing enumeration of X. For every $\alpha \in \omega + 1$ define $q_\alpha = p_{x_\alpha}$ and $G_\alpha = F_{x_\alpha}$. Pick $s \in A(q_\omega)$ and $\eta \in G_\omega \setminus R$ and define

$$B_s^\eta = \{\gamma < h(x_\omega) : s \Vdash \text{“}\gamma \in \dot{L}_\eta\text{”}\}.$$

Note that the set B_s^η is finite for every $s \in A(q_\omega)$ and every $\eta \in G_\omega$. In consequence $B = \bigcup\{B_s^\eta : s \in A(q_\omega) \wedge \eta \in G_\omega\}$ is finite as well. Hence, there exists $n \in \omega$ such that $B \cap (h(x_n), h(x_n + 1)) = \emptyset$. We shall prove that (q_ω, G_ω) and (q_n, G_n) are compatible. Define $(q', G') = (q_\omega \cup q_n, G_\omega \cup G_n)$.

Pick $\alpha \in G_n$ and $s \in dom(q') \setminus dom(q_n) = dom(q_\omega) \setminus r \subseteq (h(x_\omega), h(x_\omega + 1))$. Since $\alpha \in G_n \subseteq h(0) \cup (h(x_n), h(x_n + 1))$, we have that $\alpha < l_T(s)$ and then no $t \in A(q')$ can force that $l_T(s) \in \dot{L}_\alpha$. So $(q', G') \leq (p_n, G_n)$ trivially.

In order to see that $(q', G') \leq (q_\omega, G_\omega)$, pick $\eta \in G_\omega$ and $s \in dom(q') \setminus dom(q_\omega) = dom(q_n) \setminus r \subseteq (h(x_n), h(x_n + 1))$. If it is the case that $\eta \in R$, again we have that $l_T(s) > \eta$ and no $t \in A(q')$ can force that $l_T(s) \in \dot{L}_\eta$. On the other hand, if $\eta \in G_\omega \setminus R$ and we take $t \in A(q')$ such that $s \subseteq t$ and $l_T(t) \geq \eta > h(x_\omega)$, we have that $t \in A(q_\omega)$. By the choice of $n \in \omega$ satisfying $B \cap (h(x_n), h(x_n + 1)) = \emptyset$, it follows that $t \nVdash$ "$l_T(s) \in \dot{L}_\eta$" and we are done. □

4.4 Anti-uniformization properties

A ladder system L on a stationary $E \subseteq \omega_1$ is said to be *thin* if for each $f : \omega_1 \to \omega$, the set $\{\alpha \in E : |f(L_\alpha)| = \aleph_0\}$ is non-stationary. This is the strongest of the anti-uniformization properties introduced in [7] and indeed if a ladder system is uniformizable then it is not thin, because any function which uniformizes a sequence of one to one functions $s_\alpha : L_\alpha \to \omega$ in the strongest sense witnesses the failure of thinness.

Definition 4.4.1. A ladder system $L = \langle L_\alpha : \alpha \in S \rangle$ satisfies the property:

- (G_1) If for every function $f : \omega_1 \to \omega$, the set
$$\{\alpha \in S : |f[L_\alpha]| = \aleph_0\}$$
is not stationary.
- (G_2) If for every function $f : \omega_1 \to \omega$, the set
$$\{\alpha \in S : f \upharpoonright L_\alpha \text{ is finite to one}\}$$
is not stationary.
- (G_3) If for every function $f : \omega_1 \to \omega$, the set
$$\{\alpha \in S : f \upharpoonright L_\alpha \text{ is eventually one to one}\}$$
is not stationary.

- (H_1) If for each function $f : \omega_1 \to \omega$, the set

$$\{\alpha \in S : |f[L_\alpha]| < \aleph_0\}$$

is stationary.

- (H_2) If for each function $f : \omega_1 \to \omega$, the set

$$\{\alpha \in S : f \restriction L_\alpha \text{ is not finite to one}\}$$

is stationary.

- (H_3) If for each function $f : \omega_1 \to \omega$, the set

$$\{\alpha \in S : f \restriction L_\alpha \text{ is not eventually one to one}\}$$

is stationary.

It is easy to see that $G_i \Rightarrow H_i$, $G_i \Rightarrow G_{i+1}$ and $H_i \Rightarrow H_{i+1}$ for those G_i and H_i that are well defined. The property G_i is just the previously defined concept of being thin. A consistent example of a ladder system constructed by Shelah [51] has the property that L is H_2 and X_L is normal, so some anti-uniformization properties are consistent with relatively strong uniformization properties. As a consequence of Shelah's construction, one gets a normal space X_L which is not collectionwise Hausdorff, and the closed discrete subspace witnessing this property, is not G_δ, answering a question of P. Nyikos. In general, the subspace $E \times \{1\}$ of X_L is a G_δ subset iff L satisfies H_2.

However, it is an open question whether consistently there may be a thin ladder system that is also $\mathcal{M}_{<\omega}$. This was the main question that arose from the paper [7] where it was shown that the existence of a thin and $\mathcal{M}_{<\omega}$ ladder system would give consistent counter-examples to two notable open problems concerning separation properties of countably paracompact spaces. Namely the problem of whether every countably paracompact subspace of ω_1^2 is normal [36] and the problem of whether every countably paracompact, locally compact, screenable space is paracompact [6].

We will summarize in diagram 4.4 some important implications in the diagram 4.4 where arrows from $\diamondsuit^\sharp$ (see [16] for a definition) and $\clubsuit$ mean

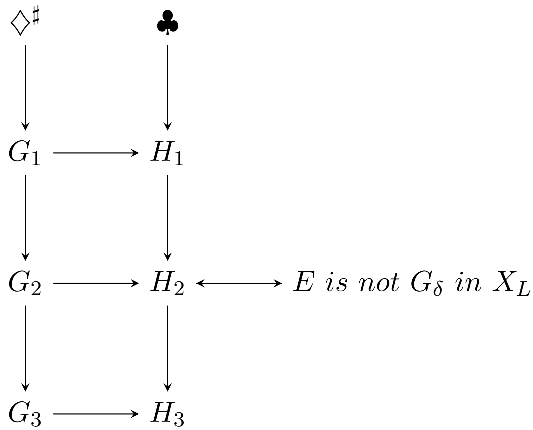

Figure 4.2: Anti uniformization properties

that we can define a ladder system satisfying G_1 and H_1 respectively from these principles.

We now show that in any model obtained by forcing with the Souslin tree over a model of $MA_{\omega_1}(\mathcal{K})$, no ladder system is even H_3.

In order to prove this, note that we only have to prove that for every total ladder system $L = \langle L_\alpha : \alpha \in \lim(\omega_1)\rangle$ there exists a function $f : \omega_1 \to \omega$ such that $f \restriction L_\alpha$ is eventually one-to-one for every $\alpha \in \lim(\omega_1)$. Also, since S does not add reals and is ccc, if $L = \langle L_\alpha : \alpha \in \lim(\omega_1)\rangle$ is a total ladder system in the extension, there exists a set $L' = \{L^n_\alpha : \alpha \in \lim(\omega_1) \wedge n \in \omega\}$ in the ground model such that $L_\alpha \in \{L^n_\alpha : n \in \omega\}$ for each α and in consequence it is suffices to prove that the following holds:

Theorem 4.4.2 ($MA_{\omega_1}(\mathcal{K})$)**.** *For every family $L = \{L^n_\alpha : \alpha \in \lim(\omega_1) \wedge n \in \omega\}$ (where each L^n_α is a ω-sequence cofinal in α) there exists a function $f : \omega_1 \to \omega$ such that $f \restriction L^n_\alpha$ is eventually one-to-one for every $\alpha \in \lim(\omega_1)$ and every $n \in \omega$.*

Proof. The proof is verbatim the same as the proof of theorem 4.3.3 but with a different poset. Let

$$\mathbb{P} = \mathbb{P}(L) = \{(p, F) : p \in Fn(\omega_1, \omega) \wedge F \in [\omega_1 \times \omega]^{<\omega}\}$$

and let $(p,F) \le (q,G)$ iff $p \supseteq q, F \supseteq G$ and $(p \setminus q) \restriction L^n_\alpha$ is one-to-one for every $(\alpha,n) \in G$. Clearly a generic filter gives us the function which we are looking for, using the ω_1-many dense sets $D_\eta = \{(p,F) : \alpha \in dom(p)\}$ and $C^n_\alpha = \{(p,F) : (\alpha,n) \in F\}$ and we can get this generic filter because $\mathbb{P}$ is Knaster. To see this let $\{(p_\alpha, F_\alpha) : \alpha \in \omega_1\} \subseteq \mathbb{P}$ and assume without loss of generality that $\{dom(p_\alpha) : \alpha \in \omega_1\}$ and $\{\pi_1[F_\alpha] : \alpha \in \omega_1\}$ both form delta systems with roots r and R respectively, where $\pi_1 : \omega_1 \times \omega \to \omega_1$ is the projection to the first coordinate. Then, we can also assume that there is a function $g : \omega_1 \to \omega_1$ such that $r \cup \pi_1[R] \subseteq g(0)$ and

$$(p_\alpha \setminus r) \cup \pi_1[F_\alpha \setminus R] \subseteq (g(\alpha), g(\alpha+1))$$

for every $\alpha \in \omega_1$.

Let $X \subseteq \omega_1$ of order type $\omega+1$ and let $\{x_\alpha : \alpha \in \omega+1\}$ be the increasing enumeration of X. Then, define $(q_\alpha, G_\alpha) = (p_{x_\alpha}, F_{x_\alpha})$ for every $\alpha \in \omega+1$.

For every $(\alpha,n) \in G_\omega \setminus R$, let $B^n_\alpha = \{\eta \in \omega_1 : \eta \in L^n_\alpha \wedge \eta < g(x_\omega)\}$, and note that $|B^n_\alpha| < \omega$ because $\alpha > g(x_\omega)$. Then define

$$B = \bigcup \{B^n_\alpha : (\alpha,n) \in G_\omega \setminus R\}.$$

Since B is finite, there exists $N \in \omega$ such that $B \cap (g(x_N), g(x_N+1)) = \emptyset$. We will see that (q_ω, G_ω) and (q_N, G_N) are compatible. Let

$$(q,G) = (q_\omega \cup q_N, G_\omega \cup G_N).$$

$(q,G) \le (q_N, G_N)$: Note that $dom(q) \setminus dom(q_N) = dom(q_\omega) \setminus r \subseteq (g(x_\omega), \omega_1)$ and $\pi_1[G_N] \subseteq g(N+1) < g(\omega)$ and in consequence $(q \setminus q_N) \restriction L^n_\alpha = \emptyset$ for every $(\alpha,n) \in G_N$.
$(q,G) \le (q_\omega, G_\omega)$: Pick $(\alpha,n) \in G_\omega$. If $(\alpha,n) \in R$, we can do the same as above. On the other hand, if $(\alpha,n) \in G_\omega \setminus R$ then $B^n_\alpha \subseteq B$ and by the choice of N again $(q \setminus q_\omega) \restriction L^n_\alpha = \emptyset$.
By lemma 4.3.2, $\mathbb{P}$ is Knaster and this finishes the proof. □

Corollary 4.4.3. *$MA(S)[S]$ implies that every ladder system is countably metacompact but not countably paracompact and fails to satisfy H_3 and $\mathcal{M}_n$ for every $n \in \omega$.* □

4.5 Walks on ladder systems

We recall some definitions from Chapter 2. Remind that a point x is an α_1*-point* if whenever we have countable many sequences S_n converging to x, there is a single sequence $S \to x$ such that $|S_n \setminus S| < \omega$ for all $n \in \omega$. A point $x \in X$ is a *Fréchet* point if whenever $x \in \overline{A}$, there is a sequence $\{x_n : n \in \omega\} \subseteq A$ converging to x. A space X is α_1 (Fréchet) if every point $x \in X$ is an α_1-point (a Fréchet point).

Definition 4.5.1. [3] A space X is *absolutely Fréchet* if every point $x \in X$ is a Fréchet point in βX (equivalently in some compactification).

Let $\mathcal{A} \subseteq \mathcal{P}(X)$ and $x \in X$. We will say that $x \in \overline{\mathcal{A}}$ if $x \in \overline{A}$ for every $A \in \mathcal{A}$. Also, for a filter base $\{G_n : n \in \omega\} \subseteq \mathcal{P}(X)$, we will say that $G_n \to x$ if for every open neighborhood U of x there is $n \in \omega$ such that $G_n \subseteq U$.

Definition 4.5.2. A space X is *bisequential* if for every filter $\mathcal{F}$ such that $x \in \overline{\mathcal{F}}$, there is a family $\{G_n : n \in \omega\}$ such that $\mathcal{F} \cup \{G_n : n \in \omega\}$ generates a filter and $G_n \to x$.

We stated the following questions in the final section of Chapter 2 and solved them using AD families (one of them only in the presence of CH):

- Is there a absolutely Fréchet space which is not bisequential?
- Is there an α_1-Fréchet space which is not bisequential?

Here we will consistently construct a space of size $\aleph_1$ that answers both questions at the same time. For our purpose, we will take the next variation of the definition of a ladder system. A *ladder system* on ω_1 is defined to be a sequence $\langle L_\alpha : \alpha \in \omega_1 \rangle$ such that:

- If $\alpha = \beta + 1$ then $L_\alpha = \{\beta\}$ and
- If α is a limit ordinal then L_α is an increasing and unbounded subset of α of order type ω.

We will use the theory of walks on ordinals developed by Todorčević (see [55]). We can walk from an ordinal α to a smaller ordinal β in ω_1 using a ladder system in the following way: Define $\alpha_0 = \alpha$ and recursively define $\alpha_{i+1} = \min(L_{\alpha_i} \setminus \beta)$ and stopping when we reach $\beta = \alpha_n$. It is well defined since $\{\alpha_i : i \leq n\}$ is a decreasing sequence of ordinals. Let $\rho_2(\beta, \alpha) = n$ denote the (uniquely determined) length of the walk from α to β. Some properties of the ρ_2 function are the following:

Fact 4.5.3. [55] The ρ_2 function satisfies the next two properties:

(∗) (Coherence) For $\alpha < \beta < \omega_1$,

$$\sup_{\xi<\alpha} |\rho_2(\xi, \alpha) - \rho_2(\xi, \beta)| < \omega$$

(∗∗) (Unboundedness) For every uncountable family $\mathcal{A} \subseteq [\omega_1]^{<\omega}$ of pairwise disjoint finite subsets of ω_1 and for every $n \in \omega$, there exists $\mathcal{B} \in [\mathcal{A}]^{\omega_1}$ such that $\rho_2(\alpha, \beta) > n$ for every $\alpha \in a$, $\beta \in b$ and $a \neq b$ in $\mathcal{B}$.

An easy consequence of (∗∗) is the following:

(∗ ∗ ∗) For every pair $A, B \in [\omega_1]^{\omega_1}$ and every $n \in \omega$ there are $\alpha \in A$ and $\beta \in B$ such that

$$\rho_2(\alpha, \beta) > n.$$

We define a topology on $\omega_1 + 1$ such that the points of ω_1 are isolated and a basic neighborhood of the point ω_1 is of the form

$$\{\omega_1\} \cup \bigcup_{\alpha \in \lim(\omega_1)} \{\xi < \alpha : \rho_2(\xi, \alpha) > n_\alpha\},$$

where $n_\alpha < \omega$.

Lemma 4.5.4. *The local base at the point ω_1 is generated by sets of the form*

$$U(\alpha, n) = [\alpha, \omega_1] \cup \{\xi < \alpha : \rho_2(\xi, \alpha) > n\}.$$

Proof. Let $V(\{n_\alpha : \alpha \in \omega_1\})$ be a basic neighborhood of ω_1. We will first prove that V contains a tail of the form $[\alpha, \omega_1]$. Assume it is not the case and let C be an uncountable set disjoint from V. Let $n \in \omega$ and $C' \in [C]^{\omega_1}$ such that $n_\alpha = n$ for every $\alpha \in C'$. Using $(**)$ of fact 4.5.3, there are $\alpha < \beta \in C'$ such that $\rho_2(\alpha, \beta) > n$ but then $\alpha \in V$, which is a contradiction.

Let $\alpha \in \omega_1$ such that $[\alpha, \omega_1] \subseteq V$. Fix $n = n_\alpha$. Thus $U(\alpha, n) \subseteq V$. It remains to prove that $U(\alpha, n)$ is open. In order to prove this, we have to find for every $\beta \in \omega_1$ an $n_\beta \in \omega$ such that $\{\xi < \beta : \rho_2(\xi, \beta) > n_\beta\} \subseteq U(\alpha, n)$. Using $(*)$, for every $\beta \in \omega_1$ we can find $N_\beta \in \omega$ such that

$$N_\beta = \sup_{\xi < \min(\{\alpha, \beta\})} \{|\rho_2(\xi, \alpha) - \rho_2(\xi, \beta)|\}.$$

Then let $n_\beta = N_\beta + n$. It follows that if $\rho_2(\xi, \beta) > n_\beta$ then $\rho_2(\xi, \alpha) > n$ and we are done. $\square$

In [12] it is proved that the analogous space for κ using a $\square(\kappa)$ sequence instead of a ladder system, is α_1 and absolutely Fréchet. Actually, this space is FU_{fin} (see [29]) for every κ [11]. Since a ladder system witnesses $\square(\omega_1)$, this space is α_1 and absolutely Fréchet for every ladder system. It remains to prove that there is a ladder system such that it is not bisequential. For this notice that $\omega_1 \in \overline{\text{Club}(\omega_1)}$ where Club is the club filter on ω_1. Then, if $\{S_n : n \in \omega\}$ is a decreasing sequence, $\{S_n : n \in \omega\} \cup Club$ generates a filter iff S_n is stationary for every $n \in \omega$.

Theorem 4.5.5. *Let $\mathbb{P}$ be the forcing for adding a ladder system generically with countable approximations. Then*

$$V^{\mathbb{P}} \models \exists X \text{ absolutely Fréchet, } \alpha_1 \text{ and non-bisequential.}$$

Proof. A sequence of stationary sets $\{S_n : n \in \omega\}$ does not converge to ω_1 iff there is an open neighborhood of ω_1 such that none of the S_n is contained in it iff there exists a closed set C not containing ω_1 such that $C \cap S_n \neq \emptyset$ for every $n \in \omega$. Hence, we will prove that if $\{S_n : n \in \omega\}$ is a sequence of stationary sets, there is a closed set $C = C(\alpha, m) = \{\beta < \alpha : \rho_2(\beta, \alpha) \leq m\}$ such that $C \cap S_n \neq \emptyset$ for every $n \in \omega$.

Let $\mathbb{P}$ be the forcing for adding a ladder system with countable conditions (i.e., $p \in \mathbb{P}$ iff $p = \langle L_\alpha : \alpha \in \lim(\eta) \rangle$ is a family of ladders for some

$\eta \in \omega_1$ and ordered by inclusion). Notice that we only have to take care of limit ordinals when defining the ladders. Let G be a $\mathbb{P}$-generic filter over V and $\{\dot{S}_n : n \in \omega\}$ a sequence of $\mathbb{P}$-names for stationary sets in $V[G]$. Take M a countable elementary submodel of $H(\theta)$ for θ large enough such that $\mathbb{P}, p, \{\dot{S}_n : n \in \omega\} \in M$. For $q \in \mathbb{P}$, we will say that $l(q) = \alpha$ if $q = \langle L_\eta : \eta \in \lim(\alpha)\rangle$. Let $\delta = M \cap \omega_1$ and $\alpha = l(p) \in M$. Define recursively $\{q_\eta : \eta \in \delta\}$ as follows:

- $q_0 = p$,
- $q_\eta = \bigcup_{\beta<\eta} q_\beta$ if η is a limit ordinal and
- $q_{\eta+1} \leq q_\eta$ is such that $q_{\eta+1}$ decides $\dot{S}_n \cap l(q_\eta)$ for every $n \in \omega$.

The last point can be done since the forcing is σ-closed and there are only countable many formulas of the form "$\alpha \in \dot{S}_n$" to decide. In V define $q = \bigcup_{\eta<\delta} q_\eta = \langle L_\alpha : \alpha \in \lim(\delta)\rangle \in \mathbb{P}$. Notice that q is a generic condition and $q \Vdash$"$\forall n \in \omega$ ($\dot{S}_n$ is unbounded in δ)". Then we can define a ladder $L_\delta = \{\delta_n : n \in \omega\} \subseteq \delta$ such that $q \Vdash$"$\delta_n \in \dot{S}_n$" for every $n \in \omega$. Define $q' = \langle L_\alpha : \alpha \in \lim(\delta + 1)\rangle \in \mathbb{P}$. Then $q' \Vdash$"$\forall n \in \omega$ ($\dot{L} \cap \dot{S}_n) \neq \emptyset$" where $\dot{L}$ is a name for the generic ladder and hence in $V[G]$ the closed set $C(\delta, 1)$ has nonempty intersection with each S_n. Thus the space is not bisequential, but it is α_1 and absolutely Frechét due to the results commented before the theorem. $\square$

Chapter 5

Questions

We list here the questions raised throughout this work. For definitions, context and motivation, see the corresponding chapter.

- (Question 1 on page 28): Is there a weakly tight AD family in ZFC?
- (Question 2 on page 28): Is there an α_3-FU AD family in ZFC which is not bisequential?
- (Question 3 on page 36):(CH) Is there a Luzin and/or MAD family which is almost-normal?
- (Question 4 on page 38): Is it consistent with MA that there are almost-normal MAD families?
- (Question 5 on page 42): Does there exist (in ZFC) an almost-normal AD family which is not normal? (an almost-normal AD family of size $\mathfrak{c}$?)
- (Question 6 on page 43): Are almost-normal AD families potentially normal?
- (Question 7 on page 43): Is it consistent that quasi-normal (partly-normal, mildly-normal) AD families are potentially normal?
- (Question 8 on page 47): Is it consistent that strongly $\aleph_0$-separated (or strongly $(\aleph_0, < \mathfrak{c})$-separated) AD families are potentially normal? Is it consistent with MA?

- (Question 9 on page 56): If X_L is countably paracompact, does L satisfies $\mathcal{P}_{<\omega}$?

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