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Linear stability of linear series and stability of syzygy bundles on smooth curves with special and general moduli

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"La belleza de las matemáticas solo se revela a los seguidores más pacientes"
-Maryam Mirzakhani

"Nuestra libertad dentro de las matemáticas radica en las preguntas que hacemos, y en cómo las abordamos, pero no en las respuestas que nos deparan"
-Steven Strogatz

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Contents

Resumen

En la presente tesis aplicamos métodos modernos de teoría de estabilidad y teoría de Brill-Noether para curvas a un problema de estabilidad de haces. Más precisamente, el problema en cuestión relaciona la estabilidad de series lineales y la estabilidad del haz de sizigias asociado a dichas series, mientras que los métodos utilizados incluyen una extensión de la teoría de Brill-Noether para curvas k-gonales, el estudio de las condiciones de estabilidad en superficies $K3$, la estabilidad de la restricción de haces de una superficie a una curva dentro de la misma.

En el Capítulo 1 introducimos aspectos básicos de la teoría de Brill-Noether, introducimos el problema principal, un par de conjeturas propuestas por Ernesto Mistretta y Lidia Stoppino en [25] que relacionan la estabilidad de series lineales, en el caso completo e incompleto y su relación con la estabilidad del haz de sizigias asociado a estas series lineales, así como las respuestas positivas o negativas que se han presentado al momento.

En los siguientes 2 capítulos estudiamos la solución a una de las conjeturas en el caso de una curva k-gonal general para gonalidad genérica e intermedias. En el Capítulo 2 proponemos una solución a la conjetura para curvas generales siguiendo las ideas propuestas por Castorena y Torres-Lopez en [11], estudiaremos una aplicación multiplicación de secciones globales obtenidos mediante manipulación del diagrama de Butler. En el Capítulo 3 presentamos una introducción a la teoría de Brill-Noether para curvas k-gonales, lo cual nos permite estudiar la dimensión de las variedades de Brill-Noether asociada a este tipo de curvas vía la no-negatividad de una modificación al número de Brill-Noether conocido en la teoría clásica. Presentaremos así condiciones para las cuales las pruebas del Capítulo 2 para curvas genéricas se pueden extender a curvas generales en los estratos k-gonales utilizando esta teoría.

En el Capítulo 4 estudiamos las condiciones de estabilidad de haces vectoriales sobre superficies $K3$. Nos enfocamos también en la estabilidad de algunos haces de sizigias muy específicos, llamados los haces de Lazersfeld-Mukai, así como la estabilidad de su restricción a curvas que viven en dichas superficies. Concluimos con algunas condiciones bajo las cuales la estabilidad de la restricción de estos haces vectoriales , aunado a la estabilidad de la serie lineal, implica la estabilidad de los haces de sizigias sobre curvas, estudiados a lo largo de esta tésis.

Palabras Clave: Estabilidad de haces, Condiciones de estabilidad, Teoría de Brill-Noether, Haces de sizigias, Moduli de curvas.

Abstract

In this thesis, we apply modern methods from stability theory and Brill-Noether theory for curves to a problem concerning the stability of bundles. More precisely, we relates the stability of linear series to the stability of the syzygy bundle associated with these linear series. The methods employed include an extension of Brill-Noether theory for k-gonal curves, the study of stability conditions on $K3$ surfaces, and the stability of restrictions of bundles from a surface to a curve within that same surface.

In Chapter 1, we introduce the fundamental aspects of Brill-Noether theory and present the main problem, along with a couple of conjectures proposed by Ernesto Mistretta and Lidia Stoppino in [25]. These conjectures relate the stability of linear series, in both complete and incomplete cases, to the stability of the syzygy bundle associated with these linear series, as well as the positive or negative responses that have been presented thus far.

In the following two chapters, we study the solution to one of the conjectures in the case of a general k-gonal curve for generic and intermediate gonality. In Chapter 2, we propose a solution to the conjecture for general curves, following the ideas of Castorena and Torres-Lopez in [11]. We will examine a multiplication application of global sections obtained through manipulation of the Butler's diagram. In Chapter 3, we present an introduction to Brill-Noether theory for k-gonal curves, which allows us to study the dimension of Brill-Noether varieties associated with these types of curves via the non-negativity of a modification to the classical Brill-Noether number. We will present conditions under which the proofs from Chapter 2 for generic curves can be extended to general curves within k-gonal strata using this theory.

In Chapter 4, we examine stability conditions for vector bundles on $K3$ surfaces. We also focus on the stability of specific syzygy bundles known as Lazarsfeld-Mukai bundles, as well as their restrictions to smooth curves residing on these surfaces. We conclude with certain conditions under which the stability of these bundles restrictions, combined with the linear stability of the linear series, implies the stability of syzygy bundles on curves studied throughout this thesis.

Keywords: Stability of vector bundles, Stability conditions, Brill-Noether theory, Syzygy bundle, Moduli of curves.

Introduction

In this thesis we provide new families of curves where the Mistretta-Stoppino conjecture holds shedding light over deep connections between linear stability of linear series and slope stability of associated syzygy bundles. Such connections are important in Brill-Noether theory and the minimal resolution conjecture.

Let C be a smooth projective curve over the complex numbers and L a globally generated line bundle of degree d on C. Consider a subspace $V \subseteq H^0(L)$ of dimension $r+1$ that generates L. The pair (L, V) is called a generated linear series of type $(d, r+1)$. Associated to this linear series is the syzygy bundle $M_{V,L}$, defined as the kernel of the evaluation map $V \otimes \mathcal{O}_C \xrightarrow{ev} L$. That is, we have the short exact sequence

$$0 \to M_{V,L} \to V \otimes \mathcal{O}_C \xrightarrow{ev} L \to 0.$$

The bundle $M_{V,L}$ is also known as the syzygy bundle, kernel bundle or dual span bundle. When $V = H^0(L)$, we denote the bundle $M_{H^0(L),L}$ by M_L. Slope-stability of the syzygy bundle $M_{V,L}$ is closely related to the study of Brill-Noether varieties and the minimal resolution conjecture, as discussed in [17]. In [31] Stoppino generalizes the notion of linear stability introduced by Mumford in [26] for projective varieties to the setting of pairs (L, V) over a curve C. It is an interesting question when the linear semi-stability of the pair (L, V) implies the slope-semistability of the associated syzygy bundle $M_{V,L}$. This is the focus of two conjectures proposed by Mistretta and Stoppino in [25]:

Conjecture (Conjecture 1.2.1)**.** *Let C be a smooth curve of genus g and (L, V) be a generated linear series of type $(d, r+1)$ over C with $V \subsetneq H^0(L)$. If $d \leqslant kr$ where k denotes the gonality of C, then linear (semi)stability of (L, V) is equivalent to slope-(semi)stability of $M_{V,L}$.*

Conjecture (Conjecture 1.2.2)**.** *For any curve C, and any line bundle L on C, linear (semi)stability of $(L, H^0(L))$ is equivalent to slope-(semi)stability of M_L.*

Conjecture 1.2.2 has a positive answer for general and hyperelliptic curves due to [11]. In this thesis we present families of curves for which Conjecture 1.2.1 holds. Considering techniques from [11] and using that for general curves when $d \leqslant g + r$ the condition $d \leqslant kr$ is satisfied, we prove the following:

Theorem (Corollary 2.1.1)**.** *Let (L, V) be a generated linear series of type $(d, r+1)$ over a general curve C of genus $g \geqslant 2$ with $codim_{H^0(L)}(V) \leqslant h^1(L)$. Then linear (semi)stability of (L, V) is equivalent to the slope-(semi)stability of $M_{V,L}$.*

We also consider k-gonal curves, for which a modified Brill-Noether number $\overline{\rho}_k(g,r,d)$ was introduced in [28]. We give bounds on the number of global sections $r+1$ and degree d of (L,V) in terms of g and k to get an analogue of Theorem for general k-gonal curves:
In [28] it is introduced a modified Brill-Noether number for k-gonal curves $\overline{\rho}_k(g,r,d)$, see Definition 3.1.1. We construct bounds for the projective dimension of V, called r and the degree of L, called d for the pair (L,V) in terms of g and k to verify that we prove Mistretta-Stoppino's conjecture for k-gonal general curves.

Theorem (Proposition 3.2.3)**.** *Let C be a general k-gonal curve of genus $g>2$, with k non-generic, and let (L,V) be a generated linear series of type $(d,r+1)$ over C. Suppose that $d \leqslant g+r$ and $d \leqslant kr$. If (r,d) satisfies at least one of the conditions:*

1. *If $g+1-k+2(r-1) \leqslant \frac{r-1}{r}d$ and satisfies one of the following conditions:*
   (a) $k \leqslant 6$.
   (b) $k>6$ *and* $g \geqslant \frac{k^2}{4}$.
   (c) $k>6$ *and* $2(r-1) \geqslant \sqrt{k^2-4g}+k-2$.
2. (a) *If $\frac{r-1}{r}d \leqslant g-1$ and satisfies all the following conditions:*
      i. $r-1 \leqslant \frac{g-k+1}{k-1}$.
      ii. $\frac{r-1}{r}d \leqslant g+2(r-1)+1-k$.
      iii. $\frac{r-1}{r}d \geqslant g+(r-1)+1-k$.
   (b) *Or if $\frac{r-1}{r}d \geqslant g-1$ and satisfies all the following conditions:*
      i. $(r-1)+2 \leqslant k$.
      ii. $\frac{r-1}{r}d \leqslant g+2(r-1)+1-k$.
      iii. $(r-1) \leqslant \sqrt{4g+5k^2-4k}-2k$.

Then the linear (semi)stability of (L,V) is equivalent to the slope-(semi)stability of $M_{V,L}$.

Finally, over a principally polarized K3 surface (X,H) we consider curves C lying in $|H|$. By studying the stability conditions on X, we prove that the Lazarsfeld-Mukai bundle $F_{V,L}$ associated to (L,V) is H-Gieseker stable. Under a degree bound, we can conclude that the restriction of $F_{V,L}$ to C is slope-stable, leading to the following,

Theorem (Proposition 4.1.2)**.** *Let (X,H) be a polarized K3 surface with the property that H^2 divides $H.D$ for any curve classes D on X. Take $C \in |H|$ a curve of genus $g>2$, let (L,V) be a generated linear series of type $(d,r+1)$ over C with $1<r<d \leqslant \min\{g-1,kr\}$ where k is the gonality of C. If (L,V) is linearly stable then $M_{V,L}$ is slope-stable.*

Chapter 1

Preliminaries

In this chapter, we introduce classical results of Brill-Noether theory for line bundles over smooth curves in order to introduce the problem we aim to study in this thesis.
We work over the complex numbers $\mathbb{C}$ and all varieties and schemes will be assumed to be projective. Also, by *curve* we mean a one-dimensional closed irreducible smooth subscheme of $\mathbb{P}^n$ for some $n \geqslant 1$. For a curve C, we denote by K_C the canonical line bundle on C.
Let $\mathcal{M}_g$ be the coarse moduli space of smooth irreducible projective curves of genus g. By abuse of notation we write $C \in \mathcal{M}_g$ to denote the class $[C] \in \mathcal{M}_g$. We say that C is general when C is an element of an open dense subset of $\mathcal{M}_g$, in this thesis we mainly work with the notion of Petri general curve or k-gonal general curve.

1.1 Brill-Noether Theory and stability

Brill-Noether theory for line bundles on curves studies projective realizations by morphisms induced by line bundles. For a smooth curve C of genus g any (non-degenerated) morphism $C \to \mathbb{P}^r$ defines a degree d line bundle L on C together with a subspace $V \subseteq H^0(L)$ of dimension $r+1$. On the other side, a line bundle L' on C with a subspace $V' \subseteq H^0(C, L')$ of sections induces a map $C \to \mathbb{P}^{\dim(V'-1)}$. Such a pair (L', V') is called a *linear series* on C of type $(\text{degree}(L'), \dim(V'))$.
A pair (L, V) is a point in the determinantal variety

$$G^r_d(C) := \{(L, V) | L \in Pic^d(C), V \subseteq H^0(L), dimV = r+1\}.$$

When it is not necessary to specify the elements of the linear series, we denote by $\mathfrak{g}^r_d$ a point in the variety $G^r_d(C)$ and we denote by $|V|$ the linear system of (L, V). For $v = (L, V) \in G^r_d(C)$ on a curve C of genus g, we have the cup-product morphism

$$\mu_{0,V} \colon V \otimes H^0(K_C \otimes L^\vee) \to H^0(K_C).$$

Following [2, Proposition 4.1 of §4] the dimension

$$\dim T_v(G^r_d(C)) = \rho(g, r, d) + \dim(\ker \mu_{0,V}),$$

where $\rho(g, r, d) = g - (r+1)(g - d + r)$ is called the *Brill-Noether number* of the $\mathfrak{g}^r_d$. In [19], D. Gieseker showed that for a general curve of genus g, the cup-product morphism $\mu_{0,V}$ is injective. The following result summarizes the relation between the Brill-Noether number and the variety $G^r_d(C)$ over a general curve.

Theorem 1.1.1 ([2, Theorem V.1.5])**.** *Let C be a general curve of genus g. Let d and r be integers such that $d \geqslant 1$, $r \geqslant 0$. Then if $\rho(g,r,d) < 0$, the variety $G^r_d(C)$ is empty. If $\rho(g,r,d) \geqslant 0$, then $G^r_d(C)$ is reduced of pure dimension $\rho(g,r,d)$.*

We say that the linear series (L,V) is a *generated (globally generated or base point free)* linear series if $V \subseteq H^0(L)$ is free of base points, that is, the evaluation map of sections $V \otimes \mathcal{O}_C \xrightarrow{ev} L$ is surjective.

Definition 1.1.1. *Let (L,V) be a generated linear series over a smooth curve C. We say that (L,V) is linear (semi)stable if any linear series of degree d' and dimension r' contained in $|V|$ satisfies*

$$\frac{d'}{r'} > \frac{d}{r}(\textit{respectively } \geqslant).$$

The associated *syzygy bundle* (*kernel bundle* or *dual span bundle*) of a generated linear series (L,V) is defined as

$$M_{V,L} = Ker(V \otimes \mathcal{O}_C \xrightarrow{ev} L).$$

This bundle fits into the exact sequence

$$0 \to M_{V,L} \to V \otimes \mathcal{O}_C \xrightarrow{ev} L \to 0.$$

The vector bundle $M_{V,L}$ has rank r and degree $deg(M_{V,L}) = -d$. When $V = H^0(L)$, we denote $M_{H^0(L),L}$ by M_L. With these invariants we can define another stability condition:

Definition 1.1.2. *Let E be a vector bundle over C, we define the slope of E as the rational number*

$$\mu(E) = \frac{deg(E)}{rk(E)}.$$

We say that a vector bundle E over C is slope-(semi)stable if for any proper subbundle $F \subseteq E$ we have

$$\mu(F) < \mu(E)(\leqslant \textit{ respectively}).$$

If E is not slope-semistable we say that is slope-unstable.

We seek to investigate the slope-stability of the bundle $M_{V,L}$ due to its connections with several research areas, including the geometry of Brill-Noether varieties, the Resolution Minimal Conjecture (see [16]), the stability of the tangent bundle of a projective space when restricted to a curve (see [9]), theta divisors associated with vector bundles on curves (see [12]), and the exploration of moduli spaces for vector bundles, among other topics. For example,

- Ein and Lazarsfeld used the stability of $M_{V,L}$ to prove the stability of the Picard bundle, see [15].
- Paranjape and Ramanan proved in [27] that M_{K_C} is semistable.
- David C. Butler showed that M_L is stable for $d > 2g$, and it is semistable for $d = 2g$, see [8].

In the study of $M_{V,L}$ it is important to consider the following situation: Let $S \subseteq M_{V,L}$ be a proper subbundle considered as a saturated sheaf. There exists a vector bundle F_S and a subspace $W \hookrightarrow V$ fitting into the commutative diagram (called Butler's diagram):

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & S \arrow[r] \arrow[d] & W \otimes \mathcal{O}_C \arrow[r] \arrow[d] & F_S \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & M_{V,L} \arrow[r] & V \otimes \mathcal{O}_C \arrow[r] & L \arrow[r] & 0
\end{tikzcd} \tag{1.1.1}$$

The bundle $F_S^\vee$ can be defined as the syzygy bundle

$$F_S^\vee = Ker(W^\vee \otimes \mathcal{O}_C \to S^\vee)$$

and we define $W \hookrightarrow V$ as the space $W^\vee = Im(V^\vee \to H^0(S^\vee))$. Note that $W^\vee$ generates $S^\vee$ because $V^\vee$ generates $M_{V,L}^\vee$.

Properties 1.1.1 (See [8])**.** *In the above notation, the following properties hold:*

1. *The sheaf F_S is globally generated and $h^0(F_S^\vee) = 0$.*

2. *The induced map $\alpha : F_S \to L$ is not the zero map.*

3. *If S is a maximal destabilizing subbundle of $M_{V,L}$, that is $\mu(S) \geqslant \mu(M_{V,L})$, then $deg(F_S) \leqslant deg(I)$ where $I := Im(\alpha)$, and the equallity holds if and only if $rank(F_S) = 1$.*

4. *The sheaf F_S has no trivial summands.*

In Butler's diagram, when $rank(F_S) = 1$ then S is exactly the syzygy bundle M_{W,F_S} and the pair (F_S, W) contradicts that (L, V) is linear (semi)stable exactly when $\mu(M_{W,F_S}) > (\geqslant)\mu(M_{V,L})$, that is:

Remark 1.1.1. *Linear (semi)stability of (L, V) is equivalent to the condition that the bundle $M_{V,L}$ can not be destabilized by subbundles of the form $M_{V',L'}$ where (L', V') is a generated linear subseries of (L, V).*

We have the following implication for a generated linear series (L, V) over C:

$$\text{Slope-(semi)stability of } M_{V,L} \Longrightarrow \text{Linear (semi)stability of } (L, V). \tag{1.1.2}$$

1.2 Brill-Noether loci

In this section, we explore properties of specific families of curves parameterized by gonality. Such families allow us to stratify the moduli space $\mathcal{M}_{\}}$.

For positive integers g, r and d, we introduce the *Brill-Noether locus*

$$\mathcal{M}^r_{g,d} \text{ as the locus of curves } C \in \mathcal{M}_g \text{ that admits a } \mathfrak{g}^r_d \text{ on } C.$$

From Theorem 1.1.1, when $\rho(g, r, d)$ is negative, $\mathcal{M}^r_{g,d}$ is a proper subvariety of $\mathcal{M}_g$ all of whose components have codimension at most $-\rho(g, r, d)$. There is a well known result by D. Eisenbud and J. Harris (see [15]) that when $\rho(g, r, d) = -1$, the variety $\mathcal{M}^r_{g,d}$ has a divisorial component (pure codimension 1) in $\mathcal{M}_g$.

1.2.1 Stratification by gonality

Now we introduce the definition of gonality of a curve. The gonality of a curve $C \in \mathcal{M}_g$, denoted by $gon(C)$ is the minimum positive integer $d > 1$ such that there exists a $\mathfrak{g}^1_d$ on C. In other words, the gonality of C is the smallest degree of a rational map from C to the projective line. This geometric invariant gives us an indication of how "far" the curve C is from being rational. For $g > 2$ consider the stratification of the moduli space $\mathcal{M}_g$ given by the gonality

$$\mathcal{M}^1_{g,2} \subseteq \mathcal{M}^1_{g,3} \subseteq ... \subseteq \mathcal{M}^1_{g,k} \subseteq ... \subseteq \mathcal{M}_g,$$

where

$$\mathcal{M}^1_{g,k} = \{C \in \mathcal{M}_g | C \text{ has a } \mathfrak{g}^1_k\},$$

is called the k-gonal locus. These strata have the following properties (see [3]):

Properties 1.2.1. *Let $g > 2$ be an integer and $\mathcal{M}_g$ the moduli of curves of genus g, the following properties holds:*

- *For $k \leqslant \frac{g+2}{2}$, the k-gonal locus is an irreducible variety of dimension $2g + 2k - 5$.*
- *For $k \geqslant [\frac{g+3}{2}]$, the dimension of $\mathcal{M}^1_{g,k}$ is equal to the dimension of $\mathcal{M}_g$.*
- *The number $[\frac{g+3}{2}]$ is called the **generic gonality** of curves of genus g.*

In [25] the authors study the converse of 1.1.2 formulating the following conjecture for non-complete linear series on a curve:

Conjecture 1.2.1 ([25, Conjecture 8.6])**.** *Let C be curve of genus g and (L, V) be a generated linear series of type $(d, r + 1)$ over C with $V \subsetneq H^0(L)$. If $d \leqslant kr$ where $k = gon(C)$, then linear (semi)stability of (L, V) is equivalent to slope-(semi)stability of $M_{V,L}$.*

Condition $d \leqslant kr$ for non-complete linear series (L, V) seems to be natural due to:

Proposition 1.2.1 ([25, Proposition 8.4])**.** *On any curve C there exists a non-complete linear system $V \subseteq H^0(L)$ such that (L, V) is linearly stable and $M_{V,L}$ is slope-unstable.*

To prove this result authors produce a linear series satisfying $d > kr$ such that $V \subseteq H^0(L)$ is not general. For complete linear series, they also formulate the following conjecture:

Conjecture 1.2.2 ([25, Conjecture 8.7])**.** *For any curve C and any line bundle L on C linear (semi)stability of $(L, H^0(L))$ is equivalent to slope-(semi)stability of M_L.*

Conjecture 1.2.2 has a positive answer:

Theorem 1.2.1 ([11, Corollary 4.1])**.** *Let $L \in Pic^d(C)$ be a generated line bundle over a general curve C of genus $g \geqslant 2$. Suppose that $h^0(L) = r + 1$. Then*

1. *Linear (semi)stability of $(L, H^0(L))$ is equivalent to slope-(semi)stability of M_L.*

2. *M_L fails to be stable if and only if all the following three conditions hold:*

   (a) *$h^1(L) = 0$.*

   (b) *$deg(L) = g + r$ and r divides g.*

   (c) *There is an effective divisor Z with $h^0(L(-Z)) = h^0(L) - 1$ and $deg(Z) = 1 + \frac{g}{r}$.*

In [10] authors gived a counterexample for Conjecture 1.2.2 for plane curves of degree 7:

Theorem 1.2.2 ([10, Theorem 4.1])**.** *For any smooth plane curve C of degree 7, the general element in $W^2_{15}(C)$ satisfies*

- *The complete linear series $(L, H^0(L))$ is generated and linearly stable.*
- *The vector bundle M_L is not slope-stable.*

This shows the importance to study when the equivalence between linear (semi)-stability of linear series and stability of syzygy bundles holds for incomplete linear series over a curve. In this thesis we aim to state results concerning this equivalence for general curves and for different families of special curves.

Chapter 2

Linear stability for non-complete linear series on general curves

In this Chapter we use classical Brill-Noether theory to prove that the Conjecture 1.2.1 holds for non-complete generated linear series on general curves. In order to prove this result we follow the approach used by Castorena and Torres-Lopez to prove Conjecture 1.2.2 for complete generated linear series on general curves. Moreover, we prove that for non-complete generated linear series on general curves the slope-stability of $M_{V,L}$ is equivalent to another kind of stability, called cohomological stability.

2.1 Determinant bundles

In this section, we present similar results to those given by Castorena and Torres-Lopez in [11] emphasizing the diferences with our case. As a first approach to understand the Conjecture 1.2.1, we have the following result of [25] giving conditions for the bundle F_S and for its determinant bundle $det(F_S)$.

Lemma 2.1.1 ([25, Lemma 4.3])**.** *Let C be a k-gonal curve. With the notation of Butler's diagram of the pair (L, V) by S, suppose that $rank(F_S) \geqslant 2$. If F_S fits in an exact sequence*

$$0 \to \bigoplus^{rank(F_S)-1} \mathcal{O}_C \to F_S \to det(F_S) \to 0$$

which is also exact on global sections, then the following properties hold:

1. *If $deg(L) \leqslant k(dim(V) - 1)$ where $k = gon(C)$, then $\mu(S) \leqslant \mu(M_{V,L})$. Furthermore, we have equality if and only if*
   - $W = H^0(F_S)$
   - $k = \frac{deg(det(F_S))}{h^0(det(F_S))-1}$
   - $k = \frac{deg(L)}{dim(V)-1}$
2. *If $deg(L) < k(dim(V) - 1)$ where $k = gon(C)$, then $\mu(S) < \mu(M_{V,L})$*

By dualizing the first exact row in Butler's diagram of the pair (L, V) by S $(0 \to S \to W \otimes \mathcal{O}_C \to F_S \to 0)$ and twisting by K_C, we get a map at sections

$$m_W : W^\vee \otimes H^0(K_C) \to H^0(S^\vee \otimes K_C).$$

In contrast to [11] we consider the incomplete case, that is (L, V) a generated linear series with $V \in Gr(r+1, H^0(L))$ generating space with $r+1 < h^0(L)$ and the Butler's diagram of the pair (L, V) by S.

Proposition 2.1.1. *Let $Q = M_{V,L}/S$*

i) If the multiplication map

$$m_W : W^\vee \otimes H^0(K_C) \to H^0(S^\vee \otimes K_C)$$

is surjective, then $H^0(Q) = 0$.

ii) If m_W is surjective, then $W = H^0(F_S)$.

iii) If $S \subseteq M_{V,L}$ is stable of maximal slope, then $H^0(Q) = 0$.

Proof. i) We follow [11]. By dualizing Butler's diagram of (L, V) by S as shown in 1.1.1, twisting by K_C and taking cohomology,

$$\begin{tikzcd}
H^0(L^\vee \otimes K_C) \arrow[r] & V^\vee \otimes H^0(K_C) \arrow[r, "m_1"] \arrow[d, "p_1"] & H^0(M_{V,L}^\vee \otimes K_C) \arrow[r, "A"] \arrow[d, "p_2"] & H^1(L^\vee \otimes K_C) \arrow[r, "D"] & V^\vee \otimes H^1(K_C) \arrow[r] & 0 \\
H^0(F_S^\vee \otimes K_C) \arrow[r] & W^\vee \otimes H^0(K_C) \arrow[r, "m_W"] & H^0(S^\vee \otimes K_C) \arrow[r, "E"] \arrow[d, "a"] & H^1(F_S^\vee \otimes K_C) \arrow[r, "B"] & W^\vee \otimes H^1(K_C) \arrow[r] & 0 \\
& & H^1(Q^\vee \otimes K_C) \arrow[d, "b"] & & & \\
& & H^1(M_{V,L}^\vee \otimes K_C) & & &
\end{tikzcd}$$

Since m_W is surjective and $W \hookrightarrow V$, it follows that p_1 is also surjective, and $m_W \circ p_1$ is surjective as well. By commutativity of the diagram $m_W \circ p_1 = p_2 \circ m_1$ is surjective and this implies that p_2 is also surjective, which is equivalent to stating that the morphism a is equal to zero.

On the other hand, by Serre duality,

$$H^1(Q^\vee \otimes K_C) \cong H^0(Q)^\vee \quad \text{and} \quad H^1(M_{V,L}^\vee \otimes K_C) \cong H^0(M_{V,L})^\vee.$$

Furthermore, since $H^0(M_{V,L}) \cong 0$ then $a \equiv 0$ and b is an isomorphism. We conclude that $H^0(Q)^\vee \cong 0$, equivalently $H^0(Q) = 0$.

ii) If m_W is surjective then the map E vanishes and B is an isomorphism. By Serre duality $H^1(F_S^\vee \otimes K_C) \cong H^0(F_S)^\vee$. Thus $H^0(F_S)^\vee \cong W^\vee \otimes H^1(K_C)$, leading to the isomorphisms,

$$H^0(F_S)^\vee \cong W^\vee \otimes H^1(K_C) \cong W^\vee.$$

By dualizing we obtain $H^0(F_S) \cong W$.

iii) This proof is analogous to [11, Theorem 1.1] which is based on the study of Q when $M_{V,L}$ is slope-semistable (for which Q is slope-semistable of negative degree) or when $M_{V,L}$ is slope-unstable (in this case, the maximal slope of the Harder-Narasimhan filtration for Q is negative, leading to $H^0(Q) = 0$).

■

Remark 2.1.1. *In Theorem 1.1 of [11] the case i) is an **if and only if** result. However, in the non-complete case as in Proposition 2.1.1, the proof of $H^0(Q) = 0$ implies that the map m_W is surjective is non true in general. This is due to the fact that the morphism $D : H^1(L^\vee \otimes K_C) \to V^\vee \otimes H^1(K_C)$ (following the notation of the proof) is not an isomorphism since $r + 1 < h^0(L)$. Consequently, when $V \subsetneq H^0(L)$ we cannot assert that the morphisms m_1 and p_2 are surjective as the authors do in [11]. This indicates that the results established for the complete case cannot be directly extended to the incomplete case.*

Using this construction, for a general curve we can give a proof of the slope-semistability of $M_{V,L}$ and conditions for the strictly slope-semistability of $M_{V,L}$.

Proposition 2.1.2. *Let C be a general curve of genus $g \geqslant 2$. Let (L, V) be a generated linear series of type $(d, r+1)$ on C, and consider $c\colon = codim_{H^0(L)} V \leqslant h^1(L)$. Then $M_{V,L}$ is slope-semistable. Moreover, if there exists a proper subbundle $S \subseteq M_{V,L}$ with $\mu(S) = \mu(M_{V,L})$, then*

- $h^1(L) - c = 0$.
- $s := rank(S) = r - 1$.
- $d = g + r$ *with* $r|g$.

Proof. This proof is analogous to [11, Lemma 4.1]; we will refer to this proof later but for completeness we will provide the entire argument here. Consider a proper subbundle $S \subseteq M_{V,L}$ with inclusion $0 \to S \to M_{V,L}$. By dualizing the sequence $M_{V,L}^\vee \to S^\vee \to 0$, we obtain $S^\vee$ as a quotient of $M_{V,L}^\vee$. Consequently, for any $U \in Gr(s+1, H^0(S^\vee))$, we have the following exact sequence,

$$0 \to S \to U^\vee \otimes \mathcal{O}_C \to det(S^\vee) \to 0$$

inducing the exact sequence in cohomology

$$0 \to H^0(S) \to U^\vee \to H^0(det(S^\vee)) \to \dots$$

Since $h^0(M_{V,L}) = 0$ and $S \hookrightarrow M_{V,L}$, it follows that $H^0(S) = 0$, leading to $h^0(det(S^\vee)) \geqslant dim(U) = s+1$. Using that C is general we get $h^0(det(S^\vee)) \geqslant s+1$ and $deg(det(S^\vee)) = deg(S^\vee)$, the last argue implies that $det(S^\vee)$ is a degree $deg(S^\vee)$ line bundle with at least $s + 1$ global sections. The Brill-Noether number for $det(S^\vee)$ is given by

$$\rho(g, s, deg(S^\vee)) = g - (s+1)(g - deg(S^\vee) + s) \geqslant 0$$

where

$$deg(S^\vee) \geqslant \frac{s(s+g+1)}{s+1}. \tag{2.1.1}$$

Thus,

$$\mu(S) = \frac{-deg(S^\vee)}{s} \leqslant -\frac{\frac{s(s+g+1)}{s+1}}{s} = -\frac{s+g+1}{s+1} = -1 - \frac{g}{s+1}$$

Now, from Riemann-Roch theorem for the line bundle L and letting $h = h^1(L)$, we have

$$\mu(M_{V,L}) = -\frac{d}{r} = \frac{r + h^0(L) - (r+1) - h + g}{r} = \frac{r + c - h + g}{r} = -1 + \frac{h - c - g}{r}.$$

Consequently,

$$\mu(S) - \mu(M_{V,L}) \leqslant -1 - \frac{g}{s+1} + 1 - \frac{h-c-g}{r} = g\left(\frac{1}{r} - \frac{1}{s+1}\right) - \frac{h-c}{r}. \qquad (2.1.2)$$

Since $h - c$ is greater than 0, inequality 4.1.4 is less or equal to 0 which implies that $\mu(S) \leqslant \mu(M_{V,L})$. Therefore, we conclude that $M_{V,L}$ semistable. For the last inequality in 4.1.4, if $\mu(S) = \mu(M_{V,L})$, then $h^1 = c$ and $r = s+1$. Moreover, from Riemann-Roch theorem we get that $d = g + r$ and $\mu(M_{V,L}) = deg(M_{V,L}/S) \in \mathbb{Z}$ and we deduce that r divides g. ■

At this point, we turn our attention to the bundle F_S associated with the subbundle S rather than $S^\vee$, as F_S appears in the Butler's diagram of (L, V) by S. To effectively apply Lemma 2.1.1, we consider the case in which $M_{V,L}$ is strictly slope-semistable, with S being a subbundle that shares the same slope as $M_{V,L}$, specifically $\mu(S) = \mu(M_{V,L})$. This condition allows us to compute the dimension of global sections for the bundle $det(F_S)$, which for the case of $rank(F_S) \geqslant 2$ will provide valuable insights into the stability properties of $M_{V,L}$.

Proposition 2.1.3. *Let C be a general curve of genus $g \geqslant 2$, (L, V) be a generated linear series over C and let $c = codim_{H^0(L)}(V) \leqslant h^1(L)$. If S is a proper subbundle of $M_{V,L}$ with $\mu(S) = \mu(M_{V,L})$. Then*

$$h^0(det(F_S)) = s + 1.$$

Proof. As in [11] we have that

$$\mu(S) = -\frac{deg(F_S)}{s} = \frac{deg(S)}{s} = -\frac{deg(L)}{r} = \mu(M_{V,L}).$$

From Proposition 2.1.2 then $deg(F_S) = s + g - \frac{g}{r} \in \mathbb{Z}$. Notice that $h^0(det(F_S)) = h^0(det(S^\vee)) \geqslant s + 1 = r$. Assume that $h^0(det(F_S)) \geqslant r + 1$. Using that C is general and $det(F_S)$ is a degree $deg(F_S)$ line bundle with at least $r + 1$ sections, it follows that the corresponding Brill-Noether number $\rho = \rho(g, r, deg(F_S))$ is non-negative. However, we have,

$$0 \leqslant \rho = g - (r+1)(r - deg(F_S) + g) = g - (r+1)\left(1 + \frac{g}{r}\right) = -r - 1 - \frac{g}{r} < 0$$

contradicting the generality of C. Therefore $h^0(det(F)) = s + 1 = r$. ■

We aim to prove that under the conditions outlined in Proposition 2.1.2 we can conclude the same result as in part *ii)* of Proposition 2.1.1. From Remark 2.1.1, this conclusion does not hold in general for the non-complete case. By establishing the semistability of $M_{V,L}$, we can analyze the properties of the bundle F_S associated with the subbundle S. This connection is essential, as it allows us to leverage the results from the semistability of $M_{V,L}$ to draw conclusions about the behavior of F_S.

Proposition 2.1.4. *Let C be a general curve of genus $g \geqslant 2$, (L, V) be a generated $\mathfrak{g}^r_d$ over C, if S is a proper subbundle of $M_{V,L}$ with $\mu(S) = \mu(M_{V,L})$ and $rank(F_S) > 1$, then $W = H^0(F_S)$.*

Proof. Since $M_{V,L}$ is strictly slope-semistable from Proposition 2.1.2 we conclude that $s = rank(S) = r - 1$. From the inclusion $W \hookrightarrow V$ the dimension $w = dim(W)$ can take only two possible values: either $w = r+1$ or $w = r$. First, consider the case where $w = r$. In this case we have $rank(F_S) = 1$, which does not fall within the cases we are currently considering. Next, we assume $w = r + 1$. In this case $W \cong V$ leading to $rank(F_S) = 2$. Since $W \subseteq H^0(F_S)$, it follows that $w \leqslant h^0(F_S)$. Moreover, according to [25, Proposition 4.1], F_S fits into the following exact sequence

$$0 \to \mathcal{O}_C \to F_S \to det(F_S) \to 0.$$

By taking cohomology

$$0 \to H^0(\mathcal{O}_C) \to H^0(F_S) \xrightarrow{\varphi} H^0(det(F_S)) \to \cdots . \tag{2.1.3}$$

From Proposition 2.1.3 we get $h^0(det(F_S)) = s + 1 = r$ and from dimension theorem,

$$h^0(F_S) = dim(Im(\varphi)) + dim(Ker(\varphi)).$$

Using the exactness of the sequence 2.2.1 and the fact that $Im(\varphi)$ is a subspace of $H^0(det(F_S))$, we obtain the following inequality

$$h^0(F_S) = dim(Im(\varphi)) + dim(Ker(\varphi)) = dim(Im(\varphi)) + 1 \leqslant r + 1.$$

We conclude that $r + 1 = w \leqslant h^0(F_S) \leqslant r + 1$, which implies that $W = H^0(F_S)$. ■

We recall that linear semistability of (L, V) is equivalent to the fact that the bundle $M_{V,L}$ cannot be destabilised by subbundles of the form $M_{W,L'}$. Next we show that it is possible to construct a specific linear subseries associated with a proper subbundle $S \subseteq M_{V,L}$ that shares the same slope.

Theorem 2.1.1. *Let C be a general curve of genus $g \geqslant 2$ and let (L, V) a globally generated linear series such that $dim(V) = r + 1$ with $c = codim_{H^0(L)}(V) \leqslant h^1(L)$. Consider $S \subseteq M_{V,L}$ a proper subbundle with $\mu(S) = \mu(M_{V,L})$. Then, there exists a line bundle F_S which fits into the commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r] \arrow[d] & W \otimes \mathcal{O}_C \arrow[r] \arrow[d] & F_S \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_{V,L} \arrow[r] & V \otimes \mathcal{O}_C \arrow[r] & L \arrow[r] & 0
\end{tikzcd}$$

Proof. This proof is analogous to [11, Theorem 4.1]. From Proposition 2.1.2 S is a stable bundle, since $\mu(S) = \mu(M_{V,L})$, we conclude that $h = h^1(L) = c$ and $s = r-1$. Since $W \subseteq V$ and $dim(W) \geqslant rank(S)$, it follows that the rank of F_S is either 1 or 2. Consider the case $r = 2$, then $d = g+2$ and $h = c$. We have that $\mu(S) = \mu(M_{V,L}) = -\frac{g+2}{2}$. From Proposition 2.1.3 $h^0(S^\vee) = 2$ and since $W^\vee := Im(V^\vee \xrightarrow{\phi} H^0(S^\vee))$, which generates $S^\vee$, we conclude $F_S = S^\vee$ is a line bundle and $W = H^0(S^\vee)$ has dimension 2. This completes the case $r = 2$.
Now, consider $r > 2$. By hypothesis, the gonality of C is $k = gon(C) \geqslant \frac{g+2}{2}$, leading to the inequality

$$d = g + r < \frac{g+2}{2} r \leqslant kr.$$

Assuming that $rank(F_S) = 2$, we apply Proposition 2.1.4 to find that $h^0(F_S) = dim(W) = r+1$. Additionally, since $h^0(det(F_S)) = r$, it follows that the exact sequence

$$0 \to \mathcal{O}_C \to F_S \to det(F_S) \to 0 \tag{2.1.4}$$

is exact on global sections. To see this, consider the cohomology of sequence 2.1.4,

$$0 \longrightarrow H^0(\mathcal{O}_C) \xrightarrow{f_1} H^0(F_S) \xrightarrow{f_2} H^0(det(F_S)) \xrightarrow{\delta} H^1(\mathcal{O}_C) \longrightarrow \cdots$$

From exactness, we notice that $\dim(Im(f_1)) = \dim(Ker(f_2)) = 1$ then,

$$r + 1 = h^0(F_S) = \dim(Ker(f_2)) + \dim(Im(f_2)) = 1 + \dim(Im(f_2)).$$

This implies that $\dim(Im(f_2)) = r$ meaning that f_2 is surjective. Consequently, the sequence 2.1.4 is exact on global sections. This leads to $\mu(S) < \mu(M_{V,L})$ (see Lemma 2.1.1), which is a contradiction. Therefore F_S is a line bundle. ■

From above results, in the context of Theorem 2.1.1, we know properties of the elements that appear in the Butler's diagram of (L, V) by S when S has the same slope as $M_{V,L}$. This allows us to leverage these properties to conclude that the linear stability of (L, V) implies the slope-stability of $M_{V,L}$.

Corollary 2.1.1. *Let (L, V) be a generated linear series of type $(d, r+1)$ over a general curve C of genus $g \geqslant 2$ with $c \leqslant h^1(L)$. Then linear (semi)stability of (L, V) is equivalent to slope-(semi)stability of $M_{V,L}$*

Proof. If $M_{V,L}$ is stable, then (L, V) is linearly stable. Suppose that $M_{V,L}$ is strictly slope-semistable and let $S \subseteq M_{V,L}$ be a subbundle with $\mu(S) = \mu(M_{V,L})$. By Theorem 2.1.1 the rank of F_S is equal to 1 and this implies that (L, V) is strictly linear semistable. ■

Finally, with the same assumptions as in the Corollary, we aim to study under which conditions $M_{V,L}$ fails to be slope-stable in terms of the elements of the linear series (L, V). Specifically, we explore how the properties of the elements in the Butler's diagram of (L, V) by S when S has the same slope as $M_{V,L}$ influence the slope-stability of the vector bundle $M_{V,L}$. The proof is similar to that given in Castorena and Torres-Lopez with the difference that in our case we have to consider the linear series $(L(-Z), V(-Z))$ where the space $V(-Z) := H^0(L(-Z)) \cap V$ for a specific effective divisor Z on C.

Proposition 2.1.5. *Let C be a general curve of genus $g \geqslant 2$ and let (L, V) be a globally generated linear series such that dim$(V) = r+1$ and deg$(L) = d$. Then, $M_{V,L}$ fails to be stable if and only if the following three conditions hold:*

i) $c := codim_{H^0(L)}(V) = h^1(L) = h^1$.

ii) $d = g + r$ *with* $r|g$.

iii) *There is an effective divisor Z, with $h^0(L(-Z)) = h^0(L) - 1$ and $dim(V(-Z)) = r$, where $V(-Z) = V \cap H^0(L(-Z)) \subseteq H^0(L)$ and $deg(Z) = 1 + \frac{g}{r}$.*

Proof. $\Leftarrow$) The evaluation map $ev : H^0(L(-Z)) \otimes \mathcal{O} \to L(-Z)$ is surjective. Notice that $V \subsetneq H^0(L(-Z))$ because $dim(V \cap H^0(L(-Z))) < dim(V)$. We know that $L(-Z)$ is generated by $V(-Z)$; otherwise, there exists a point $p \in C$ such that $ev|_p : V(-Z) \otimes \mathcal{O}_p \to L(-Z)_p$ is not surjective. This leads to two cases:

- If $p \in C$ and $p \notin Z$, since V generates L, the evaluation map $ev|_p : V(-Z) \otimes \mathcal{O}_p \to L(-Z)_p$ is surjective.
- If $p \in Z$, given that the map $ev : H^0(L(-Z)) \otimes \mathcal{O} \to L(-Z)$ is surjective, it follows that $ev|_p : V(-Z) \otimes \mathcal{O}_p \to L(-Z)_p$ is also surjective.

Moreover, we have $F_S = L(-Z)$ and $W = V(-Z) = H^0(L(-Z)) \cap V$ in the Butler's diagram of (L, V) by $S = M_{V(-Z),L(-Z)}$, with $\mu(M_{V(-Z),L(-Z)}) = \mu(M_{V,L})$. Hence, $M_{V,L}$ is slope-semistable but not slope-stable.

$\Rightarrow$) Following the ideas in [11, Corollary 4.3]. If $M_{V,L}$ is strictly slope-semistable. According to Proposition 2.1.2, there exists a subbundle $S \subseteq M_{V,L}$ of rank $r-1$ such that $c = h^1(L)$ and $d = g + r$ with $r|g$.

Since F is a line bundle and the morphism $\alpha : F \to L$ is non-zero, there exists an effective divisor Z such that $F = L(-Z)$ and $dim(W) = r$. We have the inclusion $W \hookrightarrow V$, since F is generated $W \hookrightarrow H^0(L(-Z))$ making W a subspace of $H^0(L(-Z)) \cap V = V(-Z)$ of maximal dimension, thus $W = V(-Z)$.

Since $deg(F) = -deg(S)$ and $\mu(S) = \mu(M_{V,L})$, it follows that $deg(Z) = 1 + \frac{g}{r}$. Note that $Q := M_{V,L}/S = \mathcal{O}_C(-Z)$, so $h^0(Q) = 0$ and $\dim(H^0(L(-Z)) \cap V) = \dim W = r$, which gives condition *iii*).

■

2.2 Cohomological stability

In this section, we consider another type of stability for vector bundles. In [25] authors studied the relation between this type of stability (that results stronger than the slope-stability) for $M_{V,L}$, the slope-stability of $M_{V,L}$ and linear stability of (L, V). We present similar results of to those presented by Castorena and Torres-Lopez in [12], we show sketches of the proofs where we emphatize the diferences with our case.

Definition 2.2.1. *Let E be a vector bundle over a curve C. We say that E is cohomologically (semi)stable if for any $A \in Pic^a(C)$, and for every $t < rank(E)$, we have that*

$$h^0\left(\bigwedge^t E \otimes A\right) = 0$$

whenever $a \leqslant t \cdot \mu(E)$ (respectively $a < t \cdot \mu(E)$).

In [15], authors show that cohomological semistability is equivalent to slope-semistability, and that cohomological stability implies slope-stability. We want to look for precise conditions for when such stabilities are equivalent since we know they satisfy the following implications:

Cohomological stability of $M_{V,L} \Rightarrow$ Slope-stability of $M_{V,L} \Rightarrow$ Linear stability of (L, V)

In order to find precise conditions for the equivalence between the first two stabilities, we study the following property for syzygy bundles associated to linear subseries related by divisors.

Lemma 2.2.1 ([25, Lemma 7.4])**.** *Let (L, V) be a $\mathfrak{g}^r_d$ on a smooth curve C, which induces a birational morphism, and let $D_k = p_1 + \ldots + p_k$ be a general effective divisor on C, with $k < r$. The kernel bundle associated to the linear series lies in the following exact sequence of sheaves*

$$0 \to M_{V(-D_k),L(-D_k)} \to M_{V,L} \to \bigoplus_{i=1}^{k} \mathcal{O}_C(-p_i) \to 0.$$

Remark 2.2.1.

- *With notation and conditions in Lemma 2.2.1, if we consider a general effective divisor D of maximal degree $r-1$, we have that $M_{V(-D),L(-D)}$ is a line bundle which is dual to $\mathcal{O}_C(p_r + \ldots + p_d)$ and*

$$M_{V(-D),L(-D)} \cong \mathcal{O}_C(-p_r - \ldots - p_d).$$

- *Let (L, V) and $x_1, \ldots, x_{r-1}$ be as in Lemma 2.2.1 and let $F = \oplus_{j=1}^{r-1} \mathcal{O}_C(-x_j)$. For any integer $t < r$, we get the following short exact sequence of exterior powers*

$$0 \to \bigwedge^{t-1} F \otimes L^{\vee}\left(\sum_{j=1}^{r-1} x_j\right) \to \bigwedge^t M_{V,L} \to \bigwedge^t F \to 0. \tag{2.2.1}$$

Now, we follow [12] in the incomplete case in direction to give conditions for which bundle $M_{V,L}$ is cohomological (semi)stable.

Proposition 2.2.1. *Let (L, V) be a generated $\mathfrak{g}^r_d$ on a smooth curve C which induces a birational morphism. Let $A \in Pic^a(C)$ such that $a \leqslant t \cdot \frac{d}{r}$ and $h^0(A) \leqslant t$ with integers t, d and r satisfying $0 < t < r < d$. Then $h^0(\wedge^t M_{V,L} \otimes A) = 0$.*

Proof. In a similar way as in [12, Proposition 3.4] $M_{V,L}$ fits into the sequence 2.2.1. Moreover, since $h^0(A) \leqslant t$, we can take points $x_1, \ldots, x_{r-1}$ satisfying the property: for any $i_1, \ldots, i_t \in \{1, \ldots, r-1\}$, with $1 \leqslant i_1 < \cdots < i_t \leqslant r-1$, we have $h^0(A(-x_{i_1} - \cdots - x_{i_t})) = 0$. By twisting 2.2.1 by A, we get at cohomology

$$\bigoplus H^0\left(L^\vee \otimes A\left(\sum_{j=1}^{r-1} x_j - \sum_{j=1}^{t} x_{i_j}\right)\right) \to H^0(\bigwedge^t M_{V,L} \otimes A) \to \bigoplus H^0\left(A\left(-\sum_{j=1}^{t} x_{i_j}\right)\right). \tag{2.2.2}$$

Next we prove that first and the last terms of this sequence are trivial. Each summand of the right side is zero due to the properties on A and on the points $\{x_{i_1}, \ldots, x_{i_t}\}$. The left side is zero since each summand is the global sections of line bundles with degrees

$$deg\left(L^\vee \otimes A\left(\sum_{j=1}^{r-1} x_j - \sum_{j=1}^{t} x_{i_j}\right)\right) = (r-t)\left(1 - \frac{d}{r}\right) < 0,$$

since $r > t$ and $d > r$. Hence $h^0(\wedge^t M_{V,L} \otimes A) = 0$. ■

From now on, let us assume that C is a general curve. We aim to establish a bound on the dimension of $H^0(A)$ for a line bundle A, as above.

Remark 2.2.2. *It follows from Riemman-Roch formula that hypothesis*

$$c := codim_{H^0(L)}(V) \leqslant h^1(L)$$

is equivalent to condition $d \leqslant g + r$.

Proposition 2.2.2. *Let C be a general curve of genus g. Let $A \in Pic^a(C)$ with $a \leqslant t \cdot \frac{d}{r}$ with integers t, d and r satisfying $0 < t < r < d \leqslant g + r$, then $h^0(A) \leqslant t+1$. Moreover, $h^0(A) = t+1$ if and only if $a = t \cdot \frac{d}{r}$, $d = g + r$ and $t + 1 = r$.*

Proof. This proof is analogous to [12, Proposition 3.6], which is based in the study of Brill-Noether numbers associated with the line bundle A and the property that C is general. Assuming that $h^0(A) \geqslant t+2$, the corresponding Brill-Noether number $\rho(g, t+1, a)$ satisfies

$$\begin{aligned} 0 \leqslant \rho(g, t+1, a) &\leqslant -g - (t+2)(t+1) + t(t+2)\left(\frac{g+r}{r}\right) - tg \\ &= -g - (t+2) - tg\left(1 - \frac{t+2}{r}\right) \\ &< 0. \end{aligned}$$

giving a contradiction since C is general,thus we conclude that $h^0(A) \leqslant t+1$.
Assuming now that $h^0(A) = t+1$, it follows that

$$\begin{aligned} 0 \leqslant \rho(g, t, a) &= g - (t+1)(t - a + g) \\ &= -t(t+1) + a(t+1) - tg \\ &\leqslant t(t+1)\frac{d}{r} - t(t+1) - tg \\ &\leqslant t(t+1)\left(\frac{g+r}{r}\right) - t(t+1) - tg \\ &= gt\left(\frac{t+1}{r} - 1\right) \leqslant 0 \end{aligned}$$

Thus, we find that $h^0(A) = t+1$ if and only if $\rho(g, t, a) = 0$, which is equivalent to $a = \frac{d}{r}t$, $d = g + r$, and $t + 1 = r$. ■

To establish the cohomological semistability of $M_{V,L}$, notice that from Proposition 2.1.2 and the results presented in [15], cohomological semistability is equivalent to slope-semistability. This leads us to conclude that $M_{V,L}$ is cohomologically semistable. We aim to characterize the conditions under which this cohomological semistability holds. Specifically, we analyze the implications of the Brill-Noether numbers associated with A, as well as the dimensions of the relevant cohomology groups, to provide a comprehensive understanding of the stability properties of the sheaf.

Proposition 2.2.3. *Let (L, V) be a generated linear series of type $(d, r+1)$ which induces a birational morphism over a smooth general curve C. Then $M_{V,L}$ is cohomologically semistable.*

Proof. This proof is analogous to [12, Theorem 3.7], let $t < r$ and consider $A \in Pic^a(C)$ with $a < t\frac{d}{r}$. From Propositions 2.2.1 and 2.2.2 $h^0(A) \leqslant t$ and $h^0(\wedge^t M_{V,L} \otimes A) = 0$. Hence $M_{V,L}$ is cohomologically semistable. ■

As for slope-semistability, we want to find suitable conditions for the cohomological stability of $M_{V,L}$. A first step in this direction is the following result.

Corollary 2.2.1. *Let (L, V) be a generated linear series of type $(d, r+1)$ which induces a birational morphism over a general curve C with $c \leqslant h^1(L)$. Then $M_{V,L}$ is cohomologically stable if one of the following conditions holds:*

i) $c < h^1(L)$.

ii) $c = h^1(L)$ *and* r *does not divide* g.

Proof. In a similar way as in [12, Corollary 3.8] for $t < r$ and $A \in Pic^a(C)$ with $a \leqslant \frac{d}{r}t$. In case i), since $c < h^1(L)$ then $d < g + r$. From Proposition 2.2.2 $h^0(A) \leqslant t$ and case i) follows from Proposition 2.2.1.
If $h^1(L) = c$ and $t = r - 1$, then $d = g + r$ and

$$t\frac{d}{r} = (r-1)\frac{g+r}{r} = g + r - 1 - \frac{g}{r}.$$

Assume that r does not divide g then the condition $a \leqslant (r-1)\frac{d}{r}$ implies $a < (r-1)\frac{d}{r}$ and hence $h^0(A) \leqslant t = r - 1$. This proves ii). ■

Next result states that the slope-stability of $M_{V,L}$ is equivalent to its cohomological stability. To achieve this, we generalize the results presented in [12] to the setting of the sheaf $M_{V,L}$. We show that the conditions under which $M_{V,L}$ is slope-stable imply its cohomological stability, and viceversa. This equivalence allows us to leverage the powerful tools of Brill-Noether theory and the study of Brill-Noether varieties to draw conclusions about the stability properties of $M_{V,L}$.

Theorem 2.2.1. *Let (L, V) be a generated $\mathfrak{g}^r_d$ over a general curve C which induces a birational morphism with $c \leqslant h^1(L)$. Then*

1. *$M_{V,L}$ is strictly slope-semistable if and only if the following three conditions hold:*

   (a) $h^1(L) = c$.

(b) $d = g + r$ *and* $r|g$.

(c) There is a line bundle A *with degree* $deg(A) = g + r - 1 - \frac{g}{r}$ *and* $h^0(A) = r$ *such that* $h^0(\wedge^{r-1} M_{V,L} \otimes A) = 1$.

2. If $M_{V,L}$ *is slope-stable, then* $M_{V,L}$ *is cohomologically stable.*

Proof. The proof is the same as in [12, Corollary 3.10]. ■

It follows from Proposition 2.1.2 and Theorem 2.2.1 that for general curves satisfying assumptions of Theorem 2.2.1, the linear stability of the pair (L, V) is equivalent to the slope-stability of $M_{V,L}$, and this slope-stability is equivalent to the cohomological stability of $M_{V,L}$.

Chapter 3

Mistretta-Stoppino's conjecture on k-gonal curves

In this chapter, we focus in Brill-Noether theory for general curves inside the strata $\mathcal{M}^1_{g,k}$, that is, curves with non-generic gonality k. In this direction, we review some key results from Ballico-Keem [4], Coppens-Martens [13], and Pflueger [28]. We aim to estimate the dimension of the Brill-Noether varieties $W^r_d(C)$ associated with these k-gonal curves. We introduce the number $\overline{\rho}_k(g, r, d)$ defined by Pflueger, which seeks to extend the classical Brill-Noether number. We also recall an important result of Jensen and Ranganathan, who proved that the dimension of $W^r_d(C)$ indeed coincides with $\overline{\rho}_k(g, r, d)$ for general k-gonal curves. This solves a conjecture proposed by Pflueger.
The goal of this chapter is to address the Mistretta- Stoppino conjecture for general k-gonal curves using techniques as above.

3.1 Brill-Noether theory for gonal curves

In this section, we use a version of Theorem 1.1.1 for k-gonal general curves for non-generic values of the gonality k. Let C be a curve of gonality k. For each possible value of k, when $d \geqslant kr$, we can always construct a $\mathfrak{g}^r_d$ by adding $d - kr$ base point to $r \cdot \mathfrak{g}^1_k$. From Riemann-Roch we can estimate the dimension of $W^r_d = \{L \in Pic^d(C) : h^0(L) \geqslant r + 1\}$ as follows,

$$\begin{aligned}\dim W^r_d(C) &\geqslant \max\{d - kr, (2g - 2 - d) - k(g - d + r - 1)\} \\ &= \rho(g, r - r', d) - r'k\end{aligned}$$

where $r' = \min\{r, g - d + r - 1\}$.
In [4], Ballico and Keem, show that under the assumption $g \leqslant 4k - 4$ the dimension $\dim W^r_d(C)$ can exceed $\rho(g, r, d)$ by at most $g - 2k + 2$. In [13], Coppens and Martens exhibit components of $W^r_d(C)$ of dimension $\rho(g, r - l, d) - lk$ for $l \in \{0, 1, r'\}$. Moreover, in [14], they expand the results for the case where $r' + 1 - l$ divides either r' or $r' + 1$ and is smaller than k. This result, with the upper bound in [28] determines the dimension of $W^r_d(C)$ for general trigonal and tetragonal curves, they also define a number namely $\overline{\rho}_k(g, r, d)$ to generalize the notion of the Brill-Noether number for k-gonal curves.

Definition 3.1.1. *Fix an integer $k \geqslant 2$ and integers $d, r \geqslant 1, g > 2$ with $d \leqslant g + r$. Let $r' = \min\{r, g - d + r - 1\}$. Define*

$$\overline{\rho}_k(g, r, d) := \max_{l \in \{0, \dots, r'\}} \{\rho(g, r - l, d) - lk\} \tag{3.1.1}$$

The expresion of r' comes from the fact that $W^r_d(C) \cong W^{g-d+r-1}_{2g-2-d}(C)$. This ensures that definition of $\overline{\rho}_k(g, r, d)$ is invariant under this duality. Pflueger shows in [28] that $\dim W^r_d(C) \leqslant \overline{\rho}_k(g, r, d)$ and conjetures that the equality holds. Later, in [21], Jensen and Ranganathan prove the following theorem that gives an affirmative answer to Pflueger's conjecture.

Theorem 3.1.1 ([21, Theorem 9.4]). *Let C be a general k-gonal curve of genus g. Then*

$$\dim W^r_d(C) = \overline{\rho}_k(g, r, d).$$

For a general k-gonal curve the scheme $W^r_d(C)$ can have irreducible components of different dimensions. For example, if C is a general trigonal curve of genus 6, then $W^1_4(C)$ has a 1-dimensional component and an isolated point, see [22, Lemma 2.1]. Thus, Theorem 3.1.1 says that $W^r_d(C)$ has a component of maximal possible dimension $\overline{\rho}_k(g, r, d)$.

3.2 Quadratic expression associated to $\overline{\rho}_k$

In this section, we present a quadratic expression that allows us to study the dimension of the Brill-Noether varieties associated to k-gonal curves. In Proposition 2.1.2 we consider a subbundle S of $M_{V,L}$ with $rank(S) = s$ and we use this bundle to exhibit a linear series in $G^s_{deg(S^\vee)}(C)$ to make sure that the Brill-Noether number $\rho(g, s, deg(S^\vee))$ is non-negative. We are interested in studying the expression inside the definition of $\overline{\rho}_k(g, r, d)$, particularly $\rho(g, r-l, d) - lk$. Let $f_1(l) := \rho(g, r-l, d) - lk$, this is a quadratic function of a real variable l with expression

$$f_1(l) = g - rg + rd - r^2 - g + d - r + (r + g - d + r + 1 - k)l - l^2$$

which reach its maximum at $l_1 = \frac{1}{2}(g - d + 2r + 1 - k)$, say $f_1(l_1) = \left(\frac{d-g+k-1}{2}\right)^2 + d - kr$. Pflueger defines the following Brill-Noether $\overline{\rho}_k(g, r, d)$ with respect the gonality of C and $f_1(l)$:

$$\overline{\rho}_k(g, r, d) = \max_{l \in \{0, \dots, r'\}} f_1(l).$$

Let $d' = deg(S^\vee)$ be and let $f_2(l) := \rho(g, s - l, d') - lk$ the quadratic function which reach its maximum at $l_2 = \frac{1}{2}(g - d' + 2s + 1 - k)$, say $f_2(l_2) = \left(\frac{d'-g+k-1}{2}\right)^2 + d' - ks$. Now we analize both of these quadratic functions under the assumptions that $d \leqslant kr$, $d \leqslant g + r$, $s < r$ and for more specific properties of S. Since we are exhibiting $(L, V) \in G^r_d(C)$ and an element of $G^s_{d'}(C)$, using that C is a k-gonal general curve, we know that $\overline{\rho}_k(g, r, d)$ and $\overline{\rho}_k(g, s, d')$ are both non-negative. Thus, if we call the intervals $I_1 = [0, r']$ and $I_2 = [0, s']$, we can make sure that the sets $f_1(I_1) \cap [0, \infty)$ and $f_2(I_2) \cap [0, \infty)$ are non-empty.

Remark 3.2.1. *With notation as in Definition 3.1.1, let* $s' = \min\{s, g-(d')+s-1\}$*. The condition* $s' = s$ *is equivalent to* $d' \leqslant g-1$*. Similarly,* $s' = g-d'+s-1$ *is equivalent to* $d' \geqslant g-1$*.*

The roots of the quadratic functions $f_i(l)$ are $r_i^{\pm} = l_i \pm \frac{\sqrt{\Delta_i}}{2}$ where Δ_i is the discriminant of f_i, explicitly

$$\Delta_1 = d^2 - 2dg + 2dk + 2d + g^2 - 2gk + 2g + k^2 - 4kr - 2k + 1$$

equivalently $\Delta_1 = (d-g+k-2r-1)^2 + 4d(r+1) - 4r(g+r+1)$ and for $f_2(l)$ we have

$$\Delta_2 = (d')^2 - 2(d')g + 2(d')k + 2(d') + g^2 - 2gk + 2g + k^2 - 4ks - 2k + 1.$$

Remark 3.2.2. *The expression 2.1.1 that we aim to get in Proposition 2.1.2, said* $d'(s+1) \geqslant s(s+g+1)$*, is equivalent using above notation to* $\frac{\sqrt{\Delta_2}}{2} \geqslant l_2$*.*

Notice that the expression $\frac{\sqrt{\Delta_2}}{2} \geqslant l_2$ at the same time is equivalent to

$$((d') - g + k - 2s - 1)^2 + 4(d')(s+1) - 4s(g+s+1) \geqslant (g - (d') + 2s + 1 - k)^2.$$

Solving for d' and sustracting $((d') - g + k - 2s - 1)^2$ at both sides,

$$4(d')(s+1) - 4s(g+s+1) \geqslant 0,$$

which is equivalent to the condition in Proposition 2.1.2.

Now, we want to study the possible values of s and d' for which the condition in Remark 3.2.2 holds. We focus on the numerical conditions pertaining to s and d' without considering their potential relations with the invariants r and d of the linear series (L, V). For this, we consider the following cases for l_2, only in the interval $l_2 \leqslant s' = \min\{s, g-d'+s-1\}$:

Proposition 3.2.1. *Let C be a general k-gonal curve of genus g, if the variety $G^s_{d'}(C)$ is non empty and the pair (d', s) satisfies one of the following conditions:*

1. *If $g+1-k+2s \leqslant d'$ and satisfies one of the following conditions:*
   (a) *$k \leqslant 6$.*
   (b) *$k > 6$ and $g \geqslant \frac{k^2}{4}$.*
   (c) *$k > 6$ and $2s \geqslant \sqrt{k^2-4g} + k - 2$.*
2. (a) *If $d' \leqslant g-1$ and satisfies all the following conditions:*
      i. *$s \leqslant \frac{g-k+1}{k-1}$.*
      ii. *$d' \leqslant g+2s+1-k$.*
      iii. *$d' \geqslant g+s+1-k$.*
   (b) *Or if $d' \geqslant g-1$ and satisfies all the following conditions:*
      i. *$s+2 \leqslant k$.*
      ii. *$d' \leqslant g+2s+1-k$.*
      iii. *$s \leqslant \sqrt{4g+5k^2-4k} - 2k$.*

3. (a) *If $d' \leqslant g-1$ and all the following conditions hold:*
   i. $g+1-k \leqslant d'$.
   ii. $d'-s \leqslant g+1-k$.
   iii. $s \leqslant \frac{1}{2}\sqrt{4g+(k-2)^2}-k$.

   (b) *Or if $d' \geqslant g-1$ and all the following conditions hold:*
   i. $k-2 \leqslant s$.
   ii. $d' \leqslant g-3+k$.
   iii. $2s \leqslant \sqrt{4g+2k^2-4k}-k$.

Then $\rho(g,s,d') \geqslant 0$.

Proof. Since C is a general k-gonal curve and there exists an element in the variety $G^s_{d'}(C)$, it follows from Theorem 3.1.1 that $\overline{\rho}_k(g,s,d') = dimW^s_{d'} = dimG^s_{d'}$ is non-negative. Note that condition $\rho(g,s,d') \geqslant 0$ is equivalent to the expression in Remark 3.2.2. Now we study the possible values for l_2:

Case 1 Consider $l_2 \leqslant 0$, which implies $g-d'+2s+1-k \leqslant 0$ or equivalent

$$g+1-k+2s \leqslant d'.$$

Since $f_2(I_2) \cap [0,\infty) \neq \emptyset$, we have that $\rho(g,s,d') = f_2(0) \geqslant 0$. Given that $\rho(g,s)(d')$ is increasing for $d' \geqslant -1$ and that $g+2-k+2s$ is greater than zero, it follows that

$$\rho(g,s)|_{d'=d'} \geqslant \rho(g,s)|_{d'=g+1-k+2s}$$

The expression on the right side is greater or equal to zero for either $k \leqslant 6$ or $k > 6$ or for $g \geqslant \frac{k^2}{4}$; another case for $k > 6$ is $2s \geqslant \sqrt{k^2-4g}+k-2$ and $4g < k^2$

In this case, conditions that must be satisfied are $g+1-k+2s \leqslant d'$ and one of the following:

$$k \leqslant 6 \tag{3.2.1}$$

$$k > 6 \text{ and } g \geqslant \frac{k^2}{4} \tag{3.2.2}$$

$$k > 6 \text{ and } 2s \geqslant \sqrt{k^2-4g}+k-2. \tag{3.2.3}$$

Case 2 Now consider $0 \leqslant l_2 \leqslant \frac{1}{2}s'$:

1. For $s' = s$, the condition in this case is equivalent to

$$0 \leqslant g-d'+2s+1-k \leqslant s$$

which leads to $g+s+1-k \leqslant d'$. The condition in Remark 3.2.2 holds if $\sqrt{\Delta_2} \geqslant s$, and considering the bound $g+s+1-k \leqslant d'$ in the expression of Δ_2 (noting that $\Delta_2(d')$ is increasing for $d' \geqslant g-k-1$), we find that $\sqrt{\Delta_2} \geqslant s$ is satisfied for s such that

$$s \leqslant \frac{g-k+1}{k-1}.$$

Thus, if $s' = s$ the conditions that must be satisfied are

$$s \leqslant \frac{g-k+1}{k-1} \tag{3.2.4}$$
$$d' \leqslant g + 2s + 1 - k \tag{3.2.5}$$
$$d' \geqslant g + s + 1 - k \tag{3.2.6}$$

2. When $s' = g - d' + s - 1$, the condition is equivalent to the following inequalities
$$0 \leqslant g - d' + 2s + 1 - k \leqslant g - d' + s - 1.$$
From the right side we get $g - d' + 2s + 1 - k \leqslant g - d' + s - 1$ or equivalently, $s + 2 \leqslant k$. From the left side, $0 \leqslant g - d' + 2s + 1 - k$ or equivalently, $d' \leqslant g + 2s + 1 - k$. The condition in Remark 3.2.2 holds if $\Delta_2 - (g - d' + s - 1)^2 \geqslant 0$, which is equivalent to
$$d'(2 + 2s) - 2gk - 2gs + 4g + k^2 - 4ks - 2k - s^2 + 2s \geqslant 0$$
Notice that this expression, as a function in the variable d is increasing, since $d' \geqslant g - 1$, we can use
$$(\Delta_2 - (g - d' + s - 1)^2)|_{d'=d'} \geqslant (\Delta_2 - (g - d' + s - 1)^2)|_{d'=g-1}.$$
The right side of this inequality evaluates to $4g + k^2 - 4ks - 4k - s^2$ and is greater than zero for $s \leqslant \sqrt{4g + 5k^2 - 4k} - 2k$.
Thus, if $s' = g - d' + s - 1$, the conditions that must to be satisfied are:
$$s + 2 \leqslant k \tag{3.2.7}$$
$$d' \leqslant g + 2s + 1 - k \tag{3.2.8}$$
$$s \leqslant \sqrt{4g + 5k^2 - 4k} - 2k. \tag{3.2.9}$$

Case 3 In this case, we consider that l_2 satisfies $\frac{1}{2}s' \leqslant l_2 \leqslant s'$. For each possible value of s', the conditions for $\frac{1}{2}\sqrt{\Delta_2} \geqslant l_2$ are as follows:

1. When $s' = s$. The bounds for l_2 lead to the following two inequalities
$$s \leqslant g - d' + 2s + 1 - k$$
$$2s \geqslant g - d' + 2s + 1 - k.$$
Simplifying,
$$g + 1 - k \leqslant d'$$
$$d' - s \leqslant g + 1 - k.$$
The condition in Remark 3.2.2 holds if $\sqrt{\Delta_2} \geqslant 2s$, which is equivalent to $\Delta_2 - 4s^2 \geqslant 0$. This expression, as a function of the variable d' is increasing for $d' \geqslant g - k - 1$, and since $d' \geqslant g - k + 1$ satisfies this condition, it follows that
$$4g - 4ks - 4k - 4s^2 + 4 = (\Delta_2 - 4s^2)|_{d'=g-k+1} \leqslant (\Delta_2 - 4s^2)|_{d'=d'}$$

The left side of this inequality is greater than zero if $s \leqslant \frac{1}{2}\sqrt{4g+(k-2)^2} - k$. Thus, for $s' = s$ and $\frac{1}{2}s' \leqslant l_2 \leqslant s'$ the conditions that must be satisfied for s and d' to ensure $\sqrt{\Delta_2} \geqslant 2l_2$ are:

$$g + 1 - k \leqslant d' \tag{3.2.10}$$

$$d' - s \leqslant g + 1 - k \tag{3.2.11}$$

$$s \leqslant \frac{1}{2}\sqrt{4g + (k-2)^2} - k. \tag{3.2.12}$$

2. When $s' = g - d' + s - 1$. The bounds for l_2 can be expressed as the following two inequalities:

$$g - d' + s - 1 \leqslant g - d' + 2s + 1 - k \leqslant 2(g - d' + s - 1).$$

From the left side inequality, $k - 2 \leqslant s$. Similarly, the right side simplifies to $d' \leqslant g - 3 + k$. Thus, we have the following inequalities for s and d'

$$k \leqslant s + 2$$
$$d' \leqslant g - 3 + k.$$

To establish conditions under which Remark 3.2.2 holds, we note that it holds if $\Delta_2 - 4(s')^2 \geqslant 0$, which is equivalent to

$$d'(2 + 2s) - 2gk - 2gs + 4g + k^2 - 4ks - 2k - s^2 + 2s \geqslant 0.$$

This expression, as a function of the variable d' is increasing for $3g + k + 3s - 3 \geqslant 3d'$ and since $d' = g - 1$ satisfies this condition, then

$$(\Delta_2 - 4(s')^2)|_{d'=d'} \geqslant (\Delta_2 - 4(s')^2)|_{d'=g-1} = 4g + k^2 - 4ks - 4k - 4s^2.$$

In this situation the expression on the right side is greater than or equal to zero for $s \leqslant \frac{1}{2}\sqrt{4g + 2k^2 - 4k} - \frac{k}{2}$. Therefore, for $s' = g - d' + s - 1$ and $\frac{1}{2}s' \leqslant l_2 \leqslant s'$, the conditions that must be satisfied for s and d' to ensure $\sqrt{\Delta_2} \geqslant 2l_2$ are:

$$k - 2 \leqslant s \tag{3.2.13}$$

$$d' \leqslant g - 3 + k \tag{3.2.14}$$

$$s \leqslant \frac{1}{2}\sqrt{4g + 2k^2 - 4k} - \frac{k}{2}. \tag{3.2.15}$$

■

Now that we have established conditions on the pair (s, d') that ensure the non-negativity of the Brill-Noether number $\rho(g, s, d')$, the next step is to understand how these conditions constrain the values of the projective dimension r and degree d of the linear series (L, V). By deriving these restrictions, we will be able to leverage the stability results obtained previously and apply them to a broader class of linear series on k-gonal curves.

3.2.1 Extremal cases

In this section, we study the extremal cases of the quadratic expression. We aim to relate the conditions on (d') and s as the above with conditions on d and r, on this direction, we aim to compare this conditions with the worst cases for (d') and s subject to the properties of $S \hookrightarrow M_{V,L}$ and the Butler's diagram of (L, V) by S.

Since we know that $S \hookrightarrow M_{V,L}$, we get that $s \leqslant r - 1$. For the degree of S the worst case is considering it a maximal destabilizing subbundle of $M_{V,L}$, for which we have $(d') \leqslant d\frac{s}{r}$. In this way, for condition 1. in Proposition 3.2.1 we have the following inequalities

$$\begin{aligned} g + 1 - k &\leqslant d' - 2s \\ &\leqslant d\frac{s}{r} - 2s \\ &\leqslant s\left(\frac{d}{r} - 2\right) \\ &\leqslant (r-1)\left(\frac{d}{r} - 2\right) \\ &= \frac{r-1}{r}d - 2(r-1). \end{aligned}$$

conditions $k \leqslant 6$, $k > 6$ and $g \geqslant \frac{k^2}{4}$ stayed as before. For this case last condition can be expressed as $k > 6$ and the inequalities $2(r-1) \geqslant 2s \geqslant \sqrt{k^2 - 4g} + k - 2$.

In a similar way, for condition 2. in Proposition 3.2.1, it follow that

a) If $\frac{r-1}{r}d \leqslant g - 1$ and satisfies all of the following conditions:

   i. $r - 1 \leqslant \frac{g-k+1}{k-1}$.

   ii. $\frac{r-1}{r}d \leqslant g + 2(r-1) + 1 - k$.

   iii. $\frac{r-1}{r}d \geqslant g + (r-1) + 1 - k$.

b) Or if $\frac{r-1}{r}d \geqslant g - 1$ and satisfies all of the following conditions:

   i. $(r-1) + 2 \leqslant k$.

   ii. $\frac{r-1}{r}d \leqslant g + 2(r-1) + 1 - k$.

   iii. $(r-1) \leqslant \sqrt{4g + 5k^2 - 4k} - 2k$.

Furthermore, for condition 3. of Proposition 3.2.1, we estimate the conditions using $\Delta_2 - 4(s')^2$, notice that this is not the best estimation since $d - kr \leqslant 0$ and $d - kr$ is the value of $f_1(r')$. At this point, we only consider the first two conditions to estimate the values of r and d.

We summarize the work in this section in thefollowing result that extends Proposition 2.1.2:

Proposition 3.2.2. *Let C a general k-gonal curve of genus $g > 2$, with k non-generic gonality, and (L, V) be a generated linear series of type $(d, r+1)$ over C, let $c := codim_{H^0(L)}(V)$ with $d \leqslant \min\{g + r, kr\}$. If the pair (r, d) satisfies at least one of the following conditions:*

1. *If* $g+1-k+2(r-1) \leqslant \frac{r-1}{r}d$ *and satisfies one of the following conditions:*
   (a) $k \leqslant 6$.
   (b) $k > 6$ *and* $g \geqslant \frac{k^2}{4}$.
   (c) $k > 6$ *and* $2(r-1) \geqslant \sqrt{k^2-4g}+k-2$.
2. (a) *If* $\frac{r-1}{r}d \leqslant g-1$ *and satisfies all the following conditions:*
      i. $r-1 \leqslant \frac{g-k+1}{k-1}$.
      ii. $\frac{r-1}{r}d \leqslant g+2(r-1)+1-k$.
      iii. $\frac{r-1}{r}d \geqslant g+(r-1)+1-k$.
   (b) *Or if* $\frac{r-1}{r}d \geqslant g-1$ *and satisfies all the following conditions:*
      i. $(r-1)+2 \leqslant k$.
      ii. $\frac{r-1}{r}d \leqslant g+2(r-1)+1-k$.
      iii. $(r-1) \leqslant \sqrt{4g+5k^2-4k}-2k$.

Then $M_{V,L}$ *is semistable. Moreover, if there exists a proper subbundle* $S \subsetneq M_{V,L}$ *with* $\mu(S) = \mu(M_{V,L})$*, then*

- $h^1(L) - c = 0$.
- $s := rank(S) = r-1$.
- $d = g + r$ *with* $r|g$.

Proof. In notation as above, following the proof of Proposition 2.1.2, consider the line bundle $det(S^\vee)$, for which we have $h^0(det(S^\vee)) \geqslant s+1$; thus, it follows that $\overline{\rho}_k(g,s,deg(S^\vee)) = \max_{l_2\in\{0,\dots,s\}} f_2(l_2) \geqslant 0$. If (g,k,r,d) satisfies condition *1*, then $l_2 \leqslant 0$; furthermore, since $f_2(0) \geqslant 0$, we conclude that $f_2(0) = \rho(g,s,deg(S^\vee))$.
If the values of (g,k,r,d) satisfy condition *2*, then, following the computations above, we know from Proposition 3.2.1 that these inequalities imply $\frac{\sqrt{\Delta_2}}{2} \geqslant l_2$. As noted in Remark 3.2.2, this is equivalent to condition $deg(S^\vee)(s+1) \geqslant s(s+g+1)$. In either case, we have established that $deg(S^\vee)(s+1) \geqslant s(g+s+1)$ and we can proceed with the proof of Proposition 2.1.2 to reach our conclusion. ■

For the complete case we can extend Proposition 3.2.2 following [11, Lemma 4.1] instead of Proposition 2.1.2, in which case we get:

Corollary 3.2.1. *Let* C *a general* k*-gonal curve of genus* $g > 2$*, with* k *non-generic gonality, and* $L \in Pic^d(C)$ *be a generated line bundle over* C *with* $h^0(L) = r+1$*,* $d \leqslant g+r$ *and* $d \leqslant kr$*. If the pair* (r,d) *satisfies one of the following conditions:*

1. *If* $g+1-k+2(r-1) \leqslant \frac{r-1}{r}d$ *and it satisfies one of the following conditions:*
   (a) $k \leqslant 6$.
   (b) $k > 6$ *and* $g \geqslant \frac{k^2}{4}$.
   (c) $k > 6$ *and* $2(r-1) \geqslant \sqrt{k^2-4g}+k-2$.
2. (a) *If* $\frac{r-1}{r}d \leqslant g-1$ *and satisfies all the following conditions:*

i. $r-1 \leqslant \frac{g-k+1}{k-1}$.

ii. $\frac{r-1}{r}d \leqslant g+2(r-1)+1-k$.

iii. $\frac{r-1}{r}d \geqslant g+(r-1)+1-k$.

(b) If $\frac{r-1}{r}d \geqslant g-1$ *and satisfies all the following conditions:*

i. $(r-1)+2 \leqslant k$.

ii. $\frac{r-1}{r}d \leqslant g+2(r-1)+1-k$.

iii. $(r-1) \leqslant \sqrt{4g+5k^2-4k}-2k$.

Then M_L *is semistable. Moreover, if there exists a proper subbundle* $S \subsetneq M_L$ *with* $\mu(S)=\mu(M_L)$, *then*

- $h^1(L)=0$.
- $s := rank(S) = r-1$.
- $d = g+r$ *with* $r|g$.

Proof. Using the conditions established in *1* and *2*, along with Proposition 3.2.1, we see from Proposition 3.2.2 that in both cases

$$deg(S^{\vee})(s+1) \geqslant s(g+s+1).$$

Thus, we can follow the proof of [11, Lemma 4.1] to reach our conclusion. ■

Now we aim to compare the linear stability of the linear series (L,V) (possibly complete) with the slope-stability of $M_{V,L}$.

Proposition 3.2.3. *Let* C *be a general* k*-gonal curve of genus* $g>2$*, with* k *non-generic gonality, and let* (L,V) *be a generated linear series of type* $(d,r+1)$ *over* C*. Suppose that* $d \leqslant g+r$ *and* $d \leqslant kr$*. If* (r,d) *satisfies one of the conditions of Proposition 3.2.2 then the linear (semi)stability of* (L,V) *is equivalent to the slope-(semi)stability of* $M_{V,L}$.

Proof. Let $S \subsetneq M_{V,L}$ be a proper subbundle with $\mu(S)=\mu(M_{V,L})$, by Corollary 3.2.1 and Proposition 3.2.2 the bundle S is stable (for complete and non-complete case, respectively). Consider the Butler's diagram of (L,V) by S as in 1.1.1

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r] \arrow[d] & W \otimes \mathcal{O}_C \arrow[r] \arrow[d] & F_S \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & M_{V,L} \arrow[r] & V \otimes \mathcal{O}_C \arrow[r] & L \arrow[r] & 0,
\end{tikzcd}$$

from Proposition 2.1.1, we get that $W = H^0(F_S)$. Again, from Corollary 3.2.1 and Proposition 3.2.2 follows that $h = h^1(L) = c$ and $s = r-1$. Since $W \subseteq V$ (with V possibly equal to $H^0(L)$) and the dimension of W is greater than s, it follows that $rank(F_S)=1$ or $rank(F_S)=2$. Using that C is a k-gonal general curve and r,d satisfies at least one of the conditions of Corollary 3.2.1 (for complete case) or Proposition 3.2.2 (for non-complete case), Proposition 2.1.3 holds. Then $h^0(det(F_S)) = r$ and from the proof of Theorem 2.1.1, we conclude that F_S is a line bundle, and the Butler's diagram of (L, V) by S is

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r] \arrow[d] & H^0(F) \otimes \mathcal{O}_C \arrow[r] \arrow[d] & F_S \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & M_{V,L} \arrow[r] & V \otimes \mathcal{O}_C \arrow[r] & L \arrow[r] & 0.
\end{tikzcd}$$

Now, in order to prove the main result of the proposition, recall that if $M_{V,L}$ is stable then (L, V) is linearly stable. Suppose that $M_{V,L}$ is strictly semistable and $S \subseteq M_{V,L}$ is a subbundle with $\mu(S) = \mu(M_{V,L})$. By the above computations, F_S is of rank one and this implies that (L, V) is strictly linear semistable. ■

We provide concrete examples to exhibit the non-emptiness of the numerical bounds on the rank r and degree d of the linear series (L, V) in Proposition 3.2.2. We consider specific values for the genus g and gonality k that satisfy the hypotheses of cases (1.) and (2.) in the proposition. These examples will illustrate how the restrictions on (r, d) ensure the existence of linear series (L, V) that are linear semistable, with their associated syzygy bundles $M_{V,L}$ being slope semistable on general k-gonal curves.

Example 3.2.1. *Consider the moduli space of smooth projective curves of genus $g = 15$. Within this moduli space, we can examine non-generic values of k and construct pairs of (r, d) that satisfies each condition of the Proposition 3.2.2.*

- *A general curve C in $\mathcal{M}^1_{15,8}$ does admit an element $(L, V) \in G^5_{20}(C)$ for which the associated syzygy bundle $M_{V,L}$ is semistable. Specifically, the parameters $g = 15$, $k = 8$, $r = 5$ and $d = 20$ satisfy the condition 1. in Proposition 3.2.2:*

$$15 + 1 - 8 = 8 \leqslant \frac{5-1}{5}20 - 2(5-1) = 8.$$

  and since $k > 6$ and $2(5-1) = 8 \geqslant \sqrt{64-60} + 8 - 2 = 8$.

- *A general curve C in $\mathcal{M}^1_{15,6}$ admits an element $(L, V) \in G^3_{18}(C)$ for which the syzygy bundle $M_{V,L}$ is semistable. Here, the parameters $g = 15$, $k = 6$, $r = 3$ and $d = 18$ satisfy the condition 2. in Proposition 3.2.2, note that $\frac{3-1}{3}18 = 12 < 14 = 15 - 1$ and we can verify the following inequality:*

$$15 + 2 + 1 - 6 = 12 \leqslant \frac{3-1}{3}18 = 12 \leqslant 15 + 2(2) + 1 - 6 = 14.$$

The existence of values (g, k, r, d) that satisfy each of the specified conditions in Proposition 3.2.2 can be examined by using computational tools. To facilitate this analysis, Appendix A provides resources and examples that enhance the discussion in this chapter.

Chapter 4

Curves in K3 surfaces

In this chapter, we consider a pair (X, H) where X is a smooth K3 surface over $\mathbb{C}$ and H is an hyperplane section of X such that

$$(**) \ \ H^2 \text{ divides } H.D \text{ for all curve classes } D \text{ on } X$$

An example of such pairs is the case when H is an hyperplane section of X such that $Pic(X) \cong \mathbb{Z}.H$. For a more detailed discussion of the stability conditions in this chapter, please see Appendix B. Throughout this chapter, we will reference the necessary theory to clarify the notation.

Let $d, g \in \mathbb{Z}$ with $g > 1$. The moduli space of H-Gieseker stable sheaves with Mukai vector $v = (0, H, d+1-g)$ is $M_H(v)$ (see B.0.1), the moduli space $M_H(v)$ parametrizes one-dimensional sheaves F of Euler characteristic $d + 1 - g$ where the support $|F|$ corresponds to a curve in the linear system $|H|$.

We consider the stability condition $\sigma_{\beta,\alpha}$ associated to $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$. We study the wall-crossing for the moduli space $M_{\sigma_{\beta,\alpha}}(v)$ (see B.0.3), with $v = (0, H, d+1-g)$ as above. From Theorem B.0.4 we have $M_{\sigma_{\beta,\alpha}}(v) = M_H(v)$ for $\alpha >> 0$, and we want to find the walls that bounds this chamber called the Gieseker-chamber, for wall-chamber structure see B.0.7.

Consider $\beta = 0$. In this case $\mathfrak{Im}(Z_{0,\alpha}(\mathcal{O}_X)) = 0$, this means we have stability conditions for

$$\alpha > \alpha_0 = \sqrt{\frac{2}{H^2}},$$

for details see B.0.1. For these stability conditions, notice that $\mathcal{O}_X[1]$ is an object in the category $Coh^0(X)$ with

$$\mathfrak{Im}(Z_{0,\alpha}(\mathcal{O}_X[1])) = 0,$$

i.e. of slope $\mu_0(\mathcal{O}_X[1]) = +\infty$ (see B.0.1), therefore it is automatically semistable. Using Proposition B.0.3 $\mathcal{O}_X[1]$ has no subobjects in $Coh^0(X)$ (see B.0.2), and so $\mathcal{O}_X[1]$ is stable for $\beta = 0$.

Lemma 4.0.1 ([5, Lemma 6.1])**.** *For $\alpha > \alpha_0$ and $\beta = 0$, we have an isomorphism $M_{\sigma_{0,\alpha}}(v) = M_H(v)$ identifying the stable objects with stable sheaves.*

In other words, there is no wall intersecting the line segment $\beta = 0, \alpha = \left(\frac{2}{H^2}, \infty\right)$. The interplay between Brill-Noether theory and wall-crossing theory is fundamental in this section. While Brill-Noether theory allows us to study properties of the Lazarsfeld-Mukai sheaves $F_{V,L}$ associated to a linear series (L, V), wall-crossing theory provides the tools to analyze the stability of these sheaves when restricted to curves living on K3 surfaces. The conditions presented in Lemma 4.0.2 complement the result presented in Lemma 4.0.1.

Lemma 4.0.2 ([5, Lemma 6.2])**.** *There is a wall bounding the Gieseker-chamber where $Z_{\beta,\alpha}(\mathcal{O}_X)$ aligns with $Z_{\beta,\alpha}(v)$. The sheaves $L \in M_{\sigma_{\beta,\alpha}}(v)$ getting destabilised are exactly those with $h^0(L) > 0$, and the destabilising short exact sequence are given by*

$$H^0(L) \otimes \mathcal{O}_X \hookrightarrow L \twoheadrightarrow W \tag{4.0.1}$$

for some object W that remains stable at the wall.

Proof. The locus where the central charges of all objects in 4.0.1 are aligned is the line segment between v and $v(\mathcal{O}_X)$. In Figure 4.1, we observe an arc of a circle ending at $(0, \alpha_0)$. Next, consider the path in the upper half-plane in Figure 4.1, which begins at $\beta = 0$ and $\alpha >> 0$, proceeds straight to the point $(0, \alpha_0 + \varepsilon)$ a bit above $(0, \alpha_0)$, and then turns left until it hits the above mentioned semicircle. The visualization of walls as lines clearly indicates that if this path were to encounter any other wall beforehand to reach the semicircle, that wall would also intersects the straight line segment defined by $\beta = 0$ and $\alpha \in (\alpha_0, \infty)$, contradicting Lemma 4.0.1. Additionally, along this path, the sheaf $\mathcal{O}_X$ cannot be destabilized: for values of (β, α) close to $(0, \alpha_0)$, we get that $|Z_{\beta,\alpha}(\mathcal{O}_X)| << 1$, making it the only stable object with that property.

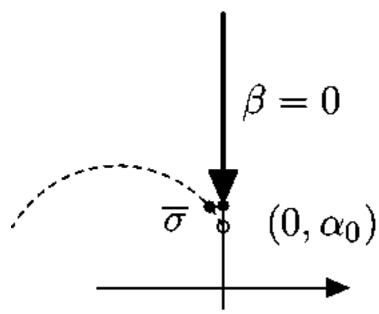

Figure 4.1: Path to construct $\overline{\sigma}$. [5, Figure 2]

Let $\overline{\sigma} = (Coh^{\overline{\beta}}(X), \overline{Z})$ be the stability condition at the wall. In the abelian category of $\hat{\sigma}$-semistable objects with central charge aligned with $\overline{Z}(v)$, the object $\mathcal{O}_X$ is a simple object, hence the natural map $H^0(L) \otimes \mathcal{O}_X \to L$ must necessarily be an injective map, and the quotient W must be semistable. It remains to show that W is stable. Notice that $Hom(W, \mathcal{O}_X) = 0$ as W is a quotient of L. Moreover, $Hom(\mathcal{O}_X, W) = 0$ follows by applying $Hom(\mathcal{O}_X, \cdot)$ to the short exact sequence

$$H^0(L) \otimes \mathcal{O}_X \hookrightarrow L \twoheadrightarrow W.$$

Hence stability of the object W follows from the next lemma. ■

Lemma 4.0.3. *Let $\overline{\sigma}$ be a stability condition on the wall constructed above. Let W be an object of class $v - tv(\mathcal{O}_X)$ for some $t \in \mathbb{Z}$, and assume that W is $\overline{\sigma}$-semistable. Then, W is stable if and only if $Hom(\mathcal{O}_X, W) = 0 = Hom(W, \mathcal{O}_X)$.*

When L is globally generated, the object W in 4.0.1 is the shift $F_L[1]$ of the kernel bundle F_L of the evaluation map $H^0(L) \otimes \mathcal{O}_X \xrightarrow{ev} L$, called the Lazarsfeld-Mukai bundle. This result establishes the stability of the Lazarsfeld-Mukai bundle F_L associated to a complete linear series $(L, H^0(L))$ on a curve $C \in |H|$. For a non-complete generated linear series (L, V) on $C \in |H|$, we can construct the Lazarsfeld-Mukai bundle associated to the evaluation map $V \otimes \mathcal{O}_X \xrightarrow{ev} L$, and we have the the following,

Corollary 4.0.1. *Suppose $\beta < 0$. If L is generated by $V \in Gr(r+1, H^0(L))$ then 4.0.1 have the form*

$$V \otimes \mathcal{O}_X \hookrightarrow L \twoheadrightarrow F_{V,L}[1]$$

with $F_{V,L}[1]$ strictly-semi-stable on the wall and stable on the side of the wall where L is stable.

Proof. First, notice that $\mathcal{O}_X \otimes H^0(L) \hookrightarrow L \twoheadrightarrow W$. If L is globally generated then the evaluation map ev is surjective as a map in $Coh(X)$ leading to the inclusion $\mathcal{O}_X \otimes H^0(L) \hookrightarrow L \twoheadrightarrow W$, where $W = F_L[1]$ and $W_1 = F_{V,L}[1]$. Let $\iota : V \to H^0(L)$ be the inclusion, from Lemma 4.0.3 $Hom(W, \mathcal{O}_X) = 0$ since W_1 and W are quotient of L, which implies $Hom(W, \mathcal{O}_X) = Hom(W_1, \mathcal{O}_X) = 0$ as well. Consider the diagram

$$\begin{tikzcd}
V \otimes \mathcal{O}_X \arrow[r] & L \arrow[r] & W_1 \\
H^0(L) \otimes \mathcal{O}_X \arrow[r] \arrow[u, "\iota\otimes Id"] & L \arrow[r] \arrow[u, "Id"] & W \arrow[u]
\end{tikzcd}$$

By applying $Hom(\mathcal{O}_X, \cdot)$ we get

$$\begin{tikzcd}
Hom(\mathcal{O}_X, V \otimes \mathcal{O}_X) \arrow[d, "Hom(\mathcal{O}_X,\iota)"] & Hom(\mathcal{O}_X, L) \arrow[l] \arrow[d, "Hom(\mathcal{O}_X,Id)"] & Hom(\mathcal{O}_X, W_1) \arrow[l] \arrow[d] \\
Hom(\mathcal{O}_X, H^0(L) \otimes \mathcal{O}_X) & Hom(\mathcal{O}_X, L) \arrow[l] & Hom(\mathcal{O}_X, W) \arrow[l]
\end{tikzcd}$$

Let us denote the dimension of $dim(Hom(\cdot))$ by $hom(\cdot)$. By Theorem 4.0.2, we have $Hom(\mathcal{O}_X, W) = 0$ and $hom(\mathcal{O}_X, H^0(L) \otimes \mathcal{O}_X) = h^0(L) = hom(\mathcal{O}_X, L)$. Therefore,

$$hom(O_X, W_1) = h^0(L) - dim(V) = codim_{H^0(L)}(V).$$

By Lemma 4.0.3, it follows that W_1 is strictly semistable and it is stable at the same side where L is stable. ■

Now that we have established the basic results for studying moduli spaces of stable objects with respect to a given stability condition, we aim to take advantage of these tools to relate the stability of vector bundles to the stability of their restrictions to divisors. By employing wall-crossing techniques, we are able to construct precise connections between the (semi)stability of a vector bundle on a surface X and the (semi)stability of its restriction to a curve C lying on X. This approach allows us to relate the stability of linear series on curves to the stability of the associated syzygy bundles, which is the main objective of this thesis.

4.1 Cohomological conditions for the stability of $M_{V,L}$ and restriction theorem

In this section, we explore the relation among the stability of vector bundles on a K3 surface X and the stability of their restrictions to curves C inside X. Specifically, we focus on Lazarsfeld-Mukai bundles $F_{V,L}$ associated to a generated linear series (L, V) on C. By leveraging the results on stability conditions for sheaves on K3 surfaces developed in the previous sections, we establish conditions under which the restriction of $F_{V,L}$ to C remains stable.

We recall results of stability for restrictions from the literature, such as those appearing in [18]. These results provide sufficient conditions to ensure that the restriction of a stable bundle remains stable when restricted to a curve. By adapting and extending these conditions to the specific setting of Lazarsfeld-Mukai bundles, we are able to relate their stability to the stability of the associated syzygy bundles $M_{V,L}$ on the curve C. The insights gained from this analysis are crucial to state the main results in this thesis, which aim to connect the stability of linear series to the stability of syzygy bundles. By understanding the interplay between the geometry of K3 surfaces and the stability of restrictions, we aim to use the stability of Lazarsfeld-Mukai bundles to draw conclusions about the stability of syzygy bundles.

Let X be a smooth complex projective variety of dimension $n \geqslant 2$ with an ample divisor H. For a μ-stable coherent sheaf E of positive rank on X, we have

$$\tilde{\Delta}(E) = \left(\frac{ch_1(E).H^{n-1}}{ch_0(E)H^n}\right)^2 - 2\frac{ch_2(E).H^{n-2}}{ch_0(E)H^n}$$

- $\mu^{max}(E) = max\{\mu(F) : F \text{ is a subsheaf of } E \text{ with } \mu(F) < \mu(E)\}$
- $\mu^{min}(E) = min\{\mu(F') : F' \text{ is a proper quoatient sheaf of } E\}$

and $\delta(E) = min\{\mu^{min}(E) - \mu(E), \mu(E) - \mu^{max}(E)\}$.

Theorem 4.1.1 ([18, Theorem 1.1]). *Let E be a μ-stable reflexive sheaf on X of rank $r > 0$. The restricted sheaf $E|_D$ for any irreducible divisor $D \in |mH|$ is μ-semistable on D if*

$$m \geqslant \frac{r+2}{\sqrt{r+1}}\sqrt{\tilde{\Delta}(E)} \text{ and } \frac{m}{2} + \sqrt{\frac{m^2}{4} - \tilde{\Delta}(E)} \geqslant \frac{\tilde{\Delta}(E)}{\delta(E)}.$$

Moreover, $E|_D$ is μ-stable if the inequalities are both strict.

When $r > 1$, then $\delta(E) \geqslant \frac{1}{H^n r(r-1)}$ and we can restrict the conditions as:

Proposition 4.1.1 ([18, Proposition 4.6]). *Let E be a μ-stable reflexive sheaf as above with rank $r > 1$. The restricted sheaf $E|_D$ for any irreducible divisor $D \in |mH|$ is μ-(semi)stable on D if*

$$m > (\geqslant) r(r-1)\tilde{\Delta}(E) + \frac{1}{r(r-1)}. \tag{4.1.1}$$

For a smooth $K3$ surface X, a smooth curve $C \subseteq X$ and a generated linear series $(L, V) \in G^r_d(C)$, the Lazarsfeld-Mukai bundle is defined via the following elementary modification on X

$$0 \to F_{C,V,L} \to V \otimes \mathcal{O}_S \to L \to 0$$

For short we write $F_{V,L}$ when the context is understood. Now, let $C \in |H|$ be a smooth curve with a $(L, V) \in G^r_d(C)$ a generated linear series $\mathfrak{g}^r_d$ on C such that $d \leqslant g - 1$, and consider the corresponding Lazarsfeld-Mukai bundle $F_{V,L}$ on X. Let's compute the condition 4.1.1 for this case: First observe that $ch(F_{V,L}) = (r + 1, H, g - 1 - d)$, $n = 2$ and

$$\begin{aligned}\tilde{\Delta}(F_{V,L}) = \left(\frac{H.H^1}{r.H^2}\right)^2 - 2\frac{(g-1-d).H^0}{(r+1)H^2} &= \frac{1}{(r+1)^2} - 2\frac{g-1-d}{(r+1)(2g-2)} \\ &= \frac{1}{r+1}\left(\frac{d}{g-1} - \frac{r}{r+1}\right),\end{aligned}$$

by substituting the right side of 4.1.1 for $m = 1$,

$$\begin{aligned}r(r+1)\cdot\frac{1}{r+1}\left(\frac{d}{g-1} - \frac{r}{r+1}\right) + \frac{r}{r+1} &= \frac{rd}{g-1} - \frac{r^2}{r+1} + \frac{1}{r(r+1)} \\ &= \frac{rd}{g-1} + \frac{1-r^3}{r(r+1)}\end{aligned}$$

Since $d \leqslant g - 1$ then $\frac{d}{g-1}r \leqslant r$, then for $r \geqslant 2$,

$$\begin{aligned}\frac{rd}{g-1} + \frac{1-r^3}{r(r+1)} &\leqslant r + \frac{1-r^3}{r(r+1)} \\ &= \frac{r^3+r^2+1-r^3}{r(r+1)} \\ &= \frac{r^2+1}{r^2+r} \\ &< 1.\end{aligned}$$

We conclude that the restriction of the bundle $F_{V,L}$ to the curve C, denoted by $F_{V,L}|_C$, is stable. The following short exact sequence relates $F_{V,L}|_C$ with $M_{V,L}$ (see [1]). We denote $K_C^{-1}L$ for the line bundle $L \otimes K_C^{-1}$:

$$0 \to K_C^{-1}L \to F_{V,L}|_C \to M_{V,L} \to 0$$

Proposition 4.1.2. *We have the following correspondence between the subbundles of $M_{V,L}$ and the subbundles of $F_{V,L}|_C$:*

$$\{S \subseteq M_{V,L}\} \Longleftrightarrow \left\{S' \subseteq F_{V,L}|_C : K_C^{-1}L \subseteq S'\right\}$$

Proof. $(\Rightarrow)$ Taking $S \subseteq M_{V,L}$, consider the following diagram:

$$\begin{tikzcd} & K_C^{-1}L \arrow[d, "="] & & S \arrow[d, "\iota"] & \\ 0 \arrow[r] & K_C^{-1}L \arrow[r] & F_{V,L}|_C \arrow[r] & M_{V,L} \arrow[r] & 0 \end{tikzcd}$$

Where ι is the inclusion, we can complete the diagram by using the pullback of the diagram $F_{V,L}|_C \to M_{V,L} \xleftarrow{\iota} S$ in the abelian category $Coh(C)$ (see [29, Lemma 7.29] for the case of modules, the proof is analogue for abelian categories), called S' and we obtain:

$$\begin{tikzcd}
0 \arrow[r] & K_C^{-1}L \arrow[r] \arrow[d, "="] & S' \arrow[r] \arrow[d] & S \arrow[r] \arrow[d, "\iota"] & 0 \\
0 \arrow[r] & K_C^{-1}L \arrow[r] & F_{V,L}|_C \arrow[r] & M_{V,L} \arrow[r] & 0
\end{tikzcd}$$

by uniqueness of the pullback, the correspondence $S \mapsto (K_C^{-1} \hookrightarrow S')$ is injective.

($\Leftarrow$) Taking $S' \subseteq F_{V,L}|_C$ with $K_C^{-1}L \hookrightarrow S'$, consider S as the quotient S'/K_C^{-1}, since $S' \hookrightarrow F_{V,L}|_C$ then $S \hookrightarrow M_{V,L}$ and the following diagram is commutative

$$\begin{tikzcd}
0 \arrow[r] & K_C^{-1}L \arrow[r] \arrow[d, "="] & S' \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_C^{-1}L \arrow[r] & F_{V,L}|_C \arrow[r] & M_{V,L} \arrow[r] & 0
\end{tikzcd}$$

■

Consider the following diagram under the correspondence above, with the snake's lemma we have that the last row is an isomorphism.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_C^{-1}L \arrow[r] \arrow[d, "="] & S' \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_C^{-1}L \arrow[r] & F_{V,L}|_C \arrow[r] \arrow[d] & M_{V,L} \arrow[r] \arrow[d] & 0 \\
 & & F_{V,L}|_C/S' \arrow[r, "="] \arrow[d] & M_{V,L}/S \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{4.1.2}$$

From now on we denote by Q the quotient $M_{V,L}/S$ or equivalently $F_{V,L}|_C/S'$. Given the relation between the bundles $M_{V,L}$ and $F_{V,L}|_C$, we aim to take advantage of this connection and the fact that the restriction of $F_{V,L}|_C$ is stable to construct conditions under which this stability implies the stability of $M_{V,L}$. We recall a result that connects the stability of a stable vector bundle E with the global sections of the elements that appear in a short exact sequence involving E.

Lemma 4.1.1 ([30, Lemma 1.1])**.** *Let E be a slope-stable vector bundle. Assume that we have an exact sequence*

$$0 \to E' \to E \to E'' \to 0.$$

Then $h^0((E'')^\vee \otimes E') = 0$.

This result allows us to conclude that in the case when $d \leqslant g-1$ then $H^0(Q^\vee \otimes S') = 0$ and $H^0(M_{V,L}^\vee \otimes K_C^{-1}L) = 0$, by considering the exact sequences presented in diagram 4.1.2 and the stability of $F_{V.L}|_C$ in this case.

Lets recall that we consider polarized $K3$ surfaces (X, H) that satisfy the next property:

$$(**) \;\; H^2 \text{ divides } H.D \text{ for all curve classes } D \text{ on } X$$

Theorem 4.1.2. *Let (X, H) be a polarized K3 surface satisfying property $(**)$ and consider $C \in |H|$ a curve of genus $g > 2$, let (L, V) be a generated linear series of type $(d, r+1)$ over C, with $1 < r < d \leqslant \min\{g-1, kr\}$ where k is the gonality of C. If (L, V) is linearly stable then $M_{V,L}$ is slope-stable.*

Proof. Let $F_{V,L}$ the Lazarsfeld-Mukai bundle associated to (L, V). We recall that with our hyphotesis the restriction to C, $F_{V,L}|_C$, is a slope-stable bundle. Suposse that $M_{V,L}$ is not slope-stable and let S be a slope-stable maximal destabilizing subbundle of $M_{V,L}$. We denote by s and d_S the rank and degree of the bundle S, respectively.

Since S is slope-stable with $\mu(S) \geqslant \mu(M_{V,L})$ then $-\frac{d_S}{r_S} \leqslant \frac{d}{r}$, and $d \leqslant g-1$. We can compare the slopes $\mu(F_{V,L}|_C)$ and $\mu(S)$, given that $\mu(F_{V,L}|_C) = -\frac{2g-2}{r+1}$,

$$\begin{aligned}
\mu(F_{V,L}|_C) - \mu(S) &= -\frac{2g-2}{r+1} - \frac{d_S}{r_S} \\
&\leqslant -\frac{2g-2}{r+1} + \frac{d}{r} \\
&\leqslant -\frac{2g-2}{r+1} + \frac{g-1}{r} \\
&= \frac{g-1}{r} - \frac{2}{r+1}(g-1) \\
&= (g-1)\frac{1}{r(r+1)}(r+1-2r) \\
&= \frac{g-1}{r(r+1)}(1-r) \\
&< 0.
\end{aligned}$$

Thus, we conclude that $\mu(F_{V,L}|_C) < \mu(S)$, which implies that $Hom(S, F_{V,L}|_C) = 0$, or equivalently, $H^0(S^\vee \otimes F_{V,L}|_C) = 0$.

Next, using the short exact sequence $0 \to S' \to F_{V,L}|_C \to Q \to 0$ and that $F_{V,L}|_C$ is slope-stable, from Lemma 4.1.1 we obtain $H^0(S' \otimes Q^\vee) = 0$ and $H^0(S' \otimes M_{V,L}^\vee) \hookrightarrow H^0(S' \otimes S^\vee)$, where this space satisfies $H^0(S' \otimes S^\vee) \hookrightarrow H^0(F_{V,L}|_C \otimes S^\vee)$.

We aim to establish conditions under which the space $H^0(S' \otimes M_{V,L}^\vee)$ is non-zero. By Serre duality, $H^0(S' \otimes M_{V,L}^\vee) \cong H^1(M_{V,L} \otimes (S')^\vee \otimes K_C)^\vee$. To compute the dimension of this space we use [1, Theorem 2.28], for which we aim to compute $H^1(M_{V,L} \otimes (S')^\vee \otimes K_C \otimes L)$. Additionally, from Serre duality $H^1(M_{V,L} \otimes (S')^\vee \otimes K_C \otimes L) \cong H^0(M_{V,L}^\vee \otimes S' \otimes L^{-1})^\vee$.

<u>Claim:</u> $H^0(M_{V,L}^\vee \otimes S' \otimes L^{-1}) = 0$.

Proof of Claim: If $H^0(M_{V,L}^\vee S' \otimes L^{-1}) \neq 0$, then $L \hookrightarrow M_{V,L}^\vee \otimes S'$, or equivalently, $r+1 < h^0(L) \leqslant h^0(M_{V,L}^\vee \otimes S')$. Now, consider the short exact sequence obtained by twisting the first exact row of diagram 4.1.2 by $S^\vee$

$$0 \to K_C^{-1}L \otimes S^\vee \to S' \otimes S^\vee \to S \otimes S^\vee \to 0.$$

Since S is slope-stable and $K_C^{-1}L$ is a line bundle, we have that $K_C^{-1}L \otimes S^\vee$ is slope-stable with degree

$$deg(K_C^{-1}L \otimes S^\vee) = r_S(d+2-2g) - d_S = r_S(d+1-g) + r_S(1-g) - d_S < r_S(1-g) - d_S < 0.$$

Therefore, $h^0(K_C^{-1}L \otimes S^\vee) = 0$ and $h^0(S' \otimes S^\vee) \leqslant h^0(S \otimes S^\vee) = 1$ by stability of S.

On the other hand, by dualizing the right exact column of diagram 4.1.2 and twisting by S',

$$0 \to S' \otimes Q^\vee \to S' \otimes M_{V,L}^\vee \to S' \otimes S' \otimes S^\vee \to 0.$$

Since $F_{V,L}|_C$ is slope-stable, from Lemma 4.1.1 we get $h^0(S' \otimes Q^\vee) = 0$ and $h^0(S' \otimes M_{V,L}^\vee) \leqslant h^0(S' \otimes S^\vee) \leqslant 1$, which leads to a contradiction, since we assumed that $h^0(M_{V,L}^\vee \otimes S') > r+1$. Therefore, we conclude that $h^0(M_{V,L}^\vee \otimes S' \otimes L^{-1}) = 0$. $\square$

Since $H^0(M_{V,L}^\vee \otimes S' \otimes L^{-1}) = 0$ and $H^1(M_{V,L} \otimes (S')^\vee \otimes K_C \otimes L) \cong H^0(M_{V,L}^\vee \otimes S' \otimes L^{-1})^\vee$, it follows that $H^1(M_{V,L} \otimes (S')^\vee \otimes K_C \otimes L) = 0$. From [1, Theorem 2.28], we conclude that

$$H^0(S' \otimes M_{V,L}^\vee)^\vee \cong H^1(M_{V,L} \otimes (S')^\vee \otimes K_C) \cong K_{r-1,2}(C, M_{V,L} \otimes (S')^\vee \otimes K_C, L, V).$$

The latter space is isomorphich to (see [1]):

$$coker\left(\bigwedge^r V \otimes H^0(M_{V,L} \otimes (S')^\vee \otimes K_C \otimes L) \to H^0\left(M_{V,L} \otimes (S')^\vee \otimes K_C \otimes \bigwedge^{r-1} M_{V,L} \otimes L^2\right)\right). \tag{4.1.3}$$

To compute the dimension of this vector space, we use the fact that $M_{V,L}^\vee \cong \bigwedge^{r-1} M_{V,L} \otimes L$ and apply Riemann-Roch formula for the bundles $M_{V,L} \otimes (S')^\vee \otimes K_C \otimes L$ and $M_{V,L} \otimes (S')^\vee \otimes K_C \otimes \bigwedge^{r-1} M_{V,L} \otimes L^2$ to compute the Euler characteristic of this bundles, called χ_1 and χ_2 respectively. We get

$$\chi_1 = r(-d - d_S + 2g - 2) + (r_S + 1)(-d) + r(r_S + 1)(d + g - 1)$$
$$\chi_2 = r^2(-d - d_S + 2g - 2) + r^2(r_S + 1)(d + g - 1).$$

To simplify these expressions, we compute the dimensions of the first cohomology space of both vector bundles. Denote by h_1 the dimension

$$dim_{\mathbb{C}} H^1(M_{V,L} \otimes (S')^\vee \otimes K_C \otimes L).$$

From Serre duality, $h_1 = h^0(S' \otimes M_{V,L}^\vee \otimes L^{-1})$ and from our previous claim, we know that $h_1 = 0$ and $\chi_1 = h^0(M_{V,L} \otimes (S')^\vee \otimes K_C \otimes L)$. For the second vector bundle, let h_2 denote the dimension

$$dim_{\mathbb{C}} H^1(M_{V,L} \otimes (S')^\vee \otimes K_C \otimes \bigwedge^{r-1} M_{V,L} \otimes L^2).$$

From Serre duality, $h_2 = h^0(S' \otimes M_{V,L} \otimes M_{V,L}^\vee \otimes L^{-1})$. Consider the second exact row of the Butler's diagram of (L, V) by S, twisting by $S' \otimes M_{V,L}^\vee \otimes L^{-1}$ and taking cohomology, we obtain the exact sequence

$$0 \to H^0(M_{V,L} \otimes S' \otimes M_{V,L}^\vee \otimes L^{-1}) \to V \otimes H^0(S' \otimes M_{V,L}^\vee \otimes L^{-1}) \to \cdots$$

Since the second term of this sequence is zero, it follows that $h^0(S' \otimes M_{V,L} \otimes M_{V,L}^\vee \otimes L^{-1})$ vanishes as well. Thus, we conclude that $h_2 = 0$ and $\chi_2 = h^0(M_{V,L} \otimes (S')^\vee \otimes K_C \otimes \bigwedge^{r-1} M_{V,L} \otimes L^2)$. Now, the vector space in 4.1.3 is not zero if $(r+1)\chi_1 < \chi_2$ and this is equivalent to

$$r(r_S + 1)d + r(d + d_S - (2g-2)) + (r_S + 1)d - r(r_S + 1)(d + g - 1) > 0 \tag{4.1.4}$$

Notice that the expression in the left side of 4.1.4 is greater than

$$r(d + d_S - (2g-2)) + (r_S + 1)d - r(r_S + 1)(d + g - 1). \tag{4.1.5}$$

Now, we analize the last expression. Since $S \subseteq M_{V,L}$ is a slope-stable maximal destabilizing subbundle, then $0 \leqslant d_S + d$, and since we are considering $0 < r < d$, we have that expression 4.1.5 is greater than

$$-(2g-2)r + r(r_S + 1) + r(r_S + 1)(-d - g + 1) \tag{4.1.6}$$

Besides that, since we are considering $d \leqslant g - 1$, we know that $-d \geqslant -(g-1)$ and the expression in 4.1.6 is greater or equal than

$$-(2g-2)r + r(r_S+1) + r(r_S+1)(2-2g) = (2g-2)(-r - r(r_S+1)) + r(r_S+1) \tag{4.1.7}$$

Since $r_S + 1 \leqslant r$ then $-(r_S + 1) \geqslant -r$ and the expression 4.1.7 is greater or equal than

$$(2g-2)(-r + r^2) + r(r_S + 1) \tag{4.1.8}$$

From the hypothesis of $r \geqslant r_S + 1 > 1$ and $g > 2$ the expression 4.1.8 is always positive. Then $h^0(S' \otimes M_{V,L}^\vee) \neq 0$ and this implies that $h^0(S^\vee \otimes F_{V,L}|_C) \neq 0$ which is a contradiction, then $M_{V,L}$ has to be slope-stable. ■

In conclusion, Theorem 4.1.2 establishes a positive answer to the Mistretta-Stoppino conjecture for generated linear series (L, V) on smooth curves on $K3$ surfaces for $d \leqslant \min\{g-1, kr\}$, that is, under these conditions we have that linear stability of a pair (L, V) is equivalent to the slope-stability of the associated syzygy bundle $M_{V,L}$. We highlight that this proposition does not contradict the counterexample presented in Theorem 1.2.2 of Castorena, Mistretta, and Torres-López in [10]. Their counterexample involves a plane curve of degree 7, which lies outside the framework of the Martens theorem established in [24, Theorem 3.1]. This theorem asserts that a complex projective K3 surface cannot contain a curve isomorphic to a smooth plane curve of degree $\geqslant 7$. Consequently, the counterexample provided by Castorena, Mistretta and Torres-López, which relies on a plane curve of degree 7, does not apply to the context considered in Theorem 4.1.2. This distinction underscores the significance of the specific hypotheses and geometric contexts in which positive or negative results relating linear stability of linear series to slope stability of syzygy bundles can be derived.

Final Remark. As the reader can check, if the restriction $F_{V,L}|_C$ is slope-stable for $g - 1 \leqslant d \leqslant 2g - 1$, then the proof of Proposition 4.1.2 can be adapted modifying some numerical inequalities to get the slope-stability of $M_{V,L}$.

Appendix A

Computational tools

Let C be a general k-gonal curve of genus $g > 2$, consider $(L, V) \in G^r_d(C)$ a generated linear series over C.

In this appendix, we present a code in [32] to list the 4-plets (g, k, d, r) for which the general k-gonal curve C of genus g, the pair (r, d) satisfies the Proposition 3.2.3. We only consider non-generic values of k.

All examples on this appendix uses the following code:

```
def find_solutions(g):
    """
    Finds all the solutions (g, k, d, r) that satisfy the given conditions for a specific value of g.

    Args:
        g (int): The value of g to find solutions for.

    Returns:
        list: A list of tuples, where each tuple represents a solution in the format (g, k, d, r, condition).
    """
    solutions = []

    # Iterate over the possible values of r, d, and k
    for r in range(1, g+1):
        for k in range(2, int((g+3)/2)):
            for d in range(1, g+r+1):
                # Check if the triple (r, d, k) satisfies the first set of inequalities
                if (d <= g+r and d <= k*r and g+1-k <= (r-1)*d/r - 2*(r-1)):
                    if (k<7):
                        solutions.append((g,k,d,r, "Condition 1"))
                    if (k>6):
                        if(4*g > pow(k,2)):
                            solutions.append((g,k,d,r, "Condition 1"))
                        if(2*(r-1)+2-k >= pow(pow(k,2)-4*g,0.5)):
                            solutions.append((g,k,d,r, "Condition 1"))

                # Check if the triple (r, d, k) satisfies the second set of inequalities
                if (d*(r-1)/r <= g-1):
                    if (d <= g+r and d <= k*r):
```

```
                        if (r-1 <= (g-k+1)/(k-1) and g+r-k <= (r-1)*d/r and (r-1)*d/r <= g+2*(r-1) +1-k):
                            solutions.append((g, k, d, r, "Condition 2"))
                else:
                    if (g-1 <= (r-1)*d/r):
                        if (d <= g+r and d <= k*r):
                            if (r+1 <= k and (r-1)*d/r <= g+2*(r-1)+1-k and r+1 +2*k <= pow(4*g + 5*k*k - 4*k,0.5)):
                                solutions.append((g, k, d, r, "Condition 2"))
    return solutions

# Example usage
g = 15
solutions = find_solutions(g)

print(f"There are {len(solutions)} solutions for g = {g}, denoted by (g,k,d,r,Number of condition that satisfy) the solutions are:")
for solution in solutions:
    print(solution)
```

Listing A.1: Code example for genus 15

A.0.1 Small genus

For genus 3 to 8, there is no 4-plets that satisfies the Proposition 3.2.3, for genus 9, the first 4-plet that satisfy is:

```
There are 1 solutions for g = 9, denoted by (g,k,d,r,Number of condition that satisfy) the solutions are:
(9, 4, 12, 3, 'Condition 2')
```

The second 4-plet that satisfy the conditions for Proposition 3.2.3 appear in genus 12:

```
There are 1 solutions for g = 12, denoted by (g,k,d,r,Number of condition that satisfy) the solutions are:
(12, 5, 15, 3, 'Condition 2')
```

Note that for $g < 15$, there aren't 4-plets that satisfies the Condition 1 in Proposition 3.2.3.

A.0.2 Genus for condition 1

For genus 15 is the first genus where appear a 4-plet that satisfy the condition 1, said:

```
There are 2 solutions for g = 15, denoted by (g,k,d,r,Number of condition that satisfy) the solutions are:
(15, 6, 18, 3, 'Condition 2')
(15, 8, 20, 5, 'Condition 1')
```

Note that the number of solutions in each genus g is not an increasing function, for instance, the number of solutions for genus from 15 to 30 are 2, 2, 0, 2, 0 , 2, 2, 0, 0, 4, 2, 0, 2, 2, 0 and 4 respectively.

Appendix B

Bridgeland stability conditions

In this appendix, we examine the fundamental concepts of stability conditions on the bounded derived category $D^b(X)$ of coherent sheaves on a smooth projective variety X. The translation functor on $D^b(X)$ denoted by [1].
A key ingredient in our analysis is the construction of a certain abelian subcategory $Coh^\beta(X)$ within $D^b(X)$, which consists of two-term complexes. This abelian category $Coh^\beta(X)$ depends on the choice of a real parameter β.
To establish the necessary foundations for working with $Coh^\beta(X)$, first we need to introduce some basic notions from homological algebra and the theory of stability conditions. These concepts provide the framework for our subsequent results of how stability conditions on $D^b(X)$ can be utilized to study the stability of vector bundles and their restrictions on curves.

B.0.1 The heart of coherent sheaves

We recall the slope of a coherent sheaf E shifted by β:

$$\mu_\beta(E) \colon = \begin{cases} \frac{H.c_1(E)}{rk(E)} - \beta & \text{if } rk(E) > 0 \\ +\infty & \text{otherwise} \end{cases} \tag{B.0.1}$$

Definition B.0.1. *We say that $E \in Coh(X)$ is μ_β-(semi)stable if for all subsheaves $A \subsetneq E$, we have $\mu_\beta(A) < (\leqslant) \mu_\beta(E/A)$*

The introduction of the slope function μ_β generalizes the classical notion of slope that depends solely on the hyperplane section H. By definition, when $\beta = 0$, the slope μ_β reduces to the standard slope on the abelian category $Coh(X)$. This enrichment of the category of coherent sheaves $Coh(X)$ with the parameter $\beta \in \mathbb{R}$ allows us to establish a broader set of properties that extend the well-known results obtained using the classical slope. These generalized properties play a crucial role in our analysis of the stability conditions on the K3 surface X and the associated syzygy bundles.

Proposition B.0.1.
- *Every sheaf E has a (unique and functorial) Harder-Narasimhan filtration (HN-filtration):*

  $$0 = E_0 \subseteq E_1 \subseteq E_2 \subseteq ... \subseteq E_m = E$$

  of coherent sheaves where E_i/E_{i+1} is μ_β-semistable for $1 \leqslant i \leqslant m$, and with

  $$\mu_\beta^+(E) \colon = \mu_\beta(E_1/E_0) > \mu_\beta(E_2/E_1) > ... > \mu_\beta(E_m/E_{m-1}) =\colon \mu_\beta^-(E)$$

- *If E, F are slope-semistable with $\mu_\beta(E) > \mu_\beta(F)$, then $Hom(E,F) = 0$*

We use the existence of Harder-Narasimhan filtrations for μ_β to construct a specific torsion pair that decomposes the abelian category into two pieces. This torsion pair, denoted by (T^β, F^β), is crucial for our analysis as it allows us to study the stability of objects within the category. By exploiting the structure of the torsion pair, we can relate the existence of Harder-Narasimhan filtrations to the stability properties of objects in the category, providing a deeper understanding of the stability conditions.

$$
\begin{aligned}
T^\beta &= \{E \in Coh(X) : \mu_\beta^-(E) > 0\} \\
&= \{E \in Coh(X) : \text{ all HN-factors of } E \text{ satisfy } \mu_\beta(\cdot) > 0\} \\
&= \{E \in Coh(X) : \text{ all quotients } E \to Q \to 0 \text{ satisfy } \mu_\beta(Q) > 0\} \\
&= \langle E \in Coh(X) : E \text{ is slope-stable with } \mu_\beta(E) > 0 \rangle \\
F^\beta &= \{E \in Coh(X) : \mu_\beta^+(E) \leqslant 0\} \\
&= \{E \in Coh(X) : \text{ all HN-factors of } E \text{ satisfy } \mu_\beta(\cdot) \leqslant 0\} \\
&= \{E \in Coh(X) : \text{ all subobjects } 0 \to A \to E \text{ satisfy } \mu_\beta(A) \leqslant 0\} \\
&= \langle E \in Coh(X) : E \text{ is slope-stable with } \mu_\beta(E) \leqslant 0 \rangle
\end{aligned}
$$

Here, the notation $\langle \cdot \rangle$ denotes the smallest subcategory of $Coh(X)$ that includes the given objects and is closed under extensions.

Remark B.0.1. *The pair (T^β, F^β) is a torsion pair, i.e:*

- *For $T \in T^\beta, F \in F^\beta$, we have $Hom(T, F) = 0$*
- *Each $E \in Coh(X)$ fits into a (unique and functorial) short exact sequence*

$$0 \to T(E) \to E \to F(E) \to 0$$

  with $T(E) \in T^\beta, F(E) \in F^\beta$

With the aid of a torsion pair, we can use homological algebra tools, such as tilting, to construct a new abelian subcategory $\mathcal{A}$ of the bounded derived category $D^b(X)$. This abelian subcategory $\mathcal{A}$, known as the heart of the torsion pair, captures the essential information of the objects in $D^b(X)$ or their translations. An important property of this heart $\mathcal{A}$ is that its associated Grothendieck group coincides with the Grothendieck group of the entire derived category $D^b(X)$.

Proposition B.0.2 ([20, Corollary 2.2])**.** *The following (equivalent) characterization define an abelian subcategory of $D^b(X)$:*

$$
\begin{aligned}
Coh^\beta(X) &= \langle T^\beta, F^\beta[1] \rangle \\
&= \{E \in D^b(X) : H^0(E) \in T^\beta, H^{-1}(E) \in F^\beta, H^i(E) = 0 \text{ for } i \neq 0, 1\} \\
&= \{E \in D^b(X) : E \cong (F_{-1} \xrightarrow{d} F_0), ker(d) \in F^\beta, coker(d) \in T^\beta\}
\end{aligned}
$$

Since we aim to leverage the Grothendieck group of the heart $\mathcal{A} = Coh^{\beta}(X)$, which coincides with the Grothendieck group of the bounded derived category $D^b(X)$, we need to study the short exact sequences within the abelian category $\mathcal{A}$. These short exact sequences in $\mathcal{A}$ correspond precisely to the exact triangles in $D^b(X)$ of the form

$$A \xrightarrow{a} E \xrightarrow{b} B \to A[1]$$

where all the objects A, E and B belong to the heart $\mathcal{A}$, using the two-terms structure of $\mathcal{A}$ we have that the short exact sequence can be expressed as follows:

$$\begin{tikzcd}
A: \arrow[d, "a"] & A_{-1} \arrow[r, "d_A"] \arrow[d, "a_{-1}"] & A_0 \arrow[d, "a_0"] \\
E: \arrow[d, "b"] & E_{-1} \arrow[r, "d_E"] \arrow[d, "b_{-1}"] & E_0 \arrow[d, "b_0"] \\
B: & B_{-1} \arrow[r, "d_B"] & B_0
\end{tikzcd}$$

with all squares commutative, columns exact with a_i injective and b_i surjective.
Since we have $T^{\beta} \hookrightarrow Coh^{\beta}(X)$ as $T \mapsto T. = (0 \xrightarrow{0} T) \in Coh^{\beta}(X)$, if $F \in F^{\beta}$ then

$$F. : ... \xrightarrow{0} 0 \xrightarrow{0} \underbrace{F}_{\text{index } 0} \xrightarrow{0} 0 \xrightarrow{0} ... \qquad \in D^b(X)$$

$$F.[1] : ... \xrightarrow{0} \underbrace{F}_{\text{index } -1} \xrightarrow{0} 0 \xrightarrow{0} 0 \longrightarrow ... \qquad \in D^b(X)$$

and $F \mapsto F.[1] = (F \xrightarrow{0} 0) \in Coh^{\beta}(X)$ where $F \mapsto F.$ is the same morphism that before, that is, the injection in the index 0.
In particular, every object $E \in Coh^{\beta}(X)$ fits into a short exact sequence

$$H^{-1}(E). \hookrightarrow E \twoheadrightarrow H^0(E).$$

The isomorphism class of $E \in Coh^{\beta}(X)$ is determined by the extension class

$$Ext^1_{D^b(X)}(H^0(E)., H^{-1}(E).[1])$$

that is equivalent to $Ext^2_{Coh(X)}(H^0(E), H^{-1}(E))$ as:

$$\begin{tikzcd}
H^{-1}(E).[1]: \arrow[d] & H^{-1}(E) \arrow[r, "0"] \arrow[d] & 0 \arrow[d, "0"] \\
E: \arrow[d] & E_{-1} \arrow[r, "d"] \arrow[d, "0"] & E_0 \arrow[d] \\
H^0(E).: & 0 \arrow[r, "0"] & H^0(E)
\end{tikzcd}$$

Now, since we can see the objects E of T^{β} as objects in $Coh^{\beta}(X)$, denoted as $E.$, we want to study a characterization of the subobjects $A. \hookrightarrow E.$ in $Coh^{\beta}(X)$ using properties of the abelian category $Coh(X)$.

Proposition B.0.3 ([5, Proposition 2.4])**.** *Let $E \in T^\beta$ and $E.$ considered as an object of $Coh^\beta(X)$. To give a subobject $A. \hookrightarrow E.$ of E respect to the category $Coh^\beta(X)$ is equivalent of giving a sheaf $A \in T^\beta$ with a map $f : A \to E$ whose kernel (as a map in $Coh(X)$) satisfies $ker(f) \in F^\beta$.*

In order to reply the additive property of $deg(\cdot)$ and $rank(\cdot)$ under the short exact sequences, we define the Grothendieck group, denoted as $K(\mathcal{A})$, for an abelian category $\mathcal{A}$. This group is constructed as the quotient of the free abelian group generated by the objects of $\mathcal{A}$, under the relation $[B] = [B'] + [B'']$ for any short exact sequence of the form $0 \to B' \to B \to B'' \to 0$ in $\mathcal{A}$. For example, if

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

is a short exact sequence in $Coh(X)$, where C is a curve on the surface X, then $[\mathcal{O}_X] = [\mathcal{O}_X(-1)] + [\mathcal{O}_C]$ in $K(X)$. This relation reflects the additivity of the degree and rank under short exact sequences.
In our case, for the abelian category $\mathcal{A} = Coh(X)$ the Grothendieck group denoted as $K(X) := K(Coh(X))$ is generated by the classes of vector bundles $[F]$ on the variety X, modulo the relation defined above. This group, $K(X)$, provides a convenient way to encode numerical invariants of coherent sheaves, such as the rank and degree, in a linear algebraic setting.
We fix a finite rank lattice Λ (that is, a free abelian group with finite rank) and a surjective group homomorphism $\nu : K(\mathcal{A}) \to \Lambda$. This allows us to define numerical invariants of objects in $Coh(X)$ by considering their images under ν. For instance, the rank and degree of a vector bundle F can be recovered as $\nu([F]) = (rank(F), deg(F))$.

B.0.2 Stability conditions on abelian categories

In this subsection, we explore the fundamental concepts related to stability conditions on abelian categories. The formal definition of stability conditions was introduced by Bridgeland in [6].
Bridgeland's framework provides a powerful tool for studying the structure of the bounded derived category $D^b(X)$ of a variety X. Central to this approach is the notion of a stability condition, which endows the objects in $D^b(X)$ with a notion of (semi)stability. These stability conditions are parametrized by a manifold, known as the stability manifold, which exhibits a rich wall and chamber structure.
For the specific case of K3 surfaces, we focus on a 2-dimensional family of stability conditions. This specialized setting allows us to gain a more refined understanding of the structure of the walls and chambers in the stability manifold.
The key ideas and properties of Bridgeland stability conditions on abelian categories will be introduced in the following subsection, laying the groundwork for our subsequent results.

Definition B.0.2. *A weak stability function on an abelian category $\mathcal{A}$ is a group homomorphism $Z : \Lambda \to \mathbb{C}$ such that for any $E \in A$,*

$$Z(\nu(E)) = m(\nu(E))exp(i\pi\phi(\nu(E)))$$

where $m(\nu(E)) \geqslant 0$ and $0 < \phi(\nu(E)) \leqslant 1$

If for any non-trivial object E, we have $Z(\nu(E)) \neq 0$, the homomorphism Z is called a *stability function*. If $Z(\nu(E)) = 0$ for a non-trivial object $E \in A$, then we define $\phi(\nu(E)) = 1$. The real number $\phi(\nu(E)) \in (0, 1]$ is called the phase of the object E. For abuse notations we write $Z(E)$ and $\phi(E)$ instead of $Z(\nu(E))$ and $\phi(\nu(E))$.

Definition B.0.3. *A non-zero object $E \in \mathcal{A}$ is said to be Z-(semi)stable when Z is a stability function if*

$$0 \neq E' \subsetneq E \Longrightarrow \phi(E') < \phi(E) (\leqslant \text{ resp.})$$

We say that the stability function Z satisfies the Harder-Narasimhan property if every non-zero object $E \in \mathcal{A}$ has a finite filtration

$$0 = E_0 \subsetneq E_1 \subsetneq ... \subsetneq E_{n-1} \subsetneq E_n = E$$

whose factors $F_i = E_i/E_{i+1}$ are Z-semistable and

$$\phi^+(E) = \phi(F_1) > \phi(F_2) > ... > \phi(F_n) = \phi^-(E)$$

Definition B.0.4. *Pick a norm $|| \cdot ||$ on $\Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$. A (weak) stability function Z on an abelian category $\mathcal{A}$ satisfies the support property if there exists a constant $C > 0$ such that for all Z-semistable objects $0 \neq E$, we have*

$$||\nu(E)|| \leqslant C|Z(\nu(E))|$$

The support property plays a crucial role in endowing the set of stability functions with a geometric structure. By comparing stability functions to norms in an Euclidean space, the support property allows us to define a metric and topology on the space of stability functions on an abelian category $\mathcal{A}$.
Building upon this foundation, we can extend the notion of a stability function from abelian categories to the bounded derived category $D^b(X)$. The key idea is to leverage the concept of the heart $\mathcal{A}$ of a t-structure. Since the heart $\mathcal{A}$ and the entire derived category $D^b(X)$ share the same Grothendieck group, we can define a stability function on $D^b(X)$ that restricts to a stability function on $\mathcal{A}$. This extension allows us to study the stability of complexes in $D^b(X)$ using the same framework as for objects in the abelian category $\mathcal{A}$.

Definition B.0.5. *A (weak) stability condition on the bounded derived category $D^b(X) = D^b(Coh(X))$ is a pair $v = (Z, \mathcal{A})$ where $\mathcal{A}$ is the heart of a bounded t-structure on $D^b(X)$ and Z is a (weak) stability function on the abelian category $\mathcal{A}$ which satisfies the Harder-Narasimhan property and the support property.*

If $v = (Z, \mathcal{A})$ is a stability condition on $D^b(X)$ an object $E \in D^b(X)$ is said to be v-(semi)stable if a shift $E[k]$ is contained in the abelian category $\mathcal{A}$ and the object $E[k]$ is (semi)-stable with respect to the stability function Z.
The definition of stability conditions on a triangulated category, such as the bounded derived category $D^b(X)$, relies crucially on the concept of the heart of a t-structure. This abelian subcategory $\mathcal{A}$ plays a central role in Bridgeland's framework, as it provides the appropriate setting for defining a stability function and the associated notion of (semi)stability.

The construction of the heart $\mathcal{A}$ is not a trivial task and often involves the use of homological algebra tools, such as tilting. By performing a tilting procedure, we can produce new hearts of t-structures that exhibit different properties and allow for a more refined analysis of the stability conditions.
Furthermore, the concept of a slicing is intimately connected to the heart of a t-structure. A slicing is a parametrization of the subcategories of the triangulated category using the real numbers. This parametrization enables us to consider families of objects that belong to specific subcategories within given intervals. These families of objects are crucial for the definition and study of stability conditions on $D^b(X)$.

Definition B.0.6. *A slicing P of a triangulated category D consist of full additive subcategories $P(\phi)$ for each $\phi \in \mathbb{R}$ satisfying the following axioms:*

1. $\forall \phi \in \mathbb{R}$, $P(\phi+1) = P(\phi)[1]$
2. *If $\phi_1 > \phi_2$ and $A_i \in P(\phi_i)$ then $Hom_D(A_1, A_2) = 0$*
3. *for each non-zero object $E \in D$ there is a finite sequence of real numbers*

$$\phi^+(E) = \phi_1 > \phi_2 > ... > \phi_n = \phi^-(E)$$

and a collection of triangles

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & ... \arrow[r] & E_{n-1} \arrow[rr] & & E_n \arrow[dl] \\
& F_1 \arrow[ul] & & F_2 \arrow[ul] & & & & F_n \arrow[ul] &
\end{tikzcd}$$

with $F_i \in P(\phi_i)$ for all i

Any weak stability condition $v = (Z, \mathcal{A})$ defines a slicing P_v (which depends of v) of $D^b(X)$ as follows: for each $\phi \in (0,1]$, let $P_v(\phi)$ be the full additive subcategory of $D^b(X)$ of semistable objects with phase ϕ, together with 0. The part 1 of the Definition B.0.6 determines $P_v(\phi)$ for all $\phi \in \mathbb{R}$. Then, to refer to a stability condition ν, we can use the pair $(Z, \mathcal{A})$ where $\mathcal{A}$ is the hearth of a bounded t-structure on $D^b(X)$ or the pair (Z, P_ν) where P_ν is a slicing of $D^b(X)$.

Theorem B.0.1 ([7, Lemma 6.2])**.** *For each $\beta, \alpha \in \mathbb{R}$ with $\alpha > 0$, considerer the pair $\sigma_{\beta,\alpha} = (Z_{\beta,\alpha}, Coh^\beta(X))$ with $Coh^\beta(X)$ as below and with $Z_{\beta,\alpha} : K(D^b(X)) \to \mathbb{C}$ defined by:*

$$\begin{aligned}
Z_{\beta,\alpha}(E) &= \frac{\alpha^2 H^2 - b^2 H^2}{2} v_0(E) + \beta H v_1(E) - v_2(E) + i\alpha H(v_1(E) - \beta H rk(E)) \\
&= \langle exp(i\alpha H + \beta H), v(E) \rangle \\
&= \langle exp(i\alpha H + \beta H), (ch_0(E), ch_1(E), ch_2(E) + ch_0(E) \rangle
\end{aligned}$$

This pair defines a Bridgeland stability condition on $D^b(X)$ if $\mathfrak{Re}(Z_{\beta,\alpha}(\delta)) > 0$ for all roots $\delta \in H^*_{alg}(X;\mathbb{Z})$ of the form $(r, r\beta, s)$ with $r > 0$ and $s \in \mathbb{Z}$ arbitrary; in particular, this holds for $\alpha^2 H^2 \geqslant 2$.
Moreover, the family of stability conditions $\sigma_{\beta,\alpha}$ varies continuously as (β, α) vary in $\mathbb{R} \times \mathbb{R}_{>0}$.

In order to explain the notation we introduce the Mukai vector of an object E.
The Mukai vector of an object $E \in D^b(X)$ given by

$$v(E) = (v_0(E), v_1(E), v_2(E)) = (ch_0(E), ch_1(E), ch_2(E) + ch_0(E))$$

lies in the algebraic cohomology $H^*_{alg}(X;\mathbb{Z})$. The pairing $\langle\ ,\ \rangle$ is the Mukai pairing

$$\langle v(E), v(F)\rangle = -\chi(E,F) = v_1(E)v_1(F) - v_0(E)v_2(F) - v_2(E)v_0(F)$$

For each sheaf E, we have $\mathfrak{Im}(Z_{\beta,\alpha}(E)) \geqslant 0$ if and only if $\mu_\beta(E) \geqslant 0$. And $Z_{\beta,\alpha}(F[1]) = -Z_{\beta,\alpha}(F)$.
Let $Stab(X)$ be the set of stability conditions of $D^b(X)$ (with respect to the lattice Λ and the vector v). This set can be enriched with a topology as the coarsest topology such that for any $E \in D^b(X)$ the maps $(Z,\mathcal{A}) \mapsto Z$, $(Z,\mathcal{A}) \mapsto \phi^+(E)$ and $(Z,\mathcal{A}) \mapsto \phi^-(E)$ are continuous.

Theorem B.0.2 ([6, Theorem 1.2])**.** *The map $\mathcal{Z} : Stab(X) \to Hom(\Lambda,\mathbb{C})$ given by $(\mathcal{A}, Z) \mapsto Z$ is a local homeomorphism. In particular, $Stab(X)$ is a complex manifold of dimension $rk(\Lambda)$.*

We can study the behavior of an object $E \in Coh(X)$ when we let vary the stability condition in $Stab(X)$. We want to study the sets of stability conditions for which E is stable, semistable or unstable.

Definition B.0.7. *Let $v_0, w \in \Lambda - 0$ be two non-parallel vectors. A numerical wall $W_w(v_0)$ for v_0 with respect to w is a non-empty subset of $Stab(X)$ given by*

$$W_w(v_0) = \{\sigma = (Z,P) \in Stab(X) : \mathfrak{Re}(Z(v_0)) \cdot \mathfrak{Im}(Z(w)) = \mathfrak{Re}(Z(w)) \cdot \mathfrak{Im}(Z(v_0))\}$$

The wall and chamber structure of the stability manifold $Stab(X)$ is intimately related to the topology of this space. We denote by $\mathcal{W}(v_0)$ the set of numerical walls for a fixed Mukai vector v_0. These numerical walls are real codimension-1 submanifolds of $Stab(X)$, which divide the stability manifold into connected components called chambers.
This wall and chamber structure encodes crucial information about the behavior of (semi)stability for objects with fixed Mukai vector v_0. As we vary the stability condition by moving within a chamber, the (semi)stability of objects with Mukai vector v_0 remains unchanged. However, when crossing a wall, the (semi)stability can change abruptly. By understanding this wall and chamber structure, we gain a global perspective on how the stability of objects depends on the choice of stability condition.
Furthermore, the topology of $Stab(X)$ is closely related to the wall and chamber decomposition. The connected components of the complement of the union of all walls are precisely the chambers. The walls themselves form a locally finite arrangement, ensuring that the topology of $Stab(X)$ is well-behaved.
In order to study the case of K3 surfaces, we want to describe the wall and chamber structure on the (β,α)-plane for the $\sigma_{\beta,\alpha}$ stability conditions as in [23] section 6.4.
Consider $T = \{v \in K(X) : \chi(v,w) = 0 \text{ for all } w \in K(X)\}$ and the numerical Grothendieck group $K_{num}(X) = K(X)/T$ as a finitely generated $\mathbb{Z}$-lattice. In the case where X is a K3 surface, the group $K_{num}(X)$ coincides with the algebraic cohomology group $H^*_{alg}(X)$.

Proposition B.0.4 ([23, Proposition 6.22])**.** *Fix a class $v \in K_{num}(X)$.*

1. *All numerical walls are either semicircles with center on the β-axis or vertical rays.*
2. *Two different numerical walls for v cannot intersect.*
3. *For a given class $K_{num}(X)$ the hyperbola $\mathfrak{Re}(Z_{\beta,\alpha}(V)) = 0$ intersects all umerical semicircular walls at their top points.*
4. *If $ch_0(v) \neq 0$, then there is an unique numerical vertical wall defined by the equation*
$$\beta = \frac{H \cdot ch_1(v)}{H^2 \cdot ch_0(v)}$$
5. *If $ch_0(v) \neq 0$, then all semicircular walls to either side of the unique numerical vertical wall are strictly nested semicircles.*
6. *If $ch_0(v) = 0$, then there are only semicircular walls that are strictly nested.*
7. *If a wall is an actual wall at a single point, it is an actual wall everywhere along the numerical wall.*

Building upon the foundational result established in Theorem B.0.1, which states that the pair $\sigma_{\beta,\alpha} = (Z_{\beta,\alpha}, Coh^{\beta}(X))$ defines a stability condition on the bounded derived category $D^b(X)$, we now turn our attention to understanding the behavior of the stability of coherent sheaves as we navigate the wall and chamber structure of the (β, α)-plane.

Corollary B.0.1 ([5, Corollary 3.5])**.** *Given a class $v \in H^*_{alg}(X; \mathbb{Z})$. For objects of Mukai vector v, being $\sigma_{\beta,\alpha}$-(semi)stable is independent on the choice of (β, α) in any given chamber.*

Now that we have established the independence of (semi)stability on the choice of stability condition within a given chamber of the (β, α)-plane, the natural next step is to study the moduli spaces that parametrize the semistable objects for a fixed Mukai vector v. These moduli spaces encode crucial information about the behavior of (semi)stable objects as we vary the stability condition by moving between different chambers in the (β, α)-plane.

B.0.3 Moduli spaces of stable objects and restriction theorem

In this subsection, we delve into the study of moduli spaces that parametrize stable objects with respect to a given stability condition. Our goal is to understand how the structure of these moduli spaces varies as we change the stability condition. Furthermore, we investigate whether there exist stability conditions for which the associated moduli spaces coincide with the moduli spaces obtained under classical stability notions, such as Gieseker stability.

To lay the groundwork for this analysis, let us first recall some fundamental results regarding the structure and non-emptiness of moduli spaces of stable objects. These

results serve as a foundation for our subsequent investigations of how the moduli spaces depend on the choice of stability condition.
A crucial question is whether these moduli spaces are non-empty for a given stability condition. This is closely related to the existence of stable objects for that condition. By leveraging the wall-crossing techniques developed in the previous sections, we establish conditions under which the moduli spaces are non-empty and study how their structure changes as we vary the stability condition.

Theorem B.0.3 ([5, Theorem 4.1]). *Consider a vector $v \in H^*_{alg}(X;\mathbb{Z})$, and let $\sigma = \sigma_{\beta,\alpha}$ be a stability condition that is not on any of the walls for the wall and chamber decomposition with respect to v. Then the coarse moduli space $M_\sigma(v)$ of $\sigma_{\beta,\alpha}$-stable objects of Mukai vector v exists as a smooth projective irreducible holomorphic symplectic variety. It is non-empty if and only if $v^2 \geqslant -2$ and its dimension is given by $dim M_\sigma(v) = v^2 + 2$*

Consider now the moduli space of H-Giesker-stable sheaves and lets compare this set with the moduli space of $\sigma_{\beta,\alpha}$-stable objects with $\alpha >> 0$.

Theorem B.0.4 ([5, Theorem 4.4]). *Let $v = (v_0, v_1, v_2)$ a class in $H^*_{alg}(X;\mathbb{Z})$ having either positive rank $v_0 > 0$, or satisfying $v_0 = 0$ with v_1 being effective. Then there exists α_0 such that for all $\alpha \geqslant \alpha_0$ and all $\beta > \frac{H\cdot v_1}{H^2\cdot v_0}$ (or $\beta > \frac{v_2}{H\cdot v_1}$ in case $v_0 = 0$), the moduli space $M_{\sigma_{\beta,\alpha}}(v)$ is equal to the moduli space $M_H(v)$ of H-Gieseker-stable sheaves of class v. More precisely, and object $E \in D^b(X)$ with $v(E) = v$ is $\sigma_{\beta,\alpha}$-stable if and only if it is the shift of a Gieseker-stable sheaf.*

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