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Finite dimensional infinite partitions

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Resumen

Las familias *mad* (de *maximal almost disjoint*) de subconjuntos infinitos de ω y su tamaño mínimo, típicamente denotado $\mathfrak{a}$, forman un área de investigación ampliamente estudiada dentro de la Teoría de Conjuntos. Aquí se estudian generalizaciones de estos conceptos a dimensiones finitas. Se presentan algunos resultados sobre el invariante cardinal $\mathfrak{a}(A \oplus B)$, es decir, el mínimo tamaño de las particiones infinitas del producto libre de las álgebras Booleanas infinitas A y B. Estos resultados se aplican a $\mathcal{P}(\omega)/fin \oplus \mathcal{P}(\omega)/fin$, un álgebra Booleana cuyos elementos básicos son los *rectángulos* de la forma $X \times Y$, para infinitos $X, Y \subseteq \omega$. Se define un ideal sobre $\omega \times \omega$, denotado $\mathcal{NC}$, tal que dichos rectángulos son densos en el cociente $\mathcal{P}(\omega \times \omega)/\mathcal{NC}$. Se estudia la estructura combinatoria de $\mathcal{NC}$ y de su cociente, así como la de otras versiones de mayor dimensión, con énfasis en sus particiones infinitas.

Palabras Clave: Invariantes cardinales, Combinatoria infinita, Particiones infinitas, Álgebras Booleanas, Ideales.

Abstract

Mad families of infinite subsets of ω and their lowest size, typically denoted $\mathfrak{a}$, are a widely studied field of research in Set Theory. Here, generalizations of these concepts in higher finite dimensions are studied. Some results on the cardinal invariant $\mathfrak{a}(A \oplus B)$, i.e. the lowest size of the infinite partitions of the free product of some given infinite Boolean algebras A and B, are presented. These results are applied to $\mathcal{P}(\omega)/fin \oplus \mathcal{P}(\omega)/fin$, the Boolean algebra whose basic elements are *rectangles* of the form $X \times Y$, for infinite $X, Y \subseteq \omega$. An ideal on $\omega \times \omega$ is defined, denoted $\mathcal{NC}$, such that these rectangles are dense in the quotient $\mathcal{P}(\omega \times \omega)/\mathcal{NC}$. The combinatorial structure of $\mathcal{NC}$ and its quotient, as well as that of other higher-dimensional versions, with an emphasis on infinite partitions, is studied.

Keywords: Cardinal invariants, Infinite combinatorics, Infinite partitions, Boolean algebras, Ideals.

Introducción

Las familias *casi ajenas maximales* (familias *mad* por sus siglas en inglés) de subconjuntos infinitos de ω han sido una fuente de estudio en la Teoría de Conjuntos en general. Por definición son objetos combinatorios y su estudio se encuentra naturalmente en el campo de la combinatoria infinita clásica. En particular su mínimo tamaño, típicamente denotado por $\mathfrak{a}$, es una de las *características cardinales del continuo* más conocidas [3]. De estos cardinales se estudian usualmente sus relaciones de orden, tanto las que se pueden probar en ZFC como las que son independientes. Como consecuencia, las familias *mad* son también relevantes en el estudio de *forcing* y las *extensiones genéricas*.

Al igual que en muchos temas matemáticos, las familias *mad* son fuente de nuevos estudios a través de sus generalizaciones. Varias de estas generalizaciones han apuntado al estudio de las familias $\mathcal{I}$-mad [8], para ideales $\mathcal{I}$ sobre conjuntos numerables, y a las familias *mad* sobre cardinales no numerables [4]. Una vía aun más general es el estudio de *particiones* infinitas de álgebras Booleanas A y su mínimo tamaño $\mathfrak{a}(A)$. Este es el método seguido en [18].

En dicho libro se plantea la pregunta sobre cuál es la relación entre $\mathfrak{a}(A \oplus B)$ y $\mathfrak{a}(A), \mathfrak{a}(B)$, donde A y B son álgebras Booleanas infinitas y $A \oplus B$ es su *producto libre*. Esta cuestión motivó los resultados principales del Capítulo 3, donde se dan algunas cotas inferiores para el cardinal $\mathfrak{a}(A \oplus B)$, relacionadas a los cardinales invariantes de A y B. Como corolario, se obtienen algunas cotas inferiores para el cardinal $\mathfrak{a}(2)$, el mínimo tamaño de familias 2-*mad*, o *familias mad de dimensión 2*. Permanence abierta la pregunta sobre si la desigualdad $\mathfrak{a}(2) < \mathfrak{a}$ es consistente, y estos resultados delinean las dificultades sobre conseguir un modelo donde sea cierta. Sin embargo, $\mathfrak{a}(2)$ y versiones de mayor dimensión parecen ser legítimas características cardinales del continuo.

Las familias 2-*mad*, cuyos elementos se pueden describir como *rectángulos* del tipo $X \times Y$, donde X y Y son subconjuntos infinitos de ω, motivaron la definición de un ideal sobre ω^2. Fue llamado el ideal *Nunca Centrado*, denotado a lo largo de este texto por $\mathcal{NC}$, y su principal característica es que los susodichos rectángulos son densos en el cociente $\mathcal{P}(\omega \times \omega)/\mathcal{NC}$. Como consecuencia, todas las familias 2-*mad* inducen particiones infinitas de $\mathcal{P}(\omega \times \omega)/\mathcal{NC}$ del mismo tamaño.

El ideal $\mathcal{NC}$ y su cociente no ayudaron en la pregunta sobre la consistencia de $\mathfrak{a}(2) < \mathfrak{a}$, a pesar de que era su propósito original. Sin embargo, la combinatoria de su cociente dio como resultado interesante la existencia de *torres* y particiones infinitas pequeñas. Estos resultados, junto con el estudio de otras estructuras y cardinales invariantes en $\mathcal{P}(\omega \times \omega)/\mathcal{NC}$, conforman la mayor parte del Capítulo 4. Versiones de mayor dimensión del ideal $\mathcal{NC}$ también se estudian, así como su estructura como ideal.

Se asume que quien lee tiene conocimiento básico sobre conceptos y notación de teoría de conjuntos. Por ejemplo, $[X]^\kappa$ denota la familia de subconjuntos de X de tamaño κ y κ^λ se refiere a la familia de funciones del cardinal λ al cardinal κ. En este respecto se toma [15] como recurso para los conceptos y la notación. Para elementos de álgebras Booleanas e ideales se proveen secciones generosas en el Capítulo 1 aunque [19] y [17] se sugieren fuertemente como referencia para los respectivos tópicos. El capítulo 2 da una mirada extensa sobre los cardinales invariantes de los ideales y las álgebras Booleanas. Grandes recursos en ambos tópicos son [17] y [18]. Lo poco que es necesario sobre teoría descriptiva de conjuntos, árboles y *forcing* se describe también en el Capítulo 1.

Introduction

Maximal almost disjoint (mad) families of infinite subsets of ω have been a source of study for Set Theory in general. They are combinatorial objects by definition, and their study naturally lies in the scope of classic infinite combinatorics. In particular their least possible size, typically denoted as $\mathfrak{a}$, is one of the most known *cardinal characteristics of the continuum* [3]. These cardinals are usually studied in their order relations, both those provable in ZFC and those that are independent. Accordingly, mad families are also relevant in the study of *forcing* and *generic extensions*.

As in many mathematical topics, mad families have been a source of new studies through their generalizations. In this case, most of them have been to study $\mathcal{I}$-mad families [8], for ideals $\mathcal{I}$ on a countable set, and mad families on uncountable cardinals [4]. An even more general approach is the study of infinite *partitions* of Boolean algebras A and its lowest possible size $\mathfrak{a}(A)$. This is the approach taken in [18].

This book raises the question of which is the relation between $\mathfrak{a}(A \oplus B)$ and $\mathfrak{a}(A), \mathfrak{a}(B)$, where A and B are infinite Boolean algebras and $A \oplus B$ is their *free product*. This question motivated the main results of Chapter 3 where some lower bounds, related to the cardinal invariants of A and B, are given to the number $\mathfrak{a}(A \oplus B)$. As a corollary, lower bounds are given to the number $\mathfrak{a}(2)$, the least size of a 2-mad family, or a *2-dimensional mad family.* The question of whether $\mathfrak{a}(2) < \mathfrak{a}$ is consistent remains open and these results speak to the difficulty of obtaining a model where it holds. However, $\mathfrak{a}(2)$ and higher-dimensional versions seem to be legitimate cardinal invariants of the continuum.

Motivated by 2-mad families, whose elements can be described as *rectangles* of the type $X \times Y$, where X and Y are infinite subsets of ω, an ideal on ω^2 was defined. It is called the *Nowhere centered* ideal, denoted throughout the text $\mathcal{NC}$, and its defining characteristic is that the rectangles mentioned above are dense in the quotient $\mathcal{P}(\omega \times \omega)/\mathcal{NC}$. As a consequence, all 2-mad families induce infinite partitions in $\mathcal{P}(\omega \times \omega)/\mathcal{NC}$ of the same size.

The ideal $\mathcal{NC}$ and its quotient did not help in the question of the consistency of $\mathfrak{a}(2) < \mathfrak{a}$ which was its original purpose. However, the combinatorics of the quotient gave as an interesting result the existence of small *towers* and infinite partitions. These results along with the study of other structures and cardinal invariants of $\mathcal{P}(\omega \times \omega)/\mathcal{NC}$ form the majority of

Chapter 4. Higher-dimensional versions of the ideal $\mathcal{NC}$ are also studied as well as its structure as ideal.

The reader is assumed to have basic knowledge of concepts and the notation of set theory. For example, $[X]^\kappa$ denotes the family of subsets of X of size κ and κ^λ refers to the family of functions from the cardinal λ to the cardinal κ. As a model in this respect [15] is taken. For elements of Boolean algebras and ideals, generous sections are provided in Chapter 1, although [19] and [17] are strongly suggested as reference for the respective topics. Chapter 2 gives a comprehensive look at the cardinal invariants of ideals and Boolean algebras. Great resources on both topics are, respectively, [17] and [18]. The little knowledge that is needed on descriptive set theory, trees and forcing is given also in Chapter 1.

Chapter 1

Preliminaries

1.1 Boolean algebras

A *Boolean Algebra* is a structure $(A, \cdot, +, 0, 1, -)$, where A is a non-empty set, 0 and 1 are elements of A, $\cdot$ and $+$ are binary operations on A and $-$ is a function from A to A, such that for all $a, b, c \in A$

1. $(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \wedge \quad (a + b) + c = a + (b + c)$
2. $a \cdot b = b \cdot a \quad \wedge \quad a + b = b + a$
3. $a \cdot (b + c) = (a \cdot b) + (a \cdot c) \quad \wedge \quad a + (b \cdot c) = (a + b) \cdot (a + c)$
4. $a + (a \cdot b) = a \quad \wedge \quad a \cdot (a + b) = a$
5. $a + (-a) = 1 \quad \wedge \quad a \cdot (-a) = 0$

Usually $(A, +, \cdot, -, 1, 0)$ is abbreviated as A, unless there is any possible confusion. The operation $+$ will be called *sum*, the operation $\cdot$ will be called *product* and the function $-$ will be called *complement*. Observe that 0 is the neutral element for the sum and that 1 is the neutral element of the product. There is a partial ordering implicit in the definition of a Boolean algebra: we will say that $a \leqslant b$ whenever $a \cdot b = a$. Observe that both 1 is the maximal element of the partially ordered set $(A, \leqslant)$, and 0 is its minimal element. As usual, we will write $x < y$, if $x \neq y$ and $x \leqslant y$. The set $A \setminus \{0\}$ will be called the set of *positive* elements of A and will be denoted A^+. An element $a \in A^+$ will be called an *atom* if $b \leqslant a$ implies $b = a$, for all $b \in A^+$.

Basic examples of Boolean algebras include:

- *Power sets*, i.e. $(\mathcal{P}(X), \cup, \cap, X \setminus \ , X, \emptyset)$, where X is a given set, $\mathcal{P}(X)$ is its *power set*, and $\cup, \cap$ and $X \setminus$ are the usual set-theoretic operations,
- *Set algebras*, i.e. a subset of some power set $\mathcal{P}(X)$ which is closed by taking finite unions and intersections, and complements,

- *Regular open algebras*, i.e. the algebra on the set $\mathcal{RO}(X) := \{A \subseteq X \mid int(\overline{A}) = A\}$, for some topological space X, whose binary operations are $\cup$ and $\cap$, and whose complement is $int(X \setminus A)$, for all $A \in \mathcal{RO}(X)$ (where $int(B)$ refers to the interior of any set $B \subseteq X$).

Besides the binary operations $+$ and $\cdot$, some further algebraic notations are used. If $x_0, ..., x_{n-1} \in A$, the sum of those elements $x_0 + ... + x_{n-1}$ will be denoted $\sum_{i<n} x_i$. Similarly, their product will be denoted $\prod_{i<n} x_i$. Observe, from point 4 of the definition of Boolean algebra, that $x_j \leqslant \sum_{i<n} x_i$, for all $i < n$. Also if $0 \neq y \leqslant \sum_{i<n} x_i$, there exists $j < n$ such that $x_j \cdot y \neq 0$. Dually, $\prod_{i<n} x_i \leqslant x_j$, for all $j < n$, and such that $y \leqslant \prod_{i<n} x_i$, for all $y \in A$ such that $y \leqslant x_i$, for all $i < n$. These observations justify the next definitions and their notations.

Whenever $\{x_\alpha \mid \alpha < \kappa\}$ is an arbitrary subset of A, by its sum or *supremum* we will mean an element x such that $x_\alpha \leqslant x$, for all $\alpha < \kappa$, and such that for all $0 \neq y \leqslant x$, there exists $\alpha < \kappa$ such that $x_\alpha \cdot y \neq 0$. The existence of such element does not follow from the axioms defining Boolean algebras. If it exists, it will be denoted $\sum_{\alpha<\kappa} x_\alpha$ or $\bigvee_{\alpha<\kappa} x_\alpha$. Similarly if $\{x_\alpha \mid \alpha < \kappa\}$ is an arbitrary subset of A, by its product or *infimum* we will mean an element x such that $x \leqslant x_\alpha$, for all $\alpha < \kappa$, and such that $x \cdot y \neq 0$, for all y such that $y \leqslant x_\alpha$, for all $\alpha < \kappa$. If such an element exists, it will be denoted $\prod_{\alpha<\kappa} x_\alpha$ or $\bigwedge_{\alpha<\kappa} x_\alpha$.

Now we procede to define some possible relations between Boolean algebras and some kinds of Boolean algebras.

Definition 1.1.1 (Morphisms)**.** *Let A and B be Boolean algebras. A function $f : A \to B$ is a homomorphism if it preserves the operations and the elements 1 and 0. If furthermore the function f is bijective, it is an isomorphism, and A and B will be said to be isomorphic, which is denoted $A \cong B$.*

Definition 1.1.2 (Subalgebra)**.** *Let A be a Boolean algebra. If $B \subseteq A$, $1, 0 \in B$ and B is closed under the operations $+, \cdot, -$, we will say that B is a subalgebra of A and it will be denoted $B \leqslant A$. This notation will also be used when A and B are Boolean algebras and B is isomorphic to a subalgebra of A.*

Definition 1.1.3. *Let A be a Boolean algebra. A set $C \subseteq A^+$ will be called an antichain if $a \cdot b = 0$, for all distinct $a, b \in C$.*

Definition 1.1.4 (Regular and dense subalgebras)**.** [1] *Let A and B be Boolean algebras such that $A \leqslant B$. It will be said that A is a regular subalgebra of B, in symbols $A \leqslant_{reg} B$, if for all $C \subseteq A^+$ maximal antichain, it is also a maximal antichain of B. It will be said that A is a dense*

[1] Though similar to the definition of a *complete embedding* of partially ordered sets, the author uses the definition of regular subalgebra of [13] to avoid any confusions with the notion of (relatively) complete subalgebra in [18] and [19].

subalgebra of B, in symbols $A \leqslant_\pi B$, if for all $b \in B^+$ there exists $a \in A^+$ such that $a \leqslant b$.

Definition 1.1.5. *If $x \in A$, we define the restriction of A to x, a Boolean algebra with $A \restriction x := \{y \in A \mid y \leqslant x\}$ as underlying set and structure $(A \restriction x, +, \cdot, -', x, 0)$, where $-'y := (-y) \cdot x$, for all $y \leqslant x$.*

Definition 1.1.6. *Let A be a Boolean algebra. If none of its positive elements is an atom, it will be called an atomless Boolean algebra.*

Definition 1.1.7. *A Boolean algebra A will be called homogeneous if $A \restriction x$ is isomorphic to A, for all $x \in A^+$.*

Definition 1.1.8 (Algebras of clopen sets)**.** *Let X be a topological space. Then the family of its clopen (i.e. open and closed) subsets forms a subalgebra of the set algebra $\mathcal{P}(X)$. This Boolean algebra will be denoted clop(X). The space X will be called zero-dimensional if clop(X) is a base of the space.*

Among all studied operations of Boolean algebras providing new ones, the most relevant for this text is the free product of two Boolean algebras.

Definition 1.1.9. *If A and B are two Boolean algebras, their free product, denoted $A \oplus B$, is an algebra C such that there exist subalgebras $A', B' \leqslant C$, such that $A \cong A'$, $B \cong B'$,*

$$C = \left\{ \sum_{i<n} a_i \cdot b_i \mid n < \omega, a_i \in A', b_i \in B' \right\}$$

and $a \cdot b \neq 0$, for all $a \in A' \setminus \{0\}$ and all $b \in B' \setminus \{0\}$. Given two Boolean algebras A and B, this algebra exists and is unique up to isomorphisms.

The main use of duality in this text will be when dealing with free products.

Theorem 1.1.10 (Stone duality)**.** *Every Boolean algebra is isomorphic to an algebra of sets. Moreover this algebra consists of the clopen sets forming a base of a Hausdorff, compact, zero-dimensional topological space.*

Theorem 1.1.11. *Let A and B be Boolean algebras. If A is isomorphic to clop(X), the algebra of clopen basic sets of the space X, and B is isomorphic to clop(Y), then $A \oplus B$ is isomorphic to the algebra of clopen sets of the space $X \times Y$.*

Proofs for both theorems can be found in [19] (Theorem 7.8 and in Chapter 4, Section 11), as well as details and results on all of the definitions of this section.

If $\{A_\alpha \mid \alpha < \kappa\}$ is a family of Boolean algebras, then $\bigoplus_{\alpha<\kappa} A_\alpha$ will denote their free product. If $A_\alpha \cong \mathit{clop}(X_\alpha)$, for all $\alpha < \kappa$, then $\bigoplus_{\alpha<\kappa} A_\alpha \cong \mathit{clop}(\prod_{\alpha<\kappa} X_\alpha)$. It follows that $A_\alpha \leqslant \bigoplus_{\alpha<\kappa} A_\alpha$, and that $\prod_{i<n} x_i \neq 0$, for

$x_i \in A^+_{\alpha_i}$, for all $\alpha_0 < ... < \alpha_{n-1} < \kappa$. When there exists a Boolean algebra A such that $A_\alpha = A$, for all $\alpha < \kappa$, the free product $\bigoplus_{\alpha<\kappa} A_\alpha$ will simply be denoted by $\bigoplus_{\alpha<\kappa} A$.

Another usual operation between Boolean algebras is the *product*. If $\{A_\alpha \mid \alpha < \kappa\}$ is a family of Boolean algebras, their product will be the Boolean algebra on their cartesian product with operations defined coordinatewise and simply denoted $\prod_{\alpha<\kappa} A_\alpha$. In topological duality the product corresponds to topological sums.

1.2 Ideals and quotients

A subfamily $\mathcal{I} \subseteq \mathcal{P}(X)$, the power set of some set X, is called an *ideal* if

1. $X \notin \mathcal{I}$,
2. $A \cup B \in \mathcal{I}$ for all $A, B \in \mathcal{I}$, and
3. $A \in \mathcal{I}$ whenever $A \subseteq B$ and $B \in \mathcal{I}$.

Examples of ideals include $\mathcal{N}$, the ideal of *Lebesgue measure zero* subsets and $\mathcal{M}$, the ideal of *meager* subsets, both on the real line $\mathbb{R}$. Both ideals consist of *small* sets of the real line. Another important example is fin, the ideal of finite sets of ω. In general, an ideal on a set X represents some notion of smallness, of *nullity*, on the subsets of the set X. Accordingly, if $\mathcal{I}$ is an ideal on the set X, its complement $\mathcal{P}(X) \setminus \mathcal{I}$ is denoted $\mathcal{I}^+$ and its elements are called $\mathcal{I}$*-positive*. In general it will be assumed that $[X]^{<\omega} \subseteq \mathcal{I}$, for all ideal $\mathcal{I}$ on a set X.

If $\mathcal{G} \subseteq \mathcal{I}$, it will be said that $\mathcal{G}$ *generates* $\mathcal{I}$ if for all $I \in \mathcal{I}$ there exists $F \in [\mathcal{G}]^{<\omega}$ such that $I \subseteq \bigcup F$. If $\mathcal{G} \subseteq \mathcal{P}(\omega)$ it will be said that $\mathcal{G}$ generates an ideal if $\bigcup F \neq X$, for all $F \in [\mathcal{G}]^{<\omega}$. The ideal generated by $\mathcal{G}$, usually denoted $\langle \mathcal{G} \rangle_{id}$, is the set $\{I \subseteq X \mid \exists\ F \in [\mathcal{G}]^{<\omega} I \subseteq \bigcup F\}$.

If $\mathcal{I} \subseteq \mathcal{P}(X)$ is an ideal, for some countable set X, define on $\mathcal{P}(X)$ the equivalence relation $\sim_\mathcal{I}$:

$$A \sim_\mathcal{I} B \equiv A \triangle B := A \setminus B \cup B \setminus A \in \mathcal{I},$$

for all $A, B \subseteq X$. It is not hard to see that the corresponding set of equivalence classes $\{[A]_\mathcal{I} \mid A \subseteq X\}$ is a Boolean algebra with the operations induced by the Boolean operations of $\mathcal{P}(X)$. Those operations are:

- $[A]_\mathcal{I} + [B]_\mathcal{I} := [A \cup B]_\mathcal{I}$,
- $[A]_\mathcal{I} \cdot [B]_\mathcal{I} := [A \cap B]_\mathcal{I}$ and
- $-[A]_\mathcal{I} := [X \setminus A]_\mathcal{I}$.

The maximum element is $[X]_\mathcal{I}$ and the minimum element is $[\emptyset]_\mathcal{I} = \mathcal{I}$. The Boolean algebra thus defined is denoted $\mathcal{P}(X)/\mathcal{I}$ and called the *quotient* of $\mathcal{P}(X)$ *modulo* $\mathcal{I}$. When studying this structure the notation $[A]_\mathcal{I}$ is mostly avoided and positive elements of $\mathcal{P}(X)/\mathcal{I}$ are identified with their representatives in $\mathcal{I}^+$. The order relation of $\mathcal{P}(X)/\mathcal{I}$ being important and not directly induced by $\subseteq$, its interpretation on $\mathcal{I}^+$ is highlighted: $A \subseteq_\mathcal{I} B$ iff $[A]_\mathcal{I} \leqslant [B]_\mathcal{I}$ iff $A \setminus B \in \mathcal{I}$, for all $A, B \in \mathcal{I}^+$.

Except for the minor role of the ideals $\mathcal{M}$ and $\mathcal{N}$, in this text most of the ideals are over ω or any other countable set X. The bijection between X and ω witnessing the countability of X will always be tacitly acknowledged. Some important relations between two ideals on countable sets that will be helpful when studying their combinatorics is the following one.

Definition 1.2.1. *The Katětov-Blass order $\leqslant_{KB}$ on ideals is defined as follows: if $\mathcal{I}$ and $\mathcal{J}$ be two ideals on ω, then $\mathcal{I} \leqslant_{KB} \mathcal{J}$ if there exists $f : \omega \to \omega$ finite-to-one such that $f^{-1}[I] \in \mathcal{J}$ for all $I \in \mathcal{I}$.*

If the function f is not required to be finite-to-one, we get the Katětov relation $\mathcal{I} \leqslant_K \mathcal{J}$.

An operation for getting new ideals on ω from known ones is the following.

Definition 1.2.2. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω. Their Fubini product $\mathcal{I} \times \mathcal{J}$ is the ideal on $\omega \times \omega$ defined*

$$\mathcal{I} \times \mathcal{J} := \{A \subseteq \omega \times \omega \mid \{n < \omega \mid \{m < \omega \mid (n, m) \in A\} \notin \mathcal{J}\} \in \mathcal{I}\}.$$

The main use of this operation in this text is for the definition of the finite powers of the ideal fin. For $0 < k < \omega$, we will recursively define an ideal fin^k on ω^k as follows:

- $fin^1 := fin$
- $fin^k := fin \times fin^{k-1}$, for $1 < k < \omega$.

1.3 Descriptive set theory

A topological space X is called *Polish* if it is separable and completely metrizable. Classic examples of Polish spaces include the *Cantor* space 2^ω and the *Baire* space ω^ω. Every Polish space is continuous image of Baire space. If X is a Polish space by $Borel(X)$ will be denoted the least σ-algebra containing all open subsets of X. The elements of $Borel(X)$ will be called *Borel* subsets of X. In this context, Σ^0_1 denotes the family of open subsets of X. For $\alpha < \omega_1$ the following hierarchy of Borel subsets of X is recursively defined:

- $\Pi^0_\alpha := \{X \setminus Y \mid Y \in \Sigma^0_\alpha\}$

- $\Sigma^0_\alpha := \{\bigcup_{n<\alpha} Y_n \mid \forall n < \omega\ Y_n \in \bigcup_{\beta<\alpha} \Pi^0_\beta\}$
- $\Delta^0_\alpha := \Sigma^0_\alpha \cap \Pi^0_\alpha$.

Observe that $Borel(X) = \bigcup_{\alpha<\omega_1} \Sigma^0_\alpha$. The family of closed sets is Π^0_1. The elements of Σ^0_2 are usually called F_σ, and those of Π^0_2 are called G_δ. If Y is a Borel subset of X and α is the least ordinal such that $Y \in \Sigma^0_\alpha$ (or $Y \in \Pi^0_\alpha$), then by Σ^0_α (resp. Π^0_α) we will denote the *complexity* of Y, or the complexity of its *definition*. On a higher level of complexity of *definability* subsets of Polish spaces are *analytic* sets, i.e. continuous images of Borel sets, denoted Σ^1_1, and *coanalytic* sets, i.e. complements of analytic sets, denoted Π^1_1.

Note that $\mathcal{P}(X)$, for a countable set X, can be identified through characteristic functions with the Cantor space $2^X \cong 2^\omega$. Hence ideals on the set X can be described according to their definability and their complexity. The ideal fin of finite subsets of ω is an F_σ subset of $\mathcal{P}(\omega)$, for example. Borel and analytic ideals of countable sets have been extensively studied (see, for example, [17]). For general descriptive set theory see [14].

1.4 Trees

A partially ordered set $(T, \leqslant)$ will be called a *tree* if $t\downarrow := \{s \in T \mid s < t\}$ is a well-ordered set, for all $t \in T$. By definition trees have a well defined rank function called *height*, the height of t is the order-type (ordinal) of the set $t\downarrow$. The height of T is the least ordinal α such that there is no element of T of height α. A tree will be called *Hausdorff* if $t\downarrow \neq s\downarrow$, for all different $t, s \in T$.

All trees relevant to this text will be Hausdorff. If T is a Hausdorff tree, then there exist ordinals κ and λ (which is the height of T) such that T can be identified with a subtree of the tree $(\kappa^{<\lambda}, \subseteq)$, i.e. $T \subseteq \kappa^{<\lambda}$ and $\sigma \restriction \alpha \in T$, for all $\sigma \in T$ and $\alpha \in dom(\sigma)$. If $\sigma \in T$, then its height will be simply equal to $dom(\sigma)$.

Now some notations useful for Hausdorff trees is set. If $\alpha < \lambda$, the α-th level of T is the set

$$T_\alpha := \{\sigma \in T \mid dom(\sigma) = \alpha\}.$$

Accordingly the restriction to α is the subtree of

$$T_{<\alpha} := \{\sigma \in T \mid dom(\sigma) < \alpha\}.$$

The (cofinal) *branches* of T are the elements of

$$[T] := \{f \in \kappa^\lambda \mid \forall \alpha < \lambda\ f \restriction \alpha \in T\}.$$

We will say that T is *well-pruned* if for all $\sigma \in T$ and all $dom(\sigma) < \alpha < \lambda$ there exists $\tau \in T_\alpha$ such that $\sigma \subseteq \tau$.[2] A tree $T \subseteq \omega^{<\omega}$ is called *well-founded*

[2]For the purposes of this text this definition will be enough. For a more standard definition of well-pruned tree see [15].

if $(T, \supseteq)$ is a well-founded poset, i.e. all of its non-empty subsets have a minimal element.

A tree $T \subseteq \omega^{<\omega_1}$ is called an *Aronszajn* tree if $0 < |T_\alpha| \leqslant \omega$, for all $\alpha < \omega_1$, and $[T] = \emptyset$. The existence of Aronszajn trees is provable in ZFC. Basic and abundant information on this subject can be found in Chapter III of [15].

1.5 Forcing

For basics of the method of forcing, and the notation used in this text, the reader is referred to [15]. For the combinatorics of several generic extensions see [5]. Just some definitions of some of the forcing notions relevant to this text will be given.

By the *Cohen* forcing, $\mathbb{C}$ we usually understand the set $\{p; \omega \to 2 \mid |p| < \omega\}$, where $p; X \to Y$ stands for "p is a function such that $dom(p) \subseteq X$, is finite and $im(p) \subseteq Y$", with the order relation $q \leqslant p$ iff $p \subseteq q$. The fact that all atomless countable forcing notions, like $(2^{<\omega}, \supseteq)$ and $(\omega^{<\omega}, \supseteq)$, are forcing equivalent to Cohen forcing will be used in this text.

By *Mathias* forcing $\mathbb{M}$ we mean the set $\{(s, X) \mid s \in [\omega]^{<\omega} X \in [\omega]^\omega\}$ with the order relation $(t, Y) \leqslant (s, X)$ iff $t \supseteq s$, $Y \subseteq X$ and $t \setminus s \subseteq X$. If $\mathcal{U} \subseteq [\omega]^\omega$ by $\mathbb{M}_\mathcal{U}$ it will be denoted the set $\{(s, X) \mid s \in [\omega]^{<\omega} X \in \mathcal{U}\}$ with the same order relation as Mathias forcing. This is the parameterized version of Mathias forcing.

By the *Hechler* forcing $\mathbb{H}$ we understand the set $\{(s, f) \mid s \in \omega^{<\omega} f \in \omega^\omega\}$ with the order relation $(t, g) \leqslant (s, f)$ iff $t \supseteq s$, $g \geqslant f$ and $t(i) \geqslant f(i)$, for all $i \in dom(t) \setminus dom(s)$.

A forcing notion $\mathbb{P}$ will be called *σ-centered* if $\mathbb{P} = \bigcup_{n<\omega} P_n$, where P_n is *centered*, i.e. if $p_0, ..., p_{k-1} \in P_n$, then there exists $r \in \mathbb{P}$ such that $r \leqslant p_i$, for all $i < k$ and $n < \omega$. By cardinality, Cohen forcing is σ-centered. Hechler forcing is σ-centered as witnessed by the sets $\{(s, f) \mid f \in \omega^\omega\}$, for $s \in \omega^{<\omega}$. If $\mathcal{U}$ is a filter in ω, then $\mathbb{M}_\mathcal{U}$ is σ-centered as witnessed by the sets $\{(s, X) \mid X \in \mathcal{U}\}$, for $s \in [\omega]^{<\omega}$.

Chapter 2

Combinatorics of Boolean algebras and ideals

In this chapter we introduce some combinatorial concepts on Boolean algebras and ideals with focus on related cardinal invariants most of which can be found in [18] or [17]. A word is said about the classic applications of these concepts and cardinals to the Boolean algebra $\mathcal{P}(\omega)/fin$, typically known as the *cardinal characteristics of the continuum.* Finally, some known relations of these cardinal invariants are presented in the general case.

2.1 Cardinal invariants of infinite Boolean algebras

Besides the order relation $\leqslant$ defined in Section 1.1, other simple notions that help to study the combinatorial structure of a Boolean algebra are defined next. Given a Boolean algebra A, and two elements $a, b \in A^+$, we will say that a and b are *disjoint* if $a \cdot b = 0$. We will say that a *splits* b if $b \cdot a \neq 0 \neq b \cdot (-a)$. If $P \subseteq A^+$, we will say that P is a *centered family* if $\prod_{i<n} x_i \neq 0$, for any non-empty finite collection of elements $x_0, ..., x_{n_1} \in P$. It will be called a *(pairwise) disjoint family* if a and b are disjoint, for all $a, b \in P$. Whenever P is a centered family, if $x \in A^+$ and $x \leqslant a$, for all $a \in P$, x will be called a *pseudointersection* of P. Two families $C, D \subseteq A^+$ will be called *orthogonal* if x and y are disjoint, for all $x \in C$ and all $y \in D$. Some special families arising from these simple concepts are the following.

Definition 2.1.1. *Let A be a Boolean algebra.*

- *A partition of A is a disjoint family $P \subseteq A^+$ which is maximal with respect to this property.*
- *A splitting family of A is a subset $P \subseteq A^+$ such that all positive element of A is split by some element of P.*

- *An unreaped family of A is a subset $P \subseteq A^+$ such that no element of A splits every element of P.*
- *A tower of A is a subset $P \subseteq A^+$ that is well-ordered by $\geqslant$ and has no positive lower bounds in $\leqslant$ (i.e. it has no pseudointersections).*
- *A Rothberger gap of A consists of a pair (C, D) of orthogonal families of A^+ such that C is a countable disjoint family and there is no $x \in A^+$ which is disjoint to all element of C and an upper bound to all element of D.*

Each one of these kinds of subfamilies of Boolean algebras is related to a *cardinal invariant.*

Definition 2.1.2. *Let A be an infinite Boolean algebra. Some of its combinatorial cardinal invariants are the following:*

- $\mathfrak{a}(A) := \min\{|P| : P \subseteq A^+ \textit{ is an infinite partition}\}$.
- $\mathfrak{p}(A) := \min\{|P| : P \subseteq A^+ \textit{ is centered with no pseudointersection}\}$.
- $\mathfrak{t}(A) := \min\{|P| \mid P \subseteq A^+ \textit{ is a tower}\}$.
- $\mathfrak{s}(A) := \min\{|P| \mid P \subseteq A^+ \textit{ is a splitting family}\}$.
- $\mathfrak{r}(A) := \min\{|P| \mid P \subseteq A^+ \textit{ is an unreaped family}\}$.
- $\mathfrak{b}(A) := \min\{|D| : \exists C \in [A^+]^\omega\ (C, D) \textit{ is a Rothberger gap}\}$.

Relations between any pair of these cardinals will be detailed in Section 2.4. To simplify notation, in the often studied case of $A := \mathcal{P}(X)/\mathcal{I}$, where $\mathcal{I}$ is an ideal on a given countable set X, these cardinal invariants will be denoted here $\mathfrak{a}(\mathcal{I})$, $\mathfrak{p}(\mathcal{I})$, $\mathfrak{t}(\mathcal{I})$, $\mathfrak{s}(\mathcal{I})$, $\mathfrak{r}(\mathcal{I})$ and $\mathfrak{b}(\mathcal{I})$, respectively. For the case of $\mathfrak{a}(\mathcal{I})$ some more definitions will be useful.

Definition 2.1.3. *Take an ideal $\mathcal{I} \subseteq \mathcal{P}(X)$, of some countable set X, and $A, B \in \mathcal{I}^+$. If $A \cap B \in \mathcal{I}$, it will be said that A and B are $\mathcal{I}$-almost disjoint. A subfamily of $\mathcal{I}^+$ consisting of pairwise $\mathcal{I}$-almost disjoint sets is called an $\mathcal{I}$-ad family. If this family is maximal with respect to this property, it will be called an $\mathcal{I}$-mad family. The cardinal $\mathfrak{a}(\mathcal{I})$ is defined as the smallest size of an infinite $\mathcal{I}$-mad family.*

Observe that if $P \subseteq \mathcal{I}^+$ is an $\mathcal{I}$-mad family, then it consists of representatives of an infinite partition of $\mathcal{P}(X)/\mathcal{I}$. Therefore, as defined in the previous definition $\mathfrak{a}(\mathcal{I})$ is equal to $\mathfrak{a}(\mathcal{P}(X)/\mathcal{I})$. It is not uncommon to have $\mathfrak{a}(\mathcal{I}) = \omega$, when $\mathcal{I}$ is a definable ideal on ω. For example, $\mathfrak{a}(nwd) = \omega$, where *nwd* is the ideal of *nowhere dense* subsets of $\mathbb{Q}$, as witnessed by the family $\{[z, z+1] \mid z \in \mathbb{Z}\}$. Therefore, the following cardinal was defined, which better reflects the behaviour of $\mathfrak{a}$ and better contrasts it as well.

Definition 2.1.4. *Let $\mathcal{I}$ be an ideal on a countable set X. The cardinal $\bar{\mathfrak{a}}(\mathcal{I})$ is defined as the smallest size of an uncountable $\mathcal{I}$-mad family.*

$$\begin{tikzcd}
 & & cov(\mathcal{I}) \arrow[dr] & & \\
\omega \arrow[r] & add(\mathcal{I}) \arrow[ur] \arrow[dr] & & cof(\mathcal{I}) \arrow[r] & 2^{|X|} \\
 & & non(\mathcal{I}) \arrow[ur] & &
\end{tikzcd}$$

Figure 2.1: Inequalities between the cardinal invariants of an ideal $\mathcal{I}$ on a set X.

2.2 Cardinal invariants of ideals over the real line and ω

Given an ideal $\mathcal{I}$ on a set X some cardinal invariants can be defined:

- *Additivity:* $add(\mathcal{I}) := \min\{|F| \mid F \subseteq \mathcal{I}\ \bigcup F \notin \mathcal{I}\}$
- *Covering:* $cov(\mathcal{I}) := \min\{|F| \mid F \subseteq \mathcal{I}\ \bigcup F = X\}$
- *Uniformity:* $non(\mathcal{I}) := \min\{|Y| \mid Y \subseteq X\ Y \notin \mathcal{I}\}$
- *Cofinality:* $cof(\mathcal{I}) := \min\{|F| \mid F \subseteq \mathcal{I}\ \forall x \in \mathcal{I}\ \exists y \in F\ x \subseteq y\}$.

Since $\bigcup F \in \mathcal{I}$, for all finite $F \subseteq \mathcal{I}$, it follows that $\omega \leqslant add(\mathcal{I})$. Since it is assumed that $X \notin \mathcal{I}$, if $\bigcup F = X$, for some $F \subseteq \mathcal{I}$, then $\bigcup F \notin \mathcal{I}$. Therefore $add(\mathcal{I}) \leqslant cov(\mathcal{I})$. Take $Y \in \mathcal{I}^+$. Since $[X]^{<\omega} \subseteq \mathcal{I}$ and hence $F := \{\{x\} \mid x \in Y\} \subseteq \mathcal{I}$, it follows that $\bigcup F \notin \mathcal{I}$. Therefore $add(\mathcal{I}) \leqslant non(\mathcal{I})$. Again, since $[X]^{<\omega} \subseteq \mathcal{I}$, if F is a cofinal subset of $\mathcal{I}$, then $\bigcup F = X$. It follows that $cov(\mathcal{I}) \leqslant cof(\mathcal{I})$. Let $\{x_\alpha \mid \alpha < \kappa\} \subseteq \mathcal{I}$ be a cofinal family. Since $X \notin \mathcal{I}$, take $p_\alpha \in X \setminus x_\alpha$. Clearly $\{p_\alpha \mid \alpha < \kappa\}$ is not an element of $\mathcal{I}$. Therefore $non(\mathcal{I}) \leqslant cof(\mathcal{I})$.

Observe that if $\mathcal{I}$ is an ideal on a countable set X, then trivially $add(\mathcal{I}) = cov(\mathcal{I}) = non(\mathcal{I}) = \omega$. Only the cofinality of $\mathcal{I}$ can be some cardinal between ω and $\mathfrak{c}$. In this case the following versions of these cardinal invariants, more related to the Boolean algebra $\mathcal{P}(X)/[X]^{<\omega}$.

- (Additivity) $add^*(\mathcal{I}) := \min\{|\mathcal{F}| \mid \mathcal{F} \subseteq \mathcal{I}\ \neg\exists I \in \mathcal{I}\ \forall A \in \mathcal{F}\ A \not\subseteq^* I\}$
- (Covering) $cov^*(\mathcal{I}) := \min\{|\mathcal{F}| \mid \mathcal{F} \subseteq \mathcal{I}\ \forall B \in [X]^\omega\ \exists A \in \mathcal{F}\ |B \cap A| = \omega\}$
- (Uniformity) $non^*(\mathcal{I}) := \min\{|\mathcal{X}| \mid \mathcal{X} \subseteq [X]^\omega\ \neg\exists A \in \mathcal{I}\ \forall B \in \mathcal{X}\ |A \cap B| = \omega\}$
- (Cofinality) $cof^*(\mathcal{I}) := \min\{|\mathcal{F}| \mid \mathcal{F} \subseteq \mathcal{I}\ \forall A \in \mathcal{I}\ \exists B \in \mathcal{F}\ A \subseteq^* B\}$

$$\begin{tikzcd}
& & cov^*(\mathcal{I}) \arrow[dr] & & \\
\omega \arrow[r] & add^*(\mathcal{I}) \arrow[ur] \arrow[dr] & & cof^*(\mathcal{I}) \arrow[r] & \mathfrak{c} \\
& & non^*(\mathcal{I}) \arrow[ur] & &
\end{tikzcd}$$

Figure 2.2: Inequalities between the cardinal invariants of a tall ideal $\mathcal{I}$ on a countable set X.

Observe that $cof(\mathcal{I}) = cof^*(\mathcal{I})$, for all ideal $\mathcal{I}$ over a countable set X. These cardinal invariants are mostly studied when $\mathcal{I}$ is a tall ideal. In this case, there is an ideal on $\mathcal{P}(X) \cong 2^X \cong 2^\omega$ associated to $\mathcal{I}$:

$$\hat{\mathcal{I}} := \{\mathcal{Y} \subseteq \mathcal{P}(X) \mid \exists Z \in \mathcal{I}\ \forall Y \in \mathcal{Y}\ |Z \cap Y| = \omega\}.$$

As proved in [12], the following hold:

- $add(\hat{\mathcal{I}}) = add^*(\mathcal{I})$
- $cov(\hat{\mathcal{I}}) = cov^*(\mathcal{I})$
- $non(\hat{\mathcal{I}}) = non^*(\mathcal{I})$
- $cof(\hat{\mathcal{I}}) = cof^*(\mathcal{I})$.

Therefore, the inequalities in Figure 2.1 translate to those in Figure 2.2.

Definition 2.2.1. *Let $\mathcal{I}$ be an ideal on a countable set X.*

- *If $\omega < add^*(\mathcal{I})$, $\mathcal{I}$ will be called a P-ideal.*
- *If $\omega < non^*(\mathcal{I})$, $\mathcal{I}$ will be called ω-hitting.*

In general are called ω-hitting all subfamilies $\mathcal{F} \subseteq \mathcal{P}(X)$ such that for all sequence $\{Y_n \mid n < \omega\} \subseteq [X]^\omega$ there exists $Y \in \mathcal{F}$ such that $|Y \cap Y_n| = \omega$, for all $n < \omega$.

2.3 Cardinal characteristics of the continuum

The classic case for studying all cardinal invariants defined in Section 2.1 is for the Boolean algebra $\mathcal{P}(\omega)/fin$. Indeed all of them were originally defined in that case and are simply denoted $\mathfrak{a}$, $\mathfrak{p}$, $\mathfrak{t}$, $\mathfrak{s}$, $\mathfrak{r}$ and $\mathfrak{b}$. Among several others, these cardinal numbers are known as the *cardinal characteristics of the continuum*. Examples include the cardinal invariants defined on countable ideals in the previous section as well as those defined over the σ-ideals $\mathcal{N}$ and

$$\begin{tikzcd}
& & \mathfrak{s} & \\
\omega_1 \arrow[r] & \mathfrak{p} = \mathfrak{t} \arrow[ur] \arrow[dr] & & \mathfrak{a} \\
& & \mathfrak{b} \arrow[ur] \arrow[dr] & \\
& & & \mathfrak{r}
\end{tikzcd}$$

Figure 2.3: Inequalities provable in ZFC between some cardinal characteristics of the continuum.

$$\begin{tikzcd}
& cov(\mathcal{N}) \arrow[r] & non(\mathcal{M}) \arrow[r] & cof(\mathcal{M}) \arrow[r] & cof(\mathcal{N}) \arrow[r] & \mathfrak{c} \\
\omega_1 \arrow[r] & add(\mathcal{N}) \arrow[r] \arrow[u] & add(\mathcal{M}) \arrow[r] \arrow[u] & cov(\mathcal{M}) \arrow[r] \arrow[u] & non(\mathcal{N}) \arrow[u] &
\end{tikzcd}$$

Figure 2.4: Partial version of Cichon's diagram.

$\mathcal{M}$ of respectively null sets and meager sets of the real line, as well as some defined on related structures as $(\omega^\omega, \leqslant^*)$ (where $f \leqslant^* g$ if $f(n) \leqslant g(n)$, for almost all $n < \omega$, for $f, g \in \omega^\omega$). These are cardinal numbers lying between ω_1 (or ω) and $\mathfrak{c}$, which is the cardinality of the real line, i.e. the *continuum*.

Since the *Continuum Hypothesis* (CH), i.e. the equality $\omega_1 = \mathfrak{c}$, is known to be consistent and implies the equality of all cardinal characteristics of the continuum, the study of these cardinal numbers consists mainly in assessing which inequalities between them are provable in ZFC alone and which values they can consistently have.

In Figures 2.3 and 2.4 summarize most of the inequalities on the classical cardinal characteristics of the continuum used in this text. As it will be seen in Subsection 2.4 some of the inequalities in Figure 2.3 arise from the general combinatorics of infinite Boolean algebras and some of them are related to the properties of $\mathcal{P}(\omega)/fin$.

Cardinal $\mathfrak{b}$ is usually defined as the least size of an *unbounded* family of ω^ω, i.e. a family $\mathcal{X} \subseteq \omega^\omega$ such that for all $f \in \omega^\omega$ there exists $g \in \mathcal{X}$ such that $g \nleqslant^* f$.

Cardinal $\mathfrak{p}$ can also be defined as the least cardinal κ such that there exist a σ-centered forcing $\mathbb{P}$ and $\mathcal{D} \in [\mathbb{P}]^\kappa$ a family of dense sets of $\mathbb{P}$ such that there exists no $\mathcal{D}$-generic filter. A similar cardinal in this regard is $cov(\mathcal{M})$. It can be defined as the least cardinal such that there exists $\mathcal{D} \in [\mathbb{P}]^\kappa$ a family of dense sets of Cohen forcing such that there exists no $\mathcal{D}$-generic filter. Proofs of the equivalence of these definitions can be found in [3].

The provability of $\mathfrak{p} = \mathfrak{t}$ in ZFC remained a relevant open problem until solved in [16]. Every other strict inequality consistent with Figure 2.3 is consistent with ZFC. For a general overview of these and other cardinal characteristics see [3].

2.4 Inequalities between cardinal invariants of Boolean algebras

The relations of the cardinal invariants defined in Section 2.1 are studied now, as in the case of $\mathcal{P}(\omega)/fin$ discussed in the previous section. Also their conditions for being defined will be described.

If A is an infinite Boolean algebra, it always has an infinite disjoint family. Indeed, if A has infinitely many atoms, they form an infinite disjoint family. Suppose that A has finitely many atoms, so without loss of generality A is atomless. We can easily construct a strictly decreasing family $\{x_n \mid n < \omega\}$. It follows that defining $y_n := x_n \cdot (-x_{n+1})$, for $n < \omega$, then $\{y_n \mid n < \omega\}$ is a disjoint family. In any case, by Zorn's Lemma, extending any infinite disjoint family to a maximal one we get an infinite partition. Therefore $\mathfrak{a}(A)$ is always defined. Observe that if P is an infinite partition of A, then $\{-x \mid x \in P\}$ is a centered family with no pseudointersection. Also, if $P \subseteq A^+$ is a finite centered family, $\prod P$ is a pseudointersection of P. Therefore $\mathfrak{p}(A)$ is also well defined and $\omega \leqslant \mathfrak{p}(A) \leqslant \mathfrak{a}(A)$, for all infinite Boolean algebras.

When defined, $\mathfrak{t}(A)$ has to be an infinite regular cardinal. The cardinal $\mathfrak{t}(A)$ is not always defined. Consider the set algebra

$$A := \{X \subseteq \omega_1 \mid |X| < \omega \vee |\omega_1 \setminus X| < \omega\}.$$

We claim that A has no towers. Take $P := \{x_\alpha \mid \alpha < \kappa\} \subseteq A^+$ such that $x_\beta \subseteq x_\alpha$, for all $\alpha < \beta < \kappa$, for some infinite regular cardinal κ. Suppose that there exists a sequence $\{\alpha_n \mid n < \omega\}$ such that $x_{\alpha_{n+1}} \subsetneq x_{\alpha_n}$, for all $n < \omega$. If $\sup\{\alpha_n \mid n < \omega\} = \kappa$, take $\gamma \in \bigcap_{n<\omega} x_{\alpha_n} = \bigcap_{\alpha<\kappa} x_\alpha$. The set $\{\gamma\}$ witnesses that P is not a tower. If $\alpha := \sup\{\alpha_n \mid n < \omega\} < \kappa$, then x_α is a finite set and w.l.o.g $x_\beta = x_\alpha$, for all $\alpha \leqslant \beta < \kappa$. The same happens for some $\alpha < \kappa$ if no such increasing sequence exists. In either case, such x_α witnesses that P is not a tower.

For infinite Boolean algebras A with no towers, $\mathfrak{t}(A)$ is said to be equal to ∞. Since any tower of an infinite Boolean algebra A is a centered family with no pseudointersection, either if $\mathfrak{t}(A)$ is a regular infinite regular cardinal or if it is ∞, the inequality $\mathfrak{p}(A) \leqslant \mathfrak{t}(A)$ holds.

Let A be an atomless Boolean algebra and take $L \subseteq A^+$ a maximal set such that $(L, \leqslant)$ is a lineal order. Suppose that $x \in A^+$ is a lower bound for L. Since A is atomless, take $y \lneq x$. Then $(L \cup \{y\}, \leqslant)$ is a linear order, which is a contradiction. Therefore L has no lower bounds and any of its coinitial subsets T such that $(T, \geqslant)$ is a well ordered set is a tower of A, and $\mathfrak{t}(A)$ is well defined for any atomless Boolean algebra A.

Take an infinite Boolean algebra A and suppose that $x \in A$ is an atom. On the one hand x witnesses that A has no splitting subfamilies, and that $\mathfrak{s}$ is not well defined. On the other hand, $\{x\}$ is an unreaped family witnessing that $\mathfrak{r}(A) = 1$. However, if the Boolean algebra A is atomless, then A itself is a splitting family and an unreaped family. Also any finite subset of A^+ is split by one element of A^+. Therefore, both $\mathfrak{s}(A)$ and $\mathfrak{r}(A)$ are well defined and non-trivial in this case.

It will be proved that $\mathfrak{t}(A) \leqslant \mathfrak{s}(A)$. Suppose that A is an atomless Boolean algebra and take $\{x_\alpha \mid \alpha < \kappa\} \subseteq A^+$, for some $\kappa < \mathfrak{t}(A)$. We will inductively construct a family $\{y_\alpha \mid \alpha < \kappa\} \subseteq A^+$ such that $y_\beta \lneq y_\alpha$, for all $\alpha < \beta < \kappa$. Suppose we have constructed $\{y_\alpha \mid \alpha < \beta\}$, for some $\beta < \kappa$. Since $\kappa < \mathfrak{t}(A)$ and A is atomless, take y' such that $y' \lneq y_\alpha$, for all $\alpha < \beta$. Define y_β as $y' \cdot x_\beta$ if it is not equal to 0, and as $y' \cdot (-x_\beta)$ otherwise. Take $y \in A^+$ such that $y \lneq y_\alpha$, for all $\alpha < \kappa$. Then, by construction, y witnesses that $\{x_\alpha \mid \alpha < \kappa\}$ is not a splitting family.

An infinite Boolean algebra A is said to be a σ-algebra if $\sum P$ exists, for all $P \in [A]^\omega$. It is clear that if A is a σ-algebra, then there are no Rothberger gaps in A. Otherwise if A is not a σ-algebra and $C \in [A]^\omega$ witnesses it, then (C, D), where $D := \{x \in A \mid \forall y \in C\ x \cdot y = 0\}$, is a Rothberger gap. Therefore $\mathfrak{b}(A)$ is well defined if and only if A is not a σ-algebra. It is clear that $\omega \leqslant \mathfrak{b}(A)$.

Centered families, towers and pseudointersections have corresponding dual concepts which are sometimes useful.

Definition 2.4.1. *Let A be a Boolean algebra.*

- *A non-empty set $P \subseteq A^+$ will be said to have the finite union property (f.u.p) if $\sum_{i<n} x_i \neq 1$, for any non-empty finite collection $x_0, \ldots, x_{n-1} \in P$.*
- *If $P \subseteq A^+$ has the finite union property and $x \in A \setminus \{1\}$ is such that $a \leqslant x$, for all $a \in P$, x will be called a pseudounion of P.*
- *An increasing tower of A is a non-empty subset $P \subseteq A \setminus \{1\}$ that is well-ordered with $\leqslant$ and has no pseudounion.*

With these concepts alternative definitions of $\mathfrak{p}(A)$ and $\mathfrak{t}(A)$:

- $\mathfrak{p}(A) := \{|P| \mid P \subseteq A$ *has the f.u.p and no pseudounion*$\}$
- $\mathfrak{t}(A) := \{|P| \mid P \subseteq A$ *is an increasing tower*$\}$.

The cardinal invariants $\mathfrak{p}(A)$, $\mathfrak{t}(A)$ and $\mathfrak{a}(A)$ are related to the existence of suprema and infima of infinite subfamilies of A. For example, if $\{x_\alpha \mid \alpha < \mathfrak{a}(A)\}$ is a partition of A, then $\sum_{\alpha<\mathfrak{a}(A)} x_\alpha = 1$. The same happens for the other two cardinal invariants. Suppose that $\sum P = x$ exists non-trivially, for some $P \in [A]^{\geqslant\omega}$, i.e. $\sum F \neq x$, for all $F \in [P]^{<\omega}$. Then $P \cup \{-x\}$ is a

$$\begin{tikzcd}
& \mathfrak{p}(A) \arrow[r] \arrow[dr] & \mathfrak{t}(A) \arrow[r] & \mathfrak{s}(A) \\
\omega \arrow[ur] \arrow[r] \arrow[dr] & \mathfrak{r}(A) & \mathfrak{a}(A) & \\
& \mathfrak{b}(A) & &
\end{tikzcd}$$

Figure 2.5: Known inequalities for cardinal invariants for some atomless Boolean algebra A.

family with the finite union property and no pseudounion, and $\mathfrak{p}(A) \leqslant |P|$. Therefore, when no trivial, $\sum P$ is never defined for infinite P of size less than $\mathfrak{p}(A)$. Similar statements hold for $\sum P$, when $P \subseteq A \setminus \{1\}$ is a strictly increasing family with no last element, or a disjoint family.

Hitherto, if A is an atomless Boolean algebra, the known relations between the cardinal invariants defined in this section are summarized in Figure 2.6. From now on other possible lines in this diagram will be discussed. Unless stated otherwise, all cardinal invariants of the following examples were calculated by the author.

Example 2.4.2. *Let $\{X_n \mid n < \omega\}$ be a family of disjoint copies of $\beta\omega \setminus \omega$ and take $p \in X_0$. Define $X := \bigcup_{n<\omega} X_n$,*

$$U := \{a \in \mathcal{P}(X) \mid (\forall n < \omega)\,(a \cap X_n \in \mathit{clop}\,(X_n)) \wedge$$
$$(\forall^\infty n < \omega)\,(a \cap X_n = X_n) \wedge p \in a \cap X_0\}$$

and

$$I := \{a \in P(X) \mid (\forall n < \omega)\,(a \cap X_n \in \mathit{clop}\,(X_n)) \wedge$$
$$(\forall^\infty n < \omega)\,(a \cap X_n = \emptyset) \wedge p \notin a \cap X_0\}.$$

Define $A := I \cup U$ as a subalgebra of $\mathcal{P}(X)$.

Observe that $X_n \in A$, for all $0 < n < \omega$, while $X_0 \notin A$. This Boolean algebra was defined in [22], where it was proved that $\omega_1 \leqslant \mathfrak{p}(A)$.

Proposition 2.4.3. *Take A as defined in Example 2.4.2. Then $\mathfrak{r}(A) = \omega$.*

Proof. Consider the family $\{X_n \mid 1 \leqslant n < \omega\}$ and take $x \in A$. If $x \in I$, there exists $1 \leqslant n < \omega$ such that $x \cap X_n = \emptyset$. If $x \in U$, there exists $1 \leqslant n < \omega$ such that $X_n \subseteq x$. In either case x does not split every element of $\{X_n \mid n < \omega\}$, which proves that $\mathfrak{r}(A) = \omega$. ∎

If V is a model of CH and $\mathbb{C}^{\omega_2}$ is the forcing notion for adding ω_2 many Cohen reals, then $V^{\mathbb{C}^{\omega_2}} \models \mathfrak{a} = \mathfrak{b} = \mathfrak{s} = \omega_1 \wedge \mathfrak{r} = \omega_2$ (see [5], Theorem 1.13 and [15]). This fact with Example 2.4.2 and the previous proposition show that $\mathfrak{r}(A)$ is in general incomparable to $\mathfrak{s}(A), \mathfrak{t}(A), \mathfrak{a}(A), \mathfrak{p}(A)$, for all atomless Boolean algebras A.

Proposition 2.4.4. *Let A and B be two infinite Boolean algebras. Then $\mathfrak{b}(A \oplus B) = \omega$.*

Proof. Let X and Y be Boolean spaces such that $clop(X) = A$ and $clop(Y) = B$. Let $\{a_n : n < \omega\} \subseteq A^+$ and $\{b_n :< \omega\} \subseteq B^+$ be disjoint families. Then $C := \{a_n \times b_n : n < \omega\}$ and $D := \{a_n \times b_m : m, n < \omega, n \neq m\}$ are orthogonal families of $A \oplus B$. Suppose that $x := \sum_{i<k} x_i \times y_i$, with $k < \omega$, is an element of $A \oplus B$ such that $a_n \times b_n \subseteq x$, for all $n < \omega$. Then there exist $n < m < \omega$ and $i < k$ such that both $(x_i \times y_i) \cap (a_n \times b_n) \neq \emptyset$ and $(x_i \times y_i) \cap (a_m \times b_m) \neq \emptyset$. It follows that $x \cap a_n \times b_m \neq \emptyset$. Therefore (C, D) is a Rothberger gap proving that $\mathfrak{b}(A \oplus B) = \omega$. ■

Take $A := \mathcal{P}(\omega)/fin \oplus \mathcal{P}(\omega)/fin$. From the previous observation it follows that $\mathfrak{b}(A) = \omega$. The following theorem was proved in [22].

Theorem 2.4.5. *Let A and B be infinite Boolean algebras. Then $\mathfrak{p}(A \oplus B) = \min\{\mathfrak{p}(A), \mathfrak{p}(B)\}$.*

It will be proved in Section 3.1 that $\mathfrak{r} = \mathfrak{r}(A)$. Therefore A is an example of $\mathfrak{b}(A)$ being smaller than all other cardinal invariants in Figure 2.6.

Example 2.4.6. *Define $A := (\mathcal{P}(\omega)/fin)^{\omega_1}$. Then $\mathfrak{a}(A) = \omega$ and $\mathfrak{b}(A) = \mathfrak{b}$.*

Proof. For $n < \omega$, define $f_n \in A$ such that $f_n(n) = 1$ (where represents 1 is the maximum class of $\mathcal{P}(\omega)/fin$), for all $n < \omega$, and $f_n(\alpha) = 0$, for all $n < \omega$ and all $\alpha < \omega_1$ such that $n \neq \alpha$. Define also $f \in A$ such that $f(n) = 0$, for all $n < \omega$, and $f(\alpha) = 1$, for all $\omega \leqslant \alpha < \omega_1$. Then $\{f_n \mid n < \omega\} \cup \{f\}$ is a partition of A witnessing that $\mathfrak{a}(A) = \omega$[1].

Let $C := \{g_n \mid n < \omega\}$ and $D := \{h_\alpha \mid \alpha < \kappa\}$ be orthogonal families in A, where $\kappa < \mathfrak{b}$. Define

$$K_0 := \{\alpha < \kappa \mid \exists n_\alpha < \omega \ \forall m < \omega \ (g_m(\alpha) \subseteq^* \bigcup_{i<n_\alpha} g_i(\alpha))\}$$

and $K_1 := \kappa \setminus K_0$. Define $g \in A$ as follows:

- If $\alpha \in K_0$, then $g(\alpha) = \bigcup_{i<n_\alpha} g_i(\alpha)$.
- If $\alpha \in K_1$, then take $g(\alpha)$ as an upper bound of $\{g_n(\alpha) \mid n < \omega\}$, disjoint with $h_\beta(\alpha)$, for all $\beta < \kappa$. This element exists because $\kappa < \mathfrak{b}$.

Then g witnesses that (C, D) is not a gap. Therefore $\kappa < \mathfrak{b}(A)$ and $\mathfrak{b} \leqslant \mathfrak{b}(A)$.

Take $\{x_n \mid n < \omega\}$ and $\{y_\beta \mid \beta < \mathfrak{b}\}$ forming a gap in $\mathcal{P}(\omega)/fin$. Define $g_n \in A$, for $n < \omega$, such that $g_n(0) = x_n$, and $g_n(\alpha) = 0$, for all $0 < \alpha < \omega_1$. Define accordingly $h_\beta \in A$, for $\beta < \mathfrak{b}$, such that $h_\beta(0) = y_\beta$, and $h_\beta(\alpha) = 1$, for all $0 < \alpha < \omega_1$. Then $\{g_n \mid n < \omega\}$ and $\{h_\beta \mid \beta < \mathfrak{b}\}$ form a gap witnessing that $\mathfrak{b}(A) = \mathfrak{b}$. ■

[1]Observe that this is the case for all infinite product $\prod_{i<\kappa} A_i$. A proof can be found in Proposition 3.39 of [18].

If V is a ZFC model and $\mathbb{H}_{\omega_2}$ is the forcing notion for adding ω_2 many Hechler reals, then $V \models \mathfrak{s} = \omega_1 < \mathfrak{b} = \omega_2$ (see [5], Theorem 3.13). This and the previous example, with Example 2.4.4, show that $\mathfrak{b}(A)$ is in general incomparable with $\mathfrak{p}(A)$, $\mathfrak{a}(A)$ and $\mathfrak{s}(A)$.

The Hechler model with $A = \mathcal{P}(\omega)/fin$ provides an example of $\mathfrak{s}(A), \mathfrak{t}(A) < \mathfrak{a}(A)$. The consistency of $\mathfrak{a} < \mathfrak{s}$ has also been proved (see [6], Theorem 17).

Example 2.4.7. *Consider the Boolean algebra*

$$A := \{f \in (\mathcal{P}(\omega)/fin)^{\omega_1} \mid |\{\alpha < \omega_1 \mid f(\alpha) \neq 1\}| < \omega \vee$$

$$|\{\alpha < \omega_1 \mid f(\alpha) \neq 0\}| < \omega\}.$$

Then $\mathfrak{a}(A) = \omega_1$ *and* $\mathfrak{t}(A) = \mathfrak{t}$*.*

Proof. Define $f_\alpha \in A$, for $\alpha < \omega_1$, such that $f_\alpha(\alpha) = 1$ and $f_\alpha(\beta) = 0$, for all $\beta < \omega_1$ different from α. Clearly $\{f_\alpha \mid \alpha < \omega_1\}$ is a partition of A proving that $\mathfrak{a}(A) \leqslant \omega_1$. If $\{g_n \mid n < \omega\} \subseteq A^+$ is a disjoint family. Suppose that for all $\alpha < \omega_1$ there exists $F_\alpha \in [\omega]^{<\omega}$ such that

$$\bigcup_{n \in F_\alpha} g_n(\alpha) =^* \omega.$$

By definition of A, there exists $n_0 < \omega$ such that $F := \{\alpha < \omega_1 \mid f_{n_0}(\alpha) \neq 1\}$ is finite. Therefore $F_\alpha = \{n_0\}$, for all $\alpha \in \omega_1 \setminus F$. But this means that

$$\sum_{n \in \bigcup_{\alpha \in F} F_\alpha} g_n + g_{n_0} = 1,$$

which is a contradiction. Then there exists $\alpha < \omega_1$ such that

$$\bigcup_{n \in F} g_n(\alpha) \neq^* \omega,$$

for all $F \in [\omega]^{<\omega}$. Take $X \in [\omega]^\omega$ witnessing that $\{g_n(\alpha) \mid n < \omega\}$ can not be a partition of $\mathcal{P}(\omega)/fin$. Define $g \in A$, such that $g(\alpha) = X$ and $g(\beta) = 0$, for all $\beta \in \omega_1 \setminus \{\alpha\}$. Then g witnesses that $\{g_n \mid n < \omega\}$ is not a partition of A. Therefore $\mathfrak{a}(A) = \omega_1$.

Suppose that $\{X_\alpha \mid \alpha < \mathfrak{t}\} \subseteq [\omega]^\omega$ is a tower of $\mathcal{P}(\omega)/fin$. For $\alpha < \mathfrak{t}$, define $g_\alpha \in A$ such that $g_\alpha(0) = X_\alpha$ and $g_\alpha(\beta) = 0$, for all $0 < \beta < \omega_1$. Then $\{g_\alpha \mid \alpha < \mathfrak{t}\}$ forms a tower of A. Therefore $\mathfrak{t}(A) \leqslant \mathfrak{t}$.

Take $\kappa < \mathfrak{t}$ and a decreasing family $\{g_\alpha \mid \alpha < \kappa\} \subseteq A^+$. For $\alpha < \kappa$, define $K_\alpha := \{\beta < \omega_1 \mid g_\alpha(\beta) = 1\}$. Notice that $K_{\alpha_1} \subseteq K_{\alpha_0}$, for all $\alpha_0 < \alpha_1 < \kappa$. Since K_α can only be finite or cofinite, there exists $\alpha < \kappa$ such that $K_\alpha = K_\beta$, for all $\alpha \leqslant \beta < \kappa$. So, without loss of generality, $K_\alpha = K$, for some $K \subseteq \omega_1$, for all $\alpha < \kappa$. For $\beta \in \omega_1 \setminus K$ take $X_\beta \subseteq \omega$ (infinite when possible) such that $X_\beta \subseteq g_\alpha(\beta)$, for all $\alpha < \kappa$. For $\beta \in K$ put $X_\beta = \omega$. Take $g \in A$ such that $g(\beta) = X_\beta$. Then g is a positive element which is a lower bound proving that $\{g_\alpha \mid \alpha < \kappa\}$ is not a tower. Therefore, $\mathfrak{t}(A) = \mathfrak{t}$ ∎

Both of these cardinal invariants were calculated for the sake of completeness, and are specific versions of Propositions 3.38 and 4.38 of [18].

If A is the Boolean algebra of the previous example, then any model of $\omega_1 < \mathfrak{t}$ (a model of MA + ¬CH, for example), gives an example of $\mathfrak{a}(A) < \mathfrak{t}(A)$. Then for all other possible lines in Figure 2.6, there is some counterexample, with the exception of $\mathfrak{b}(A) \to \mathfrak{r}(A)$. The existence of any such counterexample is unknown for the author.

One known instance of trivial equality between these cardinal invariants is given by the following observation.

Observation 2.4.8. *Let A be an infinite Boolean algebra. Then $\mathfrak{p}(A) = \omega$ iff $\mathfrak{t}(A) = \omega$ iff $\mathfrak{a}(A) = \omega$.*

Proof. If $\{a_n \mid n < \omega\}$ is a is centered family of A with no pseudointersection, then $\{b_n \mid n < \omega\}$, where $b_n := \prod_{i \leqslant n} a_i$ for each $n < \omega$, is a tower in A; if $\{b_n \mid n < \omega\}$ is an increasing tower in A, then $\{c_n \mid n < \omega\}$, where $c_n := b_n \cdot (-\sum_{i<n} b_i)$ for each $n < \omega$, is an infinite partition of A; and if $\{c_n \mid n < \omega\}$ is an infinite partition of A, then $\{-x \mid x \in P\}$ is a centered family of A. ■

Another relation between the cardinal invariants defined in this text arises when $\omega_1 \leqslant \mathfrak{a}(A)$.

Proposition 2.4.9. *Let A be an infinite Boolean algebra which is not a σ-algebra. Then $\mathfrak{b}(A) \leqslant \mathfrak{a}(A)$, if either $\omega_1 \leqslant \mathfrak{a}(A)$ or A has only countable infinite partitions*[2].

Proof. If $\mathfrak{a}(A) \geqslant \omega_1$ and $\{a_\alpha : \alpha < \kappa\}$ is an infinite partition of size $\kappa := \mathfrak{a}(A)$, then $\mathcal{A} := \{a_n : n < \omega\}$ and $\mathcal{B} := \{a_\alpha : \alpha \in \kappa \setminus \omega\}$ form a Rothberger gap: otherwise if there is $c \in A^+$ such that $a_n \cdot c = 0$, for all $n < \omega$ and $a_\alpha \leqslant c$, for all $\alpha \in \kappa \setminus \omega$, then $\mathcal{A} \cup \{c\}$ is a partition of A, which is a contradiction. Therefore $\mathfrak{b}(A) \leqslant \mathfrak{a}(A)$.

If A has only countable partitions, take a disjoint family $C := \{a_n \mid n < \omega\}$ such that $\sum_{n<\omega} a_n$ does not exist. Since A has only countable partitions, take $D := \{b_n \mid n < \omega\}$ such that $C \cup D$ is a partition. Any element $x \in A^+$ proving that (C, D) is not a Rothberger gap would prove that $\sum_{n<\omega} a_n$ exists. Therefore, (C, D) is a Rothberger gap proving that $\mathfrak{b}(A) = \omega = \mathfrak{a}(A)$. ■

This section ends with a result on inequalities when considering certain types of subalgebras.

Proposition 2.4.10. *Let A and B be atomless Boolean algebras such that $A \leqslant_\pi B$. Then $\mathfrak{s}(B) \leqslant \mathfrak{s}(A)$ and $\mathfrak{r}(A) \leqslant \mathfrak{r}(B)$.*

---

[2] In the language of forcing notions this property is known as the *countable chain condition.* For further information see [15]

$$
\begin{tikzcd}
 & \mathfrak{b}(A) \arrow[r] & \mathfrak{a}(A) & & \\
\omega \arrow[ur] \arrow[r] \arrow[dr] & \omega_1 \arrow[r] & \mathfrak{p}(A) \arrow[u] \arrow[r] & \mathfrak{t}(A) \arrow[r] & \mathfrak{s}(A) \\
 & \mathfrak{r}(A) & & &
\end{tikzcd}
$$

Figure 2.6: Known inequalities for cardinal invariants for atomless Boolean algebra A such that $\omega_1 \leqslant \mathfrak{a}(A)$.

Proof. Let P be a splitting family of A and take $b \in B^+$. Take $a \in A^+$ such that $a \leqslant b$. Then if $x \in P$ and x splits a, x splits b. Therefore, P is a splitting family of B and $\mathfrak{s}(B) \leqslant \mathfrak{s}(A)$.

Now take $\kappa < \mathfrak{r}(A)$ and $\{b_\alpha \mid \alpha < \kappa\} \subseteq B^+$. Take $a_\alpha \in A^+$ such that $a_\alpha \leqslant b_\alpha$, for all $\alpha < \kappa$. There exist $x \in A^+$ which splits a_α, for all $\alpha < \kappa$. Then x splits b_α, for all $\alpha < \kappa$. Therefore, $\kappa < \mathfrak{r}(B)$. ■

Chapter 3

Infinite partitions on free products

Since every Boolean algebra is isomorphic to the algebra of clopen sets of some zero-dimensional compact Hausdorff space, from now on A and B will be the algebras of clopen sets of some zero-dimensional compact Hausdorff spaces X and Y, respectively. Accordingly, $A \oplus B$ will refer to the algebra of clopen sets of the product space $X \times Y$ (see Section 1.1).

If $c \in A \oplus B$, then there exist $\{a_i : i < k\} \subseteq A$ and $\{b_i : i < k\} \subseteq B$, for $k < \omega$, such that

$$c = \bigcup_{i<k} a_i \times b_i.$$

Since the following equality holds:

$$c = \bigcup_{\emptyset \neq J \subseteq k} (\bigcap_{i \in J} a_i \setminus \bigcup_{j \in k \setminus J} a_j) \times \bigcup_{i \in J} b_i,$$

we can always assume that either $\{a_i : i < k\}$ is a disjoint family or that $\{b_i : i < k\}$ is a disjoint family.

3.1 Well behaved cardinal invariants of free products of Boolean algebras

It can be easily verified that if $P \subseteq A^+$ is a partition of A, then $\{x \times Y \mid x \in P\}$ is a partition of $A \oplus B$. Since this is true for any partition of B, we have the following lemma.

Lemma 3.1.1. *Let A and B be Boolean algebras. Then both A and B (identified respectively with $\{x \times Y \mid x \in A\}$ and $\{X \times y \mid y \in B\}$) are regular subalgebras of $A \oplus B$.*

It is natural that the combinatorial structure of both A and B be imprinted in that of $A \oplus B$. For example, if A and B are homogeneous, then $A \oplus B$ is homogeneous, and if A or B are atomless, then $A \oplus B$ is atomless. The following results seem to further that intuition.

Theorem 3.1.2. $\mathfrak{p}(A \oplus B) = \min\{\mathfrak{p}(A), \mathfrak{p}(B)\}$

Theorem 3.1.3. $\mathfrak{t}(A \oplus B) = \min\{\mathfrak{t}(A), \mathfrak{t}(B)\}$.

Theorem 3.1.4. $\mathfrak{s}(A \oplus B) = \min\{\mathfrak{s}(A), \mathfrak{s}(B)\}$*, for atomless Boolean algebras* A *and* B*.*

Proofs for these three theorems can be found in [22]. The case of $\mathfrak{r}$ is more complicated and requires some extra concepts.

Definition 3.1.5. *Let* A *be an infinite Boolean algebra and take* $2 \leqslant n < \omega$*. A set* $X \subseteq A^+$ *will be called* n*-dense if for all partition* $\{a_i \mid i < n\}$ *of* A *of size* n*, there exist* $x \in X$ *and* $i < n$ *such that* $a \leqslant a_i$*. Define* $\mathfrak{r}_n := \min\{|X| \mid X \subseteq A^+ \ X \ is \ n-dense\}$*.*

Observe that $\mathfrak{r}_2 = \mathfrak{r}$. Observe also that $\mathfrak{r}_n(A) \leqslant \mathfrak{r}_{n+1}(A)$, for all $2 \leqslant n < \omega$. Therefore, one can define $\mathfrak{r}_{fin}(A) := \sup\{\mathfrak{r}_n(A) \mid 2 \leqslant n < \omega\}$ which is the least size of a family n-dense, for all $2 \leqslant n < \omega$. For $\mathfrak{r}(A \oplus B)$ we have the following bounds, originally proved in [20], though not stated explicitly. For the sake of completeness they are proved here.

Theorem 3.1.6. $\max\{\mathfrak{r}(A), \mathfrak{r}(B)\} \leqslant \mathfrak{r}(A \oplus B) \leqslant \max\{\mathfrak{r}_{fin}(A), \mathfrak{r}_{fin}(B)\}$*.*

Proof. Suppose without loss of generality that $\mathfrak{r}(B) \leqslant \mathfrak{r}(A)$. Take $\kappa < \mathfrak{r}(A)$ and

$$\{\bigcup_{i<n_\alpha} a_i^\alpha \times b_i^\alpha \mid \alpha < \kappa\} \subseteq (A \oplus B)^+.$$

There exists $x \in A^+$ such that x splits a_0^α, for all $\alpha < \kappa$. Then $x \times Y$ splits $\bigcup_{i<n_\alpha} a_i^\alpha \times b_i^\alpha$, for all α. Therefore $\max\{\mathfrak{r}(A), \mathfrak{r}(B)\} \leqslant \mathfrak{r}(A \oplus B)$.

Suppose without loss of generality that $\mathfrak{r}_{fin}(B) \leqslant \mathfrak{r}_{fin}(A)$. We will construct an unreaped subfamily of $A \oplus B$ of size $\mathfrak{r}_{fin}(A)$. Take $R \subseteq A^+$ an n-dense family, for all $n < \omega$, of size $\mathfrak{r}_{fin}(A)$. Take also $S \subseteq B^+$ an unreaped family of size $\mathfrak{r}(B) \leqslant \mathfrak{r}_{fin}(A)$. Define $RS := \{x \times y \mid x \in R \ \vee \ y \in S\}$. Clearly $|RS| = \mathfrak{r}_{fin}(A)$.

Take $c := \bigcup_{i<n} a_i \times b_i \in (A \oplus B)^+$, such that $\{a_i \mid i < n\}$ is a disjoint family. Since R is n-dense (and $n+1$-dense), there exists $x \in R$ such that either $x \cap a_i = \emptyset$, for all $i < n$, or there exists $i < n$ such that $x \subseteq a_i$. In the first case, if $y \in S$, then $c \cap x \times y = \emptyset$. In the second case, take $y \in S$ such that either $y \cap b_i = \emptyset$ or $y \subseteq b_i$. Then either $x \times y \cap c = \emptyset$ or $x \times y \subseteq c$. Therefore, RS is unreaped. ■

It turns out that both bounds are the only possible values for $\mathfrak{r}(A \oplus B)$. Firstly, we state the necessary conditions (Propositions 4.4, 4.5 and 4.6 of [20]) for both values to be obtained.

Theorem 3.1.7.

- *If* $\mathfrak{r}_{fin}(A) \leqslant \mathfrak{r}(B)$*, then* $\mathfrak{r}(A \oplus B) = \mathfrak{r}(B)$*.*
- *If* $\mathfrak{r}_{fin}(A) = \mathfrak{r}_{fin}(B)$*, then* $\mathfrak{r}(A \oplus B) = \mathfrak{r}_{fin}(B)$*.*

The next result was proved in [7] (Theorem 3.1).

Theorem 3.1.8. $\mathfrak{r}_n(A) \leqslant \mathfrak{r}(A)^+$, *for all* $n < \omega$.

Suppose that $\mathfrak{r}_{fin}(A) \leqslant \mathfrak{r}_{fin}(B)$. If the equality does not hold, then $\mathfrak{r}_{fin}(A) < \mathfrak{r}_{fin}(B)$ and, by the previous theorem, $\mathfrak{r}_{fin}(A) \leqslant \mathfrak{r}(B)$. Therefore Theorem 3.1.7 describes all possibilities for $\mathfrak{r}(A \oplus B)$. For both possibilities, there are examples given by the next result suggested in [1] and proved in [18] (Theorem 6.26).

Theorem 3.1.9. *For all infinite cardinal* κ *and all integer* $n \geqslant 2$ *there exists a Boolean algebra* A *such that* $\mathfrak{r}_n(A) = \kappa$ *and* $\mathfrak{r}_{n+1}(A) = \kappa^+$.

Taking Boolean algebras A and B such that $\mathfrak{r}(A) = \omega$, $\mathfrak{r}(B) = \omega_1$ and $\mathfrak{r}_{fin}(B) = \omega_2$, we have that

$$\mathfrak{r}(A \oplus B) = \mathfrak{r}(B) = \max\{\mathfrak{r}(A), \mathfrak{r}(B)\} < \max\{\mathfrak{r}_{fin}(A), \mathfrak{r}_{fin}(B)\}$$

and

$$\mathfrak{r}(B) < \mathfrak{r}(B \oplus B) = \mathfrak{r}_{fin}(B).$$

Therefore both bounds of Theorem 3.1.6 are possible while being different. The discussion on $\mathfrak{r}(A \oplus B)$ ends with a case, relevant to next chapters, where both bounds are always equal.

Lemma 3.1.10. *If* A *is homogeneous, then* $\mathfrak{r}(A) = \mathfrak{r}_{fin}(A)$.

Proof. Since A is homogeneous for all $x \in A^+$, there exists $R_x \subseteq A \restriction x$ of size $\mathfrak{r}(A)$ such that for all $y \subseteq x$ there exists $z \in R_x$ such that $z \subseteq y$ or $y \cap z = \emptyset$. For $n < \omega$ recursively construct:

- $R_0 := R_X$ and
- $R_n := \bigcup_{x \in R_{n-1}} R_x$, for $0 < n < \omega$.

Define $R := \bigcup_{n<\omega} R_n$. Clearly $|R| = \mathfrak{r}(A)$. We claim that R is an n-dense family, for all $n < \omega$. Take $\{a_i \mid i \leqslant n\}$, a partition of A. Take $x_0 \in R_0 \subseteq R$ such that $x_0 \subseteq a_0$ or $x_0 \cap a_0 = \emptyset$. In the first case the proof is complete. If $x_0 \cap a_0 = \emptyset$, the process can continue recursively as follows. Suppose that for some $0 < j < n$ we have defined $x_i \in R$, for all $i < j$, such that $x_{i'} \subseteq x_i$, for all $i < i' < j$, and such that $x_i \cap a_i = \emptyset$, for all $i < j$. Take $x_j \in R_{x_{j-1}} \subseteq R$ such that $x_j \subseteq a_j$ or $x_j \cap a_j = \emptyset$. In the first case the proof is complete. In the second case the process continues for $j + 1 < n$. If $j + 1 = n$, then clearly $x_j \subseteq a_n$ and the proof is complete.

∎

From this lemma and Theorem 3.1.6 follows this result.

Theorem 3.1.11. $\mathfrak{r}(A \oplus B) = \max\{\mathfrak{r}(A), \mathfrak{r}(B)\}$, *for infinite homogeneous Boolean algebras* A *and* B.

With the exception of Observation 2.4.4, all results concerning cardinal invariants and free products strongly relate the structure of $A \oplus B$ and that of A and B. However, as it will be seen in the next section this is not trivially the case for infinite partitions and the cardinal invariant $\mathfrak{a}(A \oplus B)$.

3.2 Lower bounds for $\mathfrak{a}(A \oplus B)$.

In this section we study infinite partitions and disjoint families of $A \oplus B$ and their least possible size. From the observations in the beginning of the previous section, if $\{\bigcup_{i<n_\alpha} a_i^\alpha \times b_i^\alpha \mid \alpha < \kappa\}$ is an infinite disjoint family of $A \oplus B$, then without loss of generality we can suppose that it is an infinite partition iff $\{a_i^\alpha \times b_i^\alpha \mid \alpha < \kappa\ i < n_\alpha\}$ is an infinite partition. Furthermore, the size of both families is the same. Therefore, when studying the possible sizes of infinite partitions of $A \oplus B$ it is enough to consider families of the form $\{a_\alpha \times b_\alpha \mid \alpha < \kappa\} \subseteq A \oplus B$.

Recall that both A and B are subalgebras of $A \oplus B$ by the simple embeddings $i : A \to A \oplus B$ and $j : B \to A \oplus B$ defined by $i(x) = x \times Y$ and $j(y) = X \times y$, for all $x \in A$ and $y \in B$. Furthermore, these embeddings are regular, i.e. if $P \subseteq A$ (or $P \subseteq B$) is a partition, then $i[P]$ (resp. $j[P]$) is a partition of $A \oplus B$. This leads to the following observation.

Observation 3.2.1. $\mathfrak{a}(A \oplus B) \leqslant \min\{\mathfrak{a}(A), \mathfrak{a}(B)\}$, *for all A and B infinite Boolean algebras.*

As reflected in all theorems of the previous section, a natural question to ask is whether this observation can be strengthened. In fact this question, which is still open (unless the author is mistaken), was asked in [18] (Problem 8).

Question 3.2.2. *Does*

$$\mathfrak{a}(A \oplus B) = \min\{\mathfrak{a}(A), \mathfrak{a}(B)\}$$

hold for any pair of infinite Boolean algebras A and B?

A natural way of looking for a positive answer to this question would be to take $\omega \leqslant \kappa < \min\{\mathfrak{a}(A), \mathfrak{a}(B)\}$ and an infinite disjoint family $\{a_\alpha \times b_\alpha \mid \alpha < \kappa\} \subseteq A \oplus B$ of size κ, and find an infinite $X \subseteq \kappa$ such that either $\{a_\alpha \mid \alpha \in X\}$ or $\{b_\alpha \mid \alpha \in X\}$ are disjoint families. Since in either case, the family cannot be a partition of A (or B), if $x \in A$ (resp. $x \in B$) is a witness to that fact, then $x \times Y$ (resp. $X \times x$) would be a "good" place for looking for some $c \times d$ witnessing that the initial family is not an infinite partition of $A \oplus B$. Analogous methods were used by the author for proving Theorems 3.1.2, 3.1.3 and 3.1.4.

While taking centered families, for example, in $A \oplus B$ easily induces centered families in both projections, this is not an easily analogous case with disjoint families of $A \oplus B$. However, if $a_0 \times b_0$ and $a_1 \times b_1$ are disjoint elements of $A \oplus B$, it only follows that either $a_0 \cap a_1 = \emptyset$ or $b_0 \cap b_1 = \emptyset$ but not precisely one of them. This only can be guaranteed if, for example $a_0 \cap a_1 \neq \emptyset$, in which case necessarily $b_0 \cap b_1 = \emptyset$. Therefore looking for centered families in some of the projections of a given disjoint family would be useful for proving that it is not a partition. While not answering yet

Question 3.2.2, this idea was useful for the author to give a partial answer in Theorem 13 of [22].

Theorem 3.2.3. *If A and B are infinite Boolean algebras, then*

$$\min\{\mathfrak{a}(A), \mathfrak{a}(B), \max\{\mathfrak{p}(A), \mathfrak{p}(B)\}\} \leqslant \mathfrak{a}(A \oplus B).$$

As part of the proof of this theorem one takes $\kappa < \mathfrak{a}(A), \mathfrak{a}(B), \mathfrak{p}(A)$ (assuming without loss of generality that $\mathfrak{p}(B) \leqslant \mathfrak{p}(A)$) and a disjoint family $\{a_\alpha \times b_\alpha \mid \alpha < \kappa\} \subseteq A \oplus B\}$. When possible find $E \in [\kappa]^{\geqslant\omega}$ maximal with the property that $\{a_\alpha \mid \alpha < \kappa\}$ is centered. If $a \in A^+$ is a pseudointersection of said family, take $b \in B^+$ witnessing that $\{b_\alpha \mid \alpha < \kappa\}$ is not a partition. Then $a \times b$ witnesses that $\{a_\alpha \times b_\alpha \mid \alpha < \kappa\}$ is not a partition. For more details of this proof see the proof of Theorem 13 of [22].

From Theorem 3.2.3 it follows that any instance of

$$\mathfrak{a}(A \oplus B) < \min\{\mathfrak{a}(A), \mathfrak{a}(B)\}$$

is one of $\mathfrak{p}(A), \mathfrak{p}(B) < \mathfrak{a}(A), \mathfrak{a}(B)$. By Observation 2.4.8, it is known that $\mathfrak{a}(A) = \omega$ iff $\mathfrak{p}(A) = \omega$, for all infinite Boolean algebra A. Therefore for getting such an algebra for which such equality holds we need that $\omega_1 \leqslant \mathfrak{p}(A), \mathfrak{p}(B)$, and hence that $\omega_1 \leqslant \mathfrak{a}(A \oplus B)$.

In this section a couple of lower bounds to $\mathfrak{a}(A \oplus B)$, for A and B infinite Boolean algebras, will be given, adding some nuance to the bound of Theorem 3.2.3. The first one will only work on homogeneous Boolean algebras.

Theorem 3.2.4. *Suppose that A and B are homogeneous.Then*

$$\min\{\mathfrak{a}(A), \mathfrak{a}(B), \max\{\mathfrak{s}(A), \mathfrak{s}(B)\}\} \leqslant \mathfrak{a}(A \oplus B).$$

Proof. Without loss of generality, $\mathfrak{s}(B) \leqslant \mathfrak{s}(A)$. Take an infinite cardinal $\kappa < \mathfrak{a}(A), \mathfrak{a}(B), \mathfrak{s}(A)$ and suppose that $P = \{a_\alpha \times b_\alpha : \alpha < \kappa\}$ is a disjoint family of $A \oplus B$. Since the theorem trivially holds if $\min\{\mathfrak{a}(A), \mathfrak{a}(B)\} \in \{\omega, \omega_1\}$ (see Observation 2.4.8 and Figure 2.6), suppose that $\omega_2 \leqslant \mathfrak{a}(A), \mathfrak{a}(B)$. As was highlighted in the previous partial sketch of the proof of Theorem 3.2.3, centered subfamilies of $\{a_\alpha \mid \alpha < \kappa\}$ are useful. Two cases will be proved according to the possible size of these subfamilies.

Case 1. There exists $E \in [\kappa]^\omega$ such that $\{a_\alpha : \alpha \in E\}$ is a centered family. Without loss of generality suppose that $E = \omega$. Since $\omega_1 \leqslant \mathfrak{p}(A)$, we can take $a' \in A^+$ such that $a' \subseteq a_n$, for all $n < \omega$. By homogeneity $\mathfrak{s}(A) = \mathfrak{s}(A \restriction a)$. Therefore there exists $a \in A \restriction a'$ which witnesses that $\{a_\alpha \cap a' : \alpha < \kappa\}$ is not a splitting family of $A \restriction a'$, i.e. for all $\alpha < \kappa$, either $a \cap a_\alpha = \emptyset$ or $a \subseteq a_\alpha$.

Define $G := \{\alpha < \kappa : a \subseteq a_\alpha\}$. Since $\omega \subseteq G$, it follows that G is an infinite set. Therefore $\{b_\alpha : \alpha \in G\}$ is an infinite disjoint family of B. Since $\kappa < \mathfrak{a}(B)$, there exists $b \in B$ such that $b \cap b_\alpha = \emptyset$, for all $\alpha \in G$. Take

$\alpha < \kappa$. If $\alpha \in G$, then $b \cap b_\alpha = \emptyset$. If $\alpha \notin G$, then $a \cap a_\alpha = \emptyset$. In either case, $a \times b$ is disjoint to $a_\alpha \times b_\alpha$, which means that P is not an infinite partition.

Case 2. The family $\{a_\alpha : \alpha \in E\}$ is not centered, for all $E \in [\kappa]^\omega$. Since $\kappa < \mathfrak{s}(A)$, it follows that $\{a_\alpha : \alpha < \kappa\}$ is not splitting. Note that if $c \in A^+$ witnesses this fact, so does every $0 \neq c' \subseteq c$. Since A is homogeneous, if $\lambda = |A|$, there exists $\{c_\gamma : \gamma < \lambda\} \subseteq A^+$ such that for all $\alpha < \kappa$ and all $\gamma < \lambda$ either $c_\gamma \subseteq a_\alpha$ or $c_\gamma \cap a_\alpha = \emptyset$.

Clearly the family $\{a_\alpha \mid \alpha < \kappa\ c_\gamma \subseteq a_\alpha\}$ is a centered family, for all $\gamma < \lambda$. By hypothesis of this case it follows that the set $\{\alpha < \kappa : c_\gamma \subseteq a_\alpha\}$ is finite, for all $\gamma < \lambda$. Then we can define $f : \lambda \to [\kappa]^{<\omega}$ such that $c_\gamma \subseteq a_\alpha$ iff $\alpha \in f(\gamma)$, for all $\alpha < \kappa$ and $\gamma < \lambda$.

Claim 3.2.5. *One of the following statements holds:*

1. $\emptyset \in f[\lambda]$
2. *there exists* $E \in f[\lambda]$ *such that* $b := Y \setminus \bigcup_{\alpha \in E} b_\alpha$ *is not empty*
3. *there exist* $E \in f[\lambda]$ *and* $\beta \in \kappa \setminus E$ *such that* $\{a_\alpha : \alpha \in E \cup \{\beta\}\}$ *is a centered family.*

Proof. Suppose that none of the statements holds. Take $\gamma < \delta < \lambda$ such that $f(\gamma) \neq f(\delta)$. Take without loss of generality $\beta \in f(\gamma) \setminus f(\delta)$. Since statement 3 does not hold, it follows that $\{a_\alpha \mid \alpha \in f(\delta) \cup \{\beta\}\}$ is not a centered family. Therefore,

$$(\bigcap_{\alpha \in f(\delta)} a_\alpha) \cap a_\beta = \emptyset.$$

Hence, it clearly follows that

$$(\bigcap_{\alpha \in f(\delta)} a_\alpha) \cap (\bigcap_{\alpha' \in f(\gamma)} a_{\alpha'}) = \emptyset.$$

Defining $d_E := \bigcap_{\alpha \in E}$, for $E \in f[\lambda]$, which is possible since statement 1 does not hold, it follows that $\{d_E : E \in f[\lambda]\} \subseteq A^+$ is a disjoint family.

Suppose that $f[\lambda]$ is finite and that $X = \bigcup_{E \in f[\lambda]} d_E$. Take $(x \times y) \in (A \oplus B)^+$. Take d_E, with $E \in f[\lambda]$ such that $x \cap d_E \neq \emptyset$. Since statement 2 does not hold, it follows that $\cup_{\alpha \in E} b_\alpha = Y$. Then there exists $\alpha \in E$ such that $(x \times y) \cap (a_\alpha \times b_\alpha) \neq \emptyset$. Therefore $\{a_\alpha \times b_\alpha : \alpha \in \bigcup f[\lambda]\}$ is a finite partition of $A \oplus B$, which is a contradiction. Then if $f[\lambda]$ is finite, $X \neq \bigcup_{E \in f[\lambda]} d_E$.

Suppose now that $f[\lambda]$ is infinite. Since $\kappa < \mathfrak{a}(A)$, it follows that $\{d_E \mid E \in f[\lambda]\}$ is not an infinite partition of A. This means that either if $f[\lambda]$ is finite or not, there exists $c \in A^+$ such that $c \cap d_E = \emptyset$, for all $E \in f[\lambda]$.

Since A is homogeneous and $\kappa < \mathfrak{s}(A) = \mathfrak{s}(A \restriction c)$, it follows that $\{c \cap a_\alpha \mid \alpha < \kappa\}$ is not a splitting family of $A \restriction c$. Then there exists $\emptyset \neq c' \subseteq c$ such that $c' \cap (c \cap a_\alpha) = \emptyset$ or $c' \subseteq (c \cap a_\alpha)$, for all $\alpha < \kappa$. Since

$c' \subseteq c$, it follows that $c' \cap a_\alpha = \emptyset$ or $c' \subseteq a_\alpha$, for all $\alpha < \kappa$. Then there exists $\gamma < \lambda$ such that $c_\gamma = c'$. If $E = f(\gamma)$, then $c_\gamma \subseteq d_E$, but this is a contradiction.

■

Each one of the statements of this claim will help getting a witness of $\{a_\alpha \times b_\alpha \mid \alpha < \kappa\}$ not being a partition of $A \oplus B$.

1. Take $\gamma < \lambda$ such that $f(\gamma) = \emptyset$. Since $c_\gamma \cap a_\alpha = \emptyset$, for all $\alpha < \kappa$, it follows that $(c_\gamma \times Y) \cap (a_\alpha \times b_\alpha) = \emptyset$, for all $\alpha < \kappa$.

2. Take $E \in f[\lambda]$ such that $b := Y \setminus \bigcup_{\alpha \in E} b_\alpha$ is not empty. Take $\gamma < \lambda$ such that $f(\gamma) = E$ If $\alpha \in E$, then $b \cap b_\alpha = \emptyset$. If $\alpha \in \kappa \setminus E$, then $c_\gamma \cap a_\alpha = \emptyset$. Therefore, $(c_\gamma \times b) \cap (c_\alpha \times b_\alpha) = \emptyset$, for all $\alpha < \kappa$.

3. Take $E \in f[\lambda]$ and $\beta \in \kappa \setminus E$ such that $\{a_\alpha : \alpha \in E \cup \{\beta\}\}$ is a centered family. Take $\gamma < \lambda$ such that $f(\gamma) = E$. If $\alpha \in E$, then $b_\beta \cap b_\alpha = \emptyset$. If $\alpha \in \kappa \setminus E$, then $c_\gamma \cap a_\alpha = \emptyset$. Therefore, $(c_\gamma \times b_\beta) \cap (c_\alpha \times b_\alpha) = \emptyset$, for all $\alpha < \kappa$.

■

Since $\max\{\mathfrak{p}(A), \mathfrak{p}(B)\} \leqslant \max\{\mathfrak{s}(A), \mathfrak{s}(B)\}$, for all homogeneous Boolean algebras A and B, this more specific theorem gives an improvement to Theorem 3.2.3. Concerning non-homogeneous Boolean algebras, observe that the main use of homogeneity in this proof was the implication that $\mathfrak{s}(A) = \mathfrak{s}(A \upharpoonright x)$, for all $x \in A^+$. Observe also that $\mathfrak{s}(A \upharpoonright y) \leqslant \mathfrak{s}(A \upharpoonright x)$, if $y \subseteq x$. Defining $\mathfrak{s}_{min}(A) := \min\{\mathfrak{s}(A \upharpoonright x) \mid x \in A^+\}$, this theorem can be changed to

$$\min\{\mathfrak{a}(A), \mathfrak{a}(B), \max\{\mathfrak{s}_{min}(A), \mathfrak{s}_{min}(B)\}\} \leqslant \mathfrak{a}(A \oplus B),$$

for all atomless Boolean algebras A and B.

However, the "homogeneous" case of $\mathfrak{s}_{min}(A) = \mathfrak{s}(A)$ is the relevant one. Indeed, take non-homogeneous A and B such that $\mathfrak{s}_{fin}(B) \leqslant \mathfrak{s}_{fin}(A) \leqslant \kappa = \mathfrak{a}(A \oplus B) < \mathfrak{s}(A), \mathfrak{a}(A), \mathfrak{a}(B)$, in particular both the hypothesis and conclusion of last theorem do not hold. If $\{a_\alpha \times b_\alpha \mid \alpha < \kappa\}$ is a partition of $A \oplus B$, take $x \in A$ such that $\mathfrak{s}(A \upharpoonright x) = \mathfrak{s}_{fin}(A)$. Since $\mathfrak{s}(A \upharpoonright (X \setminus x)) = \mathfrak{s}(A)$, by the previous proof if $\{\alpha < \kappa \mid a_\alpha \setminus x \neq \emptyset\}$ is infinite, then $\{(a_\alpha \setminus x) \times b_\alpha \mid \alpha < \kappa\}$ is not a partition of $A \upharpoonright (X \setminus x) \oplus B$. Therefore, $\{a_\alpha \times b_\alpha \mid \alpha < \kappa\}$ would accumulate on $A \upharpoonright x \oplus B$ and this Boolean algebra would be mainly an example of the inequality of Observation 3.2.1 being strict, thus answering Question 3.2.2, and where Theorem 3.2.4 holds.

Observe also that in most of the cases of the last proof the core idea of the proof of Theorem 3.2.3 sketched before is repeated. However it is reversed: this time we begin with an enumeration of possible pseudointersections and then choose a subfamily of $\{a_\alpha \mid \alpha < \kappa\}$ centered on one of them and maximal with respect to that property.

The main reason for considering the cardinal invariant $\mathfrak{s}(A)$ looking for lower bounds of $\mathfrak{a}(A \oplus B)$ was this heuristic. The inequality $\mathfrak{p}(A) \leqslant \mathfrak{a}(A)$ was however motivating for getting the inequality of Theorem 3.2.3. Of all cardinal invariants defined in Chapter 2.1, the only one that is sometimes a lower bound of $\mathfrak{a}(A)$ is $\mathfrak{b}(A)$. This cardinal invariant was useful for getting a lower bound of $\mathfrak{a}(A \oplus B)$.

Theorem 3.2.6. *Suppose that $\omega_1 \leqslant \mathfrak{a}(A), \mathfrak{a}(B)$. Then*

$$\min\{\mathfrak{a}(A), \mathfrak{a}(B), \max\{\mathfrak{b}(A), \mathfrak{b}(B)\}\} \leqslant \mathfrak{a}(A \oplus B).$$

Proof. Without loss of generality, $\mathfrak{b}(B) \leqslant \mathfrak{b}(A)$. Take $\kappa < \mathfrak{b}(A), \mathfrak{a}(B)$ and that $P = \{a_\alpha \times b_\alpha : \alpha < \kappa\}$ is a disjoint family of $A \oplus B$. As in the previous theorem, we have two cases.

Case 1. There exists $K \in [\kappa]^\omega$ such that $\{b_\alpha : \alpha \in K\}$ is a centered family. Without loss of generality suppose that $K = \omega$. Therefore $\{a_n : n < \omega\}$ is a disjoint family. By Observation 2.4.8 $\omega_1 \leqslant \mathfrak{p}(B)$. Then there exists $b \in B^+$ such that $b \subseteq b_n$, for all $n < \omega$.

Define $E := \{\alpha \in \kappa \setminus \omega : b_\alpha \cap b \neq \emptyset\}$. It is clear that $a_n \cap a_\alpha = \emptyset$, for all $n < \omega$ and $\alpha \in E$. Therefore $(\{a_n \mid n < \omega\}, \{a_\alpha \mid \alpha \in E\})$ form a pregap. Since $|E| \leqslant \kappa < \mathfrak{b}(A)$, there exists $c \in A$ such that $a_\alpha \subseteq c$, for all $\alpha \in E$, and $a_n \cap c = \emptyset$, for all $n < \omega$. The disjoint family $\{a_n : n < \omega\} \cup \{c\}$ is not an infinite partition of A. Take $a \in A^+$ such that $a \cap c = \emptyset$ and $a \cap a_n = \emptyset$, for all $n < \omega$. Take $\alpha < \kappa$. If $\alpha \in \omega \cup E$, then $a \cap a_\alpha = \emptyset$. If $\alpha \notin \omega \cup E$, then $b \cap b_\alpha = \emptyset$. Therefore, $(a \times b) \cap (a_\alpha \times b_\alpha) = \emptyset$, for all $\alpha < \kappa$.

Case 2. Suppose that $\{b_\alpha : \alpha \in E\}$ is not centered, for all $E \in [\kappa]^\omega$. Let $\{F_\gamma : \gamma < \kappa\}$ be the family of all $F \in [\kappa]^{<\omega}$, maximal with the property $\{b_\alpha : \alpha \in F\}$ is a centered family. Define $d_\gamma := \bigcap_{\alpha \in F_\gamma} b_\alpha$, for $\gamma < \kappa$. Take $\gamma < \delta < \kappa$. Without loss of generality there exists $\beta \in F_\gamma \setminus F_\delta$. Because of maximality

$$(\bigcap_{\alpha \in F_\delta} b_\alpha) \cap b_\beta = \emptyset.$$

Then $d_\delta \cap d_\gamma = \emptyset$. Therefore, the family $\{d_\gamma : \gamma < \kappa\}$ is pairwise disjoint.

Since $\kappa < \mathfrak{a}(B)$, take $d \in B^+$ such that $d \cap d_\gamma = \emptyset$, for all $\gamma < \kappa$. Take $F \in [\kappa]^{<\omega}$ maximal with the property $\{b_\alpha : \alpha \in F\} \cup \{d\}$ is a centered family. Extending F to some F' maximal with the property that $\{b_\alpha \mid \alpha \in F'\}$ is a centered family, find $\gamma < \kappa$ such that $F' = F_\gamma$. Since $d \cap d_\gamma = \emptyset$, it follows that $F \subsetneq F_\gamma$. Take $\beta \in F_\gamma \setminus F$ and define $b := \bigcap_{\alpha \in F} b_\alpha \cap d$. Take $\alpha < \kappa$. If $\alpha \in F$, as $\alpha, \beta \in F_\gamma$, then $b_\alpha \cap b_\beta \neq \emptyset$. Hence $a_\alpha \cap a_\beta = \emptyset$. If $\alpha \notin F$, then by maximality $\{b_{\alpha'} \mid \alpha' \in F\} \cup \{d, b_\alpha\}$ is not centered. It follows that $b_\alpha \cap b = \emptyset$. Therefore, $(a_\beta \times b) \cap (a_\alpha \times b_\alpha) = \emptyset$, for all $\alpha < \kappa$. ∎

This result is not precisely an improvement of Theorem 3.2.3 for a broad class of infinite Boolean algebras. Nevertheless, as it will be seen in the following section, some of its applications definitely are.

3.3 Combinatorics and partitions of $\mathcal{P}(\omega)/fin \oplus \mathcal{P}(\omega)/fin$

Now the results proved in this chapter will be applied to the more familiar case when $A = B = \mathcal{P}(\omega)/fin$. The Boolean algebra $\mathcal{P}(\omega)/fin$ being of foundational importance in all infinite combinatorics, all cardinal invariants defined in Section 2.1 applied to its simple product with itself will be simply denoted as $\mathfrak{p}(2) := \mathfrak{p}(\mathcal{P}(\omega)/fin \oplus \mathcal{P}(\omega)/fin)$, and so on.

Nevertheless, Theorems 3.1.2 through 3.1.11 trivialize the need for special notation for these cardinal characteristics of the continnum, since $\mathfrak{p}(2) = \mathfrak{p}$, $\mathfrak{t}(2) = \mathfrak{t}$, $\mathfrak{s}(2) = \mathfrak{s}$ and $\mathfrak{r}(2) = \mathfrak{r}$. The strange but trivial equality $\mathfrak{b}(2) = \omega$ implies that $\mathfrak{b}(2)$ could be considered to not be a "real" cardinal characteristic of the continuum.

However, as signaled by the theorems in Section 3.2, the cardinal invariant $\mathfrak{a}(2)$, and even the cardinal invariants $\mathfrak{a}(n) := \mathfrak{a}(\bigoplus_{i<n} \mathcal{P}(\omega)/fin)$, for $2 \leqslant n < \omega$, could be considered legitimate cardinal characteristics of the continuum, alongside the first but not least $\mathfrak{a}(1) := \mathfrak{a}$. Since this fact was not left unnoticed it has been already proved in [24] that $\mathfrak{b} \leqslant \mathfrak{a}(n)$, for all $1 \leqslant n < \omega$. This is a generalization of the well known result that $\mathfrak{b} \leqslant \mathfrak{a}$. The same result and a new lower bound for $\mathfrak{a}(n)$, lacking in that article, is presented here as a consequence of the more general results of the previous section.

Corollary 3.3.1. *The following statements hold:*

1. $\omega_1 \leqslant \mathfrak{a}(n+1) \leqslant \mathfrak{a}(n) \leqslant \mathfrak{a}$, *for all* $1 \leqslant n < \omega$.

2. $\min\{\mathfrak{s}, \mathfrak{a}(n-1)\} \leqslant \mathfrak{a}(n)$, *for all* $2 \leqslant n < \omega$.

3. $\mathfrak{b} \leqslant \mathfrak{a}(n)$, *for all* $1 \leqslant n < \omega$.

Proof. 1. From Observation 3.2.1 it follows that $\mathfrak{a}(2) \leqslant \mathfrak{a}(1) \leqslant \mathfrak{a}$. Taking this fact as base case suppose that $\mathfrak{a}(n) \leqslant \mathfrak{a}(n-1) \leqslant \mathfrak{a}$ has been proved for some $2 \leqslant n < \omega$. Since $\bigoplus_{i<n+1} \mathcal{P}(\omega)/fin = (\bigoplus_{i<n} \mathcal{P}(\omega)/fin) \oplus \mathcal{P}(\omega)/fin$, from the same observation it follows that $\mathfrak{a}(n+1) \leqslant \min\{\mathfrak{a}, \mathfrak{a}(n)\} = \mathfrak{a}(n)$. Since $\omega_1 \leqslant \mathfrak{p} = \mathfrak{p}(n) \leqslant \mathfrak{a}(n)$, for all $1 \leqslant n < \omega$, statement 1 is proved.

2. Observe that $\mathfrak{s}(n) = \mathfrak{s}$, for all $1 \leqslant n < \omega$. From Theorem 3.2.4 and statement 1 it follows that $\min\{\mathfrak{a}, \mathfrak{a}(n-1), \mathfrak{s}\} = \min\{\mathfrak{a}(n-1), \mathfrak{s}\} \leqslant \mathfrak{a}(n)$, for all $2 \leqslant n < \omega$.

3. The inequality $\mathfrak{b} \leqslant \mathfrak{a}$ is a well known fact (see Section 2.1.2). Taking this as the base case, suppose that $\mathfrak{b} \leqslant \mathfrak{a}(n)$ has been proved for some $1 \leqslant n < \omega$. From Theorem 3.2.6 and Observation 2.4.4. it follows that $\min\{\mathfrak{a}, \mathfrak{a}(n), \max\{\mathfrak{b}, \omega\}\} = \min\{\mathfrak{a}(n), \mathfrak{b}\} = \mathfrak{b} \leqslant \mathfrak{a}(n+1)$.

∎

Since all results in this section and Sec 3.2 were motivated by Question 3.2.2 let us restate it in the special case of the finite free products of $\mathcal{P}(\omega)/fin$.

Question 3.3.2. *1. Is it consistent with ZFC that $\mathfrak{a}(n) < \mathfrak{a}(n-1)$, for any $2 \leqslant n < \omega$?*

2. Even more: is it consistent with ZFC that $\omega_1 = \mathfrak{b} = \mathfrak{s} = \mathfrak{a}(n) < \mathfrak{a}(n-1) = \omega_2$, for any $2 \leqslant n < \omega$?

An affirmative answer to any of these questions would answer Question 3.2.2 in the negative in the context of core infinite combinatorics. Most classical models of $\mathfrak{b} = \omega_1$, like Cohen, Random, Sacks and Miller, are also models of $\mathfrak{a} = \omega_1$. While $\mathfrak{s} = \omega_1 < \mathfrak{a} = \omega_2$ holds in Hechler and Laver models, in both of them we also have that $\mathfrak{b} = \omega_2$ holds (see [3] for a summary on these models).

Forcing iterations along templates have given models of $\mathfrak{b}, \mathfrak{s} < \mathfrak{a}$. In [23] this technique was developed by Shelah for proving the consistency of this statement. By an *average* argument on ultraproducts, similar to that used in that article for making $\mathfrak{a}$ *large*, $\mathfrak{a}(2) = \mathfrak{a}$ holds in that model. Other models can be found in [9], for example. In all of them $\mathfrak{b}$ and $\mathfrak{s}$ are bigger than ω_1. This motivated the following question attributed to Brendle and Raghavan in [10].

Question 3.3.3. *Does $\mathfrak{b} = \mathfrak{s} = \omega_1$ imply that $\mathfrak{a} = \omega_1$?*

This question is clearly related to the second part of Question 3.3.2. Therefore even in the case of a model of ZFC + $\mathfrak{a}(n) < \mathfrak{a}(n-1)$, for some $n < \omega$, this question could remain relevant. Current forcing techniques seem to be not enough for giving an answer to any of them. Since everything proved in this chapter about the infinite partitions of $\bigoplus_{i<n} \mathcal{P}(\omega)/fin$ has been through the general language of Boolean algebras, some final words will be said about how to study them in a setting related to standard infinite combinatorics.

As described in Definition 2.1.3, instead of studying properly infinite partitions of $\mathcal{P}(\omega)/fin$, focus is directed towards ($fin-$)mad families. For its importance in infinite combinatorics and even in set theory their definition is recalled here. A family $\mathcal{A} \subseteq [\omega]^\omega$ is said to be an *almost disjoint* family, abbreviated *ad* family, if $|A \cap B| < \omega$, for all different $A, B \in \mathcal{A}$. Such a family is called a *mad (maximal almost disjoint)* family if for all $X \in [\omega]^\omega$ there exists $A \in \mathcal{A}$ such that $|X \cap A| = \omega$, i.e. if it is maximal with the property of being an ad family.

From the observations at the beginning of Section 3.2, when dealing with disjoint families and infinite partitions of the finite free products $\bigoplus_{i<n} P(\omega)/fin$, for $2 \leqslant n < \omega$, we can always assume that they are of the type

$$\{\prod_{i<n} X_i^\alpha : \alpha < \kappa\},$$

where each X_i^α is a non-empty clopen set of $\beta\omega \setminus \omega$. Since

$$\prod_{i<n} X_i \cap \prod_{i<n} Y_i = \emptyset$$

iff there exists $i < n$ such that $X_i \cap Y_i = \emptyset$, translating each n-dimensional *cube* of clopen sets of $\beta\omega \setminus \omega$ to a sequence of infinite sets motivates the following definition.

Definition 3.3.4. *Take* $2 \leqslant n < \omega$. *An infinite family* $\{(X_\alpha^0, ..., X_\alpha^{n-1}) : \alpha < \kappa\} \subseteq ([\omega]^\omega)^n$ *is called an* n*-ad family if for all* $\alpha < \beta < \kappa$ *there exists* $i < n$ *such that* $|X_\alpha^i \cap X_\beta^i| < \omega$. *It will be called an* n*-mad family if it is maximal with this property. Observe that* $\mathfrak{a}(n)$ *is the smallest size of an* n*-mad family.*

Chapter 4

The Nowhere Centered Ideal

4.1 The square $\omega \times \omega$ and its "modulo finite" structure.

Since from now on most of this work is set in $\omega \times \omega$, some notation on its subsets is established at this point. For $A \subseteq \omega \times \omega$ and $n < \omega$

$$A(n) := \{m < \omega \mid (n, m) \in A\},$$

i.e. $A(n)$ is the nth slice of A. Also, for $X \subseteq \omega$ and $\{Y_n \mid n \in X\} \subseteq \mathcal{P}(\omega)$ define

$$\coprod_{n \in X} Y_n := \bigcup_{n \in X} \{n\} \times Y_n.$$

Observe that $A = \coprod_{n<\omega} A(n)$, for all $A \subseteq \omega \times \omega$. Hence both notations help picture $\mathcal{P}(\omega \times \omega)$ as a structure consisting of sequences of elements in $\mathcal{P}(\omega)$. This notation helps, for example, to define some ideals on this particular set. Some examples of them are the following:

- $fin \times \emptyset := \{A \subseteq \omega \times \omega \mid \forall^\infty n < \omega \; A(n) = \emptyset\}$,
- $\emptyset \times fin := \{A \subseteq \omega \times \omega \mid \forall n < \omega \; |A(n)| < \omega\}$,
- $\mathcal{ED} := \{A \subseteq \omega \times \omega \mid \exists n < \omega \; \forall n \leqslant m < \omega \; |A(m)| \leqslant n\}$ and
- $fin \times fin := \{A \subseteq \omega \times \omega \mid \exists n < \omega \;\; \forall m < \omega \; |A(m)| < \omega\}$.

From their definition, it is clear that these ideals are Borel subsets of $\mathcal{P}(\omega)$. Observe that all of them are generated by families which are easy to describe and imagine. For example, $fin \times \emptyset$ is generated by the slices $\{\{n\} \times \omega \mid n < \omega\}$, $\emptyset \times fin$ is generated by the sets of the form $\{(n, m) \in \omega \times \omega \mid m < g(n)\}$ (what is *below* the graph of the function g), for $g \in \omega^\omega$. Accordingly $\mathcal{ED}$ is defined by the sets of the form $\{(n, m) \in \omega \times \omega \mid m = g(n)\}$ (the graph of the function), for $g \in \omega^\omega$, and the columns. Finally, $fin \times fin$ is generated by $(fin \times \emptyset) \cup (\emptyset \times fin)$ and hence by their respective generators. Therefore, $fin \times \emptyset \subseteq \mathcal{ED} \subseteq fin \times fin$ and $\emptyset \times fin \subseteq fin \times fin$.

These ideals represent some kind of smallness for subsets of $\omega \times \omega$. In this case even the one dimensional *finiteness* of ω is reproduced in two dimensions. For example, $fin \times \emptyset$ is the ideal of sets covered by finitely many slices of $\omega \times \omega$ and $fin \times fin$ is the ideal of sets whose slices are finite but for finitely many of them. Concerning the respective quotients, we have the following proposition.

Proposition 4.1.1. *Consider $i : \mathcal{P}(\omega) \to \mathcal{P}(\omega \times \omega)$ and $j : \mathcal{P}(\omega) \to \mathcal{P}(\omega \times \omega)$ defined respectively by $i(X) := \omega \times X$ and $j(X) := X \times \omega$. Then:*

1. *i induces an embedding from $\mathcal{P}(\omega)/fin$ to $\mathcal{P}(\omega\times\omega)/\emptyset\times fin$, $\mathcal{P}(\omega\times\omega)/\mathcal{ED}$ and $\mathcal{P}(\omega \times \omega)/fin \times fin$. The embedding to $\mathcal{P}(\omega \times \omega)/\emptyset \times fin$ is regular, while the other are not.*

2. *j induces an embedding from $\mathcal{P}(\omega)/fin$ to $\mathcal{P}(\omega\times\omega)/fin\times\emptyset$, $\mathcal{P}(\omega\times\omega)/\mathcal{ED}$ and $\mathcal{P}(\omega \times \omega)/fin \times fin$. All three embeddings are regular.*

Proof. 1. Take $X, Y \subseteq \omega$ such that $|X \triangle Y| < \omega$. Then there exists $k < \omega$ such that $i(X) \cap i(Y) \subseteq \omega \times k$, which is clearly an element of $\emptyset \times fin$, $\mathcal{ED}$ and $fin \times fin$. Therefore, the three respective functions defined by i are well defined and are embeddings.

Take $\{X_\alpha \mid \alpha < \kappa\} \subseteq [\omega]^\omega$ a mad family and $A \in (\emptyset \times fin)^+$. Since there exists $n < \omega$ such that $|A(n)| = \omega$, take $\alpha < \kappa$ such that $|X_\alpha \cap A(n)| = \omega$. Therefore, $i(X_\alpha) \cap A \in (\emptyset \times fin)^+$, proving that the embedding to $\mathcal{P}(\omega \times \omega)/\emptyset \times fin$ is regular.

Take $\{X_\alpha \mid \alpha < \kappa\} \subseteq [\omega]^\omega$ a mad family such that $|X_n \cap X_\alpha| \leqslant 1$, for all $\alpha < \kappa$ and all $n \in \omega \setminus \{\alpha\}$, and such that $X_n \cap X_m = \emptyset$, for all $n < m < \omega$. Consider the set $X := \coprod_{n<\omega} X_n$. Clearly $X \in \mathcal{ED}^+$. Take $\alpha < \kappa$. Then $|X(n) \cap X_\alpha| \leqslant 1$, for all $n \in \omega \setminus \{\alpha\}$. Therefore $X \cap i(X_\alpha) \in \mathcal{ED}$ and it proves that $\{X_\alpha \mid \alpha < \kappa\}$ does not induce an infinite partition in $\mathcal{P}(\omega \times \omega)/\mathcal{ED}$ and the embedding is not regular.

Take $\{X_\alpha \mid \alpha < \kappa\} \subseteq [\omega]^\omega$ a mad family. Consider the set $X := \coprod_{n<\omega} X_n$. Clearly $X \in (fin \times fin)^+$. Take $\alpha < \kappa$. Then $X(n) \cap X_\alpha$ is finite, for all $n \in \omega \setminus \{\alpha\}$. Therefore $X \cap i(X_\alpha) \in fin \times fin$ and it proves that $\{X_\alpha \mid \alpha < \kappa\}$ does not induce an infinite partition in $\mathcal{P}(\omega \times \omega)/fin \times fin$ and the embedding is not regular.

2. As in the previous case, it is easy to prove that j induces an embedding to the respective quotients.

To prove that is regular in all the cases considered, take $\{X_\alpha \mid \alpha < \kappa\} \subseteq [\omega]^\omega$ a mad family and $X \in \mathcal{I}^+$, for some $\mathcal{I} \in \{fin\times\emptyset, \mathcal{ED}, fin\times fin\}$. Take $Y \in [\omega]^\omega$ such that $X(n)$ is non-empty, for all $n \in Y$ (resp. $\{|X(n)| \mid n \in Y\}$ is unbounded or $X(n)$ is infinite, for all $n \in Y$). Since there exists $\alpha < \kappa$ such that $|X_\alpha \cap Y| = \omega$, it follows that $j(X_\alpha) \cap X \in \mathcal{I}^+$. Therefore, all three embeddings induced by j are regular.

■

Once we observe that the structure of $\mathcal{P}(\omega)/fin$ is well preserved at least by "one coordinate" on the square $\omega\times\omega$ and some of its quotients, the question arises of the preservation of the structure of the "square" $\mathcal{P}(\omega)/fin \oplus \mathcal{P}(\omega)/fin$. It easily follows, from Proposition 4.1.1, that $\mathcal{P}(\omega)/fin \oplus \mathcal{P}(\omega)/fin$ is naturally embedded in $\mathcal{P}(\omega\times\omega)/\mathcal{ED}$ and $\mathcal{P}(\omega\times\omega)/fin \times fin$. Indeed, if $\mathcal{I} \in \{\mathcal{ED}, fin \times fin\}$, then the subalgebra generated by $\{[X\times Y]_{\mathcal{I}} \mid X, Y \in [\omega]^\omega\}$ is isomorphic to $\mathcal{P}(\omega)/fin \oplus \mathcal{P}(\omega)/fin$.

For both ideals, a similar argument to that of the proof of Proposition 4.1.1 proves that this embedding is not regular. The rest of this chapter will be devoted to $\mathcal{NC}$, an ideal on $\omega\times\omega$ in whose quotient the square $\mathcal{P}(\omega)/fin \oplus \mathcal{P}(\omega)/fin$ is regularly embedded.

4.2 The nowhere centered ideal $\mathcal{NC}$ and its quotient

Definition 4.2.1. *The Nowhere Centered ideal is defined on $\omega\times\omega$ as follows*

$$\mathcal{NC} := \{A \subseteq \omega\times\omega \mid \forall X \in [\omega]^\omega \textit{ the set } \{A(n) : n \in X\} \textit{ is not centered}\}.$$

Here *centered* means centered as representatives of the Boolean algebra $\mathcal{P}(\omega)/fin$. Trivially $\{\omega^2(n) \mid n < \omega\} = \{\omega\}$ is a centered family and it follows that $\omega\times\omega \notin \mathcal{NC}$. It is also trivial that if $B \subseteq A \subseteq \omega\times\omega$, $X \in [\omega]^\omega$ and $\{A(n) \mid n \in X\}$ is not a centered family, then $\{B(n) \mid n \in X\}$ is not a centered family. Therefore, $\mathcal{NC}$ is downward closed. To prove that it is an ideal, it is enough to prove that it is closed under finite unions.

Take $A_0, A_1 \subseteq \omega\times\omega$ and suppose that $A_0 \cup A_1 \notin \mathcal{NC}$. Then there exists $X \in [\omega]^\omega$ such that $\{A_0(n) \cup A_1(n) \mid n \in X\}$ is a centered family. Take $\mathcal{U} \subseteq \mathcal{P}(\omega)$, an ultrafilter extending this family. Then there exist $i < 2$ and $X' \in [X]^\omega$ such that $\{A_i(n) \mid n \in X'\}$ is subset of $\mathcal{U}$, and hence is a centered family. Therefore $A_i \notin \mathcal{NC}$, and we conclude that $\mathcal{NC}$ is closed under finite unions.

Take $A \in fin \times fin$ and $X \in [\omega]^\omega$. Since there exists n such that $|A(m)| < \omega$ for all $m \geqslant n$, it follows that the family

$$|\bigcap_{i\in X\cap(n+1)} A(i)| < \omega$$

and hence that the family $\{A(m) \mid m \in X\}$ is not centered. Therefore, $fin \times fin \subseteq \mathcal{NC}$ and is bigger than all ideals defined in Section 4.1.

The ideal $\mathcal{NC}$ is strictly bigger than $fin \times fin$. Indeed, if $\{Y_n \mid n < \omega\} \subseteq [\omega]^\omega$ is an almost disjoint family, the set

$$Y := \coprod_{n<\omega} Y_n$$

is an element of $\mathcal{NC}$. However it does not lie in $fin \times fin$. Observe that Y is similar to the set proving that the family

$$\{X_0 \times X_1 \mid X_0, X_1 \in [\omega]^\omega\}$$

does not generate a regular subalgebra of $\mathcal{P}(\omega \times \omega)/fin \times fin$. Therefore, proving that this family generates a regular subalgebra of $\mathcal{P}(\omega \times \omega)/\mathcal{NC}$ is not surprising.

Proposition 4.2.2. *Consider $i : \mathcal{P}(\omega) \to \mathcal{P}(\omega \times \omega)$ and $j : \mathcal{P}(\omega) \to \mathcal{P}(\omega \times \omega)$ defined respectively by $i(X) := \omega \times X$ and $j(X) := X \times \omega$. Then both functions induce regular embeddings from $\mathcal{P}(\omega)/fin$ to $\mathcal{P}(\omega \times \omega)/\mathcal{NC}$. Furthermore, the embedding induced by both from $\mathcal{P}(\omega)/fin \oplus \mathcal{P}(\omega)/fin$ to $\mathcal{P}(\omega \times \omega)/\mathcal{NC}$ is dense.*

Proof. Both embeddings are well defined for similar reasons as in the proof of Proposition 4.1.1. To prove that both are regular, take $\{A_\alpha \mid \alpha < \kappa\} \subseteq [\omega]^\omega$ a mad family and $A \in \mathcal{NC}^+$. Since there exists $X_0 \in [\omega]^\omega$ such that $\{A(n) \mid n \in X\}$ is a centered family, take $X_1 \in [\omega]^\omega$ which is a pseudointersection thereof. Then $X_0 \times X_1 \subseteq_{\mathcal{NC}} A$ (in fact even $X_0 \times X_1 \subseteq_{\mathcal{ED}} A$ holds). Take $\alpha_0, \alpha_1 < \kappa$ such that $|A_{\alpha_0} \cap X_0| = \omega = |A_{\alpha_1} \cap X_1|$. Therefore, both $\{\omega \times A_\alpha \mid \alpha < \kappa\}$ and $\{A_\alpha \times \omega \mid \alpha < \kappa\}$ are partitions of $\mathcal{P}(\omega \times \omega)/\mathcal{NC}$ and both embeddings are regular.

By then there is a natural embedding from $\mathcal{P}(\omega)/fin \oplus \mathcal{P}(\omega)/fin$ to $\mathcal{P}(\omega \times \omega)/\mathcal{NC}$ induced by the identity. That the embedding is dense is proved by taking the same X_0 and X_1 as in the previous paragraph for a given $A \in \mathcal{NC}^+$. ■

From the previous proof the following useful fact is abstracted.

Observation 4.2.3. *The set $\{X \times Y \mid X, Y \in [\omega]^\omega\}$ is dense in $\mathcal{NC}^+$.*

As it will be further noticed in the next section both copies of $\mathcal{P}(\omega)/fin$ in $\mathcal{P}(\omega \times \omega)/\mathcal{NC}$ given by Proposition 4.2.2 bahave differently. However, the special relation with the "vertical" embedding, that induced by function i, is introduced by the next observation.

Observation 4.2.4. *If $X \in [\omega]^\omega$ and $A \in \mathcal{NC}$, then there exists $Y \in [X]^\omega$ such that $A \cap (\omega \times Y) \in fin \times fin$.*

Proof. Indeed we claim that there exists $n < \omega$ such that $|X \setminus \bigcup_{m \in F} A(m)| = \omega$, for all $F \in [\omega \setminus n]^{<\omega}$. Suppose otherwise for all $n < \omega$ there exists $F \in [\omega \setminus n]^{<\omega}$ such that $X \subseteq^* \bigcup_{m \in F} A(m)$. Recursively define $\{F_n \mid n < \omega\} \subseteq [\omega]^{<\omega}$ such that $F_n < F_{n+1}$ and that $X \subseteq^* \bigcup_{m \in F_n} A(m)$, for all $n < \omega$. If $\mathcal{U}$ is an ultrafilter containing X we can take $f \in \omega^\omega$ such that $f(n) \in F_n$ and $A(f(n)) \in \mathcal{U}$. But then $\{A(f(n)) \mid n < \omega\}$ is a centered family, which is a contradiction.

Given such $n < \omega$, take $Y \in [X]^\omega$ almost disjoint to all element of $\{A(m) \mid n \leqslant m < \omega\}$. It is easy to verify that Y is the desired set.

■

If the word "nowhere" in the name of the ideal $\mathcal{NC}$ is a plain reference to *nullness* in the quotient $\mathcal{P}(\omega)/fin$, this observation relates the behaviour of $\mathcal{NC}$ to that of the *nowhere dense* ideal of the real line. Indeed, contrast Observation 4.2.4 with the well known fact that whenever we have $\sigma \in \omega^{<\omega}$ and a subtree $T \subseteq \omega^{<\omega}$ whose branches form a nowhere dense set of ω^ω, then there exists $\tau \in \omega^{<\omega}$, an extension of σ, such that $\langle \tau \rangle \cap T = \emptyset$. In this case given the rectangle $\omega \times X$ and $A \in \mathcal{NC}$ we get an "extension" of the rectangle which "avoids" A, modulo $fin \times fin$.

It is also easy to see that Observation 4.2.4 suggests an alternative definition of $\mathcal{NC}$. Indeed, if $A \in \mathcal{NC}^+$, then there exists $X \in [\omega]^\omega$, which is a pseudointersection of infinitely many elements of $\{A(n) \mid n < \omega\}$. Then $A \cap (\omega \times Y) \in (fin \times fin)^+$ for all $Y \in [X]^\omega$. Therefore,

$$\mathcal{NC} = \{A \subseteq \omega \times \omega \mid \forall X \in [\omega]^\omega \ \exists Y \in [X]^\omega \ A \cap (\omega \times Y) \in fin \times fin\}.$$

However, this definition is more complex than the original which is coanalytic.

Indeed, $A \in \mathcal{NC}^+$ iff there exists $Y \in [\omega]^\omega$ which is a pseudointersection of infinitely many elements of $\{A(n) \mid n < \omega\}$. In other words, $\mathcal{NC}^+$ is the second coordinate projection of the subset of $[\omega]^\omega \times \mathcal{P}(\omega \times \omega)$ consisting of all (Y, A) satisfying the Borel formula

$$\forall n < \omega \ \exists m \geqslant n \ \exists k < \omega \ \forall l \geqslant k \ (l \in Y \Rightarrow l \in A(m)).$$

It follows that the ideal $\mathcal{NC}$ is coanalytic. It turns out that this is its precise complexity. To prove this, some concepts and results on descriptive set theory will be useful (for details on the following concept, see Section 34.B in [14]).

Definition 4.2.5. *Let A be a coanalytic subset of a Polish space X. A function $\phi : A \to \omega_1$ will be called a coanalytic rank if there exist partial orders $\leqslant_\phi, \leqslant_{\phi^\wedge} \subseteq X \times X$, a coanalytic and an analytic set respectively, such that $\phi(x) \leqslant \phi(y)$ iff $x \leqslant_\phi y$ iff $x \leqslant_{\phi^\wedge} y$, for all $x, y \in A$.*

A nice characteristic of coanalytic sets is that all of them have coanlytic ranks. An example of coanalytic set with a natural coanalytic rank is the set WF of well-founded subtrees of $\omega^{<\omega}$. If $T \subseteq \omega^{<\omega}$ a well-founded subtree, then the well-founded order $(T, \supseteq)$ has a height $\rho(T) < \omega_1$. This height turns out to be a coanalytic rank. A key characteristic of WF as a coanalytic set is the following.

Definition 4.2.6. *A coanalytic subset B of a Polish space Y will be called complete coanalytic if for any coanalytic subset A of a Polish space X there exists continuous $f : X \to Y$ such that $f^{-1}[B] = A$.*

Since WF is a complete coanalytic set, the following lemma shows us how its natural rank function helps defining ranks for any coanalytic set.

Lemma 4.2.7. *Let A be a coanalytic subset of a Polish space X and take $f : A \to WF$ such that $\{x \in X \mid \tau \in f(x)\}$ is a Borel set, for all $\tau \in \omega^{<\omega}$. Then $\rho \circ f$, is a coanalytic rank for X.*

This lemma relies on the fact that Borel preimages of analytic (resp. coanalytic) sets are analytic (resp. coanalytic). The final step before proving that $\mathcal{NC}$ is not Borel is in the following result (see Theorem 35.23 in [14]).

Theorem 4.2.8. *Let A be a coanalytic subset of a Polish space X with $\phi : A \to \omega_1$ a coanalytic rank. Then A is Borel iff $\bigcup \phi[A] < \omega_1$.*

A proof of this theorem can be found in Theorem 35.23 of [14].

Proposition 4.2.9. *The ideal $\mathcal{NC}$ is not Borel.*

Proof. As suggested by Theorem 4.2.8 a coanalytic rank for $\mathcal{NC}$ will be defined. Firstly, for each $X \subseteq \omega \times \omega$ we will associate a tree:

$$T(X) := \{\tau \in \omega^{\uparrow<\omega} \mid |\bigcap_{i \in im(\tau)} X(i)| = \omega\},$$

for $X \in \mathcal{NC}$, where $\omega^{\uparrow<\omega}$ is the subset of strictly increasing elements of $\omega^{<\omega}$. Clearly, the tree $T(X)$ consists of all increasing enumerations of finite $a \subseteq \omega$ such that $\{X(i) \mid i \in a\}$ is centered. Therefore, $T(X)$ is a description of the structure of the centered families of slices of X.

Observe that $X \in \mathcal{NC}$ iff $T(X)$ is a well-founded tree. Indeed if $X \in \mathcal{NC}^+$ and $Y \in [\omega]^\omega$ is such that $\{X(n) \mid n \in Y\}$ is centered then if τ_n is the enumeration of $Y \cap n$, for all $n < \omega$, then $\bigcup_{n<\omega} \tau_n$ is an infinite branch of $T(X)$. On the other hand if f is an infinite branch if $T(X)$, then $\{X(n) \mid n \in im(f)\}$ is a centered family witnessing that $X \in \mathcal{NC}^+$.

Clearly, if $\tau \in \omega^{\uparrow<\omega}$, we have that

$$\{X \subseteq \omega \times \omega \mid \tau \in T(X)\} = \bigcap_{n<\omega} \bigcup_{n \leqslant m < \omega} \bigcap_{i \in im(\tau)} \{X \subseteq \omega \times \omega \mid m \in X(i)\},$$

which is clearly a Borel set. Then, by Lemma 4.2.7, the function $\rho \circ T$, is a coanalytic rank when restricted to $\mathcal{NC}$. For proving that the rank function $\rho \circ T$ is unbounded in ω_1 take a bijection $\phi : \omega^{\uparrow<\omega} \to \omega$ such that $\phi(\tau) \leqslant \phi(\sigma)$, for all $\tau, \sigma \in \omega^{\uparrow<\omega}$ such that $\tau \subseteq \sigma$. Take also a family

$$\{A_\sigma \mid \sigma \in \omega^{\uparrow<\omega}\} \subseteq [\omega]^\omega$$

such that $A_\sigma \subseteq A_\tau$, and $A_\tau \cap A_\sigma = \emptyset$, if $\tau \nsubseteq \sigma$ and $\sigma \nsubseteq \tau$.

For each $\alpha < \omega_1$ take $S_\alpha \subseteq \omega^{\uparrow<\omega}$, a well-founded tree, such that $\rho(S_\alpha) = \alpha$. Define

$$A_{S_\alpha} := \coprod_{n \in \phi[S_\alpha]} A_{\phi^{-1}[\{n\}]}.$$

Take $\alpha<\omega_1$ and suppose that $X\subseteq\omega$ is such that $\{A_{S_\alpha}(n)\mid n\in X\}$ is a centered family. Since clearly $X\subseteq\phi[S_\alpha]$, it follows that

$$\{A_{S_\alpha}(n)\mid n\in X\}=\{A_\sigma\mid\sigma\in\phi^{-1}[X]\}.$$

By definition of the sets A_σ, it follows that $\phi^{-1}[X]$ is a family of compatible elements of S_α. Since ϕ is a bijection, it follows that X is infinite iff S_α has an infinite branch. Therefore, $A_{S_\alpha}\in\mathcal{NC}$, for all $\alpha<\omega_1$. The following claim helps concluding the proof of $\rho\circ T$ is unbounded on ω_1, and therefore $\mathcal{NC}$ is not Borel.

Claim 4.2.10. *If $\alpha<\omega_1$, then $\alpha\leqslant\rho(T(A_{S_\alpha}))$.*

Proof. Consider the function $e:\omega^{\uparrow<\omega}\to\omega^{\uparrow<\omega}$ defined by

$$e(\sigma):=(\phi(\sigma\restriction i)\mid i\in dom(\sigma)),$$

for all $\sigma\in\omega^{\uparrow<\omega}$. Take $\sigma\subseteq\tau\in S_\alpha$. By definition of ϕ, it is clear that e is well defined and injective. Also if $\sigma\subseteq\tau\in\omega^{\uparrow<\omega}$ and $i<dom(\sigma)$, then $e(\sigma)(i)=e(\tau)(i)$. Then e is an order embedding. If $\sigma\in S_\alpha$, then $\{A_{S_\alpha}(\phi(\sigma\restriction i))\mid i<dom(\sigma)\}$ is a centered family. Hence, it follows that $e(\sigma)\in T(S_\alpha)$. Therefore, e is an order embedding from S_α to A_{S_α} and $\alpha=\rho(S_\alpha)\leqslant\rho(T(A_{S_\alpha}))$. ■

■

As has been observed elsewhere in this section, "infinite rectangles" of the kind $X\times Y$, with $X,Y\in[\omega]^\omega$, are fundamental elements of $\mathcal{NC}^+$. However, other kinds of elements are easy to describe and perhaps to imagine. For example, if $\{X_n\mid n<\omega\}\subseteq[\omega]^\omega$ is a centered family, then

$$\coprod_{n<\omega}X_n$$

is clearly an element of $\mathcal{NC}^+$. A more simple instance of this example would be if the family $\{X_n\mid n<\omega\}$ were decreasing. This kind of element will be proved to be almost as important to the study of cardinal invariants as infinite rectangles. Other less simple instance of this would be if $\{X_n\mid n<\omega\}$ were an independent family.

Having assessed some basic facts of the nowhere centered ideal such as its complexity and its basic combinatorics, in the next section the nature of its quotient is studied.

4.3 Cardinal invariants of $\mathcal{P}(\omega\times\omega)/\mathcal{NC}$

There are several ways in which the Boolean algebras $\mathcal{P}(\omega)/fin$, $\mathcal{P}(\omega)/fin\oplus\mathcal{P}(\omega)/fin$ and $\mathcal{P}(\omega\times\omega)/\mathcal{NC}$ are similar as witnessed by the couple following results concerning splitting and reaping families. Splitting families of $\mathcal{P}(\omega\times\omega)/\mathcal{NC}$ behave as it is expected.

Theorem 4.3.1. *$\mathfrak{s} = \mathfrak{s}(2) = \mathfrak{s}(\mathcal{NC})$.*

Proof. The first equality follows from Theorem 3.1.4. To prove $\mathfrak{s} \leqslant \mathfrak{s}(\mathcal{NC})$ take $\kappa < \mathfrak{s}$ and $\{A_\alpha \mid \alpha < \kappa\} \subseteq \mathcal{NC}^+$. Since the family

$$\{A_\alpha(n) \mid \alpha < \kappa,\ n < \omega\}$$

is not splitting in $\mathcal{P}(\omega)/fin$, take $Y \in [\omega]^\omega$ witnessing that fact. Define

$$K := \{\alpha < \kappa \mid \exists^\infty n < \omega\ (Y \subseteq^* A_\alpha(n))\}.$$

For each $\alpha \in K$, consider the infinite set $X_\alpha := \{n < \omega \mid Y \subseteq^* A_\alpha(n)\}$. Again by cardinality, the family $\{X_\alpha \mid \alpha \in K\}$ is not splitting in $\mathcal{P}(\omega)/fin$. Take $X \in [\omega]^\omega$ witnessing this fact.

The set $X \times Y$ witnesses that $\{A_\alpha \mid \alpha < \kappa\} \subseteq \mathcal{NC}^+$ is not splitting in $\mathcal{P}(\omega \times \omega)/\mathcal{NC}$. Indeed, take $\alpha < \kappa$. If $\alpha \in K$ and $X \subseteq^* X_\alpha$, then $X \times Y \subseteq_{\mathcal{NC}} A_\alpha$. If $\alpha \in K$ and $|X \cap X_\alpha| < \omega$, then $|Y \cap A_\alpha(n)| < \omega$ for almost all $n \in X$. On the other hand if $\alpha \notin K$, then $|Y \cap A_\alpha(n)| < \omega$ for almost all $n < \omega$. In both cases it follows that $(X \times Y) \cap A_\alpha \in \mathcal{NC}$. We conclude that $\mathfrak{s} \leqslant \mathfrak{s}(\mathcal{NC})$.

Now take a family $\{X_\alpha \mid \alpha < \mathfrak{s}\} \subseteq [\omega]^\omega$ splitting in $\mathcal{P}(\omega)/fin$. The family

$$\mathcal{S} := \{X_\alpha \times X_\beta \mid \alpha,\ \beta < \mathfrak{s}\}$$

is splitting in $\mathcal{P}(\omega \times \omega)/\mathcal{NC}$. Take $A \in \mathcal{NC}^+$, which will be proved to be split by some element of $\mathcal{S}$. By observation 4.2.3, without loss of generality $A = X \times Y$, for some $X,\ Y \in [\omega]^\omega$. Take $\alpha, \beta < \mathfrak{s}$ such that X_α splits X and X_β splits Y. It follows that both $(X \times Y) \cap (X_\alpha \times X_\beta) \in \mathcal{NC}^+$ and $(X \times Y) \setminus (X_\alpha \times X_\beta) \in \mathcal{NC}^+$, i.e. $X_\alpha \times X_\beta$ splits A. We conclude that $\mathfrak{s}(\mathcal{NC}) \leqslant \mathfrak{s}$.

■

Concerning reaping families in $\mathcal{P}(\omega\times\omega)/\mathcal{NC}$, matters are not as straightforward. Nevertheless, a non surprising upper bound was found for $\mathfrak{r}(\mathcal{NC})$ with the following cardinal invariant.

Definition 4.3.2. *A family $\mathcal{R} \subseteq [\omega]^\omega$ is said to be σ-reaping if for all $\{X_n \mid n < \omega\} \subseteq [\omega]^\omega$ there exists $A \in \mathcal{R}$ such that for all $n < \omega$ either $A \subseteq^* X_n$ or $|A \cap X_n| < \omega$. The cardinal $\mathfrak{r}_\sigma$ is defined to be the smallest cardinality of a σ-reaping family.*

Clearly, $\mathfrak{r} \leqslant \mathfrak{r}_\sigma$. Whether the equality holds or both cardinals can be consistently different has been an open question for a long time . The answer to this question would be relevant for the next result.

Theorem 4.3.3. *$\mathfrak{r} = \mathfrak{r}(2) \leqslant \mathfrak{r}(\mathcal{NC}) \leqslant \mathfrak{r}_\sigma$.*

Proof. Since $\mathcal{P}(\omega)/fin$ is an infinite homogeneous Boolean algebra, it follows that $\mathfrak{r}=\mathfrak{r}(2)$, by Theorem 3.1.11. Since $(\mathcal{P}(\omega)/fin \oplus \mathcal{P}(\omega)/fin) \leqslant_{dense} \mathcal{P}(\omega\times\omega)/\mathcal{NC}$, it follows that $\mathfrak{r}(2)\leqslant\mathfrak{r}(\mathcal{NC})$.

Now it will be proved that $\mathfrak{r}(\mathcal{NC})\leqslant\mathfrak{r}_\sigma$. Let $\mathcal{R}=\{A_\alpha \mid \alpha<\mathfrak{r}_\sigma\}\subseteq[\omega]^\omega$ be a σ-reaping family. Consider the family $\{A_\alpha\times A_\beta \mid \alpha,\beta<\mathfrak{r}_\sigma\}$ and take $B\in\mathcal{NC}^+$. Define $X:=\{n<\omega \mid |B(n)|=\omega\}$. Since $\mathcal{R}$ is σ-reaping, there exists $\alpha<\mathfrak{r}_\sigma$ such that for all $n<\omega$ either $A_\alpha\subseteq^* B(n)$ or $|A_\alpha\cap B(n)|<\omega$. Define $X_0:=\{n\in X \mid A_\alpha\subseteq^* B(n)\}$. Since $\mathcal{R}$ is a reaping family, there exists $\beta<\mathfrak{r}_\sigma$ such that either $A_\beta\subseteq^* X_0$ or $|A_\beta\cap X_0|<\omega$. In the first case it follows that $A_\beta\times A_\alpha\subseteq_{\mathcal{NC}} B$; in the second one, that $(A_\beta\times A_\alpha)\cap B\in\mathcal{NC}$. In either case we conclude that $\{A_\alpha\times A_\beta \mid \alpha,\beta<\mathfrak{r}_\sigma\}$ is a reaping family in $\mathcal{P}(\omega\times\omega)/\mathcal{NC}$. Therefore $\mathfrak{r}(\mathcal{NC})\leqslant\mathfrak{r}_\sigma$. ■

In the case of infinite partitions of $\mathcal{P}(\omega\times\omega)/\mathcal{NC}$, the following observation follows from Proposition 4.2.2.

Observation 4.3.4. $\mathfrak{a}(\mathcal{NC})\leqslant\mathfrak{a}(2)\leqslant\mathfrak{a}$.

At first glance, this observation could suggest at least some provable equality, just as in the previous theorems. Since $\mathcal{P}(\omega)/fin\oplus\mathcal{P}(\omega)/fin$ is densely embedded in $\mathcal{P}(\omega\times\omega)/\mathcal{NC}$, we could hope that $\mathfrak{a}(\mathcal{NC})=\mathfrak{a}(2)$. In such case, there would be a path for proving the consistency of $\mathfrak{a}(2)<\mathfrak{a}$. However, as it is proved with help of the following lemma, which will be very useful along this chapter, this equality does not hold in ZFC.

Lemma 4.3.5. *Take $\{Y_n : n<\omega\}\subseteq[\omega]^\omega$. If $A,B\in[\omega]^\omega$ are such that*

$$A\times B\subseteq_{\mathcal{NC}}\coprod_{n<\omega}\bigcup_{i\leqslant n}Y_i,$$

then there exists $n<\omega$ such that $B\subseteq^*\bigcup_{i\leqslant n}Y_i$.

Proof. Suppose that, on the contrary, $B\not\subseteq^*\bigcup_{i\leqslant n}Y_i$, for all $n<\omega$. Then, by a classical diagonal argument, there exists $B'\in[B]^\omega$ such that $|B'\cap Y_n|<\omega$, for all $n<\omega$. But this implies that

$$A\times B'\cap\coprod_{n<\omega}\bigcup_{i\leqslant n}Y_i\in\mathcal{NC},$$

and hence that

$$A\times B\setminus\coprod_{n<\omega}\bigcup_{i\leqslant n}Y_i\in\mathcal{NC}^+,$$

which is a contradiction. ■

Theorem 4.3.6. *If $\{Y_n : n<\omega\}\subseteq[\omega]^\omega$, then*

$$\coprod_{n<\omega}\bigcup_{i\leqslant n}Y_i=_{\mathcal{NC}}\bigvee_{n<\omega}\omega\times Y_n.$$

Proof. Define $Y := \coprod_{n<\omega} \bigcup_{i\leqslant n} Y_i$. Since $\omega \times Y_n \setminus Y \subseteq n \times Y$, it follows that $\omega \times Y_n \subseteq_{\mathcal{NC}} Y$, for all $n < \omega$. Therefore, Y is an upper bound of $\{\omega \times Y_n \mid n < \omega\}$.

To prove that Y is the least upper bound of that family, suppose that there exists $Y' \subseteq Y$ such that $Y \setminus Y' \in \mathcal{NC}^+$ and that $\omega \times Y_n \subseteq_{\mathcal{NC}} Y'$, for all $n < \omega$. Take $A \times B \subseteq_{\mathcal{NC}} Y \setminus Y'$, with $A, B \in [\omega]^\omega$. By the previous lemma, there exists $n < \omega$ such that $B \subseteq^* \bigcup_{i\leqslant n} Y_i$. But taking this n, it follows that

$$A \times B \subseteq_{\mathcal{NC}} \bigcup_{i\leqslant n} \omega \times Y_i.$$

But this means that $A\times B \subseteq_{\mathcal{NC}} Y'$, which is a contradiction. Therefore, such Y' does not exist and Y is the least upper bound the family $\{\omega\times Y_n \mid n < \omega\}$. ■

Since the slices $\{Y(n) \mid n < \omega\}$, where Y is as in the previous proof, form an increasing family, it follows that $\{\omega \setminus Y(n) \mid n < \omega\}$ is a decreasing family, and that $\omega \times \omega \setminus Y$ is as some of elements of $\mathcal{NC}^+$ described in Section 4.2. Their importance is assessed by the next corollary.

Corollary 4.3.7. *If $\{X_n \mid n < \omega\} \subseteq [\omega]^\omega$, then*

$$\coprod_{n<\omega} \bigcap_{i\leqslant n} X_i =_{\mathcal{NC}} \bigwedge_{n<\omega} \omega \times X_n.$$

Also if $\{X_n \mid n < \omega\} \subseteq [\omega]^\omega$ is an increasing (decreasing) family the set $\coprod_{n<\omega} X_n$ is the supremum (infimum) of the family $\{\omega \times X_n \mid n < \omega\}$.

Considering the last results, it could be suggested that $\mathcal{P}(\omega \times \omega)/\mathcal{NC}$ is a σ-algebra. However this is not the case and it can be proved just changing the axis. Indeed, take a disjoint family $\{Y_n \mid n < \omega\} \subseteq [\omega]^\omega$, and take $X \in \mathcal{NC}^+$ such that $Y_n \times \omega \subseteq_{NC} Y$, for all $n < \omega$. It will be proved that this upper bound is not the least one. Take A_0, B_0 such that $A_0 \times B_0 \subseteq_{\mathcal{NC}} (Y \cap Y_0 \times \omega)$. Suppose that for some $n < \omega$ infinite rectangles $A_i \times B_i$, for all $i \leqslant n$, such that $A_i \subseteq Y_i$, $B_{i+1} \subseteq B_i$ and $A_i \times B_i \subseteq_{\mathcal{NC}} (Y \cap Y_i \times \omega)$, for all $i \leqslant n$. Since $Y_{n+1} \times B_n \subseteq_{\mathcal{NC}} Y$, take $A_{n+1} \in [Y_{n+1}]^\omega$ and $B_{n+1} \in [B_n]^\omega$ such that $A_{n+1} \times B_{n+1} \subseteq (Y \cap Y_{n+1} \times B_n)$. Take now $B \in [\omega]^\omega$ such that $B \subseteq^* B_n$, for all $n < \omega$. For $n < \omega$ take $k_n \in A_n$ such that $B_n \subseteq^* Y(n)$. If $A := \{k_n \mid n < \omega\}$, then $A \times B \subseteq_{\mathcal{NC}} Y$. On the other hand $A \times B \cap Y_n \times \omega \in \mathcal{NC}$, for all $n < \omega$, witnessing that Y is not the supremum of $\{Y_n \times \omega \mid n < \omega\}$.

Considering this observation, it is worth noticing that there are other instances than those in Lemma 3.2.1 where a countable family $\{A_n \mid n < \omega\} \subseteq \mathcal{NC}^+$ has supremum in $\mathcal{P}(\omega \times \omega)/\mathcal{NC}$.

Observation 4.3.8. *Take $\{A_n \mid n < \omega\} \subseteq \mathcal{NC}^+$. If there exists $\{X_n \mid n < \omega\}$ a partition of ω such that $A_n(m) \subseteq^* X_n$ for all $n < \omega$ and for almost all $m < \omega$, there exists $A \in \mathcal{NC}^+$ such that $A = \bigvee_{n<\omega} A_n$.*

Proof. Define

$$A := \coprod_{n<\omega} \bigcup_{i \leqslant n} X_i \cap A_i(n).$$

Since $A_n \subseteq_{\mathcal{NC}} \omega \times X_n$, it follows that $A_n \subseteq_{\mathcal{NC}} A$, for all $n < \omega$. Now take $C \times D \subseteq_{\mathcal{NC}} A$. As in the proof of Lemma 4.3.6 there exists $n < \omega$ such that $D \subseteq^* \bigcup_{i \leqslant n} X_i$. Therefore there exists $i \leqslant n$ such that $C \times D \cap A_i \in \mathcal{NC}^+$, thereby proving that $A = \bigvee_{n<\omega} A_i$. ■

Another idea that could arise from Lemma 4.3.6 is that from $\{\omega \times X \mid X \in [\omega]^\omega\}$ a σ-algebra could be generated as a subalgebra of $\mathcal{P}(\omega \times \omega)/\mathcal{NC}$. However, this idea will be disproved later. Despite all this observations, Lemma 4.3.6 proves to be powerful for studying the combinatorial structure of $\mathcal{P}(\omega \times \omega)/\mathcal{NC}$ and some of its cardinal invariants.

Corollary 4.3.9. $\mathfrak{a}(\mathcal{NC}) = \omega$.

Proof. Take $\{A_n \mid n < \omega\} \subseteq [\omega]^\omega$ a partition of ω. Then

$$\{\omega \times A_n \mid n < \omega\} \cup \{\omega \times \omega \setminus \bigvee_{n<\omega} \omega \times A_n\}$$

is clearly an $\mathcal{NC}$-mad family witnessing this statement. ■

By Observation 2.4.8, we have also that $\mathfrak{t}(\mathcal{NC}) = \mathfrak{p}(\mathcal{NC}) = \omega$. Considering the ideals on $\omega \times \omega$ defined at the beginning of this chapter, this result contrasts with the simple fact that $\omega_1 \leqslant \mathfrak{a}(fin \times \emptyset), \mathfrak{a}(\mathcal{ED}), \mathfrak{a}(fin \times fin)$. The only companion of $\mathcal{NC}$ in this case is $\emptyset \times fin$. Indeed $\{\{n\} \times \omega \mid n < \omega\}$ is an infinite partition of $\mathcal{P}(\omega \times \omega)/\emptyset \times fin$. However, in that case we have that $\overline{\mathfrak{a}}(\emptyset \times fin) = \mathfrak{a}$, which is not (consistently) the case for $\mathcal{NC}$.

Theorem 4.3.10. *If $\mathfrak{p} = \mathfrak{t} = \omega_1$, then $\overline{\mathfrak{a}}(\mathcal{NC}) = \omega_1$.*

Proof. Suppose that $\mathfrak{t} = \omega_1$ and that $\{X_\alpha : \alpha < \omega_1\} \subseteq [\omega]^\omega$ is an increasing tower of $\mathcal{P}(\omega)/fin$, i.e. $X_\alpha \subseteq^* X_\beta$, for all $\alpha < \beta < \omega_1$, and for all $X \in [\omega]^\omega$, there exists $\alpha < \omega_1$ such that $|X_\alpha \cap X| = \omega$. For $\alpha < \omega_1$ define in $\mathcal{P}(\omega \times \omega)/\mathcal{NC}$

$$A_\alpha := \omega \times X_\alpha \setminus \bigvee_{\beta<\alpha} \omega \times X_\beta.$$

These elements of $\mathcal{P}(\omega \times \omega)/\mathcal{NC}$ are well defined by Lemma 4.3.6 because each $\alpha < \omega_1$ is a countable ordinal. By definition, if $\alpha < \beta < \omega_1$, then $A_\alpha \subseteq_{\mathcal{NC}} \omega \times X_\alpha$, while $A_\beta \cap (\omega \times X_\alpha) \in \mathcal{NC}$. It follows that $\{A_\alpha : \alpha < \omega_1\}$ is an $\mathcal{NC}$-ad family.

Take $A, B \in [\omega]^\omega$ and let α be the least ordinal less than ω_1 such that $|B \cap X_\alpha| = \omega$. It follows that $(A \times B) \cap (\omega \times X_\alpha) \in \mathcal{NC}^+$ and that $(A \times B) \cap (\omega \times X_\beta) \in \mathcal{NC}$, for all $\beta < \alpha$. Therefore $(A \times B) \cap A_\alpha \in \mathcal{NC}^+$. Then $\{A_\alpha \mid \alpha < \omega_1\}$ is an $\mathcal{NC}$-mad family proving that $\overline{\mathfrak{a}}(\mathcal{NC}) = \omega_1$. ■

It is known that by adding κ many Hechler reals to a GCH model, for an uncountable regular cardinal κ, we obtain a model satisfying $\mathfrak{t} = \omega_1$ and $\mathfrak{b} = \kappa = \mathfrak{c}$ (see [2] or [5]). Together with Corollary 3.3.1, this fact helps to prove the next consistency result.

Corollary 4.3.11. *If κ is an uncountable regular cardinal in a model V of ZFC + GCH, there exists a ccc generic extension of V where $\mathfrak{p} = \overline{\mathfrak{a}}(\mathcal{NC}) = \omega_1 < add(\mathcal{M}) = \mathfrak{b} = \mathfrak{a}(2) = \mathfrak{a} = \mathfrak{c} = \kappa$.*

Since $\mathfrak{b} \leqslant \mathfrak{a}(fin \times fin)$, $\mathfrak{b} \leqslant \overline{\mathfrak{a}}(\mathcal{I})$, if $\mathcal{I}$ is an analytic P-ideal[1], and $add(\mathcal{M}) \leqslant \overline{\mathfrak{a}}(nwd)$[2], where nwd is the ideal of nowhere dense subsets of the rationals, these cardinal invariants are also equal to κ in the model of Corollary 4.3.11. It is fair to say that $\mathcal{NC}$-mad families behave differently than mad families, and than $\mathcal{I}$-mad families, for several definable ideals $\mathcal{I}$. In the following theorem we will prove that in ZFC there exists a tower of $\mathcal{P}(\omega \times \omega)/\mathcal{NC}$ of size ω_1, which is also a huge contrast with classic "towers".

Theorem 4.3.12. *Let $T \subseteq \omega^{<\omega_1}$ be an Aronszajn tree. Take $\{A_\sigma \mid \sigma \in T\} \subseteq [\omega]^\omega$ such that if $\sigma \subseteq \tau \in T$, then $A_\tau \subseteq^* A_\sigma$ and $\{A_\sigma \mid \sigma \in T_\alpha\}$ is a partition of ω, for all $\alpha < \kappa$. Then the sets $X_\alpha := \bigvee_{\sigma \in T_\alpha} \omega \times A_\sigma$ for $\alpha < \omega_1$ form an $\mathcal{NC}$-tower.*

Proof. Suppose there exist $A, B \in [\omega]^\omega$ such that $A \times B \subseteq_{\mathcal{NC}} X_\alpha$, for all $\alpha < \kappa$. This means, by Lemma 4.3.5 and Theorem 4.3.6, that for all α there exists $F_\alpha \in [T_\alpha]^{<\omega}$ such that

$$B \subseteq^* \bigcup_{\sigma \in F_\alpha} A_\sigma.$$

Extending the centered family $\{\bigcup_{\sigma \in F_\alpha} A_\sigma \mid \alpha < \omega_1\}$ to an ultrafilter $\mathcal{U}$ we can choose σ_α, for all $\alpha < \omega_1$, such that $\{A_{\sigma_\alpha} \mid \alpha < \omega_1\} \subseteq \mathcal{U}$. This means that $\{\sigma_\alpha \mid \alpha < \omega_1\}$ is a set of compatible functions. Therefore, T has a cofinal branch, which is a contradiction. Therefore, there is no such lower bound and $\{X_\alpha \mid \alpha < \kappa\}$ is an $\mathcal{NC}$-tower. ■

While Corollary 4.3.11 proved the quotient of $\mathcal{NC}$ to be different from those of many definable ideals, Theorem 4.3.12 provides it with a companion in some respect: it was proved in [26] that $\mathcal{P}(\omega \times \omega)/fin \times fin$ has a tower of size ω_1.

Given this list of results on small substructures of $\mathcal{P}(\omega \times \omega)/\mathcal{NC}$, one could wonder if this tower could be used in a way similar to that of Proposition 4.3.10 to build a partition of size ω_1, whose existence were provable in ZFC. Observe that for this idea to hold it would be necessary to iterate $\bigvee$ and $\bigwedge$ over countable indices to at least some degree, i.e. to have

$$\bigwedge_{n<\omega} \bigvee_{m<\omega} \omega \times X_n^m,$$

[1] Proofs for these three inequalities can be found in [8].

[2] This was proved in [25].

in $\mathcal{P}(\omega\times\omega)/\mathcal{NC}$ at least for some families $\{X_n^m \mid n,m<\omega\}\subseteq[\omega]^\omega$. To prove the existence of such an element, it is enough to consider the case when $\{X_n^m \mid m<\omega\}$ is a disjoint family, for all $n<\omega$. Observe also that for all $n<\omega$, the family $\{X_n^m \mid m<\omega\}$ can be replaced by $\{X_n^{m_0}\cap X_i^{m_1} \mid i<n\ m_0,m_1<\omega\}$, since

$$\bigwedge_{i\leqslant n}\bigvee_{m<\omega}\omega\times X_n^m=\bigwedge_{i\leqslant n}\bigvee_{j<i\ ,m_0,m_1<\omega}\omega\times(X_i^{m_0}\cap X_j^{m_1}),$$

for all $n<\omega$. This means that the family $\{X_n^m \mid n,m<\omega\}$ can be considered to have a tree structure like that of Proposition 4.2.9 and of the previous theorem. Thus the next result uses the structure of subtrees of $\omega^{<\omega}$ for describing iterations of $\bigvee$ and $\bigwedge$ over countable indices.

Proposition 4.3.13. *Let $T\subseteq\omega^{<\omega}$ be a subtree and take $\{A_\sigma \mid \sigma\in T\}\subseteq[\omega]^\omega$ such that if $\sigma\subseteq\tau\in T$, then $A_\tau\subseteq^* A_\sigma$ and $\{A_\sigma \mid \sigma\in T_n\}$ is a partition of ω, for all $n<\omega$. Then*

$$\bigwedge_{n<\omega}\bigvee_{\sigma\in T_n}\omega\times A_\sigma$$

exists in $\mathcal{P}(\omega\times\omega)/\mathcal{NC}$ iff there exist $T^n\subseteq T$, for $n<\omega$, finite level subtrees such that

$$[T]=\bigcup_{n<\omega}[T^n].$$

Proof. First suppose that we have such decomposition $\{T^n \mid n<\omega\}$ of T. Define

$$A:=\coprod_{n<\omega}\bigcup\{A_\sigma \mid \sigma\in\bigcup_{i\leqslant n}T_n^i\}.$$

Define also

$$A_n:=\bigvee_{\sigma\in T_n}\omega\times A_\sigma,$$

which it is known to exist by Lemma 4.3.6. Take $C,B\in[\omega]^\omega$ such that $C\times B\subseteq_{\mathcal{NC}}A$. Without loss of generality, this means that $B\subseteq^* A(n)$, for all $n\in C$. It follows that for all $n\in C$ there exists $F_n\in[T_n]^{<\omega}$ such that $B\subseteq^*\bigcup_{\sigma\in F_n}A_\sigma$. Therefore $\omega\times B\subseteq_{\mathcal{NC}}A_n$, for all $n\in C$. Since $\{A_n \mid n<\omega\}$ is a decreasing family, it follows that $C\times B\subseteq_{\mathcal{NC}}A_n$, for all $n<\omega$. Since rectangles of the kind $C\times B$ are dense in $\mathcal{P}(\omega\times\omega)/\mathcal{NC}$, it follows that A is a lower bound of the family $\{A_n \mid n<\omega\}$.

Take $C,B\in[\omega]^\omega$ such that $C\times B\subseteq_{\mathcal{NC}}A_n$, for all $n<\omega$. As in previous proofs it follows that for all $n<\omega$ there exists $F_n\in[T_n]^{<\omega}$ such that $B\subseteq^*\bigcup_{\sigma\in F_n}A_\sigma$. With the help of an ultrafilter containing B one can get a branch $f\in[T]$ such that

$$\{A_{f\restriction n} \mid n<\omega\}\cup\{B\}$$

is a centered family. Take $n<\omega$ such that $f\in[T^n]$. By construction of A, this means that $\{A(m) \mid m\geqslant n\}\cup\{B\}$ is a centered family. Therefore,

$A \cap (C \times B) \in \mathcal{NC}^+$. It follows that all positive lower bound the family $\{A_n \mid n < \omega\}$ is met by A. Therefore, A is the biggest of them, in symbols $A = \bigwedge_{n<\omega} \bigvee_{\sigma \in T_n} \omega \times A_\sigma$.

Now suppose that there exists $A \in \mathcal{NC}^+$ such that

$$A = \bigwedge_{n<\omega} \bigvee_{\sigma \in T_n} \omega \times A_\sigma.$$

For $F \in [\omega]^{<\omega}$ define

$$A_F := \coprod_{n \in \omega \setminus F} \bigcap_{i \in n+1 \setminus F} A(i) \setminus \bigcup_{j \in F} A(j).$$

Claim 4.3.14.

$$A = \bigvee_{F \in [\omega]^{<\omega}} A_F.$$

Proof. By definition, it follows that $A_F \subseteq_{\mathcal{NC}} A$, for all $F \in [\omega]^{<\omega}$. For proving that A is the least upper bound of the family $\{A_F \mid F \in [\omega]^{<\omega}\}$, take $C, B \in [\omega]^\omega$ such that $C \times B \subseteq_{\mathcal{NC}} A$. Without loss of generality, it follows that $\{A(n) \mid n \in C\} \cup \{B\}$ is a centered family. Take $C' \in [\omega]^\omega$ such that $C \subseteq C'$, that $\{A(n) \mid n \in C'\} \cup \{B\}$ is a centered family and such that it is maximal with this property. Take $B' \in [B]^\omega$ such that $B' \subseteq^* A(n)$, for all $n \in C'$. Clearly $C' \times B' \subseteq_{\mathcal{NC}} A$.

Claim 4.3.15. *C' is cofinite.*

Proof. Since $C' \times B'$ is a lower bound of $\bigvee_{\sigma \in T_n} \omega \times A_\sigma$, for all $n < \omega$, we can take $F_n \in [T_n]^{<\omega}$ such that $B' \subseteq^* \bigcup_{\sigma \in F_n} A_\sigma$. Observe that $\omega \times B' \subseteq_{\mathcal{NC}} \bigvee_{\sigma \in T_n} \omega \times A_\sigma$, for all $n < \omega$. Therefore, if $\omega \setminus C'$ is infinite, the rectangle $(\omega \setminus C') \times B'$ is also a positive lower bound of $\bigvee_{\sigma \in T_n} \omega \times A_\sigma$, for all $n < \omega$. Since A is the infimum of this sets, it follows that $A \cap ((\omega \setminus C') \times B') \in \mathcal{NC}^+$.

Observe that if $n \in \omega \setminus C'$, then $|A(n) \cap B'| < \omega$: otherwise the maximality of C' would be contradicted. This means that $(\omega \setminus C') \times B'$ is $\mathcal{NC}$-almost disjoint to A, which is a contradiction. Therefore, C' is cofinite. ∎

Define $F_0 = \omega \setminus C'$. By definition of A_{F_0} and maximality of C' it follows that $B' \subseteq^* A_{F_0}(n)$, for all $n \in C'$. Therefore $C \times B' \subseteq_{\mathcal{NC}} A_{F_0} \cap (C \times B)$, whereby the claim is proved. ∎

Take $F \in [\omega]^{<\omega}$. By definition, if $n, m \in \omega \setminus F$ are such that $n < m$, then $A_F(m) \subseteq A_F(n)$. Since F is finite, with changes restricted to finitely many slices A_F can be changed to a representative in $\mathcal{NC}^+$ such that $A_F(m) \subseteq A_F(n)$, for all $n < m < \omega$. Then, by Corollary 4.3.7, there exists $\{B_n \mid n < \omega\} \subseteq [\omega]^\omega$ a decreasing family such that $A_F = \bigwedge_{n<\omega} \omega \times B_n$. By this observation and previous claim it follows that there exists a family $\{B_n^m \mid n, m < \omega\} \subseteq [\omega]^\omega$ such that $B_{n+1}^m \subseteq B_n^m$, for all $m, n < \omega$, and

$$A = \bigvee_{m<\omega} \bigwedge_{n<\omega} \omega \times B_n^m.$$

With the help of this family, the desired trees of finite levels will be built.

Claim 4.3.16. *Fix $m < \omega$. For all $k < \omega$ there exist $F_k \in [T_k]^{<\omega}$ and $n_k < \omega$ such that $B^m_{n_k} \subseteq^* \bigcup_{\sigma \in F_k} A_\sigma$.*

Proof. Suppose that, on the contrary, there exists $k < \omega$ such that for all $F \in [T_k]^{<\omega}$ and for all $n < \omega$, $B^m_n \nsubseteq^* \bigcup_{\sigma \in F} A_\sigma$. With a standard diagonal argument one can find $B \in [\omega]^\omega$, a pseudointersection of the family $\{B^m_n \mid n < \omega\}$, which is also almost disjoint to every element of the family $\{A_\sigma \mid \sigma \in T_k\}$. Therefore $\omega \times B \subseteq_{\mathcal{NC}} \bigwedge_{n<\omega} \omega \times B^m_n$ while being $\mathcal{NC}$-almost disjoint to $\bigvee_{\sigma \in T_k} \omega \times A_\sigma$. Since $\bigwedge_{n<\omega} \omega \times B^m_n \subseteq_{\mathcal{NC}} A \subseteq_{\mathcal{NC}} \bigvee_{\sigma \in T_k} \omega \times A_\sigma$, this is a contradiction.

■

Fix $m < \omega$. For $k < \omega$, define n_k as the least integer and F_k as the smallest finite subset of T_k such that $B^m_{n_k} \subseteq^* \bigcup_{\sigma \in F_k} A_\sigma$. Then

$$T^m := \bigcup_{k<\omega} F_k$$

is a subtree of T. Indeed, take $k_0 < k_1 < \omega$ and $\rho \in F_{k_1}$. First observe that $n_{k_0} \leqslant n_{k_1}$. If $F' := \{\sigma \restriction k_0 \mid \sigma \in F_{k_1}\}$, then

$$B^m_{n_{k_1}} \subseteq^* \bigcup_{\sigma \in F_{k_1}} A_\sigma \subseteq \bigcup_{\tau \in F'} A_\tau.$$

By definition also $|B^m_{n_{k_1}} \cap A_\sigma| = \omega$, for all $\sigma \in F_{k_1}$. It follows that $|B^m_{n_{k_0}} \cap A_\tau| = \omega$, for all $\tau \in F'$. Therefore, $F' \subseteq F_{k_0}$ and in particular $\rho \restriction k_0 \in F_{k_0}$. It follows that T^m is a tree.

Unfix m. Take $f \in [T]$ and define $A' := \bigwedge_{n<\omega} \omega \times A_{f\restriction n}$. Taking $C \times B \subseteq_{\mathcal{NC}} A'$, $B \subseteq^* A_{f\restriction n}$, for all $n < \omega$. Therefore $C \times B \subseteq_{\mathcal{NC}} \bigvee_{\sigma \in T_n} \omega \times A_\sigma$, for all $n < \omega$. It follows that $A' \subseteq_{\mathcal{NC}} A$. Then there exists $m < \omega$ such that

$$A' \cap \bigwedge_{n<\omega} \omega \times B^m_n \in \mathcal{NC}^+.$$

Take $k < \omega$ and let n_k and F_k be as in the previous paragraph for this new m. Since $|A_{f\restriction k} \cap B^m_{n_k}| = \omega$, it follows that $f \restriction k \in F_k$. Therefore $f \in [T^m]$ and we conclude that $[T] = \bigcup_{n<\omega} [T_n]$.

■

Before applying this proposition, contrast this result with Lemma 4.3.6. In its proof the supremum of the rectangles $\{\omega \times Y_n \mid n < \omega\}$ is approximated with the supremum of finitely many of the reals $\{Y_n \mid n < \omega\}$ in each of the slices of $\omega \times \omega$. The same happens when getting the infimum. If the purpose were to approximate the infimum of the family $\{\bigvee_{\sigma \in T_n} \omega \times A_\sigma \mid n < \omega\}$ with the same idea, there would be no obvious choice for a just one real describing the "finite" approximation $\bigwedge_{i \leqslant n} \bigvee_{\sigma \in T_i} \omega \times A_\sigma$, since there are no reals representing $\bigvee_{\sigma \in T_n} A_\sigma$ if T_n is infinite, but in trivial cases, i.e. when there exist $F \in [T_n]^{<\omega}$ such that $A_\sigma \subseteq^* \bigcup_{\tau \in F} A_\tau$, for all $\sigma \in T_n$. Then the finite level subtrees of T in Proposition 4.3.13 can be interpreted as the required code of finite operations for the infimum to exist.

Suppose that the tower of Theorem 4.3.12 associated to an Aronszajn tree T is to be used to construct a partition of size ω_1 on $\mathcal{P}(\omega \times \omega)/\mathcal{NC}$. For $\alpha < \omega_1$, a limit ordinal, the existence of $\bigwedge_{\beta<\alpha}(\bigvee_{\sigma\in T_\beta} \omega \times A_\sigma)$ would be necessary. Since α is of countable cofinality, if $\{\alpha_n \mid n < \omega\}$ is an increasing sequence of ordinals converging to α, said set exists iff $\bigwedge_{n<\omega}(\bigvee_{\sigma\in T_{\alpha_n}} \omega \times A_\sigma)$ exists. Observe that $\bigcup_{n<\omega} T_{\alpha_n} \cup \{\emptyset\}$ is isomorphic to a subtree of $\omega^{<\omega}$. With the help of Proposition 4.3.13 the conclusion is that for each limit $\alpha < \omega_1$ it would be necessary the existence of $\{S^{\alpha,m} \mid m < \omega\}$ a family of subtrees of finite levels of $T_{<\alpha}$ such that

$$[T_{<\alpha}] = \bigcup_{m<\omega} [S^{\alpha,m}].$$

The following result shows that this is not the case for any Aronszajn tree.

Proposition 4.3.17. *Let $T \subseteq \omega^{<\omega_1}$ be an Aronszajn tree. There exists $\alpha \in Lim(\omega_1)$ such that if we take $\{S^n \mid n < \omega\}$, a family of finite level subtrees of $T_{<\alpha}$, then*

$$\bigcup_{n<\omega} [S^n] \neq [T_{<\alpha}].$$

Proof. First, for $t \in T$ set the notation $t\uparrow := \{s \in T \mid t \subseteq s\}$. Notice that

$$T' := \{t \in T \mid |t\uparrow| = \omega_1\}$$

is a well-pruned Aronszajn tree and that any $\alpha \in Lim(\omega_1)$ proving the proposition for T' will prove it for T. Indeed, if α proves it for T' and $\{S^n \mid n < \omega\}$ is a family of finite level subtrees of $T_{<\alpha}$ such that $\bigcup_{n<\omega}[S^n] = [T_{<\alpha}]$, then $\bigcup_{n<\omega}[S^n \cap T'] = [T'_{<\alpha}]$, which is a contradiction. So without loss of generality, suppose that T is well-pruned.

Claim 4.3.18. *There exists $\alpha \in Lim(\omega_1)$ such that for all $t \in T_{<\alpha}$ there exists $dom(t) \leqslant \gamma < \alpha$ such that $|t\uparrow \cap T_\gamma| = \omega$.*

Proof. Suppose that it is not the case and for all $\alpha \in Lim(\omega_1)$ choose $t_\alpha \in T_{<\alpha}$ such that $|t_\alpha \uparrow \cap T_\gamma| < \omega$, for all $dom(t_\alpha) \leqslant \gamma < \alpha$. Define $g : Lim(\omega_1) \to \omega_1$ as $g(\alpha) = dom(t_\alpha)$. This is a regressive function defined on a stationary set. Therefore, there exist $\beta < \omega_1$ and $S \in [\omega_1]^{\omega_1}$ such that $g(\alpha) = \beta$, for all $\alpha \in S$ (see Lemma III.6.14 of [15]). Furthermore, since T_β is countable, there exists $t \in T_\beta$ and $S' \in [S]^{\omega_1}$ such that $t_\alpha = t$, for all $\alpha \in S'$. Since S' is a cofinal subset of ω_1 and T is well pruned, it follows that $0 < |t\uparrow \cap T_\gamma| < \omega$, for all $dom(t) \leqslant \gamma < \omega_1$. But a compacity argument gives us a cofinal branch on T, which is a contradiction. ■

Take the α given by this claim and let $\{S^n \mid n < \omega\}$ be a family of subtrees of $T_{<\alpha}$ with finite levels. Fix $\{\alpha_n \mid n < \omega\} \subseteq \alpha$ an increasing sequence converging to α. Begin with $t_0 \in T_{<\alpha}$. The claim gives us $dom(t_0) \leqslant \gamma_0 < \alpha$

such that $|t_0 \uparrow \cap T_{\gamma_0}| = \omega$. Since $|S^0 \cap T_{\gamma_0}| < \omega$ it is possible to take $t_1' \in (t_0 \uparrow \cap T_{\gamma_0}) \setminus S^0$. Then choose $t_1 \in t_1' \uparrow \cap T_{\max\{\alpha_0,\gamma_0\}}$. Suppose now that we have $\{t_0, ..., t_i\}$, an increasing chain in $T_{<\alpha}$, for some $i < \omega$. We know that there exists $dom(t_i) \leqslant \gamma_i < \alpha$ such that $|t_i \uparrow \cap T_{\gamma_i}| = \omega$. Therefore we can choose $t_{i+1} \in (t_i \uparrow \cap T_{\max\{\alpha_i,\gamma_i\}}) \setminus S^i$. If f is the cofinal branch of $T_{<\alpha}$ extending the chain $\{t_i \mid i < \omega\}$, by its construction we get that $f \notin \bigcup_{n<\omega}[S^n]$, whereby proving the proposition. ■

Up to here, some interesting small towers and partitions of $\mathcal{P}(\omega\times\omega)/\mathcal{NC}$ have been built, but also some dead ends in such topics have been found and some remarks are necessary for the path forward. First observe that all said constructions consist of sets of the type $\omega \times X$, $\bigwedge_{n<\omega} \omega \times X_n$ or $\bigvee_{n<\omega} \omega \times X_n$ where $\{X\} \cup \{X_n \mid n < \omega\} \subseteq [\omega]^\omega$.

It is easy to see that any $\mathcal{NC}$-mad family consisting only of elements of the type $\omega\times X$ or $\bigvee_{n<\omega} \omega\times X_n$ must be of size at least $\mathfrak{a}$. Indeed if $\{X_\alpha^n \mid n < \omega,\ \alpha < \kappa\} \subseteq [\omega]^\omega$, defining $Y_\alpha^n := X_\alpha^n \setminus \bigcup_{i<n} X_\alpha^i$, then $\{\bigvee_{n<\omega} \omega\times X_\alpha^n \mid \alpha < \kappa\}$ is a partition of $\mathcal{P}(\omega \times \omega)/\mathcal{NC}$ iff $\{Y_\alpha^n \mid n < \omega,\ \alpha < \kappa\}$ is a mad family. Therefore the only hope for getting a $\mathcal{NC}$-mad family of size ω_1 consisting of "nice" sets, as those described in the previous paragraph, while also having $\mathfrak{p} > \omega_1$ can only be achieved with sets of the form $\bigwedge_{n<\omega} \omega \times X_n$, as were constructed the partitions of Corollary 4.3.9 and Theorem 4.3.10.

Observe that if $\{\bigwedge_{n<\omega} \omega \times X_\alpha^n \mid \alpha < \omega_1\}$ is a partition $\mathcal{P}(\omega \times \omega)/\mathcal{NC}$ then we can suppose that

1. $X_\alpha^{n+1} \subseteq X_\alpha^n$, for all $\alpha < \omega_1$ and all $n < \omega$,
2. for all $\alpha < \beta < \omega_1$ there exists $n < \omega$ such that $|X_\alpha^n \cap X_\beta^n| < \omega$, and
3. for all $X \in [\omega]^\omega$ there exists $\alpha < \omega_1$ such that $\{X\} \cup \{X_\alpha^n \mid n < \omega\}$ is a centered family.

However, the next result says that this idea, as that of the Aronszajn tree, does not hold.

Theorem 4.3.19. *Suppose that $\omega_1 \leqslant \kappa < \mathfrak{p}$ and take $\{A_\alpha^n \mid \alpha < \kappa, n < \omega\} \subseteq [\omega]^\omega$ such that*

1. $A_\alpha^{n+1} \subseteq A_\alpha^n$, for all $\alpha < \kappa$ and all $n < \omega$, and

2. for all $\alpha < \beta < \kappa$ there exits $n < \omega$ such that $|A_\alpha^n \cap A_\beta^n| < \omega$.

Then there exists $X \in [\omega]^\omega$ such that for all $\alpha < \kappa$ there exists $n < \omega$ such that $|X \cap A_\alpha^n| < \omega$.

Proof. Without loss of generality, we can suppose that $A_\alpha^0 = \omega$, for all $\alpha < \kappa$. Also we can suppose that $\bigcap_{n<\omega} A_\alpha^n = \emptyset$, for all $\alpha < \kappa$. We will prove that there exists a function $g : \kappa \to [\omega]^{<\omega} \setminus \{\emptyset\}$ such that

$$\{ \bigcup_{i\in g(\alpha)} A_\alpha^i \setminus A_\alpha^{i+1} \mid \alpha < \kappa\}$$

is a centered family. Since $\kappa < \mathfrak{p}$, if such function exists take $X \in [\omega]^\omega$ a pseudointersection of the corresponding family. For $\alpha < \kappa$ put $i_\alpha := \max(g(\alpha))$. Therefore $|X \cap A_\alpha^{i_\alpha}| < \omega$, for all $\alpha < \kappa$, and X would be the desired set.

Firstly set some notations:

- by $p; \kappa \to \omega$ it will be meant that p is a function with a finite subset of κ as domain an images in ω, and
- for $p; \kappa \to \omega$ define

$$A(p) := \bigcap_{\alpha \in dom(p)} A_\alpha^{p(\alpha)} \setminus A_\alpha^{p(\alpha)+1}$$

Observe that if the forcing notion $\{A(p) \mid |A(p)| = \omega\}$, ordered by $\subseteq$, had the appropriate dense sets, specifically if $\{A(p) \mid \alpha \in dom(p) \wedge |A(p)| = \omega\}$ were dense, any generic filter would be the desired centered family. However, this is not necessarily a σ-centered forcing. Besides the sets $A(p)$ being infinite will not be enough for the having the required dense sets. We will need the following predicate for these intersections being as big as they need to:

$$Q(p) \equiv \exists \mathcal{X} \in [\kappa]^{\geqslant \omega_1} \ \forall \alpha \in \mathcal{X} \ \exists Y_\alpha \in [A(p)]^\omega \ \forall n < \omega (Y_\alpha \subseteq^* A_\alpha^n),$$

for $p; \kappa \to \omega$. In plain words, $Q(p)$ holds if the restriction of $\{A_\alpha^n \mid \alpha < \kappa, n < \omega\}$ to $A(p)$ is uncountable. Hence, if $Q(p)$ holds, then in $A(p)$ the hypotheses of the theorem hold. This is important since we already know it to be false in the case $\kappa = \omega$ (Corollary 4.3.9). In other words, if $Q(p)$ does not hold, it is possible that $A(p)$ cannot be "extended" to our desired centered family.

Consider the forcing notion

$$\mathbb{P} := \{A(p) \mid p; \kappa \to \omega \wedge Q(p)\},$$

ordered by inclusion. Since $Q(\emptyset)$ holds, with $\mathcal{X} = \kappa$ as a witness, $\mathbb{P}$ is non-empty. Take the sets $D_F := \{p \in \mathbb{P} \mid F \subseteq dom(p)\}$, for non-empty $F \in [\kappa]^{<\omega}$. Observe that if they were dense, then a generic filter in $\mathbb{P}$ give us the centered family. The next claim proves that those sets are indeed dense.

Claim 4.3.20. *Suppose that $Q(p)$ holds. Then for all $F \in [\kappa]^{<\omega}$ there exists $q \supseteq p$ such that $F \subseteq dom(q)$ and $Q(q)$ holds.*

Proof. Let $\mathcal{X}_p$ be the uncountable subset of κ given by $Q(p)$ and enumerate $F \setminus dom(p) := \{\alpha_0, ..., \alpha_{k-1}\}$. We will recursively define $q \subseteq q_0 \subseteq ... \subseteq q_i \subseteq q_{k-1}$, such that $Q(q_i)$ holds and $\alpha_i \in dom(q_i)$, for $i < k$.

First observe that the family $\{A(p) \cap (A_{\alpha_0}^n \setminus A_{\alpha_0}^{n+1}) \mid n < \omega\}$ is an at most countable partition of $A(p)$. Take $\alpha \in \mathcal{X}_p \setminus \{\alpha_0\}$ and $Y_\alpha \in [A(p)]^\omega$ which is a pseudointersection of $\{A_\alpha^n \mid n<\omega\}$. Since $\alpha\neq\alpha_0$, then there exists n_α such that $Y_\alpha\subseteq A(p)\cap(\cup_{i<n_\alpha}A^i_{\alpha_0}\setminus A^{i+1}_{\alpha_0})$. Since there exists $n<\omega$ such that $n=n_\alpha$ for uncountably many $\alpha\in\mathcal{X}_p\setminus\{\alpha_0\}$, there exists $i<n$ such that if $q_0:=p\cup(\alpha_0,i)$, then $Q(q_0)$ holds. This process can be repeated finitely many times and we can get the desired q. ■

However useful this claim would seem for getting the centered family described at the beginning of this proof, this poset does not seem to be ccc, much less σ-centered, and the hypothesis $\kappa<\mathfrak{p}$ is not helpful in this case. Nevertheless, this claim is useful and a countable subposet of $\mathbb{P}$ is enough for proving the theorem.

Set the notation

$$a(p,\alpha):=\{n<\omega \mid Q(p\cup(\alpha,n)))\},$$

for all $p;\kappa\to\omega$ such that $A(p)\in\mathbb{P}$ and $\alpha<\kappa$. By Claim 4.3.20, for all $A(p)\in\mathbb{P}$ and all $\alpha<\kappa$, the set $a(p,\alpha)$ is not empty. Observe also that $a(q,\alpha)\subseteq a(p,\alpha)$, if $p\subseteq q$. This notation will help to divide the proof in two cases.

Suppose that there exists $A(p)\in\mathbb{P}$ such that $a(p,\alpha)$ is finite for all $\alpha<\kappa$. In this case the function $g:\kappa\to[\omega]^{<\omega}\setminus\{\emptyset\}$ is defined by $g(\alpha)=a(p,\alpha)$. For proving that $\{\bigcup_{i\in a(p,\alpha)}A^i_\alpha\setminus A^{i+1}_\alpha \mid \alpha<\kappa\}$ is a centered family, take $F\in[\kappa]^{<\omega}$. If q is as given by Claim 4.3.20 for p and F, then $Q(q)$ holds and clearly $q(\alpha)\in a(p,\alpha)$, for all $\alpha\in F$. Since $A(q)$ is infinite, it follows that

$$\bigcap_{\alpha\in F}\bigcup_{j\in a(p,\alpha)}A^j_\alpha\setminus A^{j+1}_\alpha$$

is infinite, and therefore the family is centered.

Suppose now that for all $A(p)\in\mathbb{P}$, there exists $\alpha_p<\kappa$ such that $a(p,\alpha_p)$ is infinite. We will recursively define a countable "tree" $\mathbb{T}\subseteq\mathbb{P}$. Its root will be $\omega=A(\emptyset)$. If we have already defined that $A(p)\in\mathbb{T}$, then its set of successors in $\mathbb{T}$ will be $\{A(p\cup(\alpha_p,n)) \mid n\in a(p,\alpha_p)\}$. Observe that $\mathbb{T}$, ordered by the inclusion, is isomorphic to $\omega^{<\omega}$, and therefore forcing equivalent to Cohen forcing. For $\alpha<\kappa$ define

$$\mathcal{D}_\alpha:=\{A(p)\in\mathbb{T} \mid |a(p,\alpha)|<\omega\}.$$

Claim 4.3.21. *$\mathcal{D}_\alpha$ is open dense in $\mathbb{T}$, for all $\alpha<\kappa$.*

Proof. Fix $\alpha<\kappa$ and $A(p)\in\mathbb{T}$. If $\alpha\in dom(p)$, then trivially $a(p,\alpha)=\{p(\alpha)\}$, and $A(p)\in\mathcal{D}_\alpha$. If $\alpha=\alpha_p$ and $A(q)$ is some "successor" of $A(p)$ in $\mathbb{T}$, then, as in the previous case, $A(q)\in\mathcal{D}_\alpha$.

Suppose now that $\alpha\notin dom(p)$ and $\alpha\neq\alpha_p$. Then there exists $n\in a(p,\alpha_p)$ such that $a(p\cup(\alpha_p,n),\alpha)$ is finite. Otherwise for infinitely many $n<\omega$ (all $n\in a(p,\alpha_p)$) there would be infinitely many $m<\omega$ (all $m\in a(p\cup(\alpha_p,n))$ such that $A(p\cup(\alpha_p,n)\cup(\alpha,m))$ is infinite. This would easily give us an infinite pseudointersection of both $\{A^n_{\alpha_p} \mid n < \omega\}$ and $\{A^n_\alpha \mid n < \omega\}$, which is a contradiction. Take $n \in a(p, \alpha_p)$ such that $a(p \cup (\alpha_p, n), \alpha)$ is finite. Then $A(p \cup (\alpha_p, n)) \in \mathcal{D}_\alpha$. ■

Let $\{A(p_n) \mid n < \omega\} \subseteq \mathbb{T}$ be a $\{\mathcal{D}_\alpha \mid \alpha < \kappa\}$-generic branch, which exists because $\kappa < \mathfrak{p}$ and $\mathbb{T}$ is trivially σ-centered. For $\alpha < \kappa$, if $n < \omega$ is the least integer such that $A(p_n) \in \mathcal{D}_\alpha$, define $g(\alpha) := a(p_n, \alpha)$. The family $\{\bigcup_{i \in g(\alpha)} A^i_\alpha \setminus A^{i+1}_\alpha \mid \alpha < \kappa\}$ is a centered family. Indeed, take $F \in [\kappa]^{<\omega}$, and $k < \omega$ such that $A(p_k) \in \bigcap_{\alpha \in F} \mathcal{D}_\alpha$. By Claim 4.3.20, there exists $p \supseteq p_k$ such that $F \subseteq dom(p)$ and that $Q(p)$ holds. Since $p(\alpha) \in a(p_k, \alpha)$, for all $\alpha \in F$, and $a(p_k, \alpha) \subseteq g(\alpha)$, for all $\alpha \in F$, by definition of g, the set $A(p)$ witnesses that

$$\bigcap_{\alpha \in F} \bigcup_{i \in g(\alpha)} A^i_\alpha \setminus A^{i+1}_\alpha$$

is infinite.

■

Before the concluding remarks of this section, some observations on the previous proof will be given. Suppose that CH holds. Since $\mathfrak{p} = \omega_1$ holds in this model, take a tower $\{X_\alpha \mid \alpha < \omega_1\}$. For $\alpha < \omega_1$, with the help of a sequence $\{\alpha_n \mid n < \omega\}$ converging to α define $X^n_\alpha := X_\alpha \setminus X_{\alpha_n}$. Observe on one hand, the ω_1 many sequences thus defined code the partition given by Theorem 4.3.10. On the other hand, points 1 and 2 of the previous theorem hold on these sequences. Observe that for getting the X concluding said theorem it would be enough to diagonalize the tower. In terms similar of the last proof, sets of the form $\omega \setminus X_\alpha$ are the natural interpretation of the sets $A(p)$ and are a natural centered family already in the base model.

However, there are some families of the form of Theorem 4.3.19 in this model for which matters are not as simple, and extending the model beforehand with Cohen forcing is necessary. Consider for example $\{X_\sigma \mid \sigma \in 2^{<\omega}\} \subseteq [\omega]^\omega$ such that as usual $X_\tau \subseteq X_\sigma$, for $\sigma \subseteq \tau$, and that $\{X_\sigma \mid \sigma \in 2^n\}$ is a partition of ω, for all $n < \omega$. For $f \in 2^{<\omega}$ define $A_f := \bigwedge_{n<\omega} \omega \times A_{f\restriction n}$. Clearly this is always a partition of $\mathcal{P}(\omega \times \omega)/\mathcal{NC}$ and if CH holds its size is naturally ω_1. Observe that if $\mathcal{F}$ is a filter in our base model then there exists $f \in 2^\omega$ such that $\{A_{f\restriction n} \mid n < \omega\} \subseteq \mathcal{F}^+$. Therefore, by genericity any new pseudointersection X of $\mathcal{F}$ is also centered with all elements of $\{A_{f\restriction n} \mid n < \omega\}$, which means that X is not as given in Theorem 4.3.19. Also any new real, Cohen or not, would give us a countable centered family whose any pseudointersection is the X desired. Accordingly, in the notation of the previous proof, $Q(p)$ holds for all infinite $A(p)$.

In conclusion both cases in the previous proof would have to be considered for getting an extension of model of CH where Theorem 4.3.19. Although some good ideas for getting a $\mathcal{NC}$-mad family of size ω_1 in ZFC were finally rejected by Propositions 4.3.17 and 4.3.19, the next question remains open:

Question 4.3.22. *Is it provable in ZFC that there exists a $\mathcal{NC}$-mad family of size ω_1? Is its existence at least consistent with $\mathfrak{p} > \omega_1$?*

4.4 Higher dimensional relatives of the Nowhere Centered ideal

Recall the ideals fin^k on $\mathcal{P}(\omega^k)$, for $2 \leqslant k < \omega$, defined at the end of Section 1.2. Since Fubini products of Borel ideals are Borel, all of them are Borel. To some extent they represent a finite dimensional generalization of the ideal fin. The following result works as an example relevant to the main subject of this text (see [21], where an even stronger result is proved).

Theorem 4.4.1. *$\mathfrak{b} \leqslant \mathfrak{a}(fin^k)$, for all $1 \leqslant k < \omega$.*

Their use in this section will be helping to generalize the ideal $\mathcal{NC}$ to higher dimensions. The operation is hinted by the following easy observation.

Observation 4.4.2. *Take $A \subseteq \omega \times \omega$. Then $A \in \mathcal{NC}^+$ iff there exist $X, Y \in [\omega]^\omega$ such that $X \times Y \subseteq_{fin \times fin} A$.*

The idea for the generalization of $\mathcal{NC}$ is implicit in this definition, but first setting some notation and knowing some of the behaviour of the ideals fin^k will be useful.

Notation 4.4.3. *Take $2 \leqslant k < \omega$ and $A \subseteq \omega^k$. If $n < \omega$ define*

$$A(n) := \{\overline{x} \in \omega^{k-1} \mid (n) \frown \overline{x} \in A\}.$$

Lemma 4.4.4. *Take $2 \leqslant k < \omega$. Then the following statements hold:*

1. *$\{\overline{x} \in \omega^k \mid \overline{x}(i) < n\} \in fin^k$, for all $i < k$ and $n < \omega$.*
2. *If $A \subseteq \omega^k$ and $|\{n < \omega \mid \overline{x} \frown (n) \in A\}| < \omega$, for all $\overline{x} \in \omega^{k-1}$, then $A \in fin^k$.*
3. *If $1 \leqslant j < k$, $e_0 < ... < e_{j-1} < e < k$ and $g : \omega^j \to \omega$, then*

$$\{\overline{x} \in \omega^k \mid \overline{x}(e) < g(\overline{x}(e_0), ..., \overline{x}(e_{j-1}))\} \in fin^k.$$

Proof. First observe that statements 1 and 2 are provable from statement 3. Indeed if $e = j = k - 1$, and $g : \omega^j \to \omega$ is the function with constant value n, then 1 follows from 3. The same is the case for statement 2 with the function g defined by

$$g(\overline{x}) := \max\{n < \omega \mid \overline{x} \frown (n) \in A\},$$

for $\overline{x} \in \omega^{k-1}$.

So only statement 3 will be inductively proved for $1 \leqslant j < k < \omega$. For the basic case when $j = 1$ and $k = 2$, as was mentioned in Section 4.1, it is easy to see that $\{(x, y) \in \omega^2 \mid y < g(x)\} \in fin^2$, for all $g \in \omega^\omega$.

Now suppose that we have proved the statement for all $j' < k'$ such that either $j' < j$ or $k' < k$, for some $1 \leqslant j < k < \omega$. Take $e_0 < ... < e_{j-1} < e < k$ as well as $g : \omega^j \to \omega$. Define

$$A_g := \{\overline{x} \in \omega^k \mid \overline{x}(e) < g(\overline{x}(e_0), ..., \overline{x}(e_{j-1}))\}.$$

Suppose that $e_0 = 0$. Observe that with $j-1$, $e_1 - 1, < ... < e_{j-1} - 1 < e-1$ and $g_n : \omega^{j\setminus 1} \to \omega$, defined $g_n(\overline{x}) = g((n) \frown \overline{x})$, statement 3 holds, for all $n < \omega$. Therefore $A_g(n) \in fin^{k-1}$, by hypothesis induction for $(j-1, k-1)$. More directly if $0 < e_0$, equally $A_g(n) \in fin^{k-1}$, for all $n < \omega$, by hypothesis induction for $(j, k-1)$. Either way, $A \in fin^k$ and the lemma is proved. ■

Definition 4.4.5. *For $2 \leqslant k < \omega$, define*

$$\mathcal{NC}^k := \{A \subseteq \omega^k \mid \forall\ A_0, ..., A_{k-2} \in [\omega]^\omega\ \forall^\infty n < \omega\ \prod_{i<k-1} A_i \nsubseteq_{fin^{k-1}} A(n)\}.$$

Clearly $\mathcal{NC}^2 = \mathcal{NC}$. Since $\prod_{i<k} A_i \in (fin^k)^+$, for $(A_i)_{i<k} \in ([\omega]^\omega)^k$, it follows that $\mathcal{NC}^+ \subseteq (fin^k)^+$, and hence that $fin^k \subseteq \mathcal{NC}^k$, for all $2 \leqslant k < \omega$. Also observe that $\mathcal{NC}^k$ is a downward closed coanalytic subset of $\mathcal{P}(\omega^k)$. In order to prove that they are ideals we will need the folowing result.

Lemma 4.4.6. *Take $k \geqslant 1$ and a family $\{A_n^0, A_n^1 \mid n < \omega\} \subseteq (fin^k)^+$. If there exists $(X_0, ..., X_{k-1}) \in ([\omega]^\omega)^k$ such that*

$$\prod_{l<k} X_l \subseteq_{fin^k} A_n^0 \cup A_n^1,$$

for all $n < \omega$, then there exist $(Y_0, ..., Y_{k-1}) \in \prod_{l<k}[X_l]^\omega$, $Z \in [\omega]^\omega$ and $i < 2$ such that

$$\prod_{l<k} Y_l \subseteq_{fin^k} A_n^i,$$

for all $n \in Z$.

Proof. It will be inductively proved for $k \geqslant 1$. Suppose that $k = 1$ and that we have $\{A_n^0, A_n^1 \mid n < \omega\} \subseteq [\omega]^\omega$ and $X \in [\omega]^\omega$ such that $X \subseteq^* A_n^0 \cup A_n^1$ for all $n < \omega$. Then any ultrafilter $\mathcal{U}$ containing X will choose $i_n < 2$ such that $A_n^{i_n} \in \mathcal{U}$, for all $n < \omega$. There exist $Z \in [\omega]^\omega$ and $i < 2$ such that $i = i_n$, for all $n \in Z$. Take $Y \in [\omega]^\omega$, a pseudointersection of $\{X\} \cup \{A_n^i \mid n \in Z\}$ and the basic case is proved.

Suppose now that the lemma is proved for some $k \geqslant 1$ and that we have $\{A_n^0, A_n^1 \mid n < \omega\} \subseteq (fin^{k+1})^+$ and $(X_0, ..., X_k) \in ([\omega]^\omega)^{k+1}$ such that

$$\prod_{l\leqslant k} X_l \subseteq_{fin^{k+1}} A_n^0 \cup A_n^1,$$

for all $n < \omega$.

We will construct, for $n < \omega$, sequences $\{(Y_0^i, ..., Y_k^n) \mid n < \omega\} \subseteq ([\omega]^\omega)^{k+1}$ and $\langle i_n \mid n < \omega\rangle \in 2^\omega$ such that

1. $Y_l^0 \in [X_l]^\omega$, for all $l \leqslant k$
2. $Y_l^{n+1} \in [Y_l^n]^\omega$, for all $l \leqslant k$ and $n < \omega$, and
3. $$\prod_{1 \leqslant l \leqslant k} Y_l^n \subseteq_{fin^k} A_n^{i_n}(m),$$
   for all $m \in Y_0^n$, for all $n < \omega$.

The construction will be recursive. Observe that

$$\prod_{1 \leqslant l \leqslant k} X_l \subseteq_{fin^k} A_0^0(m) \cup A_0^1(m),$$

for almost all $m \in X_0$. Applying the hypothesis induction to the family $\{A_0^0(m), A_0^1(m) \mid m \in X_0\}$ and its pseudointersection $\prod_{1 \leqslant l \leqslant k} X_l$, we get $(Y_1^0, ..., Y_k^0) \in \prod_{1 \leqslant l \leqslant k} [X_l]^\omega$, $Y_0^0 \in [X_0]^\omega$ and $i_0 < 2$ such that

$$\prod_{1 \leqslant l \leqslant k} Y_l^0 \subseteq_{fin^k} A_0^{i_0}(m),$$

for all $m \in Y_0^0$.

Suppose now that for some $1 \leqslant n < \omega$ the sequences $\{(Y_0^m, ..., Y_k^m) \mid m < n\} \subseteq ([\omega]^\omega)^{k+1}$ and $\langle i_m \mid m < n \rangle \in 2^n$ for which 1, 2 and 3 hold. Since

$$\prod_{1 \leqslant l \leqslant k} Y_l^{n-1} \subseteq_{fin^k} \prod_{1 \leqslant l \leqslant k} X_l \subseteq_{fin^k} A_n^0(m) \cup A_n^i(m),$$

for almost all $m \in Y_0^{n-1}$, hypothesis induction can be applied again for obtaining $(Y_1^n, ..., Y_k^n) \in \prod_{1 \leqslant l \leqslant k} [Y_l^{n-1}]^\omega$, $Y_0^n \in [Y_0^{n-1}]^\omega$ and $i_n < 2$ such that

$$\prod_{1 \leqslant l \leqslant k} Y_l^n \subseteq_{fin^k} A_0^{i_n}(m),$$

for all $m \in Y_0^n$. Therefore, the recursion is completed.

For $l \leqslant k$, take $Y_l \in [\omega]^\omega$, a pseudointersection of the decreasing family $\{Y_l^n \mid n < \omega\}$. There exist $Z \in [\omega]^\omega$ and $i < 2$ such that $i_n = i$, for all $n \in Z$. If $n \in Z$, then

$$\prod_{1 \leqslant l \leqslant k} Y_l^n \subseteq_{fin^k} A_n^i(m),$$

for almost all $m \in Y_0$. Therefore,

$$\prod_{l \leqslant k} Y_l \subseteq_{fin^{k+1}} A_n^i,$$

for all $n \in Z$. ■

Proposition 4.4.7. *For all $k \geqslant 2$ the family $\mathcal{NC}^k$ is a proper ideal of ω^k.*

Proof. Since $\omega^k \notin \mathcal{NC}^k$ and it is downward closed, it only remains to be proved that $A_0 \cup A_1 \in \mathcal{NC}^k$, for all $A_0, A_1 \in \mathcal{NC}^k$. Take A_0 and A_1 subsets of ω^k such that $A_0 \cup A_1 \in \mathcal{P}(\omega^k) \setminus \mathcal{NC}^k$. By definition of $\mathcal{NC}^k$, there exists $(A_0, ..., A_{k-2}) \in ([\omega]^\omega)^{k-1}$ such that

$$\prod_{l<k-1} A_l \subseteq_{fin^k} A_0(n) \cup A_1(n),$$

for infinitely many $n < \omega$. Applying Lemma 4.4.6 we get $(A'_0, ...A'_{k-2}) \in ([\omega]^\omega)^{k-1}$ and $i < 2$ such that

$$\prod_{l<k-1} A'_l \subseteq_{fin^k} A_i(n),$$

for infinitely many $n < \omega$, which means that $A_i \notin \mathcal{NC}^k$. Therefore, if $A_0, A_1 \in \mathcal{NC}^k$, then $A_0 \cup A_1 \in \mathcal{NC}^k$.

■

Observe that for all $A \in (\mathcal{NC}^k)^+$ there exists $(X_0, ..., X_{k-1}) \in ([\omega]^\omega)^k$ such that $\prod_{i<k} X_i \subseteq_{fin^k} A$. It follows that

$$\bigoplus_{i<k} \mathcal{P}(\omega)/fin \leqslant_{dense} \mathcal{P}(\omega^k)/\mathcal{NC}^k.$$

Obviously, the set

$$\{\omega^{k-1} \times X \mid X \in [\omega]^\omega\}$$

forms a regular subalgebra of $\mathcal{P}(\omega^k)/\mathcal{NC}^k$ isomorphic to $\mathcal{P}(\omega)/fin$. Indeed, take a mad family $\{X_\alpha \mid \alpha < \kappa\} \subseteq [\omega]^\omega$ and $A \in (\mathcal{NC}^k)^+$. Take $B_0, ..., B_{k-1} \in [\omega]^\omega$ such that $\prod_{i<k} B_i \subseteq_{fin^k} A$. If $|X_\alpha \cap B_{k-1}| = \omega$, clearly $(\omega^{k-1} \times X) \cap A \in \mathcal{NC}^+$, which means that $\{\omega^{k-1} \times X_\alpha \mid \alpha < \kappa\}$ is a partition of $\mathcal{P}(\omega^k)/\mathcal{NC}^k$.

All results from Lemma 4.3.6 to Theorem 4.3.12 have a version on these quotients. But of more importance to this section is the fact that the greater the dimension the better the improvement of Lemma 4.3.6 we get. Since we are in higher dimensions, we have to set some new notations.

Notation 4.4.8. *Take* $1 \leqslant j < k < \omega$ *and* $E \in [k-1]^j$*. If* $\{A_{\overline{n}} \mid \overline{n} \in \omega^E\} \subseteq \mathcal{P}(\omega)$*, define*

$$\coprod_{\overline{n} \in \omega^E} A_{\overline{n}} := \{\overline{x} \in \omega^k \mid \overline{x}(k-1) \in A_{\overline{x}\restriction E}\}.$$

This is a generalization of the notation set at the beginning of Section 4. That notation allowed to copy the elements of a sequence of subsets of integers $\{A_n \mid n < \omega\}$ in the column $\{n\} \times \omega$ corresponding to the index of each set. The second coordinate of a point in $\coprod_{n<\omega} A_n$, is determined by the set of possibilities indexed by its first coordinate.

A slightly more complicated situation is described here. The last coordinate of a point in $\coprod_{\overline{n} \in \omega^E} A_{\overline{n}}$ is determined by a subset, possibly strict, of the previous coordinates. The rest of them are free to take any value. In a way with this last notation an array of subsets of integers is distributed "on top" of the corresponding index on ω^E. Since Theorem 4.4.11 deals with iterations of finite arbitrary length of Boolean operations, the following lemmas will help describing them. The first one will tell us how these operations work on the quotients $\mathcal{P}(\omega^k)/\mathcal{NC}^k$. The next one will focus on their finite approximations.

Lemma 4.4.9. *Take $1 \leqslant j < k < \omega$ and $\{A_{\overline{n}} \mid \overline{n} \in \omega^j\} \subseteq [\omega]^\omega$. Suppose that*

$$\bigwedge_{n_0<\omega} \bigvee_{n_1<\omega} \dots \omega^{k-1} \times A_{n_0,n_1,\dots,n_{j-1}}$$

exists in $\mathcal{P}(\omega)/\mathcal{NC}^k$ with representative A. If $\prod_{i<k} B_i \subseteq_{\mathcal{NC}^k} A$, for some $(B_0, \dots, B_{k-1}) \in ([\omega]^\omega)^k$, then there exist $g : \omega \to \omega$ and $B'_i \in [B_i]^\omega$, for $i < k$, such that

$$\prod_{i<k} B'_i \subseteq_{\mathcal{NC}^k} \bigwedge_{n_2<\omega} \bigvee_{n_3<\omega} \dots \omega^{k-1} \times A_{(n_0,g(n_0),n_2,n_3\dots,n_{j-1})},$$

for all $n_0 < \omega$.

Proof. The function g will be defined inductively for $n_0 < \omega$. Since

$$\prod_{i<k} B_i \subseteq_{\mathcal{NC}^+} \bigvee_{n_1<\omega} \bigwedge_{n_2<\omega} \dots \omega^{k-1} \times A_{(0,n_1,n_2,\dots,n_{j-1})},$$

there exists $n_1^0 < \omega$ such that

$$\prod_{i<k} B_i \cap \bigwedge_{n_2<\omega} \bigvee_{n_3<\omega} \dots \omega^{k-1} \times A_{(0,n_1^0,n_2,n_3,\dots,n_{j-1})} \in (\mathcal{NC}^k)^+.$$

Define $g(0) := n_1^0$ and take $B_i^0 \in [B_i]^\omega$, for $i < k$, such that

$$\prod_{i<k} B_i^0 \subseteq_{\mathcal{NC}^+} \bigwedge_{n_2<\omega} \bigvee_{n_3<\omega} \dots \omega^{k-1} \times A_{(0,n_1^0,n_2,n_3,\dots,n_{j-1})}.$$

Suppose that for some $n_0 < \omega$, both $g(m)$ and $B_i^m \in [B_i]^\omega$, for $i < k$, such that

$$\prod_{i<k} B_i^m \subseteq_{\mathcal{NC}^+} \bigwedge_{n_2<\omega} \bigvee_{n_3<\omega} \dots \omega^{k-1} \times A_{(m,g(m),n_2,n_3,\dots,n_{j-1})},$$

for $m \leqslant n_0$, have been defined. Put $n = n_0 + 1$. Observe that

$$\prod_{i<k} B_i^{n_0} \subseteq_{\mathcal{NC}^+} \bigvee_{n_1<\omega} \bigwedge_{n_2<\omega} \dots \omega^{k-1} \times A_{(n,n_1,n_2,\dots,n_{j-1})}.$$

As in the basic case $B_i^n \in [B_i^{n_0}]^\omega$ and $g(n)$ can be easily defined.

For $i < k$, take B'_i, a pseudointersection of the family $\{B_i^{n_0} \mid n_0 < \omega\}$, which concludes the proof.

■

Lemma 4.4.10. *Take* $1 \leqslant j < \omega$, $\{A_{\overline{n}} \mid \overline{n} \in \omega^j\} \subseteq [\omega]^\omega$ *and* $X \in [\omega]^\omega$. *Suppose that for odd* $i < j$ *there exists* $g_i : \omega^{\frac{i+1}{2}} \to \omega$ *such that*

$$X \subseteq^* A_{(n_0, g_0(n_0), n_2, g_3(n_0, n_2), \ldots)},$$

for all $n_0, n_2, n_4 < \omega$. *Define* $h_i : \omega^{\frac{i+1}{2}} \to \omega$ *as*

$$h_i(n_0, n_1, \ldots) := \max\{g_i(i_0, i_1, \ldots) \mid i_0 \leqslant n_0, i_1 \leqslant n_1, \ldots\}.$$

Then

$$X \subseteq^* \bigcap_{i_0 \leqslant n_0} \bigcup_{i_1 \leqslant n_1} \ldots A_{(i_0, i_1, \ldots)},$$

for $n_1 \geqslant h_1(n_0)$, $n_3 \geqslant h_3(n_0, n_2)$, *...*

Proof. Take $(n_0, n_1, \ldots, n_{j-1}) \in \omega^j$ such that $n_1 \leqslant h_1(n_0)$, $n_3 \leqslant h_3(n_0, n_2)$, and so forth. Suppose that

$$X \nsubseteq^* \bigcap_{i_0 \leqslant n_0} \bigcup_{i_1 \leqslant n_1} \ldots A_{(i_0, i_1, \ldots, i_j)}.$$

Take $Y \in [X]^\omega$ such that

$$Y \subseteq^* \bigcup_{i_0 \leqslant n_0} \bigcap_{i_1 \leqslant n_1} \ldots (\omega \setminus A_{(i_0, i_1, \ldots, i_j)}).$$

Take $Y_0 \in [Y]^\omega$ and $i_0 \leqslant n_0$ such that

$$Y_0 \subseteq^* \bigcap_{i_1 \leqslant n_1} \bigcup_{i_2 \leqslant n_2} \ldots (\omega \setminus A_{(i_0, i_1, i_2, \ldots, i_{j-1})}),$$

and hence that

$$Y_0 \subseteq^* \bigcup_{i_2 \leqslant n_2} \bigcap_{i_3 \leqslant n_3} \ldots (\omega \setminus A_{(i_0, g_1(i_0), i_2, i_3, \ldots, i_j)}).$$

Following this path, in finitely many steps we get $Y' \in [Y]^\omega$ and $i_0 \leqslant n_0$, $i_2 \leqslant n_2$, for odd $< j$ such that

$$Y' \subseteq^* \omega \setminus A_{(i_0, g_1(i_0), i_2, g_3(i_0, i_2) \ldots)},$$

which is clearly a contradiction. Therefore,

$$X \subseteq^* \bigcap_{i_0 \leqslant n_0} \bigcup_{i_1 \leqslant n_1} \ldots A_{(i_0, i_1, \ldots, i_j)}.$$

■

Recall that countable subfamilies of $\{\omega \times X \mid X \in [\omega]^\omega\}$ have both infimum and supremum in $\mathcal{P}(\omega^2)/\mathcal{NC}$. Also there were limits to the possibility of iterate this operations. In this new quotient the first result holds, i.e. countable subfamilies of $\{\omega^{k-1} \times X \mid X \in [\omega]^\omega\}$ have both infimum and supremum in $\mathcal{P}(\omega^k)/\mathcal{NC}^k$, while at least one step further is possible with each dimension added.

Theorem 4.4.11. *Take $1 \leqslant j < k < \omega$ and $l_0 < ... < l_{j-1} < k-1$, and define $E := \{l_i \mid i < j\}$. Then for all family $\{A_{\overline{n}} \mid \overline{n} \in \omega^j\} \subseteq [\omega]^\omega$ the following equality holds in $\mathcal{P}(\omega^k)/\mathcal{NC}^k$:*

$$\coprod_{\overline{n}\in\omega^E} \Big(\bigcap_{i_0 \leqslant \overline{n}(l_0)} \bigcup_{i_1 \leqslant \overline{n}(l_1)} ... A_{(i_0,...,i_{j-1})} \Big) =$$

$$= \bigwedge_{n_0<\omega} \bigvee_{n_1<\omega} ... \ \omega^{k-1} \times A_{(n_0,...,n_{j-1})}.$$

Proof. It will be inductively proved for $j \geqslant 1$. For the case $j = 1$ take $k \geqslant 2$, $1 \leqslant l < k-1$ and a family $\{A_n \mid n < \omega\} \subseteq [\omega]^\omega$. Define

$$A := \coprod_{\overline{n}\in\omega^{\{l\}}} \bigcap_{i \leqslant \overline{n}(l)} A_i.$$

Take $n < \omega$, and consider $\overline{x} \in A$. If $\overline{x}(l) \geqslant n$, then $\overline{x}(k-1) \in \bigcap_{i \leqslant \overline{x}(l)} A_i \subseteq A_n$, and hence it follows that $\overline{x} \in \omega^{k-1} \times A_n$. Therefore $A \setminus \omega^{k-1} \times A_n \subseteq \{\overline{x} \in \omega^k \mid \overline{x}(l) < n\}$ which is an element of $\mathcal{NC}^k$, as proved in 4.4.4. It follows that $A \subseteq_{\mathcal{NC}^k} \omega^{k-1} \times A_n$, for all $n < \omega$.

Take now $B_0, ..., B_{k-1} \in [\omega]^\omega$ such that

$$\prod_{i<k} B_i \subseteq_{fin^k} \omega^{k-1} \times A_n,$$

for all $n < \omega$. If $|B_{k-1} \setminus A_n| = \omega$, for some $n < \omega$, then

$$\prod_{i<k-1} B_i \times (B_{k-1} \setminus A_n) \subseteq (\prod_{i<k} B_i) \setminus \omega^{k_1} \times A_n \in \mathcal{NC}^+$$

which is a contradiction. Therefore $B_{k-1} \subseteq^* A_n$, for all $n < \omega$. Define $g : \omega \to \omega$ as follows $g(n) := \max(B_{k-1} \setminus \bigcap_{i \leqslant n} A_n)$. If $\overline{x} \in \prod_{i<k} B_i \setminus A$, then $\overline{x}(k-1) \in B_{k-1} \setminus \bigcap_{i \leqslant \overline{x}(j)} A_i$. Therefore

$$\prod_{i<k} B_i \setminus A \subseteq \{\overline{x} \in \omega^k \mid \overline{x}(k-1) \leqslant g(\overline{x}(l))\} \in \mathcal{NC}^k,$$

by Lemma 4.4.4. We conclude that $A = \bigwedge_{n<\omega} A_{(n)}$.

Suppose that the theorem holds for some $j' \geqslant 1$. For $j = j'+1$, take $k \geqslant j+1$, $l_0 < ... < l_{j-1} < k-1$, and a family $\{A_{\overline{n}} \mid \overline{n} \in \omega^j\} \subseteq [\omega]^\omega$.With $E := \{l_i \mid i < j\}$, define

$$A := \coprod_{\overline{n}\in\omega^E} \bigcap_{i_0 \leqslant \overline{n}(l_0)} \bigcup_{i_1 \leqslant \overline{n}(l_1)} ... A_{(i_0,...,i_{j-1})}$$

and

$$A_n := \coprod_{\overline{n}\in\omega^{E\setminus\{l_0\}}} \bigcup_{i_1 \leqslant \overline{n}(l_1)} \bigcap_{i_2 \leqslant \overline{n}(l_2)} ... A_{(n,i_1,...,i_{j-1})},$$

for all $n < \omega$. By hypothesis induction

$$A_n = \bigvee_{n_1<\omega} \bigwedge_{n_2<\omega} ... \ \omega^{k-1} \times A_{(n,n_1,...,n_{j-1})},$$

for all $n < \omega$. We need to prove that $A = \bigwedge_{n<\omega} A_n$. Take $n < \omega$ and $\overline{x} \in A$ such that $\overline{x}(l_0) \geqslant n$. Immediately

$$\overline{x}(k-1) \in \bigcap_{i_0 \leqslant \overline{x}(l_0)} \bigcup_{i_1 \leqslant \overline{x}(l_1)} \dots A_{(i_0,\dots,i_{j-1})},$$

and hence it follows that

$$\overline{x}(k-1) \in \bigcup_{i_1 \leqslant \overline{x}(l_1)} \bigcap_{i_2 \leqslant \overline{x}(l_2)} \dots A_{(n,i_1,\dots,i_{j-1})}.$$

and that $\overline{x} \in A_n$. Therefore

$$A \setminus A_n \subseteq \{\overline{x} \in \omega^k \mid \overline{x}(l_0) < n\} \in \mathcal{NC}^k.$$

We conclude that $A \subseteq_{\mathcal{NC}^k} A_n$, for all $n < \omega$.

Now take $B_0, \dots, B_{k-1} \in [\omega]^\omega$ such that

$$\prod_{i<k} B_i \subseteq_{fin^k} A_n,$$

for all $n < \omega$. Apply Lemma 4.4.9 for obtaining $g_1 : \omega \to \omega$ and $B_i^1 \in [B_i]^\omega$, for $i < k$, such that

$$\prod_{i<k} B_i^1 \subseteq_{fin^k} \bigwedge_{\substack{n_0<\omega \\ n_2<\omega}} \bigvee_{n_3<\omega} \dots \omega^k \times A_{(n_0,g_1(n_0),n_2,\dots,n_{j-1})}.$$

With the same lemma get B_i^3, for $i < k$, and $g_3 : \omega^2 \to \omega$ such that

$$\prod_{i<k} B_i^3 \subseteq_{fin^k} \bigwedge_{\substack{n<0<\omega \\ n_2<\omega \\ n_4<\omega}} \bigvee_{n_5<\omega} \dots \omega^k \times A_{(n_0,g_1(n_0),n_2,g_3(n_0,n_1),n_4,n_5,\dots,n_{j-1})}.$$

Following this recursion for as many odd integers $< j$, we finally get $B_i' \in [B_i]^\omega$, for $i < k$, and $g_1 : \omega \to \omega$, $g_3 : \omega^2 \to \omega$, $g_5 : \omega^3 \to \omega$,..., for odd numbers $< j$, such that for all $n_0, n_2, n_4, \dots < \omega$ (as many variables as pair numbers $< j$), we have that

$$\prod_{i<k} B_i' \subseteq_{fin^k} \omega^{k-1} \times A_{(n_0,g_1(n_0),n_2,g_3(n_0,n_2),n_4,g_5(n_0,n_2,n_4),\dots.)}$$

and hence that

$$B_{k-1}' \subseteq^* A_{(n_0,g_1(n_0)),n_2,g_3(n_0,n_2),n_4,g_5(n_0,n_2,n_4),\dots.)}.$$

If $C := \prod_{i<k} B_i' \setminus A$, we need to prove that $C \in \mathcal{NC}^k$ to finish the proof. For odd $i < j$ define $h_i : \omega^{\frac{i+1}{2}} \to \omega$ as follows:

$$h_i(n_0, n_1, \dots) = \max\{g_i(i_0, i_1, \dots) \mid i_0 \leqslant n_0, i_1 \leqslant n_1 \dots\}.$$

Since

$$\{\overline{x} \in \omega^k \mid [\overline{x}(l_1) < h_1(\overline{x}(l_0))] \vee [\overline{x}(l_3) < h_3(\overline{x}(l_0), \overline{x}(l_2))] \vee \dots\}$$

is an element of $\mathcal{NC}^k$, to prove that $C \in \mathcal{NC}^k$ we just need to prove that

$$C' := C \cap \{\overline{x} \in \omega^k \mid [\overline{x}(l_1) \geqslant h_1(\overline{x}(l_0))] \wedge [\overline{x}(l_3) \geqslant h_3(\overline{x}(l_0), \overline{x}(l_2))] \wedge \dots\}$$

lies in $\mathcal{NC}^k$. If $\overline{y} \in \omega^{k-1}$ and $\overline{y}(l_1) \geqslant h_1(\overline{y}(l_0))$, $\overline{y}(l_3) \geqslant h_3(\overline{y}(l_0), \overline{y}(l_2))$ and so forth, then

$$\{n < \omega \mid \overline{y} \frown (n) \in C'\} \subseteq B'_{k-1} \setminus \bigcap_{i_0 \leqslant \overline{y}(l_0)} \bigcup_{i_1 \leqslant \overline{y}(l_1)} \dots A_{(i_0,\dots,i_{j-1})}.$$

which is a finite set by Lemma 4.4.10. Therefore, $C' \in \mathcal{NC}^k$ and we conclude the proof.

■

Theorem 4.4.11 is powerful enough to settle Question 4.3.22 on $\mathcal{P}(\omega^k)/\mathcal{NC}^k$, for $k > 2$, while it is left open on $\mathcal{P}(\omega^2)/\mathcal{NC}$. Whether these results give some light or cast shadows on Question 4.3.22 is not known.

Theorem 4.4.12. *Let $T \subseteq \omega^{<\omega_1}$ be an Aronszajn tree. Take $\{A_\sigma \mid \sigma \in T\} \subseteq [\omega]^\omega$ such that if $\sigma \subseteq \tau \in T$, then $A_\tau \subseteq^* A_\sigma$ and that the set $\{A_\sigma \mid \sigma \in T_\alpha\}$ is a partition of ω for all $\alpha < \omega_1$. If $3 \leqslant k < \omega$, then*

1. *the sets $X_\alpha := \bigvee_{\sigma \in T_\alpha} \omega^{k-1} \times A_\sigma$, for $\alpha < \omega_1$, form a tower in $\mathcal{P}(\omega^k)/\mathcal{NC}^k$ and*
2. *the sets $Y_\alpha := (\bigwedge_{\beta<\alpha} X_\beta) \setminus X_\alpha$, for $0 < \alpha < \omega_1$, form a partition in $\mathcal{P}(\omega^k)/\mathcal{NC}^k$.*

The main step for proving this theorem is the existence of the elements X_α and Y_α, for $\alpha < \omega_1$, justified by Theorem 4.4.11. The rest of the argument for 1 is identical to the proof of Theorem 4.3.12. The main implication of Theorem 4.4.11 is the next ZFC result on cardinal invariants.

Corollary 4.4.13. $\overline{\mathfrak{a}}(\mathcal{NC}^k) = \omega_1$*, for all* $3 \leqslant k < \omega$*.*

From the proof of Theorem 4.4.11 it is clear how the first $k-1$ dimensions of ω^k give us room enough to finitely approximate in the last coordinate the desired algebraic operation. It is also clear that the same idea cannot be equally applied to a family indexed by ω^k. If anything, Proposition 4.3.13 hints to the possibility that there is a limit on the times that $\bigvee$ and $\bigwedge$, with countable indexes, can be iterated on $\mathcal{P}(\omega^k)/\mathcal{NC}^k$.

Conjecture 4.4.14. *For all $2 \leqslant k < \omega$, there exists $\{A_{\overline{x}} \mid \overline{x} \in \omega^k\} \subseteq [\omega]^\omega$ such that*

$$\bigwedge_{n_0<\omega} \bigvee_{n_1<\omega} \dots \omega^{k-1} \times A_{(n_0,n_1,\dots,n_{k-1})}$$

does not exist in $\mathcal{P}(\omega^k)/\mathcal{NC}^k$.

We finish this section with a question about similar results for $\omega \leqslant \alpha < \omega_1$. Take X a countable set and an ideal $\mathcal{I}$ such that there exists $\mathcal{A} \leqslant \mathcal{P}(X)/\mathcal{I}$ isomorphic to $\mathcal{P}(\omega)/fin$. For $\alpha < \omega_1$, recursively define

- $\Sigma^0_0(\mathcal{A}) := \mathcal{A}$
- $\Pi^0_\alpha(\mathcal{A}) := \{A \in \mathcal{P}(X)/\mathcal{I} \mid X \setminus A \in \Sigma^0_\alpha(\mathcal{A})\}$
- $\Sigma^0_\alpha(\mathcal{A}) := \{A \in \mathcal{P}(X)/\mathcal{I} \mid \exists \{A_n \mid n < \omega\} \subseteq \bigcup_{\beta<\alpha} \Pi^0_\beta(\mathcal{A})\ A = \bigvee_{n<\omega} A_n\}$.

For $\alpha < \omega_1$, we will say that $\mathcal{A}$ is *α-σ-closed*, if it is *β-σ-closed*, for all $\beta < \alpha$, and for all $\{A_n \mid n < \omega\} \subseteq \bigcup_{\beta<\alpha} \Pi^0_\beta(\mathcal{A})$ there exists $A \in \mathcal{P}(X)/\mathcal{I}$ such that $A = \bigvee_{n<\omega} A_n$. With this notation, Theorem 4.4.11 says that $\{\omega^{k-1} \times X \mid X \in [\omega]^\omega\}$ is $k-1$-σ-closed while Conjecture 4.4.14 says that it is not k-σ-closed, for all $2 \leqslant k < \omega$.

Question 4.4.15. *Does there exist a sequence $\{\mathcal{I}_\alpha \mid \alpha < \omega_1\}$, of definable ideals on countable sets X_α, and a sequence $\{\mathcal{A}_\alpha \mid \alpha < \omega_1\}$, of subalgebras of $\mathcal{P}(X_\alpha)/\mathcal{I}_\alpha$ isomorphic to $\mathcal{P}(\omega)/fin$, such that $\mathcal{A}_\alpha$ is α-σ-closed, but not $\alpha+1$-σ-closed, for all $\alpha < \omega_1$?*

4.5 Combinatorics of the ideal $\mathcal{NC}$

In this section we will briefly study $\mathcal{NC}$ as such not considering its quotient. Some close ideals and Katětov relations between them will be helpful.

Definition 4.5.1.
- $\Delta := \{(n, m) \in \omega \times \omega \mid m \leqslant n\}$
- $\mathcal{ED}_{fin} := \{X \subseteq \Delta \mid \exists n < \omega\ \forall m < \omega\ |X(m)| \leqslant n\}$
- *A graph is a pair (X, E), where $E \subseteq [X]^2$. We say that $Y \subseteq X$ induces a complete subgraph of (X, E) if $[Y]^2 \subseteq E$.*
- $\mathcal{G}_c := \{E \subseteq [\omega]^2 \mid \forall Y \in [\omega]^\omega\ [Y]^2 \nsubseteq E\}$.

The Borel ideal $\mathcal{ED}_{fin}$ is the restriction of the ideal $\mathcal{ED}$ to the set Δ. Hence the graphs of the functions dominated by the identity function form a generating set of $\mathcal{ED}_{fin}$. The coanalytic set $\mathcal{G}_c$ is an ideal as a consequence of, and in fact is equivalent to, the infinite Ramsey Theorem. The following proposition gives some Katětov-Blass relations (see Definition 1.2.1) to $\mathcal{NC}$.

Proposition 4.5.2. *The following relations hold:*

- $fin \times fin \leqslant_{KB} \mathcal{NC}$
- $\mathcal{ED}_{fin} \leqslant_{KB} \mathcal{NC}$
- $\mathcal{NC} \leqslant_{KB} \mathcal{G}_c$.

Proof. Since $fin \times fin \subseteq \mathcal{NC}$, the Katětov-Blass inequality follows.

For the second relation, take the function $f : \omega \times \omega \to \Delta$ defined by the rule

$$f(n,m) = (\max\{n,m\}, \min\{n,m\}),$$

for all $n, m < \omega$. Let $A \subseteq \Delta$ be the graph of a function. The set $f^{-1}[A]$ is clearly the union

$$A \cup \{(A(n), n) \mid n < \omega\}.$$

The set A is an element of $fin \times fin \subseteq \mathcal{NC}$. Notice that the second term of the union is equal to the set

$$B := \coprod_{m<\omega} A^{-1}(m).$$

Since $\{A^{-1}(m) \mid m < \omega\}$ is a pairwise disjoint family, it follows that $B \in \mathcal{NC}$. We conclude that f is a finite-to-one Katětov function.

For the last inequality, consider the function $f : [\omega]^2 \to \omega \times \omega$ defined by the rule

$$f(\{n,m\}) = (\min\{n,m\}, \max\{n,m\}).$$

Take $A \in \mathcal{NC}$ and $X \in [\omega]^\omega$. Since $A \in \mathcal{NC}$, it follows that there exists non-empty $F \in [X]^{<\omega}$ such that $|\bigcap_{i\in F} A(i)| < \omega$. In particular $|\bigcap_{i\in F} A(i) \cap X| < \omega$. Take $k \in X \setminus \bigcap_{i\in F} A(i)$ bigger than every element of F. Then there exists $i \in F$ such that $k \notin A(i)$. Since $f(\{i,k\}) = (i,k) \notin A$, it follows that $\{i,k\} \notin f^{-1}[A]$. Therefore, X does not induce a complete subgraph of $(\omega, f^{-1}[A])$, and we conclude that $f^{-1}[A] \in \mathcal{G}_c$. Since the function f is one-to-one, the Katětov-Blass relation holds. ∎

Both the additivity and the cofinality of the Nowhere Centered ideal are well-known cardinals. In order to prove that its cofinality is equal to $\mathfrak{c}$ we will use the following fact.

Fact 4.5.3. *Let $\kappa < \mathfrak{c}$ and take $\{X_\alpha \mid \alpha < \kappa\} \subseteq [\omega]^\omega$ such that $\omega \setminus X_\alpha$ is infinite, for all $\alpha < \kappa$. Then there is coinfinite $X \in [\omega]^\omega$ such that $X \not\subseteq^* X_\alpha$ for every $\alpha < \kappa$.*

Proof. Consider a family $\{A_\beta \mid \beta < \mathfrak{c}\} \subseteq [\omega]^\omega$ such that $\{\omega \setminus A_\beta \mid \beta < \mathfrak{c}\}$ is a mad family. Suppose that for all $\beta < \mathfrak{c}$ there exists $\alpha < \kappa$ such that $A_\beta \subseteq^* X_\alpha$. Since $\kappa < \mathfrak{c}$, there exist $\beta_0 < \beta_1 < \mathfrak{c}$ and $\alpha < \kappa$ such that $A_{\beta_0} \cup A_{\beta_1} \subseteq^* X_\alpha$, but this means that $\omega \subseteq^* X_\alpha$, which is a contradiction. ∎

Theorem 4.5.4. *$add^*(\mathcal{NC}) = \omega$ and $cof^*(\mathcal{NC}) = \mathfrak{c}$.*

Proof. Take the family $\mathcal{A} := \{\{n\} \times \omega \mid n < \omega\}$ which is a subset of $\mathcal{NC}$. Clearly, if $A \subseteq \omega \times \omega$ and $\{n\} \times \omega \subseteq^* A$, for all $n < \omega$, then $A \in \mathcal{NC}^+$. Therefore, $\mathcal{A}$ witnesses that $add^*(\mathcal{NC}) = \omega$.

Now take $\kappa < \mathfrak{c}$ and $\mathcal{C} := \{A_\alpha \mid \alpha < \kappa\}$ a subfamily of $\mathcal{NC}$. Consider the family $K_0 := \{\alpha < \kappa \mid A_\alpha(0) \not\supseteq^* \omega\}$ and define $C_0 := \omega$. Since $\kappa < \mathfrak{c}$, there exists $B_0 \subseteq \omega$ such that $|B_0| = |\omega \setminus B_0| = \omega$ and that $B_0 \nsubseteq^* A_\alpha(0)$, for all $\alpha \in K_0$. Suppose that for some $0 < m < \omega$ we have defined a disjoint family $\{B_i \mid i < m\} \subseteq [\omega]^\omega$ such that

$$C_m := \omega \setminus \bigcup_{i<m} B_i$$

is infinite. Take $K_m := \{\alpha < \kappa \mid A_\alpha(m) \nsupseteq^* C_m\}$. Since $\kappa < \mathfrak{c}$, there exists $B_m \subseteq C_m$ such that $|B_m| = |C_m \setminus B_m| = \omega$ and that $B_m \nsubseteq^* A_\alpha(m)$ for all $\alpha \in K_m$. Thus we can recursively construct a centered family $\{C_m \mid m < \omega\}$ and a disjoint family $\{B_m \mid m < \omega\}$ such that $B_m \subseteq C_m$, for all $m < \omega$, and such that $B_m \nsubseteq^* A_\alpha(m)$, for all $\alpha \in K_m$, for all $m < \omega$.

Define $B := \coprod_{m<\omega} B_m$, which is clearly an element of $\mathcal{NC}$. Take $\alpha < \kappa$. Since $\{C_m \mid m < \omega\}$ is a centered family and A_α is an element of $\mathcal{NC}$, there exists $n < \omega$ such that $\alpha \in K_n$. Then $B_n \nsubseteq^* A_\alpha(n)$, and hence it follows that $B \nsubseteq^* A_\alpha$. Therefore, B witnesses that $\mathcal{C}$ is not a cofinal subset of $\mathcal{NC}$ and we conclude that $cof^*(\mathcal{NC}) = \mathfrak{c}$. ■

Since the covering and the uniformity of $\mathcal{NC}$ are not as straightforward as the other two cardinal invariants, we will use the following result (see Theorem 1.5.2 in [17]) to help us to bound them.

Lemma 4.5.5. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on a countable set. If $\mathcal{I} \leqslant_K \mathcal{J}$, then $cov^*(\mathcal{J}) \leqslant cov^*(\mathcal{I})$ and $non^*(\mathcal{I}) \leqslant non^*(\mathcal{J})$.*

Theorem 4.5.6. $\min\{\mathfrak{b}, \mathfrak{s}\} \leqslant cov^*(\mathcal{NC}) \leqslant \mathfrak{b}$.

Proof. Since $\min\{\mathfrak{b}, \mathfrak{s}\} \leqslant cov^*(\mathcal{G}_c)$ and $cov^*(fin \times fin) = \mathfrak{b}$ (see Theorems 1.6.19 and 1.6.26 of [17]), the theorem follows from Proposition 4.5.2 and Lemma 4.5.5. ■

To prove the consistency of $cov^*(\mathcal{NC}) < \mathfrak{b}$ from a base model of ZFC+CH, it would be convenient to have a subfamily $\{X_\alpha \mid \alpha < \omega_1\} \subseteq \mathcal{NC}$ such that for all $Y \in [\omega \times \omega]^\omega$ there exists $\alpha < \omega_1$ such that $|Y \cap Y_\alpha| = \omega$. Furthermore, this property should be preserved while adjoining dominating reals. This characteristic is often strengthened as follows for preservation purposes. A family $\mathcal{F} \subseteq [\omega]^\omega$ is said to be ω-*hitting* if for all countable family $\{X_n \mid n < \omega\} \subseteq [\omega]^\omega$ there exists $X \in \mathcal{F}$ such that $|X \cap X_n| = \omega$, for all $n < \omega$.

Theorem 4.5.7. *It is consistent that $\omega_1 = \mathfrak{s} = cov^*(\mathcal{NC}) < \mathfrak{b} = \omega_2$.*

Proof. Let V be a model of ZFC. The model where the consistency result will hold is $V[G]$, where G is a generic filter on $\mathbb{H}_{\omega_2}$, i.e. the finite support iteration of length ω_2 of Hechler forcing. It is known that $V[G] \models \mathfrak{b} = \omega_2$ and that finite support iterations of Hechler forcing preserve ω-hitting families (see, for example, Lecture 3 in [5]). It will be enough to prove that $\mathbb{H}_{\omega_1}$ adds an ω-hitting subfamily of $\mathcal{NC}$ of size ω_1.

It is known that in each step of this iteration a Cohen real is added (see for example Lemma 3.5 in [5]). A version of Cohen forcing (due to Hechler [11]) is the following one:

$$\mathbb{P} := \{p; \omega \times \omega \to 2 \mid dom(p) = n_p \times m_p \ for \ some \ n_p, \ m_p < \omega\}$$

where $p \leqslant q$ iff

- $p \supseteq q$ and
- $|\{j \in n_q \mid p(j,i) = 1\}| \leqslant 1$, for all $i \in m_p \setminus m_q$.

Since $\mathbb{P}$ is an atomless countable forcing notion, it is equivalent to Cohen forcing. If H is a $(V, \mathbb{P})$-generic filter for some model M of ZFC, it follows that $NC_H := \{(j,i) \mid \exists p \in H \ p(j,i) = 1\}$ is an element of $\mathcal{NC} \cap V[H]$. Indeed, if $p \in H$ and $i < j < n_p$, then p forces that $H(i) \cap H(j) \subseteq m_p$. Then H is a typical element of $\mathcal{NC}$.

Furthermore, H splits all elements of $[\omega \times \omega]^\omega \cap V$. Therefore, after adding ω_1 many Cohen reals, we get in the generic extension an ω-hitting subfamily of $\mathcal{NC}$ of size ω_1. ■

It is not yet known whether the other inequality consistently holds.

Question 4.5.8. *Is it consistent that* $\min\{\mathfrak{b}, \mathfrak{s}\} < cov^*(\mathcal{NC})$*?*

While, as noticed in the last proof, Cohen forcing adds a new element of $\mathcal{NC}$ which hits all reals in the base model, observe that Mathias forcing (see 1.5) adds an infinite subset of $\omega \times \omega$ which has finite intersection with all elements of $\mathcal{NC}$ in the base model.

Observation 4.5.9. *Let $\dot{m}$ be the name of the generic real added by $\mathbb{M}$, i.e. whenever $G \subseteq \mathbb{M}$ is a generic filter, then $\dot{m}_G = \bigcup\{s \mid \exists X \in [\omega]^\omega \ (s, X) \in G\}$. If $\dot{f}$ is the name of (the graph of) the enumeration of $\dot{m}$ and $A \in \mathcal{NC}$, then $\Vdash_{\mathbb{M}} |A \cap \dot{f}| < \omega$.*

Proof. Take $(s, X) \in \mathbb{M}$. By Observation 4.2.4 there exist $Y \in [X]^\omega$, $n \geqslant |s|$ and $g \in \omega^\omega$ such that $A(m) \cap Y \subseteq g(m)$, for all $m \geqslant n$. If necessary, extend s to have $|s| = n$. Now take $Z \in [Y]^\omega$ such that for all $i < \omega$ the i-th element of Z is strictly greater than $g(n+i)$. Then

$$(s, Z) \Vdash \forall i < \omega \ \dot{f}(n+i) \in Y \setminus A(n+i)$$

and, therefore, (s, Z) forces that the intersection of A and $\dot{f}$ is finite. ■

As a consequence of this observation, any countable support iteration of the Mathias forcing would lift $cov^*(\mathcal{NC})$. However, in the mentioned extension $\mathfrak{b} = \mathfrak{s} = cov^*(\mathcal{NC}) = \mathfrak{c}$ holds, which does not give a positive answer to Question 4.5.8.

We conclude the section talking about the uniformity. A family $\mathcal{R} \subseteq [\omega]^\omega$ is called a *hereditarily reaping* if $\mathcal{R} \cap [X]^\omega$ is reaping in X, for every $X \in \mathcal{R}$. It is not hard to see that there exists a hereditarily reaping family of size $\mathfrak{r}$.

Theorem 4.5.10. $\max\{cov(\mathcal{M}), \mathfrak{b}\} \leqslant non^*(\mathcal{NC}) \leqslant \mathfrak{r}$.

Proof. First we will prove that $cov(\mathcal{M}) \leqslant non^*(\mathcal{NC})$. Take $\mathcal{F} \subseteq [\omega \times \omega]^\omega$ of size $\kappa < cov(\mathcal{M})$. Consider the version of Cohen forcing $\mathbb{P}$ defined in the proof of Theorem 4.5.7. Clearly there exists $\mathcal{D} := \{\mathcal{D}_\alpha \mid \alpha < \kappa\}$, a family of dense subsets of $\mathbb{P}$, such that if H is a $\mathcal{D}$-generic filter, then NC_H infinitely intersects any element of $\mathcal{F}$. Since $\kappa < cov(\mathcal{M})$, there exists such a filter.

Now suppose that $\kappa < \mathfrak{b}$ and that $\mathcal{F} := \{A_\alpha \mid \alpha < \kappa\}$ is a family of infinite subsets of $\omega \times \omega$. For all $A \in [\omega \times \omega]^\omega$ there exists either $n < \omega$ such that $|(\{n\} \times \omega) \cap A| = \omega$ or $f \in \omega^\omega$ such that $|A \cap f| = \omega$. Therefore, without loss of generality, there exists a partition $K_0 \cup K_1 = \kappa$, such that

- for all $\alpha \in K_0$ there exists $n_\alpha < \omega$ such that $A_\alpha \subseteq \{n_\alpha\} \times \omega$ and
- for all $\alpha \in K_1$ the set A_α is a partial function from ω to ω.

Since $\kappa < \mathfrak{b}$, there exists $f \in \omega^\omega$ such that $A_\alpha \leqslant^* f$, for all $\alpha \in K_1$. Therefore $B_1 := \{(n, m) \mid m \leqslant f(n)\}$ is an element of $\mathcal{NC}$ such that $B_1 \cap A_\alpha$ is infinite, for all $\alpha \in K_1$.

Recall that $\mathfrak{b} \leqslant \mathfrak{r}$. The set $\{A_\alpha(n_\alpha) \mid \alpha \in K_0\}$ is split by some real $C_0 \in [\omega]^\omega$. Suppose that for some $n < \omega$ we have defined a disjoint family $C_0, ..., C_n \in [\omega]^\omega$ such that $D_n := \bigcup_{i \leqslant n} C_i$ splits $A_\alpha(n_\alpha)$, for all $\alpha \in K_0$. The family

$$\{A_\alpha(n_\alpha) \setminus D_n \mid \alpha \in K_0\}$$

is split by some $C_{n+1} \in [\omega \setminus D_n]^\omega$. It follows that both $\bigcup_{i \leqslant n+1} C_i$ and C_{n+1} split $A_\alpha(n_\alpha)$, for all $\alpha \in K_0$. Thus we can recursively construct a disjoint family $\{C_n \mid n < \omega\} \subseteq [\omega]^\omega$ such that C_n splits $A_\alpha(n_\alpha)$, for all $n < \omega$ and all $\alpha \in K_0$. Take

$$B_0 := \coprod_{n<\omega} C_n,$$

which is an element of $\mathcal{NC}$, and $\alpha \in K_0$. Since C_{n_α} splits $A_\alpha(n_\alpha)$, it follows that $B_0 \cap A_\alpha$ is infinite. Therefore $B := B_0 \cup B_1$ is an element of $\mathcal{NC}$ such that $B \cap A_\alpha$ is infinite, for all $\alpha < \kappa$. We conclude that $\mathfrak{b} \leqslant non^*(\mathcal{NC})$.

Now we prove that $non^*(\mathcal{NC}) \leqslant \mathfrak{r}$. Let $\mathcal{R}$ be a hereditarily reaping family of size $\mathfrak{r}$. We claim that

$$\{\{n\} \times X \mid X \in \mathcal{R},\ n < \omega\}$$

witnesses that $non^*(\mathcal{NC}) \leqslant \mathfrak{r}$. Suppose that $A \in [\omega \times \omega]^\omega$ is such that $A \cap (\{n\} \times X)$ is infinite, for all $n < \omega$ and all $X \in \mathcal{R}$. Take $X \in \mathcal{R}$. Since $A \cap (\{0\} \times X)$ is infinite, there exists $X_0 \in [X]^\omega \cap \mathcal{R}$ such that $X_0 \subseteq^* A(0)$ or $X_0 \cap A(0)$ is finite. Since this last case cannot happen, we conclude that $X_0 \subseteq^* A(0)$. Suppose that for some $n < \omega$ we have defined a decreasing family $X_0, ..., X_n$ of elements of $\mathcal{R}$ such that $X_i \subseteq^* A(i)$, for all $i \leqslant n$. Since $A \cap (\{n+1\} \times X_n)$ is infinite, there exists $X_{n+1} \in [X_n]^\omega \cap \mathcal{R}$ such that $X_{n+1} \subseteq^* A(n+1)$ (for the case when $X_{n+1} \cap A(n+1)$ is finite cannot happen).

So we can recursively construct a decreasing subfamily of $\mathcal{R}$ witnessing that $\{A(n) \mid n < \omega\}$ is a centered family. Therefore, $A \notin \mathcal{NC}$. We conclude that $non^*(\mathcal{NC}) \leqslant \mathfrak{r}$.

■

Thus far, one of the inequalities of the previous theorem is known to be consistently strict.

Theorem 4.5.11. *It is consistent that* $\max\{cov(\mathcal{M}), \mathfrak{b}\} < non^*(\mathcal{NC})$.

Proof. In Theorem 1.6.12 of [17] we have a model of $cof(\mathcal{M}) = \omega_1$ and $non^*(\mathcal{ED}_{fin}) > \omega_1$. It follows from Proposition 4.5.2 that in this model also $non^*(\mathcal{NC}) > \omega_1$ holds. Since $cov(\mathcal{M}), \mathfrak{b} \leqslant cof(\mathcal{M})$, the inequality holds in this model. ■

Concerning the uniformity of the Nowhere Centered ideal, a question remains open:

Question 4.5.12. *Is it consistent that* $non^*(\mathcal{NC}) < \mathfrak{r}$*?*

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Finite dimensional infinite partitions.pdf

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