Universidad Nacional Autonoma de Mexico Y
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Posgrado Conjunto en Ciencias Matemáticas

THE INVERSE SCATTERING PROBLEM FOR THE JACOBI OPERATOR

TESIS

QUE PARA OBTENER EL TÍTULO DE
MAESTRO EN CIENCIAS MATEMÁTICAS

PRESENTA
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To my parents, for being my greatest example of love, strength, and perseverance; to my brothers, for their unwavering support and companionship in every stage of my life; and to Dylan and Vale, whose joy and spontaneity inspire me to dream bigger.

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Abstract

We explore the solution to the scattering problem for the Jacobi operator on the half-line with the Dirichlet boundary condition. Based on the work *Inverse problems in scattering theory for second-order self-adjoint difference operators*, PhD Thesis in Russian, Lomonosov Moscow State University, 1976 by G. Sh. Guseinov, we study the reconstruction of the Jacobi operator from the scattering data. Among others, we also refer the paper *Inverse problem with transmission eigenvalues for the discrete Schrödinger equation*, Journal of Mathematical Physics 56 (2015), 082101 by T. Aktosun and V. G. Papanicolaou.

Chapter 1 focuses on deriving scattering data from direct problems, while Chapter 2 develops the solution to the inverse scattering problems.

Keywords: Jacobi operator, scattering, Hilbert spaces, asymptotic behavior, Parseval's equality.

Resumen

Exploramos la solución al problema de dispersión para el operador de Jacobi en la semirrecta con la condición de frontera de Dirichlet. Basándonos en el trabajo *Inverse problems in scattering theory for second-order self-adjoint difference operators*, Tesis Doctoral en ruso, Universidad Estatal de Moscú Lomonósov, 1976, de G. Sh. Guseinov, estudiamos la reconstrucción del operador de Jacobi a partir de los datos de dispersión. Entre otros, también hacemos referencia al artículo *Inverse problem with transmission eigenvalues for the discrete Schrödinger equation*, Journal of Mathematical Physics 56 (2015), 082101, de T. Aktosun y V. G. Papanicolaou.

El Capítulo 1 se centra en la obtención de datos de dispersión a partir de problemas directos, mientras que el Capítulo 2 desarrolla la solución a los problemas inversos de dispersión.

Palabras clave: Operador de Jacobi, dispersión, espacios de Hilbert, comportamiento asintótico, igualdad de Parseval.

Contents

Introduction

The theory of inverse problems originates from the need to determine an operator or system based on observable data, such as spectra or asymptotics (scattering data). In the 20th century, inverse problems were developed within the framework of spectral theory for differential and difference equations, which were inspired by foundational studies such as those by Sturm-Liouville and G. Weyl.

In 1929, V.A. Ambartsumian formally introduced the study of inverse problems, demonstrating that the spectrum alone is insufficient to define an operator. Researchers like Marchenko, Gelfand, and Levitan developed effective methods to address these problems, especially in reconstructing operators from spectral functions.

In quantum mechanics, inverse scattering problems focus on recovering a potential field from the asymptotics of wave functions.

The theory was eventually extended to second-order difference equations to address discrete analogs of inverse problems using spectral data, with significant contributions from Case, Kac, and others. Challenges remain in establishing necessary and sufficient conditions for complete solutions, however.

The primary goal of this thesis, which consists of two chapters, is to determine such conditions.

In Chapter 1, we study the system

$$\begin{aligned} a_2\, y_2 + b_1\, y_1 &= \lambda\, y_1, \\ a_{n+1}\, y_{n+1} + b_n\, y_n + a_n\, y_{n-1} &= \lambda\, y_n, \qquad n \geq 2, \end{aligned} \tag{0.0.1}$$

where $\{y_n\}_{n=1}^{\infty}$ is the desired solution, λ is a complex parameter, and

$$a_n > 0, \quad \operatorname{Im}(b_n) = 0, \quad n \geq 1, \qquad \sum_{n=1}^{\infty} n\,(|1 - a_n| + |b_n|) < \infty. \tag{0.0.2}$$

Equation (0.0.1) is equivalent to finding the vector $\{y_n\}_{n=0}^{\infty}$ that satisfies

$$a_{n+1}\, y_{n+1} + b_n\, y_n + a_n\, y_{n-1} = \lambda\, y_n, \qquad n \geq 1, \quad a_1 = 1, \tag{0.0.3}$$

with the initial condition

$$y_0 = 0. \tag{0.0.4}$$

Let $\lambda = z + z^{-1}$, where

$$z = e^{i\theta}, \quad \theta \in [0, \pi], \tag{0.0.5}$$

so that Equation (0.0.3) becomes

$$a_{n+1}\, y_{n+1} + b_n\, y_n + a_n\, y_{n-1} = (z + z^{-1})\, y_n, \qquad n \geq 1, \quad a_1 = 1. \tag{0.0.6}$$

The system (0.0.6) allows a unique solution of the form

$$f_n(z) = \sum_{\mu=n}^{\infty} K_{n\mu}\, z^{\mu}, \qquad n \geq 0. \tag{0.0.7}$$

The goal of the direct scattering problem is to establish fundamental relationships that allow the scattering data to be obtained, such as the scattering matrix $S(z) = f_0(z^{-1})/f_0(z)$ and certain parameters associated with the function $f_0(z)$.

Chapter 2 focuses on the inverse scattering problem, which consists of constructing a function of the form (1.1.10) from the scattering data and determining its associated system (0.0.6).

Chapter 1

The direct scattering problem on the half-line

This chapter is based on the PhD thesis work of Guseinov G. Sh. [13]. In this work, I reformulate Guseinov's results using my new parameter z. While Guseinov develops his results in the upper half-plane of the complex plane, I present them in the unit disk centered at the origin. Specifically, I further elaborate on the proofs of the necessary conditions for the scattering data of the Jacobi system, established in [13], by introducing the parameter z related to λ. This approach provides a new perspective and enhances the comprehension of Guseinov's results within the framework of the unit disk.

1.1 Special solutions

Let us consider the infinite system of equations

$$\begin{aligned} a_2\, y_2 + b_1\, y_1 &= \lambda\, y_1, \\ a_{n+1}\, y_{n+1} + b_n\, y_n + a_n\, y_{n-1} &= \lambda\, y_n, \qquad n \geq 2, \end{aligned} \tag{1.1.1}$$

where $\{y_n\}_{n=1}^{\infty}$ is the desired solution, λ is a complex parameter, and

$$a_n > 0, \quad \operatorname{Im} b_n = 0, \quad n \geq 1, \qquad \sum_{n=1}^{\infty} n\,(|1 - a_n| + |b_n|) < \infty. \tag{1.1.2}$$

Equation (1.1.1) is equivalent to finding the vector $\{y_n\}_{n=0}^{\infty}$ that satisfies the equation

$$a_{n+1}\, y_{n+1} + b_n\, y_n + a_n\, y_{n-1} = \lambda\, y_n, \qquad n \geq 1, \quad a_1 = 1, \tag{1.1.3}$$

with the initial condition

$$y_0 = 0. \tag{1.1.4}$$

We define

$$\sigma(n) := \sum_{p=n}^{\infty} (|1 - a_p| + |b_p|)\,. \tag{1.1.5}$$

In Equation (1.1.3), let $\lambda := z + z^{-1}$, where

$$z = e^{i\theta}, \quad \theta \in [0, \pi]. \tag{1.1.6}$$

Thus, Equation (1.1.3) becomes

$$a_{n+1}\, y_{n+1} + b_n\, y_n + a_n\, y_{n-1} = (z + z^{-1})\, y_n, \qquad n \geq 1, \quad a_1 = 1. \tag{1.1.7}$$

From Equations (1.1.7) and (1.1.2), in the limit as $n \to \infty$, we obtain the recurrence relation

$$y_{n+1} + y_{n-1} = (z + z^{-1})\, y_n, \tag{1.1.8}$$

which corresponds to a second-order linear homogeneous difference equation.

The general solution to Equation (1.1.8) is a linear combination of the two particular solutions

$$y_n^{(1)}(z) = z^n, \qquad y_n^{(2)}(z^{-1}) = z^{-n}, \qquad |z| = 1. \tag{1.1.9}$$

Theorem 1.1.1. *Under condition (1.1.2), Equation (1.1.7), where z is given by (1.1.6) has a unique solution $\{f_n(z)\}_{n=0}^{\infty}$ given by*

$$f_n(z) = \sum_{\mu=n}^{\infty} K_{n\mu}\, z^{\mu} = K_{nn}\, z^n \left(1 + \sum_{m=1}^{\infty} \frac{K_{n,n+m}}{K_{nn}}\, z^m\right), \qquad n \geq 0, \tag{1.1.10}$$

where $K_{n\mu}$ are real numbers with $\lim_{n\to\infty} K_{nn} = 1$, $\lim_{m\to\infty} K_{n,n+m} = 0$ for $n \geq 0$, and $K_{n,n+m}/K_{nn} = 0$ for $m \leq 0$. Moreover,

A) K_{nn} and $K_{n,n+m}$ satisfy the following relations:

$$\begin{aligned} 0 = a_{n+1}^2 \frac{K_{n+1,n+m+1}}{K_{n+1,n+1}} - \frac{K_{n,n+m}}{K_{nn}} + b_n \frac{K_{n,n+m+1}}{K_{nn}} \\ + \frac{K_{n-1,n+m+1}}{K_{n-1,n-1}} - \frac{K_{n,n+m+2}}{K_{nn}}, \qquad n, m \geq 1, \end{aligned} \tag{1.1.11}$$

$$a_{n+1}^2 = 1 + \frac{K_{n,n+2}}{K_{nn}} - \frac{K_{n-1,n+1}}{K_{n-1,n-1}} - b_n \frac{K_{n,n+1}}{K_{nn}}, \qquad n \geq 1, \tag{1.1.12}$$

$$b_n = \frac{K_{n,n+1}}{K_{nn}} - \frac{K_{n-1,n}}{K_{n-1,n-1}}, \qquad n \geq 1, \tag{1.1.13}$$

$$a_n = \frac{K_{nn}}{K_{n-1,n-1}}, \qquad n \geq 1. \tag{1.1.14}$$

Conversely, if $K_{n,n+m}/K_{nn}$ satisfies (1.1.11)-(1.1.13), and K_{nn} satisfies (1.1.14), then $\{f_n(z)\}_{n=0}^{\infty}$ given by (1.1.10) is a solution to Equation (1.1.7).

B) K_{nn} and $K_{n,n+m}$ for $n \geq 0$ are determined by a_n and b_n as follows:

$$K_{nn} = \left(\prod_{p=n+1}^{\infty} a_p\right)^{-1}, \tag{1.1.15}$$

$$K_{n,n+1} = -K_{nn} \sum_{p=n+1}^{\infty} b_p, \tag{1.1.16}$$

$$K_{n,n+2} = K_{nn} \left[\sum_{p=n+2}^{\infty} \left(1 - a_p^2\right) + \sum_{p=n+1}^{\infty} \sum_{q=p+1}^{\infty} b_p\, b_q\right], \tag{1.1.17}$$

$$K_{n,n+2k+1} = K_{nn}\left[-\sum_{p=n+1+k}^{\infty} b_p - \sum_{q=1}^{k}\sum_{p=n+1+k-q}^{\infty} b_p \frac{K_{p,p+2q}}{K_{pp}} + \sum_{q=0}^{k-1}\sum_{p=n+1+k-q}^{\infty}(1-a_p^2)\frac{K_{p,p+2q+1}}{K_{pp}}\right], \qquad k \geq 1, \tag{1.1.18}$$

$$K_{n,n+2k} = K_{nn}\left[\sum_{p=n+k}^{\infty}\sum_{q=p+1}^{\infty} b_p\, b_q + \sum_{p=n+k+1}^{\infty}(1-a_p^2) - \sum_{q=1}^{k-1}\sum_{p=n+k-q}^{\infty} b_p \frac{K_{p,p+2q+1}}{K_{pp}} + \sum_{q=1}^{k-1}\sum_{p=n+1+k-q}^{\infty}(1-a_p^2)\frac{K_{p,p+2q}}{K_{pp}}\right], \qquad k \geq 2. \tag{1.1.19}$$

The following inequality holds:

$$\left|\frac{K_{n,n+m}}{K_{nn}}\right| \leq c\,\sigma\left(n + \left[\frac{m}{2}\right]\right), \qquad m \geq 1, \tag{1.1.20}$$

where $[\cdot]$ *denotes the integer part, and* c *is a non-negative constant independent of* n *and* m.

C) The solution $\{f_n(z)\}_{n=0}^{\infty}$ *has the following properties.*

1. *The function* $f_n(z)$ *is continuous in* z, *where* z *is as described in* (1.1.6), *and can be analytically extended to the region* $|z| \leq 1$ *given by*

$$f_n(z) = K_{nn}\, z^n \left(1 + \sum_{m=1}^{\infty}\frac{K_{n,n+m}}{K_{nn}}\, z^m\right), \tag{1.1.21}$$

and, for each n, $f_n(z)$ *has the asymptotic behavior*

$$f_n(z) = K_{nn}\, z^n\,[1 + o(1)], \qquad z \to 0. \tag{1.1.22}$$

2. *For a fixed* z, $f_n(z)$ *has the asymptotic form*

$$f_n(z) = z^n\,[1 + o(1)], \qquad |z| \leq 1, \quad n \to \infty. \tag{1.1.23}$$

Proof.

Part A) is verified by directly substituting (1.1.10) into (1.1.7). The formulas (1.1.15)-(1.1.19) are derived from (1.1.11)-(1.1.14). To prove the Inequality (1.1.20), we express $K_{n,n+m}/K_{nn}$ as

$$\frac{K_{n,n+m}}{K_{nn}} := \sum_{l=0}^{\infty}\frac{K_{n,n+m}^{(l)}}{K_{nn}^{(l)}}, \qquad m \geq 3, \tag{1.1.24}$$

where

$$\frac{K_{n,n+2k+1}^{(0)}}{K_{nn}^{(0)}} := -\sum_{p=n+1+k}^{\infty} b_p, \qquad k \geq 1, \tag{1.1.25}$$

$$\frac{K_{n,n+2k}^{(0)}}{K_{nn}^{(0)}} := -\sum_{p=n+k}^{\infty}\sum_{q=p+1}^{\infty} b_p\, b_q + \sum_{p=n+k+1}^{\infty}(1-a_p^2), \qquad k \geq 2, \tag{1.1.26}$$

$$\begin{aligned}\frac{K^{(l)}_{n,n+2k+1}}{K^{(0)}_{nn}} &:= -\sum_{q=1}^{k}\sum_{p=n+1+k-q}^{\infty} b_p \frac{K^{(l-1)}_{p,p+2q}}{K^{(l-1)}_{pp}} \\ &\quad +\sum_{q=0}^{k-1}\sum_{p=n+1+k-q}^{\infty} \left(1-a_p^2\right) \frac{K^{(l-1)}_{p,p+2q+1}}{K^{(l-1)}_{pp}}, \qquad k\geq 1, \quad l\geq 1,\end{aligned} \tag{1.1.27}$$

$$\begin{aligned}\frac{K^{(l)}_{n,n+2k}}{K^{(0)}_{nn}} &:= -\sum_{q=1}^{k-1}\sum_{p=n+k-q}^{\infty} b_p \frac{K^{(l-1)}_{p,p+2q+1}}{K^{(l-1)}_{pp}} \\ &\quad +\sum_{q=1}^{k-1}\sum_{p=n+1+k-q}^{\infty} \left(1-a_p^2\right) \frac{K^{(l-1)}_{p,p+2q}}{K^{(l-1)}_{pp}}, \qquad k\geq 2, \quad l\geq 1.\end{aligned} \tag{1.1.28}$$

We introduce the following notation:

$$\eta(n) := \sum_{p=n}^{\infty} \left|1-a_p^2\right|, \qquad \eta_1(n,k) := \sum_{p=n}^{\infty} (p-k)\left|1-a_p^2\right|, \qquad k\leq n,$$

$$\jmath(n) := \sum_{p=n}^{\infty} |b_p|, \qquad \jmath_1(n,k) := \sum_{p=n}^{\infty} (p-k)|b_p|, \qquad k\leq n.$$

By applying mathematical induction to l, we establish that

$$\left|\frac{K^{(l)}_{n,n+2k+1}}{K^{(l)}_{nn}}\right|, \ \left|\frac{K^{(l)}_{n,n+2k}}{K^{(l)}_{nn}}\right| \leq \left[\eta(n+k)+\jmath(n+k)+\jmath^2(n+k)\right]\left\{\sum_{m=0}^{l}\binom{l}{m}\frac{\jmath_1^{l-m}(n+1,n)\eta_1^m(n+1,n)}{(l-m)!m!}\right\}, \tag{1.1.29}$$

where $\binom{l}{m} = \frac{l(l-1)\cdots(l-m+1)}{m!}$. Since $(l-m)!m! \geq \left[\frac{l}{2}\right]!$ for $m\geq 0$, Equation (1.1.29) can be rewritten as

$$\left|\frac{K^{(l)}_{n,n+2k+1}}{K^{(l)}_{nn}}\right|, \ \left|\frac{K^{(l)}_{n,n+2k}}{K^{(l)}_{nn}}\right| \leq \left[\eta(n+k)+\jmath(n+k)+\jmath^2(n+k)\right]\frac{\left[\eta_1(n+1,n)+\jmath_1(n+1,n)\right]^l}{\left[\frac{l}{2}\right]!}. \tag{1.1.30}$$

From (1.1.30) and (1.1.24), as well as by using (1.1.2), we obtain the estimate (1.1.20) for $K_{n,n+m}/K_{nn}$, $m\geq 3$. From (1.1.16) and (1.1.17), it is clear that this estimate also holds for $m=1,2$. Part C) follows from the estimate (1.1.20) and the representation (1.1.21). ■

1.2 The direct scattering problem

Definition 1.2.1. *Let $\{\psi_n(\lambda)\}_{n=0}^{\infty}$ and $\{g_n(\lambda)\}_{n=0}^{\infty}$ be two solutions to Equation (1.1.3) for the same value of λ. The Wronskian of these solutions is defined as*

$$W\left[\psi_n(\lambda), g_n(\lambda)\right] := a_{n+1}\left[\psi_n(\lambda)\, g_{n+1}(\lambda) - \psi_{n+1}(\lambda)\, g_n(\lambda)\right]. \tag{1.2.1}$$

Lemma 1.2.1. *Let $\{\psi_n(\lambda)\}_{n=0}^{\infty}$ and $\{g_n(\mu)\}_{n=0}^{\infty}$ be two solutions of Equation (1.1.3). The identity*

$$\begin{aligned}a_n\left[\psi_{n-1}(\lambda)\, g_n(\mu) - \psi_n(\lambda)\, g_{n-1}(\mu)\right] - a_{n+1}\left[\psi_n(\lambda)\, g_{n+1}(\mu) - \psi_{n+1}(\lambda)\, g_n(\mu)\right] \\ = (\lambda-\mu)\,\psi_n(\lambda)\, g_n(\mu)\end{aligned} \tag{1.2.2}$$

holds for $n \geq 1$.

Proof.
Since

$$a_{n+1}\,\psi_{n+1}(\lambda) + b_n\,\psi_n(\lambda) + a_n\,\psi_{n-1}(\lambda) = \lambda\,\psi_n(\lambda), \quad n \geq 1 \text{ and}$$
$$a_{n+1}\,g_{n+1}(\mu) + b_n\,g_n(\mu) + a_n\,g_{n-1}(\mu) = \mu\,g_n(\mu), \quad n \geq 1,$$

we can multiply the first equation by $g_n(\mu)$ and the second by $\psi_n(\lambda)$; we then subtract the second from the first to obtain (1.2.2). ■

By substituting $\mu = \lambda$ in (1.2.2), we obtain the corollary below.

Corollary 1.2.1. *The Wronskian of two solutions to Equation (1.1.3) is independent of* n.

From (1.2.1), we can derive the following results.

Lemma 1.2.2. *For the solutions* $\{\psi_n(\lambda_0)\}_{n=0}^{\infty}$ *and* $\{g_n(\lambda_0)\}_{n=0}^{\infty}$ *of Equation (1.1.3) to be linearly independent, it is both necessary and sufficient that* $W\left[\psi_n(\lambda_0), g_n(\lambda_0)\right] \neq 0$.

Since a_n and b_n are real, it follows that for $|z| = 1$ the sequences $\{f_n(z)\}_{n=0}^{\infty}$ and $\{\overline{f_n(z)}\}_{n=0}^{\infty} = \{f_n(z^{-1})\}_{n=0}^{\infty}$ are both solutions to Equation (1.1.7). Since the Wronskian of the two solutions is independent of n, this Wronskian remains constant as $n \to \infty$. Therefore, by considering (1.1.23), we obtain

$$W\left[f_n(z), f_n(z^{-1})\right] = -(z - z^{-1}). \tag{1.2.3}$$

Let $\{\varphi_n(z)\}_{n=0}^{\infty}$ denote the solution to Equation (1.1.7) that satisfies the initial conditions

$$\varphi_0(z) = 0, \quad \varphi_1(z) = 1. \tag{1.2.4}$$

Lemma 1.2.3. *For* $|z| = 1$ *with* $z \neq \pm 1$, *the following identity holds:*

$$-\frac{z - z^{-1}}{f_0(z)}\varphi_n(z) = f_n(z^{-1}) - S(z) f_n(z), \tag{1.2.5}$$

where

$$S(z) := \frac{f_0(z^{-1})}{f_0(z)}. \tag{1.2.6}$$

The function $S(z)$ *satisfies the following propertie.*

1.

$$\overline{S(z)} = S(z^{-1}) = (S(z))^{-1}. \tag{1.2.7}$$

Proof.
From (1.2.3) and Lemma 1.2.2, it follows that for $|z| = 1$ with $z \neq \pm 1$ the sequences $\{f_n(z)\}_{n=0}^{\infty}$ and $\{f_n(z^{-1})\}_{n=0}^{\infty}$ form a fundamental system of solutions to Equation (1.1.7). Thus,

$$\varphi_n(z) = c_1\,f_n(z) + c_2\,f_n(z^{-1}). \tag{1.2.8}$$

By using (1.2.1) and (1.2.3), we derive the following:

$$W[\varphi_n(z), f_n(z^{-1})] = -c_1\,(z - z^{-1}),$$
$$W[\varphi_n(z), f_n(z)] = c_2\,(z - z^{-1}),$$

which leads to the coefficients c_1 and c_2 in (1.2.8) as

$$c_1 = -\frac{W[\varphi_n(z), f_n(z^{-1})]}{z - z^{-1}}, \quad c_2 = \frac{W[\varphi_n(z), f_n(z)]}{z - z^{-1}}.$$

Moreover,

$$W[\varphi_n(z), f_n(z^{-1})] = a_1\,[\varphi_0(z)\,f_1(z^{-1}) - \varphi_1(z)\,f_0(z^{-1})] = -f_0(z^{-1}), \quad \text{and}$$
$$W[\varphi_n(z), f_n(z)] = a_1\,[\varphi_0(z)\,f_1(z) - \varphi_1(z)\,f_0(z)] = -f_0(z).$$

Thus, by using (1.2.8), we obtain (1.2.5). Note $f_0(z) \neq 0$ for $|z| = 1$ with $z \neq \pm 1$, which follows from the fact that $\overline{f_0(z)} = f_0(z^{-1})$ and

$$f_0(z)f_1(z^{-1}) - f_1(z)f_0(z^{-1}) = -(z - z^{-1}).$$

Propertie 1 follows from (1.2.6) and the condition $\overline{f_0(z)} = f_0(z^{-1})$. ■

Lemma 1.2.4. *The function $f_0(z)$ for $|z| < 1$ can only have a finite number of simple zeros located in $(-1, 0)\cup(0, 1)$. Furthermore, the function $(z-z^{-1})/(f_0(z))$ is bounded in a neighborhood of $z = \pm 1$, $|z| \leq 1$. The equation*

$$M_j^2 := \frac{1 - \frac{1}{z_j^2}}{\dot{f}_0(z_j)\,f_1(z_j)} = \frac{1}{\sum_{n=1}^{\infty} f_n^2(z_j)}, \qquad j = 1, 2, \ldots, N \tag{1.2.9}$$

holds, where the dot denotes the derivative with respect to z, and z_j are the zeros of the function $f_0(z)$ in the region $|z| < 1$.

Proof.

First, let us clarify the distribution of the zeros of the function $f_0(z)$ for $|z| < 1$. Suppose $f_0(z_0) = 0$, where $z_0 = |z_0|\,e^{i\theta_0}$ with $|z_0| < 1$. The sequence $\{f_n(z_0)\}_{n=0}^{\infty}$ is a power decreasing solution of Equation (1.1.7) as $n \to \infty$. Because of the symmetry of (1.1.3), $\lambda_0 = z_0 + z_0^{-1}$ must be real, leading to the equation

$$\lambda_0 = z_0 + z_0^{-1} = |z_0|\,e^{i\,\theta_0} + \frac{1}{|z_0|}\,e^{-i\,\theta_0} = |z_0|[\cos\theta_0 + i\sin\theta_0] + \frac{1}{|z_0|}[\cos\theta_0 - i\sin\theta_0]$$
$$= \left(|z_0| + \frac{1}{|z_0|}\right)\cos\theta_0 + i\left(|z_0| - \frac{1}{|z_0|}\right)\sin\theta_0.$$

Hence, $\theta_0 = 0$ or π. Therefore, the function $f_0(z)$ can only have zeros in the intervals $(-1, 0)$ and $(0, 1)$. Next we will show that these zeros are simple. From Equation (1.1.7), we derive the relations

$$a_{n+1}\,f_{n+1}(z) + b_n\,f_n(z) + a_n\,f_{n-1}(z) = (z + z^{-1})\,f_n(z),$$
$$a_{n+1}\,\dot{f}_{n+1}(z) + b_n\,\dot{f}_n(z) + a_n\,\dot{f}_{n-1}(z) = \left(1 - \frac{1}{z^2}\right)\,f_n(z) + (z + z^{-1})\dot{f}_n(z).$$

By multiplying the first equation by $\dot{f}_n(z)$ and the second by $f_n(z)$ and then subtracting them, we get

$$a_{n+1}\left[f_{n+1}(z)\dot{f}_n(z) - \dot{f}_{n+1}(z)f_n(z)\right] - a_n\left[f_n(z)\dot{f}_{n-1}(z) - \dot{f}_n(z)f_{n-1}(z)\right] = -(1 - \frac{1}{z^2})f_n^2(z).$$

By summing over $n \geq 1$ and noting that $f_n(z) \to 0$ and $\dot{f}_n(z) \to 0$ as $n \to \infty$, we obtain

$$-f_1(z)\,\dot{f}_0(z) + \dot{f}_1(z)\,f_0(z) = -\left(1 - \frac{1}{z^2}\right) \sum_{n=1}^{\infty} f_n^2(z).$$

By setting $z = z_0$, we get

$$-\dot{f}_0(z_0)\,f_1(z_0) = -\left(1 - \frac{1}{z_0^2}\right) \sum_{n=1}^{\infty} f_n^2(z_0).$$

From this, we conclude that $\dot{f}_0(z_0) \neq 0$ (since $f_1(z_0) \neq 0$. If not, given $f_0(z_0) = 0$, we would have $f_n(z_0) \equiv 0$) and simultaneously, we obtain formula (1.2.9). Now let us demonstrate that the number of zeros of the function $f_0(z)$ in the intervals $(-1, 0)$ and $(0, 1)$ is finite. Let d denote the infimum of the distances between two consecutive zeros of the function $f_0(z)$ in $(-1, 0)$ or $(0, 1)$. It is sufficient to prove that $d > 0$. Conversely, assume that $d = 0$. We can then extract a sequence of zeros $\{z_k\} \subset (0, 1)$ of the function $f_0(z)$ such that $\lim_{k\to\infty}(z'_k - z_k) = 0$, where $z'_k > z_k > 0$. From (1.1.23), it follows that for sufficiently large n_0 the inequality $f_n(z_k) > (1/2)\,z_k^n$ holds uniformly for $n \geq n_0$. Consequently,

$$\sum_{n=n_0}^{\infty} f_n(z_k)\,f_n(z'_k) > \frac{1}{4} \sum_{n=n_0}^{\infty} z_k^n\,z_k'^n > 0. \tag{1.2.10}$$

Since the eigenvectors of the symmetric Equation (1.1.7) corresponding to different eigenvalues are orthogonal, however, we have

$$0 = \sum_{n=1}^{\infty} f_n(z_k)\,f_n(z'_k) = \sum_{n=1}^{n_0-1} f_n(z_k)\,\left[f_n(z'_k) - f_n(z_k)\right] + \sum_{n=1}^{n_0-1} f_n^2(z_k) + \sum_{n=n_0}^{\infty} f_n(z_k)\,f_n(z'_k).$$

By taking the limit as $k \to \infty$ and since $f_n(z)$ as $n \to n_0$ converges uniformly, we obtain

$$0 \geq \lim_{k\to\infty} \sum_{n=n_0}^{\infty} f_n(z_k)\,f_n(z'_k). \tag{1.2.11}$$

The Inequalities (1.2.10) and (1.2.11) contradict each other. Hence, the assumption $d = 0$ is incorrect, implying that $d > 0$. Therefore, the function $f_0(z)$ in $(0, 1)$ can only have a finite number of zeros. The same holds for the interval $(-1, 0)$. Let us now prove that the function $\Phi(z) = (z - z^{-1})/f_0(z)$ is bounded in a neighborhood of $z = 1$, $|z| \leq 1$. The case for $z = -1$ can be proven with a similar argument. Consider the region $D_r = \{z : |z| \leq 1, \;\; |z - 1| \leq r\}$ for a sufficiently small r. If $f_0(1) = 0$ and $r < (1/2)d$, then within the region D_r the function $f_0(z)$ has no additional zeros. Thus, $\Phi(z)$ is analytic at the interior points of D_r and bounded on the chord $z = 1 - r + i\tau$, $\;-\sqrt{1 - (1 - r)^2} \leq \tau \leq \sqrt{1 - (1 - r)^2}$. Since $|S(z)| = 1$, by setting $n = 1$ in (1.2.5), we get $|\Phi(z)| \leq |f_1(z)|$. Furthermore, because $f_1(z)$ is bounded on the arc of circle defined by D_r, $\Phi(z)$ is then also bounded there. Since the continuity of $\Phi(z)$ as $z \to 1$ is not established, we cannot apply the maximum principle to prove the boundedness of $\Phi(z)$ within the region D_r. Therefore, we now consider the system

$$a_{n+1,\beta}\,y_{n+1} + b_{n,\beta}\,y_n + a_{n,\beta}\,y_{n-1} = (z + z^{-1})\,y_n, \quad n \geq 1, \tag{1.2.12}$$

where β is a natural number and

$$a_{n,\beta} = \begin{cases} a_n, & n \le \beta, \\ 1, & n > \beta, \end{cases} \qquad b_{n,\beta} = \begin{cases} b_n, & n \le \beta, \\ 0, & n > \beta. \end{cases} \tag{1.2.13}$$

The system (1.2.12) converges to the system given in (1.1.7) as $\beta \to \infty$. Moreover,

$$f_{n,\beta}(z) = \sum_{\mu=n}^{\infty} K_{n\mu}^{(\beta)} z^{\mu} \tag{1.2.14}$$

is the solution to (1.2.12) and has the same properties as $f_n(z)$. Furthermore, $f_{n,\beta}(z) \to f_n(z)$ as $\beta \to \infty$. Therefore, there exist numbers $0 < r_0 < r$ such that for any β the function $f_{0,\beta}(z)$ has no more than one zero inside the region D_{r_0}, which is a neighborhood of $z = 1$. From (1.2.14), we have

$$0 = f_{0,\beta}(1) = \sum_{\mu=0}^{\infty} K_{0\mu}^{(\beta)} \Rightarrow 0 = K_{00}^{(\beta)} + \sum_{\mu=1}^{\infty} K_{0\mu}^{(\beta)} \Rightarrow K_{00}^{(\beta)} = -\sum_{\mu=1}^{\infty} K_{0\mu}^{(\beta)}.$$

Thus,

$$f_{0,\beta}(z) = (z-1)\, A_{\beta}(z), \tag{1.2.15}$$

where

$$A_{\beta}(z) := K_{00}^{(\beta)} + \sum_{\mu=1}^{\infty} \left[K_{0\mu}^{(\beta)} \sum_{p=0}^{\mu-1} z^p \right].$$

It is easy to see that the functions

$$\Phi_{\beta}(z) = \frac{z - z^{-1}}{(z-1)\, A_{\beta}(z)} = \frac{z^2 - 1}{z\,(z-1)\, A_{\beta}(z)} = \frac{z+1}{z\, A_{\beta}(z)}$$

are continuous at $z = 1$. We now define

$$\widetilde{\Phi}_{\beta}(z) := \Phi_{\beta}(z) \frac{z - z_{\beta}}{z + z_{\beta}},$$

where z_{β} is the zero of the function $f_{0,\beta}$ inside D_{r_0}. If $f_{0,\beta}$ has no zeros inside D_{r_0}, we set $z_{\beta} = 1$. It follows that the functions $\widetilde{\Phi}_{\beta}(z)$ are uniformly bounded on the boundary of the region D_{r_0}, regular within D_{r_0}, and continuous on the boundary of the region D_{r_0}. In addition,

$$\sup_{z \in \partial D_{r_0}} \left| \Phi_{\beta}(z)\, \frac{z - z_{\beta}}{z + z_{\beta}} \right| = c < \infty,$$

from which we obtain through the maximum modulus theorem for a regular function

$$\sup_{z \in D_{r_0}} \left| \Phi_{\beta}(z)\, \frac{z - z_{\beta}}{z + z_{\beta}} \right| = c < \infty.$$

Since $\lim_{\beta \to \infty} z_{\beta} = 1$, $\lim_{\beta \to \infty} \Phi_{\beta}(z) = \Phi(z)$ for $z \neq 1$, and, by considering the limit in the previous equality as $\beta \to \infty$, we obtain

$$\sup_{z \in D_{r_0}} |\Phi(z)| \le c,$$

which completes the proof. ■

1.3 Derivation of the fundamental equation

We define the following:

$$\mathcal{F} = \begin{pmatrix} b_1 & a_2 & 0 & 0 & 0 & \cdots \\ a_2 & b_2 & a_3 & 0 & 0 & \cdots \\ 0 & a_3 & b_3 & a_4 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \end{pmatrix}. \tag{1.3.1}$$

Thus, Equation (1.1.1) can be rewritten as

$$\mathcal{F}\, y = \lambda\, y.$$

The matrix $\mathcal{F}$ is known as the infinite Jacobi matrix. Let $\ell^2(1,\infty)$ denote the Hilbert space of sequences $y = \{y_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} |y_n|^2 < \infty$ with the inner product $(x, y) = \sum_{n=1}^{\infty} x_n \overline{y}_n$. For finite sequences (i.e., sequences with a finite number of non-zero terms) we define the operator $\mathcal{L}_0$ as

$$\begin{cases} (\mathcal{L}_0\, y)_1 = b_1\, y_1 + a_2\, y_2, \\ (\mathcal{L}_0\, y)_n = a_{n+1}\, y_{n+1} + b_n\, y_n + a_n\, y_{n-1}, \quad n \geq 2. \end{cases} \tag{1.3.2}$$

The operator $\mathcal{L}_0$ is symmetric. Let the closure of this operator be denoted by $\mathcal{L}$, which follows the same formula as in (1.3.2). From (1.1.2), it follows that the operator $\mathcal{L}$ is self-adjoint (see [7], page 501), and its domain $D(\mathcal{L})$ consists of all $y \in \ell^2(1,\infty)$ such that $\mathcal{L}\, y \in \ell^2(1,\infty)$.

From the results of §1.2, it follows that the vectors $\{u_n(z)\}_{n=1}^{\infty}$ and $\{u_n(z_j)\}_{n=1}^{\infty}$ for $j = 1, \ldots, N$ given by the formulas

$$u_n(z) = f_n(z^{-1}) - S(z)\, f_n(z), \quad |z| = 1, \tag{1.3.3}$$

$$u_n(z_j) = M_j\, f_n(z_j), \quad z \in (-1,0) \cup (0,1), \quad j = 1, \ldots, N. \tag{1.3.4}$$

are bounded solutions to Equation (1.1.7).

Definition 1.3.1. *The set of quantities*

$$\{S(z), z_1, \ldots, z_N; M_1, \ldots, M_N\}$$

is called the scattering data of the Jacobi matrix (1.3.1) *or Equation* (1.1.1).

The inverse scattering problem consists of determining the Jacobi matrix (1.3.1): reconstructing the Equation (1.1.1) from the scattering data. An essential tool in solving this problem is the Marchenko equation, which we will now derive. By substituting the expression (1.1.10) for $f_n(z)$ into (1.2.5), we obtain

$$-\frac{z - z^{-1}}{K_{nn}\, f_0(z)}\, \varphi_n(z) + z^n - z^{-n} = [1 - S(z)]\; z^n + \sum_{k=1}^{\infty} \frac{K_{n,n+k}}{K_{nn}}\, z^{-(k+n)} - \sum_{k=1}^{\infty} \frac{K_{n,n+k}}{K_{nn}}\, z^{k+n} + \sum_{k=1}^{\infty} [1 - S(z)]\; \frac{K_{n,n+k}}{K_{nn}}\, z^{k+n}.$$

By multiplying both sides of this equation by $\frac{1}{2i\pi}\, z^{(m+n-1)}$, $n \geq 0$, $m \geq 1$, and by integrating with respect to z over the closed curve $\mathbf{T} := \{z \in \mathbb{C} : |z| = 1\}$ in a counterclockwise direction,

we get

$$\frac{1}{2i\pi}\oint_{\mathbf{T}}\left[-\frac{z-z^{-1}}{K_{nn}\,f_0(z)}\,\varphi_n(z)+(z^n-z^{-n})\right]z^{(m+n-1)}\,dz=\frac{1}{2i\pi}\oint_{\mathbf{T}}\left[1-S(z)\right]z^n\,z^{(m+n-1)}\,dz$$
$$+\sum_{k=1}^{\infty}\frac{K_{n,n+k}}{K_{nn}}\,\frac{1}{2i\pi}\oint_{\mathbf{T}}z^{-(n+k)}\,z^{(m+n-1)}\,dz-\sum_{k=1}^{\infty}\frac{K_{n,n+k}}{K_{nn}}\,\frac{1}{2i\pi}\oint_{\mathbf{T}}z^{(n+k)}\,z^{(m+n-1)}\,dz$$
$$+\sum_{k=1}^{\infty}\frac{K_{n,n+k}}{K_{nn}}\,\frac{1}{2i\pi}\oint_{\mathbf{T}}\left[1-S(z)\right]z^{(n+k)}\,z^{(m+n-1)}\,dz.$$

Let us note that

$$\frac{1}{2i\pi}\oint_{\mathbf{T}}z^{-(n+k)}\,z^{(m+n-1)}\,dz=\frac{1}{2i\pi}\oint_{\mathbf{T}}z^{m-k-1}\,dz=\begin{cases}0, & k\le m-1,\\ 1, & k=m,\\ 0, & k\ge m+1.\end{cases}$$

On the other hand,

$$\frac{1}{2i\pi}\oint_{\mathbf{T}}z^{(n+k)}\,z^{(m+n-1)}\,dz=0.$$

Thus, we obtain

$$\frac{1}{2i\pi}\oint_{\mathbf{T}}\left[-\frac{z-z^{-1}}{K_{nn}\,f_0(z)}\,\varphi_n(z)+(z^n-z^{-n})\right]z^{(m+n-1)}\,dz=F^{(1)}_{2n+m}+\frac{K_{n,n+m}}{K_{nn}}+\sum_{k=1}^{\infty}\frac{K_{n,n+k}}{K_{nn}}\,F^{(1)}_{k+m+2n},\tag{1.3.5}$$

where

$$F^{(1)}_m=\frac{1}{2i\pi}\oint_{\mathbf{T}}\left[1-S(z)\right]z^{m-1}\,dz.$$

Now let us calculate the integral on the left side of (1.3.5)

$$\frac{1}{2i\pi}\oint_{\mathbf{T}}\left[-\frac{z-z^{-1}}{K_{nn}\,f_0(z)}\,\varphi_n(z)+(z^n-z^{-n})\right]z^{(m+n-1)}\,dz=-\frac{1}{2i\pi}\oint_{\mathbf{T}}\frac{z-z^{-1}}{K_{nn}\,f_0(z)}\,\varphi_n(z)\,z^{(m+n-1)}\,dz$$
$$=-\sum_{j=1}^{N}\operatorname{Res}\left(\frac{z-z^{-1}}{K_{nn}\,f_0(z)}\,\varphi_n(z)\,z^{(m+n-1)},z_j\right),$$

where

$$\operatorname{Res}\left(\frac{z-z^{-1}}{K_{nn}\,f_0(z)}\,\varphi_n(z)\,z^{(m+n-1)},z_j\right)=\lim_{z\to z_j}(z-z_j)\frac{z-z^{-1}}{K_{nn}\,f_0(z)}\,\varphi_n(z)\,z^{(m+n-1)}$$

$$=\frac{z_j-z_j^{-1}}{K_{nn}}\,\frac{f_n(z_j)}{f_1(z_j)}\,z_j^{(m+n-1)}\lim_{z\to z_j}\frac{z-z_j}{f_0(z)}=\frac{z_j-z_j^{-1}}{K_{nn}}\,\frac{f_n(z_j)}{f_1(z_j)}\,z_j^{(m+n-1)}\lim_{z\to z_j}\frac{1}{\dot f_0(z)}$$
$$=\frac{z_j-z_j^{-1}}{K_{nn}}\,\frac{f_n(z_j)}{f_1(z_j)}\,z_j^{(m+n-1)}\frac{1}{\dot f_0(z_j)}=\frac{1-\frac{1}{z_j^2}}{K_{nn}}\,\frac{f_n(z_j)}{\dot f_0(z_j)\,f_1(z_j)}\,z_j^{(n+m)}$$
$$=M_j^2\,\frac{f_n(z_j)}{K_{nn}}\,z_j^{(m+n)}=M_j^2\,z_j^{(2n+m)}\left(1+\sum_{k=1}^{\infty}\frac{K_{n,n+k}}{K_{nn}}\,z_j^k\right).$$

Here we use the fact that $\varphi_n(z_j) = f_n(z_j)/f_1(z_j)$. Therefore,

$$\frac{1}{2i\pi}\oint_{\mathbf{T}}\left[-\frac{z-z^{-1}}{K_{nn}\,f_0(z)}\,\varphi_n(z)+(z^n-z^{-n})\right]\,z^{(m+n-1)}\,dz = -F^{(0)}_{2n+m}-\sum_{k=1}^{\infty}\frac{K_{n,n+k}}{K_{nn}}\,F^{(0)}_{k+m+2n},$$

where

$$F^{(0)}_m = \sum_{j=1}^{N} M_j^2\, z_j^m.$$

Thus, as a result of (1.3.5), we obtain

$$F_{2n+m}+\frac{K_{n,n+m}}{K_{nn}}+\sum_{k=1}^{\infty}\frac{K_{n,n+k}}{K_{nn}}\,F_{k+m+2n}=0,\qquad m\geq 1,\quad n\geq 0, \tag{1.3.6}$$

where

$$F_m = F^{(0)}_m + F^{(1)}_m = \sum_{j=1}^{N} M_j^2\, z_j^m + \frac{1}{2i\pi}\oint_{\mathbf{T}}[1-S(z)]\;z^{m-1}\,dz. \tag{1.3.7}$$

Let us note that the integral on the left-hand side of (1.3.5) can be expressed as the sum of two integrals. Since the points z_j are located in $(-1,0)\cup(0,1)$ and the distance between them is greater than zero, there exists $\epsilon>0$ such that T_ϵ is the circle of radius ϵ centered at 0, and within T_ϵ there are no points z_j. Given that

$$\varphi_n(z) = -\frac{f_n(z^{-1})\,f_0(z)}{z-z^{-1}}+\frac{f_n(z)\,f_0(z^{-1})}{z-z^{-1}}$$

when $z\to 0$, we have

$$\varphi_n(z) = \frac{K_{00}\,z}{K_{nn}}[1+o(1)].$$

Therefore, the integral on the left-hand side of (1.3.5) equals

$$\begin{aligned}\frac{1}{2i\pi}\oint_{\mathbf{T}}\left[-\frac{z-z^{-1}}{K_{nn}\,f_0(z)}\,\varphi_n(z)+(z^n-z^{-n})\right]\,z^{(m+n-1)}\,dz+\frac{1}{2\pi i}\oint_{T_\epsilon}\left[\frac{z^{m-1}}{K_{nn}^2}-z^{m-1}\right]dz\\ = -F^{(0)}_{2n+m}-\sum_{k=1}^{\infty}\frac{K_{n,n+k}}{K_{nn}}\,F^{(0)}_{k+m+2n}+\frac{1}{2\pi i}\oint_{T_\epsilon}\frac{z^{m-1}}{K_{nn}^2}\,dz-\frac{1}{2\pi i}\oint_{T_\epsilon}z^{m-1}\,dz,\end{aligned}$$

where the integral over the closed curve T_ϵ is taken in the counterclockwise direction. By considering (1.3.6), from the last equality we obtain

$$F_{2n+m}+\frac{K_{n,n+m}}{K_{nn}}+\sum_{k=1}^{\infty}\frac{K_{n,n+k}}{K_{nn}}\,F_{k+m+2n}=\frac{1}{2\pi i}\oint_{T_\epsilon}\frac{z^{m-1}}{K_{nn}^2}\,dz-\frac{1}{2\pi i}\oint_{T_\epsilon}z^{m-1}\,dz, \tag{1.3.8}$$

in which the right-hand side of (1.3.8) is zero for $m\geq 1$. If we set $m=0$ in (1.3.8), we obtain

$$F_{2n}+\sum_{k=1}^{\infty}\frac{K_{n,n+k}}{K_{nn}}\,F_{k+2n}+1=\frac{1}{K_{nn}^2},\qquad n\geq 0. \tag{1.3.9}$$

In this way, we obtain that $K_{n,n+m}/K_{nn}$ satisfies Equation (1.3.6), and F_m is constructed solely from the scattering data. Equation (1.3.6) is called the fundamental equation of the inverse problem.

1.4 Additional necessary properties of the scattering data

At this stage, we use the fundamental equation (1.3.6) to derive estimates for F_m.

Theorem 1.4.1. *The sequence $\{F_m\}_{n=1}^{\infty}$, constructed by the formula (1.3.7), satisfies the properties below.*

1) Each F_m is real.

2) The following holds:

$$\{F_m^{(1)}\}_{m=1}^{\infty}, \{F_m\}_{m=1}^{\infty} \in \ell^2(1,\infty). \tag{1.4.1}$$

3) The following inequality holds:

$$|F_m| \le c\,\sigma\left(\left[\frac{m}{2}\right]\right), \tag{1.4.2}$$

where c is a positive constant independent of m and $\sigma(n)$ is defined by the formula (1.1.5).

4) The inequality

$$\sum_{m=1}^{\infty} m\ |F_{m+2} - F_m| < \infty \tag{1.4.3}$$

holds.

5) The following summations hold:

$$\sum_{m=1}^{\infty} |F_m| < \infty, \qquad \sum_{m=1}^{\infty} mF_m^2 < \infty. \tag{1.4.4}$$

Proof.

1) In fact, from the nature of the values z_j and by considering (1.2.7), it follows that F_m is real.

2) The condition (1.4.1) follows directly from (1.2.7) and (1.3.7).

3) To obtain the estimate (1.4.2), let us use the following lemma.

Lemma 1.4.1 (Grönwall's inequality)**.** *Let $g_m \ge 0$, $\psi_m \ge 0$ for $1 \le m < \infty$ and*

$$\sum_{m=1}^{\infty} \psi_m < \infty, \qquad \sum_{m=1}^{\infty} g_m\,\psi_m < \infty.$$

If

$$g_m \le c + \sum_{k=m+1}^{\infty} g_k\,\psi_k, \qquad 1 \le m < \infty, \tag{1.4.5}$$

where c is a positive number, then

$$g_m \le c\,\exp\left(\sum_{k=m+1}^{\infty} \psi_k\right) \qquad 1 \le m < \infty. \tag{1.4.6}$$

Now let us proceed to the proof of estimate (1.4.2). From the fundamental equation (1.3.6), we have

$$F_{m+2n} = -\frac{K_{n,n+m}}{K_{nn}} - \sum_{s=m+1}^{\infty} \frac{K_{n,n+s-m}}{K_{nn}} F_{s+2n},$$

Therefore, by virtue of (1.1.20), we obtain

$$|F_{m+2n}| \le c\,\sigma\left(n + \left[\frac{m}{2}\right]\right) + c \sum_{s=m+1}^{\infty} \sigma\left(n + \left[\frac{s-m}{2}\right]\right) |F_{s+2m}|\,. \tag{1.4.7}$$

We demonstrate that the infinite sum on the right-hand side of (1.4.7) converges for each n and m. Indeed, by the Bunyakovsky-Cauchy-Schwarz inequality, we have

$$\left(\sum_{s=m+1}^{\infty} \sigma\left(n + \left\lfloor\frac{s-m}{2}\right\rfloor\right) |F_{s+2m}|\right)^2 \le \sum_{s=m+1}^{\infty} \sigma^2\left(n + \left\lfloor\frac{s-m}{2}\right\rfloor\right) \sum_{s=m+1}^{\infty} F_{s+2m}^2. \tag{1.4.8}$$

On the other hand,

$$\begin{aligned}
\sum_{s=m+1}^{\infty} \sigma^2\left(n + \left[\frac{s-m}{2}\right]\right) &\le 2\sum_{s=0}^{\infty} \sigma^2(n+s) \le 2\,\sigma(n) \sum_{s=0}^{\infty} \sigma(n+s) \\
&= 2\,\sigma(n) \sum_{s=0}^{\infty} \sum_{p=n+s}^{\infty} (|1-a_p| + |b_p|) \\
&= 2\,\sigma(n) \sum_{s=0}^{\infty} \sum_{k=s}^{\infty} (|1-a_{n+k}| + |b_{n+k}|) \\
&= 2\,\sigma(n) \sum_{k=0}^{\infty} (k+1)\,(|1-a_{n+k}| + |b_{n+k}|) \\
&= 2\,\sigma(n) \sum_{p=n}^{\infty} (p-n+1)\,(|1-a_p| + |b_p|) < \infty.
\end{aligned} \tag{1.4.9}$$

By virtue of (1.4.1), we have

$$\sum_{s=m+1}^{\infty} F_{s+2n}^2 = \sum_{p=m+1+2n}^{\infty} F_p^2 < \infty, \tag{1.4.10}$$

consequently, the left-hand side of (1.4.8) is finite. In principle, suppose that $\sigma(n) \ne 0$ for all n, $n = 1, 2, 3, \ldots$. From (1.4.7), by dividing both sides by $\sigma\left(n + \left\lfloor\frac{m}{2}\right\rfloor\right)$, we obtain

$$|F_{m+2n}|\,\sigma^{-1}\left(n + \left[\frac{m}{2}\right]\right) \le c + c\,\sigma^{-1}\left(n + \left[\frac{m}{2}\right]\right) \sum_{s=m+1}^{\infty} \sigma\left(n + \left[\frac{s-m}{2}\right]\right) |F_{s+2n}|$$

$$= c + c\,\sigma^{-1}\left(n + \left[\frac{m}{2}\right]\right) \sum_{s=m+1}^{\infty} \sigma\left(n + \left[\frac{s-m}{2}\right]\right) \sigma\left(n + \left[\frac{s}{2}\right]\right) \left[|F_{s+2n}|\,\sigma^{-1}\left(n + \left[\frac{s}{2}\right]\right)\right].$$

Since the sequence

$$\psi_s = c\,\sigma^{-1}\left(n + \left[\frac{m}{2}\right]\right) \sigma\left(n + \left[\frac{s-m}{2}\right]\right) \sigma\left(n + \left[\frac{s}{2}\right]\right)$$

and $|F_{s+2n}|\,\sigma^{-1}\left(n+\left[\frac{m}{2}\right]\right)\psi_s$ are summable with respect to s ($s \geq m+1$) for each n and m (as appear in (1.4.8)–(1.4.10)) by applying Lemma 1.4.1 to the last equality, we then obtain

$$|F_{m+2n}|\;\sigma^{-1}\left(n+\left[\frac{m}{2}\right]\right) \leq \exp\left(\sum_{s=m+1}^{\infty}\psi_s\right). \tag{1.4.11}$$

On the other hand,

$$\sum_{s=m+1}^{\infty}\psi_s = c\,\sigma^{-1}\left(n+\left[\frac{m}{2}\right]\right)\sum_{s=m+1}^{\infty}\sigma\left(n+\left[\frac{s-m}{2}\right]\right)\sigma\left(n+\left[\frac{s}{2}\right]\right)$$
$$\leq c\sum_{s=m+1}^{\infty}\sigma\left(n+\left[\frac{s-m}{2}\right]\right)<\infty.$$

See the derivation of estimate (1.4.9). Therefore, from (1.4.11),

$$|F_{m+2n}| \leq c\,\sigma\left(n+\left\lfloor\frac{m}{2}\right\rfloor\right),$$

which is equivalent to inequality (1.4.2). Note that we have denoted positive numbers for c, which are generally different and do not depend on n or m. Let us prove that the estimate (1.4.2) holds even in the case when $\sigma(n)$ vanishes. In fact, let $\sigma(n)=0$ for $n \geq n_0$ and $\sigma(n)>0$ for $n \leq n_0-1$. Since $\sigma(n)=\sum_{p=n}^{\infty}(|1-a_p|+|b_p|)$, this implies that $a_p=1$ and $b_p=0$ for $p \geq n_0$. In this case, from (1.1.20) it follows that $K_{n,n+m}/K_{nn}=0$ for $n+\left\lfloor\frac{m}{2}\right\rfloor \geq n_0$, and, from the fundamental equation (1.3.6), we conclude that $F_{2n+m}=0$ for $n \geq n_0$ and for all $m \geq 1$. Hence, it follows that there exists a natural number M such that for $m \geq M$ we have $F_m=0$. Since for $m \geq M$ the estimate (1.4.2) will hold and because the number of values F_m, $1 \leq m \leq M$ is finite, the estimate (1.4.2) applies to all F_m. Finally, if $\sigma(n) \equiv 0$, then from (1.1.20) we have $K_{n,n+m}/K_{nn} \equiv 0$. Moreover, from the fundamental equation (1.3.6) we obtain $F_m \equiv 0$, so the estimate also holds in this case.

4) We prove the estimate (1.4.3). From the fundamental equation (1.3.6), we easily obtain

$$\left(F_{m+2(n+1)}-F_{m+2n}\right)+\left(\frac{K_{n+1,n+1+m}}{K_{n+1,n+1}}-\frac{K_{n,n+m}}{K_{nn}}\right)=\frac{K_{n,n+1}}{K_{nn}}\,F_{m-1+2(n+1)}$$

$$+\frac{K_{n,n+2}}{K_{nn}}\,F_{m+2(n+1)}-\sum_{k=1}^{\infty}\left(\frac{K_{n+1,n+1+k}}{K_{n+1,n+1}}-\frac{K_{n,n+k+2}}{K_{nn}}\right)F_{k+m+2(n+1)}. \tag{1.4.12}$$

We estimate the difference $K_{n+1,n+1+k}/K_{n+1,n+1}-K_{n,n+k+2}/K_{nn}$. From (1.1.18) we obtain

$$\frac{K_{n+1,n+1+2k+1}}{K_{n+1,n+1}}-\frac{K_{n,n+2k+3}}{K_{nn}}=\sum_{p=n+1}^{\infty}b_p\,\frac{K_{p,p+2(k+1)}}{K_{pp}}-\sum_{p=n+2}^{\infty}\left(1-a_p^2\right)\frac{K_{p,p+2k+1}}{K_{pp}},$$

and then, by considering (1.1.20),

$$\left|\frac{K_{n+1,n+1+2k+1}}{K_{n+1,n+1}}-\frac{K_{n,n+2k+3}}{K_{nn}}\right|\le c\sum_{p=n+1}^{\infty}|b_p|\,\sigma(p+k+1)+c\sum_{p=n+2}^{\infty}|1-a_p^2|\,\sigma(p+k)$$
$$\le c\,\sigma(n+k+2)\left(\sum_{p=n+1}^{\infty}|b_p|+\sum_{p=n+2}^{\infty}|1-a_p^2|\right)$$
$$\le c\,\sigma(n+k+2)\,\sigma(n)\le c\,\sigma(n+k)\,\sigma(n). \quad (1.4.13)$$

Similarly, from (1.1.19) we obtain

$$\frac{K_{n+1,n+1+2k}}{K_{nn}}-\frac{K_{n,n+2k+2}}{K_{nn}}=\sum_{p=n+1}^{\infty}b_p\,\frac{K_{p,p+2k+1}}{K_{pp}}-\sum_{p=n+2}^{\infty}\left(1-a_p^2\right)\,\frac{K_{p,p+2k}}{K_{pp}},$$

from which

$$\left|\frac{K_{n+1,n+1+2k}}{K_{nn}}-\frac{K_{n,n+2k+2}}{K_{nn}}\right|\le\sum_{p=n+1}^{\infty}|b_p|\left|\frac{K_{p,p+2k+1}}{K_{pp}}\right|+\sum_{p=n+2}^{\infty}|1-a_p^2|\left|\frac{K_{p,p+2k}}{K_{pp}}\right|$$
$$\le c\sum_{p=n+1}^{\infty}|b_p|\,\sigma(p+k)+c\sum_{p=n+2}^{\infty}|1-a_p^2|\,\sigma(p+k)$$
$$\le c\,\sigma(n+k)\,\sigma(n). \quad (1.4.14)$$

From (1.4.13) and (1.4.14), it follows that

$$\left|\frac{K_{n+1,n+1+k}}{K_{n+1,n+1}}-\frac{K_{n,n+k+2}}{K_{nn}}\right|\le c\,\sigma\left(n+\left[\frac{k}{2}\right]\right)\,\sigma(n). \quad (1.4.15)$$

Now, from (1.4.12) and by using (1.1.20), (1.4.2), and (1.4.15), we have

$$\left|\left(F_{m+2(n+1)}-F_{m+2n}\right)+\left(\frac{K_{n+1,n+1+m}}{K_{n+1,n+1}}-\frac{K_{n,n+m}}{K_{nn}}\right)\right|$$
$$\le\left|\frac{K_{n,n+1}}{K_{nn}}\right|\,\left|F_{m-1+2(n+1)}\right|+\left|\frac{K_{n,n+2}}{K_{nn}}\right|\,\left|F_{m+2(n+1)}\right|$$
$$+\sum_{k=1}^{\infty}\left|\frac{K_{n+1,n+1+k}}{K_{n+1,n+1}}-\frac{K_{n,n+k+2}}{K_{nn}}\right|\,\left|F_{k+m+2(n+1)}\right|$$
$$\le c\,\sigma(n)\,\sigma\left(n+1+\left[\frac{m-1}{2}\right]\right)+c\,\sigma(n+1)\,\sigma\left(n+1+\left[\frac{m}{2}\right]\right)$$
$$+c\,\sigma(n)\sum_{k=1}^{\infty}\sigma\left(n+\left[\frac{k}{2}\right]\right)\,\sigma\left(n+1+\left[\frac{k+m}{2}\right]\right)\le c\,\sigma(n)\,\sigma\left(n+\left[\frac{m}{2}\right]\right). \quad (1.4.16)$$

In (1.4.16), by assuming $m=1$ the first time and $m=2$ the second time, as well as by taking into account (1.1.13) and (1.1.12), we obtain

$$\left|(F_{2n+3}-F_{2n+1})+b_{n+1}\right|\le c\,\sigma^2(n), \quad (1.4.17)$$

$$\left|(F_{2n+4}-F_{2n+2})+a_{n+2}^2-1+b_{n+1}\,\frac{K_{n+1,n+2}}{K_{n+1,n+1}}\right|\le c\,\sigma^2(n). \quad (1.4.18)$$

From (1.4.17) we obtain

$$\sum_{n=0}^{\infty}(n+1)\,|F_{2n+3}-F_{2n+1}| = \sum_{n=0}^{\infty}(n+1)\,|F_{2n+3}-F_{2n+1}+b_{n+1}-b_{n+1}|$$
$$\leq \sum_{n=0}^{\infty}(n+1)\,|F_{2n+3}-F_{2n+1}+b_{n+1}| + \sum_{n=0}^{\infty}(n+1)|b_{n+1}|$$
$$\leq c\sum_{n=0}^{\infty}(n+1)\,\sigma^2(n) + \sum_{n=0}^{\infty}(n+1)|b_{n+1}| < \infty \tag{1.4.19}$$

since

$$\sum_{n=0}^{\infty}(n+1)\,\sigma^2(n) = \sum_{n=0}^{\infty}\sigma(n)\left[(n+1)\sum_{p=n}^{\infty}(|1-a_p|+|b_p|)\right]$$
$$\leq \sum_{n=0}^{\infty}\sigma(n)\sum_{p=n+1}^{\infty}(p+1)\,(|1-a_p|+|b_p|) \leq \sum_{p=1}^{\infty}(p+1)\,(|1-a_p|+|b_p|)\sum_{n=0}^{\infty}\sigma(n),$$
$$\sum_{n=0}^{\infty}\sigma(n) = \sum_{n=0}^{\infty}\sum_{p=n}^{\infty}(|1-a_p|+|b_p|) = \sum_{p=0}^{\infty}(p+1)\,(|1-a_p|+|b_p|) < \infty.$$

Similarly, from (1.4.18) and by taking into account (1.1.20), we obtain

$$\sum_{n=0}^{\infty}(n+2)\,|F_{2n+4}-F_{2n+2}|$$
$$\leq \sum_{n=0}^{\infty}(n+2)\left|(F_{2n+4}-F_{2n+2})+a_{n+2}^2-1+b_{n+1}\frac{K_{n+1,n+2}}{K_{n+1,n+1}}\right|$$
$$+\sum_{n=0}^{\infty}(n+2)|a_{n+2}^2-1| + \sum_{n=0}^{\infty}(n+2)|b_{n+1}|\left|\frac{K_{n+1,n+2}}{K_{n+1,n+1}}\right|$$
$$\leq c\sum_{n=0}^{\infty}(n+2)\,\sigma^2(n) + \sum_{n=0}^{\infty}(n+2)|a_{n+1}^2-1| + c\sum_{n=0}^{\infty}(n+2)|b_{n+1}|\,\sigma(n+1) < \infty. \tag{1.4.20}$$

Both estimates (1.4.19) and (1.4.20) are equivalent to the estimate (1.4.3).

5) The estimates (1.4.4) follow from (1.4.2). In fact,

$$\sum_{m=1}^{\infty}|F_m| \leq c\sum_{m=1}^{\infty}\sigma\left(\left[\frac{m}{2}\right]\right) \leq 2c\sum_{m=0}^{\infty}\sigma(m) < \infty,$$
$$\sum_{m=1}^{\infty}mF_m^2 \leq c\sum_{m=1}^{\infty}m\,\sigma^2\left(\left[\frac{m}{2}\right]\right) \leq 4c\sum_{m=0}^{\infty}(m+1)\,\sigma^2(m) < \infty.$$

■

Corollary 1.4.1. *According to (1.3.7), since F_m differs from $F_m^{(1)}$ by a term that decreases as $m\to\infty$, the estimate in (1.4.3) also holds for $F_m^{(1)}$.*

Lemma 1.4.2. *If $f_0(\pm 1) = 0$, then*

$$f_0(z) = (z-1)\,A_k(z), \qquad |z| = 1, \tag{1.4.21}$$

where the function $A_k(z)$ is continuous for $|z| = 1$,is periodic with a period of 2π, admits an analytic continuation to $|z| < 1$, and satisfies $A_k(\pm 1) \neq 0$.

By using the fundamental equation (1.3.6), we can establish the continuity of the function $S(z)$ for $|z| = 1$.

Theorem 1.4.2. *The function $S(z)$ is continuous for $|z| = 1$ and*

$$S(\pm 1) = \begin{cases} 1, & \text{if} \quad f_0(\pm 1) \neq 0, \\ \mp 1, & \text{if} \quad f_0(\pm 1) = 0. \end{cases} \tag{1.4.22}$$

Proof.

The continuity of the function

$$S(z) = \frac{f_0(z^{-1})}{f_0(z)}$$

for any $|z| = 1$ and $z \neq \pm 1$ follows from the fact that the function $f_0(z)$ is continuous for $|z| = 1$ and can only have zeros at the points $z = \pm 1$. If $f_0(\pm 1) \neq 0$, it is clear that the function $S(z)$ is continuous at $z = \pm 1$, and, because of periodicity,

$$S(\pm 1) = 1.$$

Now let $f_0(\pm 1) = 0$. In this case, according to Lemma 1.4.2, the function $S(z)$ is also continuous at the points $z = \pm 1$ and

$$S(\pm 1) = \lim_{z \to \pm 1} \frac{z^{-1} - 1}{z - 1} \cdot \frac{A_k(z^{-1})}{A_k(z)} = \mp 1.$$

■

Theorem 1.4.3. *The equation*

$$h_m + \sum_{k=1}^{\infty} h_k\, F^{(1)}_{k+m} = 0, \qquad m \geq 1 \tag{1.4.23}$$

has N linearly independent solutions in the space $\ell^2(1,\infty)$, where

$$F^{(1)}_m = \frac{1}{2i\pi} \oint_{\mathbf{T}} [1 - S(z)]\, z^{m-1}\, d\, z. \tag{1.4.24}$$

Proof.

Let $h = \{h_m\}_{m=1}^{\infty}$ be a solution to Equation (1.4.23) in $\ell^2(1,\infty)$. Since $F^{(1)}_m$ is real, without loss of generality, we may assume that h_m is real. We extend h to the entire space $\ell^2(-\infty,\infty)$, by setting $h_0 = h_{-1} = h_{-2} = \cdots = 0$, and define

$$H(z) := \sum_{m=1}^{\infty} h_m\, z^m. \tag{1.4.25}$$

The function $H(z)$ admits an analytic continuation to $|z|<1$, and we obtain

$$\sum_{m=1}^{\infty} h_m^2 = \frac{1}{2i\pi}\oint_{\mathbf{T}} |H(z)|^2\,\frac{d\,z}{z}. \tag{1.4.26}$$

By multiplying both sides of (1.4.23) by h_m and summing over m from 1 to ∞, we have

$$\sum_{m=1}^{\infty} h_m^2 + \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} h_m\,h_k\,F_{k+m}^{(1)} = 0.$$

By substituting the expression for $F_m^{(1)}$ from (1.4.24) and using (1.4.26), we get

$$\oint_{\mathbf{T}} \Big(|H(z)|^2 - S(z)\,H^2(z)\Big)\,\frac{d\,z}{z} = 0,$$

or, by using (1.2.7),

$$\oint_{\mathbf{T}} \big|H(z^{-1}) - S(z)\,H(z)\big|^2\,\frac{d\,z}{z} = 0.$$

Since the integrand is periodic, we conclude

$$H(z^{-1}) - S(z)\,H(z) = 0, \qquad |z|=1. \tag{1.4.27}$$

Given that $S(z) = \frac{f_0(z^{-1})}{f_0(z)}$, from (1.4.27), we obtain

$$\frac{H(z^{-1})}{f_0(z^{-1})} = \frac{H(z)}{f_0(z)}, \qquad |z|=1. \tag{1.4.28}$$

From (1.4.28), it follows that the function

$$\psi(z) = \begin{cases} \frac{H(z)}{f_0(z)}, & \text{for } |z|\le 1,\\ \frac{H(z^{-1})}{f_0(z^{-1})}, & \text{for } |z|\ge 1 \end{cases} \tag{1.4.29}$$

is periodic and meromorphic within the contour $\mathbf{T}_{R,\varepsilon}$, where R is a sufficiently large number such that it contains the zeros of $f_0(z^{-1})$ and ε is a sufficiently small number; see Figure 1.1. Moreover, within this region, $\psi(z)$ has a finite number of simple poles at the zeros of the functions $f_0(z)$ and $f_0(z^{-1})$, specifically at the points $z_j \in (-1,0)$, $z_j^{-1}\in(-\infty,-1)$, $j=1,2,3,\dots,N_0$, $z_j\in(0,1)$, $z_j^{-1}\in(1,\infty)$, $j=N_0+1,\dots,N$. Now for a fixed z consider the function

$$g(\zeta) = \frac{z\,\zeta(\zeta-\zeta^{-1})}{(z+\zeta)(z-\zeta)}\,\psi(\zeta) \qquad \zeta\in\mathbb{C}. \tag{1.4.30}$$

This function is meromorphic with respect to ζ and periodic with a period of 2π. Inside the contour $\mathbf{T}_{R,\varepsilon}$, $g(\zeta)$ has a finite number of simple poles at $\zeta=\pm z$, $z_j\in(-1,0)$, $z_j^{-1}\in(-\infty,-1)$, $j=1,2,3,\dots,N_0$, $z_j\in(0,1)$, $z_j^{-1}\in(1,\infty)$, $j=N_0+1,\dots,N$. Therefore, we have

$$\begin{aligned}\frac{1}{2i\pi}\oint_{\mathbf{T}_{R,\varepsilon}} g(\zeta)\,d\zeta &= \mathrm{Res}(g(\zeta),z) + \mathrm{Res}(g(\zeta),-z) + \sum_{j=1}^{N}\mathrm{Res}(g(\zeta),z_j) + \sum_{j=1}^{N}\mathrm{Res}(g(\zeta),z_j^{-1})\\ &= -z\,(z-z^{-1})\,\psi(z) + \sum_{j=1}^{N}\left[\frac{z\,z_j}{(z+z_j)(z-z_j)} - \frac{z\,z_j^{-1}}{(z+z_j^{-1})(z-z_j^{-1})}\right]\cdot\frac{H(z_j)}{\dot f_0(z_j)}(z_j-z_j^{-1}).\end{aligned} \tag{1.4.31}$$

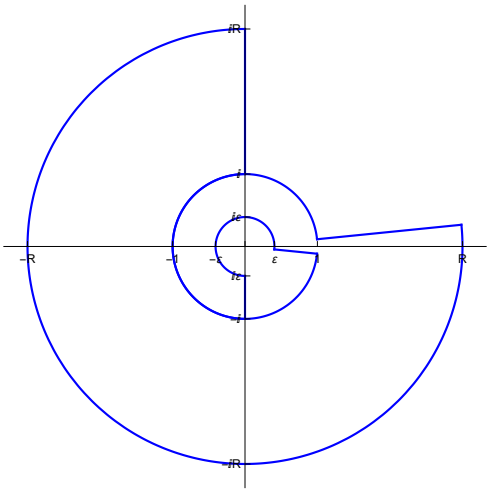

Figure 1.1: Contour $\mathbf{T}_{R,\varepsilon}$

where the integral over the closed curve $\mathbf{T}_{R,\varepsilon}$ is taken in the counterclockwise direction. On the other hand, since $g(\zeta)$ is periodic with period 2π and $g(\zeta) \to 0$ as $\zeta \to \infty$, the function $(\zeta - \zeta^{-1})/f_0(\zeta)$ through Lemma 1.2.4 is bounded near $\zeta = \pm 1$:

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \frac{1}{2i\pi} \oint_{\mathbf{T}_{R,\varepsilon}} g(\zeta)\, d\zeta = 0.$$

From this, together with (1.4.31) and by considering (1.4.29), we obtain

$$H(z) = \sum_{j=1}^{N} \left[\frac{1}{z\, z_j^{-1} - z^{-1}\, z_j} - \frac{1}{z\, z_j - z^{-1}\, z_j^{-1}} \right] \cdot \frac{H(z_j)(z_j - z_j^{-1})}{\dot{f}_0(z_j)} \cdot \frac{f_0(z)}{z\,(z - z^{-1})}. \tag{1.4.32}$$

Thus, we have shown that if $h = \{h_m\}_{m=1}^{\infty}$ is a solution to Equation (1.4.23), then its Fourier expansion (1.4.25) is represented by the form (1.4.32). Now we prove the converse: any function of the form (1.4.32), where $H(z_j) = c_j$ are arbitrarily chosen, is the Fourier expansion (1.4.25) of a solution $\{h_m\}_{m=1}^{\infty}$ to Equation (1.4.23). Indeed, since $H(\theta + 2\pi) = H(\theta)$, this function can be expanded into a Fourier series

$$H(z) = \sum_{-\infty}^{\infty} h_m\, z^m, \qquad \{h_m\}_{-\infty}^{\infty} \in \ell^2(-\infty, \infty). \tag{1.4.33}$$

On the other hand, from (1.4.32), it follows that $H(z) \to 0$ as $z \to 0$. Therefore, the series (1.4.33) must vanish for $m \le 0$, meaning that

$$H(z) = \sum_{m=1}^{\infty} h_m\, z^m, \qquad \{h_m\}_{m=1}^{\infty} \in \ell^2(1, \infty).$$

A direct verification shows that $H(z)$ satisfies Equation (1.4.27), and hence $\{h_m\}_{m=1}^{\infty}$ is a solution of Equation (1.4.23). From this analysis, it follows that the number of linearly independent solutions of Equation (1.4.23) in $\ell^2(1, \infty)$ is equal to the number of linearly independent functions

of the form

$$\left[\frac{1}{z\,z_j^{-1}-z^{-1}\,z_j}-\frac{1}{z\,z_j-z^{-1}\,z_j^{-1}}\right]\frac{c_j\,(z_j-z_j^{-1})}{\dot{f}_0(z_j)},$$

where c_j for $j=1,\dots,N$ are arbitrary constants. Therefore, this number is equal to N, as can easily be seen. ■

Theorem 1.4.4. *The system of equations*

$$\sum_{-\infty}^{m=0} g_m\,F_m^{(1)}=2g_0,\qquad \sum_{-\infty}^{m=0} g_m\,F_{m+n}^{(1)}=g_n,\qquad -\infty<n\le -1, \tag{1.4.34}$$

where $F_m^{(1)}$ is given by (1.4.24), has only the zero solution $\{g_m\}_{-\infty}^{m=0}$ in the space $\ell^2(-\infty,0)$.

Proof.

Let $g=\{g_m\}_{-\infty}^{m=0}$ be a solution of Equation (1.4.34) in $\ell^2(-\infty,0)$. We extend g to the entire space $\ell^2(-\infty,\infty)$ by setting $h_1=h_2=\ldots=0$, and we define

$$G(z):=\sum_{-\infty}^{m=0} g_m\,z^m. \tag{1.4.35}$$

The function $G(z)$ is analytic on the set $|z|\ge 1$. By using Parseval's identity, we have

$$\sum_{-\infty}^{m=0} g_m^2=\frac{1}{2i\pi}\oint_{\mathbf{T}}|G(z)|^2\,\frac{d\,z}{z}. \tag{1.4.36}$$

By multiplying the first equation in (1.4.34) by g_0 and the second by g_n, as well as by summing the latter over n from $-\infty$ to 0, we get

$$\sum_{-\infty}^{n=0}\sum_{-\infty}^{m=0} g_n\,g_m\,F_{m+n}^{(1)}=g_0^2+\sum_{n=-\infty}^{0} g_n^2.$$

By substituting $F_m^{(1)}$ from (1.4.24) and by using (1.4.35) and (1.4.36), we arrive at

$$\frac{1}{2i\pi}\oint_{\mathbf{T}}[1-S(z)]\;G^2(z)\,\frac{d\,z}{z}=g_0^2+\frac{1}{2i\pi}\oint_{\mathbf{T}}|G(z)|^2\,\frac{d\,z}{z}. \tag{1.4.37}$$

From (1.4.35), it follows that

$$\frac{1}{2i\pi}\oint_{\mathbf{T}}G^2(z)\,\frac{d\,z}{z}=g_0^2.$$

Thus, from (1.4.37), we obtain

$$\oint_{\mathbf{T}}\left(|G(z)|^2+S(z)\,G^2(z)\right)\,\frac{d\,z}{z}=0,$$

which, by using (1.2.7), leads to

$$\oint_{\mathbf{T}}|G(z^{-1})+S(z)\,G(z)|^2\,\frac{d\,z}{z}=0.$$

Since the integrand is continuous and periodic with a period of 2π, it follows that

$$G(z^{-1})+S(z)G(z)=0,\qquad |z|=1.$$

Given that $S(z) = f_0(z^{-1})/f_0(z)$, we derive from the last equation that

$$f_0(z)G(z^{-1}) = -f_0(z^{-1})G(z), \qquad |z| = 1. \tag{1.4.38}$$

The function $\psi(z) = if_0(z)G(z^{-1})$ is holomorphic in $|z| < 1$ and real on $|z| = 1$ because of (1.4.38). Since $f_0(z) \to K_{00}$ as $z \to 0$, and $G(z^{-1}) \to g_0$, we have

$$\psi(z) = i\,K_{00}\,g_0,$$

which implies

$$f_0(z)G(z^{-1}) \equiv K_{00}g_0. \tag{1.4.39}$$

Equation (1.4.38) shows that the function $f_0(z)G(z^{-1})$ is antisymmetric, while (1.4.39) contradicts this. Therefore, $G(z) \equiv 0$, and thus $g_m = 0$ for $m = 0, -1, -2, \ldots$, as required. ■

Chapter 2

The inverse scattering problem

This chapter is based on the PhD thesis of Guseinov G. Sh. [13]. Using the scattering data and their properties provided by Guseinov and presented in the previous chapter, we develop his results by incorporating our new parameter z. Specifically, Guseinov constructed an equation of the form (1.1.7) from these scattering data, and in this work, we extend his analysis by considering the new parameter z.

2.1 The solution of the inverse problem from scattering data: Fundamental theorem

From the results obtained, it follows that the scattering data

$$S(z); \quad z_1, \dots, z_N; \quad M_1, \dots, M_N$$

of Equation (1.1.7), where $z_j \in (-1, 0)$ for $j = 1, \dots, N_0$; $z_j \in (0, 1)$ for $j = N_0 + 1, \dots, N$; $M_j > 0$ for $j = 1, \dots, N$ and under the condition (1.1.2), contain the conditions described below.

I. $S(z)$ is a continuous function on $|z| = 1$ and

$$\overline{S(z)} = S(z^{-1}) = (S(z))^{-1}. \tag{2.1.1}$$

II. We have

$$\sum_{m=1}^{\infty} m \left| F_{m+2n}^{(1)} - F_m^{(1)} \right| < \infty, \tag{2.1.2}$$

where

$$F_m^{(1)} := \frac{1}{2i\pi} \oint_{\mathbf{T}} [1 - S(z)] \, z^{m-1} \, dz. \tag{2.1.3}$$

III. The equation

$$h_m + \sum_{k=1}^{\infty} h_k \, F_{k+m}^{(1)} = 0, \qquad m \geq 1 \tag{2.1.4}$$

in the space $\ell^2(1, \infty)$ has exactly N linearly independent solutions.

IV. The equation

$$\sum_{-\infty}^{m=0} g_m F_m^{(1)} = 2g_0, \quad \sum_{-\infty}^{m=0} g_m F_{m+n}^{(1)} = g_n, \qquad -\infty < n \le -1$$

in $\ell^2(-\infty, 0)$ has only the zero solution $\{g_m\}_{-\infty}^{m=0}$.

Theorem 2.1.1 (Fundamental theorem)**.** *Consider the data*

$$\{S(z), \quad z_1, z_2, \dots, z_N; \ M_1, M_2, \dots, M_N\},$$

where

- $z_j \in (-1, 0)$ *for* $j = 1, \dots, N_0$,
- $z_j \in (0, 1)$ *for* $j = N_0 + 1, \dots, N$,
- $M_j > 0$ *for* $j = 1, \dots, N$, *and the values* M_j *are distinct positive constants.*

For this data to be scattering data for the problem

$$a_{n+1}\, y_{n+1} + b_n\, y_n + a_n\, y_{n-1} = (z + z^{-1})\, y_n, \qquad n \ge 1; \quad a_1 > 0, \tag{2.1.5}$$

$$y_0 = 0, \tag{2.1.6}$$

where the coefficients belong to the class

$$a_n > 0, \quad \operatorname{Im} b_n = 0 \ \text{ for } n \ge 1, \quad \sum_{n=1}^{\infty} n\,(|1 - a_n| + |b_n|) < \infty, \tag{2.1.7}$$

it is necessary and sufficient that the given data satisfy conditions I–IV *as stated above.*

Outline of the proof.

The necessary part of the conditions I-IV has already been proven in Chapter 1. We outline the proof of the sufficient part. Consider the data $\{S(z); z_1, z_2, \dots, z_N; M_1, M_2, \dots, M_N\}$ that satisfy conditions I-IV. From this data, we construct F_m via

$$F_m := \sum_{j=1}^{N} M_j^2\, z_j^m + \frac{1}{2i\pi} \oint_{\mathbf{T}} [1 - S(z)]\, z^{m-1}\, dz$$

and consider the equation

$$F_{2n+m} + \frac{K_{n,n+m}}{K_{nn}} + \sum_{k=1}^{\infty} \frac{K_{n,n+k}}{K_{nn}} F_{k+m+2n} = 0, \qquad m \ge 1 \tag{2.1.8}$$

with the unknowns $K_{n,n+m}/K_{nn}$.

1. We must show that the Equation (2.1.8) has a unique solution for each $n \ge 0$.
2. It can be shown that

$$1 + F_{2n} + \sum_{k=1}^{\infty} \frac{K_{n,n+k}}{K_{nn}} F_{k+2n} > 0, \qquad n \ge 0,$$

where $K_{n,n+m}/K_{nn}$ is a solution of the fundamental equation (2.1.8).

3. Next we define K_{nn}, a_n, b_n, and ϱ_n by

$$\frac{1}{K_{nn}^2} := 1 + F_{2n} + \sum_{k=1}^{\infty} \frac{K_{n,n+k}}{K_{nn}} F_{k+2n},$$
$$a_n := \frac{K_{nn}}{K_{n-1,n-1}}, \qquad b_n := \frac{K_{n,n+1}}{K_{nn}} - \frac{K_{n-1,n}}{K_{n-1,n-1}},$$
$$\varrho_n := 1 + \frac{K_{n,n+2}}{K_{nn}} - \frac{K_{n-1,n+1}}{K_{n-1,n-1}} - \left(\frac{K_{n,n+1}}{K_{nn}} - \frac{K_{n-1,n}}{K_{n-1,n-1}}\right) \frac{K_{n,n+1}}{K_{nn}}.$$

We will show that

$$\varrho_n = a_{n+1}^2.$$

4. We define the function

$$f_n(z) := \sum_{\mu=n}^{\infty} K_{n\mu}\, z^{\mu}.$$

We must show that $\{f_n(z)\}_{n=0}^{\infty}$ satisfies the equation

$$a_{n+1}\, f_{n+1}(z) + b_n\, f_n(z) + a_n\, f_{n-1}(z) = (z + z^{-1})\, f_n(z), \qquad n \geq 1.$$

5. Next we must show that a_n and b_n satisfy the conditions (2.1.7).

6. It must be shown that the functions

$$u_n(z) = f_n(z^{-1}) - S(z)\, f_n(z), \quad |z| = 1,$$
$$u_n(z_j) = M_j\, f_n(z_j), \quad z_j \in (-1, 0) \cup (0, 1), \quad j = 1, \ldots, N.$$

satisfy the boundary conditions (2.1.6).

7. Finally, we must show that the given data are scattering data for the constructed problem. Therefore, we need to show that $\{u_n(z)\}_{n=1}^{\infty}$, $\{u_n(z_j)\}_{n=1}^{\infty}$, $j = 1, \ldots, N$, generate the Parseval's equality, which is equivalent to

$$\delta_{nm} = \sum_{j=1}^{N} u_n(z_j)\, \overline{u_m(z_j)} + \frac{1}{2\pi} \int_0^{\pi} u_n(e^{i\theta})\, \overline{u_m(e^{i\theta})}\, d\theta, \qquad n, m \geq 1,$$

where δ_{nm} is the Kronecker delta and for $n \to \infty$ possesses the following asymptotics

$$u_n(z) = z^{-n} - S(z)\, z^n + o(1), \qquad |z| = 1,$$
$$u_n(z_j) = M_j\, z_j^n [1 + o(1)], \qquad z_j \in (-1, 0) \cup (0, 1), \quad j = 1, \ldots, N.$$

2.2 Solvability of the fundamental equation

With the data $\{S(z);\ \ z_1, \ldots, z_N;\ \ M_1, \ldots, M_N\}$, we construct F_m as

$$F_m := F_m^{(0)} + F_m^{(1)}, \tag{2.2.1}$$

$$F_m^{(0)} := \sum_{j=1}^{N} M_j^2\, z_j^m, \tag{2.2.2}$$

$$F_m^{(1)} := \frac{1}{2i\pi} \oint_{\mathbf{T}} [1 - S(z)]\, z^{m-1}\, d\, z, \tag{2.2.3}$$

where the integral over the closed curve $\mathbf{T}$ is taken in the counterclockwise direction and we consider the equation

$$F_{2n+m} + \frac{K_{n,n+m}}{K_{nn}} + \sum_{k=1}^{\infty} \frac{K_{n,n+k}}{K_{nn}} F_{k+m+2n} = 0, \qquad m \geq 1 \tag{2.2.4}$$

with the unknowns $K_{n,n+m}/K_{nn}$ for each fixed $n \geq 0$. From the properties of z_j, M_j, and (2.1.1), it follows that F_m is real. Let us define

$$\eta(m) := \sum_{k=m}^{\infty} |F_{k+2} - F_k|\,, \tag{2.2.5}$$

and, because of (2.1.2) and (2.2.1), this series converges.

Lemma 2.2.1. *The following inequalities hold:*

1)

$$\sum_{m=1}^{\infty} \eta(m) < \infty, \tag{2.2.6}$$

2)

$$\sum_{m=1}^{\infty} m\, \eta^2(m) < \infty, \tag{2.2.7}$$

3)

$$|F_m| \leq \eta(m), \tag{2.2.8}$$

4)

$$\sum_{m=1}^{\infty} |F_m| < \infty, \qquad \sum_{m=1}^{\infty} m\, F_m^2 < \infty. \tag{2.2.9}$$

Proof.

1) By virtue of (2.1.2) and (2.2.1), (2.2.2),

$$\sum_{m=1}^{\infty} \eta(m) = \sum_{m=1}^{\infty} \sum_{k=m}^{\infty} |F_{k+2} - F_k| = \sum_{k=1}^{\infty} k\, |F_{k+2} - F_k| < \infty.$$

2)

$$\sum_{m=1}^{\infty} m\, \eta^2(m) = \sum_{m=1}^{\infty} \eta(m) \left[m \sum_{k=m}^{\infty} |F_{k+2} - F_k| \right] \leq \sum_{m=1}^{\infty} \eta(m) \sum_{k=m}^{\infty} k\, |F_{k+2} - F_k|$$
$$\leq \sum_{k=1}^{\infty} k\, |F_{k+2} - F_k| \sum_{m=1}^{\infty} \eta(m) < \infty.$$

3) By taking the limit as $m \to \infty$ in (2.2.3), we obtain

$$\lim_{m\to\infty} F_m^{(1)} = 0.$$

On the other hand, from the form of the numbers z_j and (2.2.2), it follows that $\lim_{m\to\infty} F_m^{(0)} = 0$, so $\lim_{m\to\infty} F_m = 0$. Consequently, we can express F_m as a telescopic series

$$F_m = -\sum_{k=1}^{\infty} \left(F_{m+2k} - F_{m+2(k-1)}\right).$$

Thus, by considering (2.2.5),

$$|F_m| \le \sum_{k=1}^{\infty} \left|F_{m+2k} - F_{m+2(k-1)}\right| \le \sum_{k=m}^{\infty} |F_{k+2} - F_k| = \eta(m).$$

Now (2.2.9) follows from the inequalities (2.2.6), (2.2.7), and (2.2.8).

■

We introduce the notations

$$a^{(n)} := \begin{pmatrix} \frac{K_{n,n+1}}{K_{nn}} \\ \frac{K_{n,n+2}}{K_{nn}} \\ \frac{K_{n,n+3}}{K_{nn}} \\ \vdots \end{pmatrix}, \qquad f^{(n)} := \begin{pmatrix} F_{2n+1} \\ F_{2n+2} \\ F_{2n+3} \\ \vdots \end{pmatrix}, \qquad \mathcal{F}^{(n)} := \begin{pmatrix} F_{2n+2} & F_{2n+3} & F_{2n+4} & \cdots \\ F_{2n+3} & F_{2n+4} & F_{2n+5} & \cdots \\ F_{2n+4} & F_{2n+5} & F_{2n+6} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

We rewrite Equation (2.2.4) in operator form:

$$f^{(n)} + a^{(n)} + \mathcal{F}^{(n)} a^{(n)} = 0. \tag{2.2.10}$$

Let $\ell^p(1,\infty)$, for $p = 1, 2$, denote the normed linear space of infinite sequences $y = \{y_k\}_{k=1}^{\infty}$ in which the norm is defined by

$$\|y\|_p = \left(\sum_{k=1}^{\infty} |y_k|^p\right)^{\frac{1}{p}} < \infty. \tag{2.2.11}$$

The operator norm for $A = (a_{ij})_{i,j=1}^{\infty}$ in the spaces $\ell^p(1,\infty)$, for $p = 1, 2$, is defined by

$$\|A\|_1 = \sup_j \sum_{i=1}^{\infty} |a_{ij}|, \qquad \|A\|_2 = \left(\sum_{i,j=1}^{\infty} |a_{ij}|^2\right)^{\frac{1}{2}}. \tag{2.2.12}$$

Lemma 2.2.2. *For each fixed n, the operator $\mathcal{F}^{(n)}$ is a compact operator in $\ell^{1,2}(1,\infty)$.*

Proof.

From (2.2.12) and by virtue of (2.2.9), we have

$$\begin{aligned} \|\mathcal{F}^{(n)}\|_1 &\le \sup_{m\ge 1} \sum_{k=1}^{\infty} |F_{k+m+2n}| = \sup_{m\ge 1} \sum_{s=1+m+2n}^{\infty} |F_s| \le \sum_{s=2n+2}^{\infty} |F_s| < \infty, \\ \|\mathcal{F}^{(n)}\|_2 &\le \left(\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} F_{k+m+2n}^2\right)^{\frac{1}{2}} = \left(\sum_{k=1}^{\infty}\sum_{s=k}^{\infty} F_{s+2n+1}^2\right)^{\frac{1}{2}} \end{aligned} \tag{2.2.13}$$

$$= \left(\sum_{s=1}^{\infty} s\, F_{s+2n+1}^2 \right)^{\frac{1}{2}} = \left(\sum_{2n+2}^{\infty} (k-2n-1)\, F_k^2 \right)^{\frac{1}{2}} < \infty.$$

Thus, for each n, $\mathcal{F}^{(n)}$ is a bounded operator in $\ell^{1,2}(1,\infty)$. Let us now show that this operator is compact in $\ell^{1,2}(1,\infty)$. In fact, consider in $\ell^p(1,\infty)$, for $p = 1, 2$, the unit ball $\|y\|_p \le 1$, and denote by $\mathcal{K}$ the set of all ψ of the form

$$\psi = \mathcal{F}^{(n)}\, y, \qquad \|y\|_p \le 1.$$

We must prove that $\mathcal{K}$ is a compact set in $\ell^p(1,\infty)$. To do this, let us use the compactness criterion in $\ell^p(1,\infty)$; see [15]. Because $\mathcal{F}^{(n)}$ is bounded in $\ell^p(1,\infty)$, it immediately follows that $\mathcal{K}$ is bounded. Moreover, $\forall \varepsilon > 0$ and for $m \ge m_0$, where m_0 is a sufficiently large natural number, by virtue of (2.2.8) and (2.2.6), we obtain

$$\begin{aligned}
\sum_{i=m+1}^{\infty} |\psi_i| &= \sum_{i=m+1}^{\infty} \sum_{k=1}^{\infty} |F_{2n+i+k}\, y_k| \le \sum_{i=m+1}^{\infty} \sum_{k=1}^{\infty} \eta(2n+i+k)|y_k| \\
&\le \sum_{i=m+1}^{\infty} \eta(2n+i+1) \sum_{k=1}^{\infty} |y_k| \le \sum_{i=m+1}^{\infty} \eta(2n+i+1) = \sum_{s=2n+m+2}^{\infty} \eta(s) < \varepsilon, \\
\sum_{i=m+1}^{\infty} |\psi_i|^2 &\le \sum_{i=m+1}^{\infty} \sum_{k=1}^{\infty} \eta^2(2n+i+k)|y_k|^2 \le \sum_{i=m+1}^{\infty} \eta^2(2n+i+1) \sum_{k=1}^{\infty} |y_k|^2 \\
&\le \sum_{i=m+1}^{\infty} \eta^2(2n+i+1) = \sum_{s=2n+m+2}^{\infty} \eta^2(s) < \varepsilon.
\end{aligned}$$

■

Now consider the operator equation (2.2.10). By virtue of (2.2.9), $f^{(n)} \in \ell^{1,2}(1,\infty)$. Therefore, Equation (2.2.10) can be considered as an operator equation with a compact operator in $\ell^1(1,\infty)$ and $\ell^2(1,\infty)$. Let us prove that this equation in $\ell^{1,2}(1,\infty)$ has a unique solution. For this, it is sufficient to prove that the corresponding homogeneous equation in $\ell^{1,2}(1,\infty)$ has only the zero solution.

Theorem 2.2.1. *For each $n \ge 0$, the fundamental equation (2.2.4) has a unique solution in $\ell^{1,2}(1,\infty)$.*

Proof.

Let us consider the homogeneous equation

$$h_m + \sum_{k=1}^{\infty} h_k\, F_{k+m+2n} = 0, \qquad m \ge 1. \tag{2.2.14}$$

First, we will prove that this equation has only the zero solution in $\ell^2(1,\infty)$ for each $n \ge 0$. Let us assume the contrary. Suppose that for some n Equation (2.2.14) has a non-zero solution $h = \{h_m\}_{m=1}^{\infty}$ in $\ell^2(1,\infty)$. Since F_m is real, without loss of generality, we can assume that h_m is real. Extend h to the entire space $\ell^2(-\infty,\infty)$ by assuming $h_0 = h_{-1} = h_{-2} = \ldots = 0$, and denote

$$H(z) := \sum_{m=1}^{\infty} h_m\, z^m. \tag{2.2.15}$$

The function $H(z)$ admits an analytic extension in $|z| < 1$. By Parseval's equality,

$$\sum_{m=1}^{\infty} h_m^2 = \frac{1}{2i\pi} \oint_{\mathbf{T}} |H(z)|^2\, \frac{d\,z}{z}. \tag{2.2.16}$$

By multiplying Equation (2.2.14) by h_m and by summing over m from 1 to ∞,

$$\sum_{m=1}^{\infty} h_m^2 + \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} h_m\, h_k\, F_{k+m+2n} = 0.$$

By substituting F_m for its expression (2.2.1)-(2.2.3) and using (2.2.16), we obtain

$$\sum_{j=1}^{N} M_j^2\, z_j^{2n}\, H^2(z_j) + \frac{1}{2i\pi} \oint_{\mathbf{T}} \left(|H(z)|^2 - S(z)\, z^{2n} H^2(z) \right) \frac{d\,z}{z} = 0.$$

Therefore, by virtue of (2.1.1),

$$\sum_{j=1}^{N} M_j^2\, z_j^{2n}\, H^2(z_j) + \frac{1}{4i\pi} \oint_{\mathbf{T}} \left| H(z^{-1}) - S(z)\, z^{2n} H(z) \right|^2 \frac{d\,z}{z} = 0.$$

Since all terms in this expression are positive, and by considering the continuity and periodicity of the function under the integral, we conclude that

$$H(z_j) = 0, \qquad j = 1, \ldots, N, \tag{2.2.17}$$

$$H(z^{-1}) - S(z)\, z^{2n} H(z) = 0, \qquad |z| = 1. \tag{2.2.18}$$

Let us set

$$\mathcal{H}(z) := z^n H(z); \tag{2.2.19}$$

from (2.2.17) and (2.2.18), we then obtain

$$\mathcal{H}(z_j) = 0, \qquad j = 1, \ldots, N, \tag{2.2.20}$$

$$\mathcal{H}(z^{-1}) - S(z)\mathcal{H}(z) = 0, \qquad |z| = 1. \tag{2.2.21}$$

Since the function $\mathcal{H}(z)$, because of (2.2.15) and (2.2.19) admits an analytic extension to $|z| < 1$ and $\mathcal{H}(z) \to 0$ as $z \to 0$ and is periodic with a period of 2π,

$$\mathcal{H}(z) = \sum_{m=1}^{\infty} \widetilde{h}_m\, z^m, \qquad \{\widetilde{h}_m\}_{m=1}^{\infty} \in \ell^2(1, \infty). \tag{2.2.22}$$

From (2.2.21) and (2.2.22), it follows that $\{\widetilde{h}_m\}_{m=1}^{\infty}$ is a solution to Equation (2.1.4) in $\ell^2(1, \infty)$. Moreover, consider the functions

$$\mathcal{H}_1(z) = \mathcal{H}(z), \quad \mathcal{H}_{j+1}(z) = -\frac{4\,\mathcal{H}(z)}{(z + z_j)(z - z_j) + (z^{-1} + z_j^{-1})(z^{-1} - z_j^{-1})} \tag{2.2.23}$$

for $j = 1, \ldots, N$. Since $\mathcal{H}(z)$ is analytic in $|z| < 1$, from (2.2.23) and (2.2.20) it follows that the functions $\mathcal{H}_j(z)$, $j = 1, \ldots, N+1$ are holomorphic in $|z| < 1$. Since these functions are periodic with a period of 2π and tend to be zero as $z \to 0$, then

$$\mathcal{H}_j(z) = \sum_{m=1}^{\infty} h_m^{(j)}\, z^m, \qquad \{h_m^{(j)}\}_{m=1}^{\infty} \in \ell^2(1,\infty), \qquad j = 1,\dots,N.$$

The functions $\mathcal{H}_j(z)$ thanks to (2.2.23) and (2.2.21) satisfy the equality

$$\mathcal{H}_j(z^{-1}) - S(z)\mathcal{H}_j(z) = 0, \qquad |z| = 1, \quad j = 1,\dots,N+1,$$

from which it follows that $\{h_m^{(j)}\}_{m=1}^{\infty}$, $j = 1,\dots,N+1$ are solutions to Equation (2.1.4). These solutions are linearly independent if $\mathcal{H}(z) \not\equiv 0$. Because of III, though, Equation (2.1.4) has exactly N linearly independent solutions in $\ell^2(1,\infty)$. Consequently,

$$\mathcal{H}(z) = z^n\, H(z) \equiv 0,$$

from which $H(z) \equiv 0$: $h_m \equiv 0$. Thus, the homogeneous Equation (2.2.14) for each n in $\ell^2(1,\infty)$ has only the zero solution. Since every solution of this equation in $\ell^1(1,\infty)$ belongs to $\ell^2(1,\infty)$, it follows that it also has only the zero solution in $\ell^1(1,\infty)$. ■

Corollary 2.2.1. *The operator $I + \mathcal{F}^{(n)}$ for each $n \geq 0$ in $\ell^{1,2}(1,\infty)$ has a bounded inverse operator.*

2.3 Important estimates

Theorem 2.3.1. *For $K_{n,n+m}/K_{nn}$, the following estimates hold:*

1.
$$\left|\frac{K_{n,n+m}}{K_{nn}}\right| \leq c\,\eta(2n+m), \qquad n \geq 0, \quad m \geq 1, \tag{2.3.1}$$

2.
$$\left|\left(\frac{K_{n,n+m}}{K_{nn}} - \frac{K_{n-1,n+m-1}}{K_{n-1,n-1}}\right) + \left(F_{2n+m} - F_{2(n-1)+m}\right)\right| \leq c\,\eta(2n-1)\,\eta(m+2n-1), \qquad n,m \geq 1, \tag{2.3.2}$$

3.
$$\left|\left(\frac{K_{n,n+1}}{K_{nn}} - \frac{K_{n-1,n}}{K_{n-1,n-1}}\right) + (F_{2n+1} - F_{2n-1})\right| \leq c\,\eta(2n-1)\,\eta(2n), \qquad n \geq 1, \tag{2.3.3}$$

4.
$$\left|\left(\frac{K_{n,n+2}}{K_{nn}} - \frac{K_{n-1,n+1}}{K_{n-1,n-1}}\right) + (F_{2n+2} - F_{2n})\right| \leq c\,\eta(2n-1)\,\eta(2n+1), \qquad n \geq 1, \tag{2.3.4}$$

where c is a positive number independent of n and m and $\eta(n)$ is defined in (2.2.5).

Proof.

1. By virtue of Corollary 2.2.1, the operator $\left(I+\mathcal{F}^{(n)}\right)^{-1}$ exists for each $n\geq 0$ and is bounded in $\ell^1(1,\infty)$. We will show that this family of operators is uniformly bounded with respect to $n\geq 0$:

$$\| \left(I+\mathcal{F}^{(n)}\right)^{-1} \|_1 \leq c, \qquad n\geq 0. \tag{2.3.5}$$

In fact, by virtue of (2.2.13), $\|\mathcal{F}^{(n)}\|_1\to 0$ as $n\to\infty$. Consequently, there exists a natural number n_0 such that for $n>n_0$ we have $\|\mathcal{F}^{(n)}\|_1\leq \frac{1}{2}$ and consequently

$$\| \left(I+\mathcal{F}^{(n)}\right)^{-1} \|_1 \leq \frac{1}{1-\|\mathcal{F}^{(n)}\|_1} < 2, \qquad n>n_0.$$

There remains only a finite number of operators $\left(I+\mathcal{F}^{(n)}\right)^{-1}$, $0\leq n\leq n_0$, each of which is bounded. From (2.2.10), we obtain

$$a^{(n)} = -\left(I+\mathcal{F}^{(n)}\right)^{-1} f^{(n)}.$$

Therefore,

$$\|a^{(n)}\|_1 = \| \left(I+\mathcal{F}^{(n)}\right)^{-1} \|_1 \, \|f^{(n)}\|_1,$$

or, by taking into account (2.2.11) and (2.3.5), (2.2.9)

$$\sum_{m=1}^{\infty} \left|\frac{K_{n,n+m}}{K_{nn}}\right| \leq c\sum_{m=1}^{\infty}|F_{2n+m}| = c\sum_{s=2n+1}^{\infty}|F_s| \leq \sum_{s=1}^{\infty}|F_s| = c_1<\infty. \tag{2.3.6}$$

Thus, from the fundamental equation (2.2.4) and by considering (2.2.8), we obtain

$$\left|\frac{K_{n,n+m}}{K_{nn}}\right| \leq \eta(2n+m) + \sum_{k=1}^{\infty}\left|\frac{K_{n,n+k}}{K_{nn}}\right| \eta(k+m+2n) \leq \eta(2n+m)$$
$$+\eta(1+m+2n)\sum_{k=1}^{\infty}\left|\frac{K_{n,n+k}}{K_{nn}}\right| \leq c_2\,\eta(2n+m).$$

In other words, inequality (2.3.1) holds.

2. From the fundamental equation (2.2.4), we have

$$\left(\frac{K_{n,n+m}}{K_{nn}} - \frac{K_{n-1,n+m-1}}{K_{n-1,n-1}}\right) + \left(F_{2n+m}-F_{2(n-1)+m}\right) + \sum_{k=1}^{\infty}\frac{K_{n,n+k}}{K_{nn}}F_{k+m+2n}$$
$$-\sum_{k=1}^{\infty}\frac{K_{n-1,n+k-1}}{K_{n-1,n-1}}F_{k+m+2(n-1)} = 0, \qquad n\geq 1.$$

By subtracting and adding $\sum_{k=1}^{\infty}K_{n,n+k}/K_{nn}\,F_{k+m+2(n-1)}$, we obtain

$$\left(\frac{K_{n,n+m}}{K_{nn}} - \frac{K_{n-1,n+m-1}}{K_{n-1,n-1}}\right)+\left(F_{2n+m}-F_{2(n-1)+m}\right)+\sum_{k=1}^{\infty}\frac{K_{n,n+k}}{K_{nn}}\left(F_{k+m+2n}-F_{k+m+2(n-1)}\right)$$
$$+\sum_{k=1}^{\infty}\left(\frac{K_{n,n+k}}{K_{nn}}-\frac{K_{n-1,n+k-1}}{K_{n-1,n-1}}\right)F_{k+m+2(n-1)} = 0, \qquad n\geq 1. \tag{2.3.7}$$

By denoting

$$\hat{a}^{(n)} = \left\{ \frac{K_{n,n+m}}{K_{nn}} - \frac{K_{n-1,n+m-1}}{K_{n-1,n-1}} \right\}_{m=1}^{\infty}, \tag{2.3.8}$$

$$\hat{f}^{(n)} = \left\{ \left(F_{2n+m} - F_{2(n-1)+m}\right) + \sum_{k=1}^{\infty} \frac{K_{n,n+k}}{K_{nn}} \left(F_{k+m+2n} - F_{k+m+2(n-1)}\right) \right\}_{m=1}^{\infty} \tag{2.3.9}$$

for $n \geq 1$. We rewrite (2.3.7) in the form of an operator equation:

$$\hat{a}^{(n)} + \hat{f}^{(n)} + \mathcal{F}^{(n-1)}\, \hat{a}^{(n)} = 0, \qquad n \geq 1.$$

Therefore,

$$\hat{a}^{(n)} = -\left(I + \mathcal{F}^{(n-1)}\right)^{-1} \hat{f}^{(n)},$$

and, by considering (2.3.5), we have

$$\|\hat{a}^{(n)}\| \leq \| \left(I + \mathcal{F}^{(n-1)}\right)^{-1} \|_1 \, \|\hat{f}^{(n)}\|_1 \leq c\, \|\hat{f}^{(n)}\|_1.$$

Hence, by virtue of notations (2.3.8) and (2.3.9), we obtain

$$\sum_{m=1}^{\infty} \left| \frac{K_{n,n+m}}{K_{nn}} - \frac{K_{n-1,n+m-1}}{K_{n-1,n-1}} \right| \leq c \left(\sum_{m=1}^{\infty} \left| F_{2n+m} - F_{2(n-1)+m} \right| + \right.$$
$$\left. + \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \left| \frac{K_{n,n+k}}{K_{nn}} \right| \left| F_{k+m+2n} - F_{k+m+2(n-1)} \right| \right) \tag{2.3.10}$$

for $n \geq 1$. On the other hand,

$$\sum_{m=1}^{\infty} \left| F_{2n+m} - F_{2(n-1)+m} \right| = \sum_{s=2n-1}^{\infty} |F_{s+2} - F_s| = \eta(2n-1), \tag{2.3.11}$$

and, through (2.3.1), we have

$$\sum_{k=1}^{\infty} \left| \frac{K_{n,n+k}}{K_{nn}} \right| \left| F_{k+m+2n} - F_{k+m+2(n-1)} \right| \leq c \sum_{k=1}^{\infty} \eta(2n+k) \left| F_{k+m+2n} - F_{k+m+2(n-1)} \right|$$
$$\leq c\, \eta(2n+1) \sum_{s=m+2n-1}^{\infty} |F_{s+2} - F_s| = c\, \eta(2n+1)\, \eta(m+2n-1). \tag{2.3.12}$$

By substituting (2.3.11) and (2.3.12) into (2.3.10) and by virtue of (2.2.6), we obtain

$$\sum_{m=1}^{\infty} \left| \frac{K_{n,n+m}}{K_{nn}} - \frac{K_{n-1,n+m-1}}{K_{n-1,n-1}} \right| \leq c \left(\eta(2n-1) + c\, \eta(2n+1) \sum_{m=1}^{\infty} \eta(m+2n-1) \right)$$
$$= c \left(\eta(2n-1) + c\, \eta(2n+1) \sum_{s=2n}^{\infty} \eta(s) \right) \leq c_1\, \eta(2n-1), \qquad n \geq 1. \tag{2.3.13}$$

Now, from (2.3.7) and by taking into account (2.3.1), (2.2.8), and (2.3.13), we attain

$$\sum_{m=1}^{\infty}\left|\left(\frac{K_{n,n+m}}{K_{nn}}-\frac{K_{n-1,n+m-1}}{K_{n-1,n-1}}\right)+\left(F_{2n+m}-F_{2(n-1)+m}\right)\right|$$

$$\le c\sum_{k=1}^{\infty}\eta(2n+k)\,\left|F_{k+m+2n}-F_{k+m+2(n-1)}\right|+\sum_{k=1}^{\infty}\left|\frac{K_{n,n+k}}{K_{nn}}-\frac{K_{n-1,n+k-1}}{K_{n-1,n-1}}\right|\eta(k+m+2n-2)$$

$$\le c\,\eta(2n+1)\sum_{s=m+2n-1}^{\infty}|F_{s+2}-F_s|+\eta(m+2n-1)\sum_{k=1}^{\infty}\left|\frac{K_{n,n+k}}{K_{nn}}-\frac{K_{n-1,n+k-1}}{K_{n-1,n-1}}\right|$$

$$\le c\,\eta(2n+1)\,\eta(m+2n-1)+c_1\,\eta(m+2n-1)\,\eta(2n-1)\le c_2\,\eta(2n-1)\,\eta(m+2n-1),$$

and inequality (2.3.2) holds.

By substituting (2.3.2) for $m=1$ and for $m=2$, we obtain inequalities (2.3.3) and (2.3.4), respectively. ■

2.4 Construction of the Jost solution

Theorem 2.4.1. *The following inequality holds:*

$$1+F_{2n}+\sum_{k=1}^{\infty}\frac{K_{n,n+k}}{K_{nn}}F_{k+2n}>0,\qquad n\ge 0,\tag{2.4.1}$$

where $K_{n,n+k}/K_{nn}$ is the solution to the fundamental equation (2.2.4).

Proof.

Let us denote

$$A_n(z):=\sum_{m=1}^{\infty}\frac{K_{n,n+m}}{K_{nn}}\,z^m,\qquad n\ge 0.\tag{2.4.2}$$

Because of Theorem 2.2.1, $\left\{\frac{K_{n,n+m}}{K_{nn}}\right\}_{m=1}^{\infty}\in\ell^{1,2}(1,\infty)$ and the function $A_n(z)$ is defined and admits an analytic extension in the $|z|<1$. By considering that $K_{n,n+m}/K_{nn}=0$ for $m\le 0$, from (2.4.2) and by virtue of the usual Parseval equality, we have

$$\sum_{m=1}^{\infty}\left(\frac{K_{n,n+m}}{K_{nn}}\right)^2=\frac{1}{2i\pi}\oint_{\mathbf{T}}|A_n(z)|^2\,\frac{d\,z}{z}.\tag{2.4.3}$$

Let us multiply both sides of (2.2.4) by $K_{n,n+m}/K_{nn}$ and sum over m from 1 to ∞:

$$\sum_{m=1}^{\infty}\frac{K_{n,n+m}}{K_{nn}}F_{m+2n}+\sum_{m=1}^{\infty}\left(\frac{K_{n,n+m}}{K_{nn}}\right)^2+\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{K_{n,n+m}}{K_{nn}}\frac{K_{n,n+k}}{K_{nn}}F_{k+m+2n}=0,\qquad n\ge 0.$$

We substitute in the third sum of this equality its expression (2.2.1)-(2.2.3) instead of F_m and additionally use (2.4.2) and (2.4.3) to obtain

$$\sum_{m=1}^{\infty}\frac{K_{n,n+m}}{K_{nn}}F_{m+2n}=-\sum_{j=1}^{N}M_j^2\,z_j^{2n}A_n^2(z_j)-\frac{1}{2i\pi}\oint_{\mathbf{T}}\left(|A_n(z)|^2-S(z)\,z^{2n}A_n^2(z)\right)\frac{d\,z}{z}.\tag{2.4.4}$$

On the other hand, by using (2.2.1)-(2.2.3), we have

$$\sum_{m=1}^{\infty} \frac{K_{n,n+m}}{K_{nn}} F_{m+2n} = \sum_{j=1}^{N} M_j^2 z_j^{2n} A_n(z_j) - \frac{1}{2i\pi} \oint_{\mathbf{T}} S(z) z^{2n} A_n(z) \frac{d\,z}{z}, \quad n \geq 0. \tag{2.4.5}$$

By multiplying both sides of (2.4.5) by 2 and by subtracting (2.4.4) from the resulting equation, we obtain

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{K_{n,n+m}}{K_{nn}} F_{m+2n} &= \sum_{j=1}^{N} M_j^2 z_j^{2n} \left(2A_n(z_j) + A_n^2(z_j)\right) \\ &+ \frac{1}{2i\pi} \oint_{\mathbf{T}} \left(-2\, S(z) z^{2n} A_n(z) + |A_n(z)|^2 - S(z) z^{2n} A_n^2(z)\right) \frac{d\,z}{z}, \quad n \geq 0. \end{aligned} \tag{2.4.6}$$

Moreover,

$$1 + F_{2n} = \sum_{j=1}^{N} M_j^2 z_j^{2n} + \frac{1}{2i\pi} \oint_{\mathbf{T}} \left(1 + [1 - S(z)] z^{2n}\right) \frac{d\,z}{z}. \tag{2.4.7}$$

By adding (2.4.6) and (2.4.7), we attain

$$\begin{aligned} 1 + F_{2n} + \sum_{m=1}^{\infty} \frac{K_{n,n+m}}{K_{nn}} F_{m+2n} &= \sum_{j=1}^{N} M_j^2 z_j^{2n} \left(1 + A_n(z_j)\right)^2 \\ &+ \frac{1}{2i\pi} \int_{\mathbf{T}} \left(1 + [1 - S(z)] z^{2n} - 2S(z) z^{2n} A_n(z) + |A_n(z)|^2 - S(z) z^{2n} A_n^2(z)\right) \frac{d\,z}{z}, \\ &= \sum_{j=1}^{N} M_j^2 z_j^{2n} \left(1 + A_n(z_j)\right)^2 + \frac{1}{2i\pi} \oint_{\mathbf{T}} \frac{z^{2n}}{z} d\,z \\ &+ \frac{1}{2i\pi} \int_{\mathbf{T}} \left(1 - S(z) z^{2n} - 2S(z) z^{2n} A_n(z) + |A_n(z)|^2 - S(z) z^{2n} A_n^2(z)\right) \frac{d\,z}{z}, \quad n \geq 0. \end{aligned} \tag{2.4.8}$$

By considering

$$\begin{aligned} &\left|1 - S(z)\, z^{2n} + A_n(z^{-1}) - S(z)\, z^{2n}\, A_n(z)\right|^2 \\ &= 1 - S(z^{-1})\, z^{-2n} + A_n(z) - S(z^{-1})\, z^{-2n}\, A_n(z^{-1}) - S(z)\, z^{2n} + 1 - S(z)\, z^{2n}\, A_n(z) + A_n(z^{-1}) \\ &\quad + A_n(z^{-1}) - S(z^{-1})\, z^{-2n}\, A_n(z^{-1}) + |A_n(z)|^2 - S(z^{-1})\, z^{-2n}\, A_n^2(z^{-1}) \\ &\quad - S(z)\, z^{2n}\, A_n(z) + A_n(z) - S(z)\, z^{2n}\, A_n^2(z) + |A_n(z)|^2. \end{aligned}$$

Equality (2.4.8) is rewritten in the form

$$\begin{aligned} 1 + F_{2n} + \sum_{m=1}^{N} \frac{K_{n,n+m}}{K_{nn}} F_{m+2n} &= \sum_{j=1}^{\infty} M_j^2 \left(1 + A_n(z_j)\right)^2 + \frac{1}{2i\pi} \oint_{\mathbf{T}} \frac{z^{2n}}{z} d\,z \\ &+ \frac{1}{4i\pi} \oint_{\mathbf{T}} \left|1 - S(z)\, z^{2n} + A_n(z^{-1}) - S(z)\, z^{2n} A_n(z)\right|^2 \frac{d\,z}{z}. \end{aligned} \tag{2.4.9}$$

From the form of the numbers z_j and Equation (2.4.9), Equation (2.4.1) follows. ■

By using the solution $K_{n,n+m}/K_{nn}$ of the fundamental equation (2.2.4), we construct K_{nn},

a_n, b_n, and ϱ_n as follows:

$$\frac{1}{K_{nn}^2} := 1 + F_{2n} + \sum_{k=1}^{\infty} \frac{K_{n,n+k}}{K_{nn}} F_{k+2n}, \qquad n \geq 0, \tag{2.4.10}$$

$$a_n := \frac{K_{nn}}{K_{n-1,n-1}}, \qquad n \geq 1, \tag{2.4.11}$$

$$b_n := \frac{K_{n,n+1}}{K_{nn}} - \frac{K_{n-1,n}}{K_{n-1,n-1}}, \qquad n \geq 1, \tag{2.4.12}$$

$$\varrho_n := 1 + \frac{K_{n,n+2}}{K_{nn}} - \frac{K_{n-1,n+1}}{K_{n-1,n-1}} - b_n \frac{K_{n,n+1}}{K_{nn}}, \qquad n \geq 1. \tag{2.4.13}$$

By virtue of (2.4.1) and (2.4.10), we have $K_{nn} > 0$ for $n \geq 0$. Since $K_{n,n+m}/K_{nn}$ is real, then b_n and ϱ_n, which are defined by (2.4.12) and (2.4.13), respectively, are also real. Furthermore, from (2.4.10) and (2.4.11), we have $a_n > 0$. Let us prove that the inequality

$$\sum_{n=1}^{\infty} n \left(|1 - \varrho_n| + |b_n| \right) < \infty \tag{2.4.14}$$

is valid. In fact, from (2.4.12) and by considering (2.3.3), we have

$$\begin{aligned}
\sum_{n=1}^{\infty} n\,|b_n| &= \sum_{n=1}^{\infty} n \left| \frac{K_{n,n+1}}{K_{nn}} - \frac{K_{n-1,n}}{K_{n-1,n-1}} \right| \\
&= \sum_{n=1}^{\infty} n \left| \left(\frac{K_{n,n+1}}{K_{nn}} - \frac{K_{n-1,n}}{K_{n-1,n-1}} \right) + (F_{2n+1} - F_{2n-1}) - (F_{2n+1} - F_{2n-1}) \right| \\
&\leq \sum_{n=1}^{\infty} n \left| \left(\frac{K_{n,n+1}}{K_{nn}} - \frac{K_{n-1,n}}{K_{n-1,n-1}} \right) + (F_{2n+1} - F_{2n-1}) \right| + \sum_{n=1}^{\infty} n\,|F_{2n+1} - F_{2n-1}| \\
&\leq c \sum_{n=1}^{\infty} n\,\eta(2n-1)\,\eta(2n) + \sum_{n=1}^{\infty} n\,|F_{2n+1} - F_{2n-1}| < \infty
\end{aligned} \tag{2.4.15}$$

because, by virtue of condition II and (2.2.1),

$$\sum_{n=1}^{\infty} n\,|F_{2n+1} - F_{2n-1}| \leq \sum_{n=1}^{\infty} (2n-1)\,|F_{2n+1} - F_{2n-1}| < \infty,$$

and, by (2.2.7),

$$\sum_{n=1}^{\infty} n\,\eta(2n-1)\,\eta(2n) \leq \sum_{n=1}^{\infty} (2n-1)\,\eta^2(2n-1) < \infty.$$

Additionally, from (2.4.13), we have

$$\begin{aligned}
\sum_{n=1}^{\infty} n\,|1 - \varrho_n| &= \sum_{n=1}^{\infty} n \left| \frac{K_{n,n+2}}{K_{nn}} - \frac{K_{n-1,n+1}}{K_{n-1,n-1}} - b_n \frac{K_{n,n+1}}{K_{nn}} \right| \\
&\leq \sum_{n=1}^{\infty} n \left| \frac{K_{n,n+2}}{K_{nn}} - \frac{K_{n-1,n+1}}{K_{n-1,n-1}} \right| + \sum_{n=1}^{\infty} n\,|b_n| \left| \frac{K_{n,n+1}}{K_{nn}} \right|.
\end{aligned}$$

On the other hand, by virtue of (2.3.4), (2.2.7), and condition II,

$$\begin{aligned}\sum_{n=1}^{\infty} n\left|\frac{K_{n,n+2}}{K_{nn}}-\frac{K_{n-1,n+1}}{K_{n-1,n-1}}\right| &= \sum_{n=1}^{\infty} n\left|\left(\frac{K_{n,n+2}}{K_{nn}}-\frac{K_{n-1,n+1}}{K_{n-1,n-1}}\right)+(F_{2n+2}-F_{2n})-(F_{2n+2}-F_{2n})\right| \\ &\le \sum_{n=1}^{\infty} n\left|\left(\frac{K_{n,n+2}}{K_{nn}}-\frac{K_{n-1,n+1}}{K_{n-1,n-1}}\right)+(F_{2n+2}-F_{2n})\right|+\sum_{n=1}^{\infty} n\,|F_{2n+2}-F_{2n}| \\ &\le c\sum_{n=1}^{\infty} n\,\eta(2n-1)\,\eta(2n+1)+\sum_{n=1}^{\infty} n\,|F_{2n+2}-F_{2n}|<\infty.\end{aligned}$$

Because of (2.3.1) and (2.4.15),

$$\sum_{n=1}^{\infty} n\,|b_n|\left|\frac{K_{n,n+1}}{K_{nn}}\right| \le c\sum_{n=1}^{\infty} n\,|b_n|\,\eta(2n+1) \le c\,\eta(3)\sum_{n=1}^{\infty} n\,|b_n|<\infty.$$

Consequently,

$$\sum_{n=1}^{\infty} n\,|1-\varrho_n|<\infty. \tag{2.4.16}$$

From (2.4.15) and (2.4.16), the Inequality (2.4.14) follows. From (2.4.10), it follows that

$$\lim_{n\to\infty} K_{nn}=1. \tag{2.4.17}$$

Now we define the function $f_n(z)$ as

$$f_n(z) := K_{nn}\,z^n\left(1+\sum_{m=1}^{\infty}\frac{K_{n,n+m}}{K_{nn}}\,z^m\right), \qquad n\ge 0. \tag{2.4.18}$$

Theorem 2.4.2. *The function $f_n(z)$ has the following properties.*

1. *$f_n(z)$ is continuous on $|z|=1$ and allows an analytic extension to $|z|<1$ given by*

$$f_n(z)=K_{nn}\,z^n\left(1+\sum_{m=1}^{\infty}\frac{K_{n,n+m}}{K_{nn}}\,z^m\right), \qquad |z|<1, \tag{2.4.19}$$

and, for each fixed $n\ge 0$, $f_n(z)$ has the asymptotic behavior

$$f_n(z)=K_{nn}\,z^n\,[1+o(1)], \qquad z\to 0, \tag{2.4.20}$$

and this estimate is uniform with respect to $n\ge 0$.

2. *For $|z|<1$ with respect to n, $f_n(z)$ has the asymptotic behavior*

$$f_n(z)=z^n\,[1+o(1)], \quad n\to\infty, \tag{2.4.21}$$

with $o(1)\to 0$ as $n\to\infty$, which is uniform with respect to z.

Proof.

By virtue of (2.3.1) and (2.2.6), we have

$$\sum_{m=1}^{\infty}\left|\frac{K_{n,n+m}}{K_{nn}}\right| \le c\sum_{m=1}^{\infty}\eta(2n+m)=c\sum_{s=2n+1}^{\infty}\eta(s)\le c\sum_{s=1}^{\infty}\eta(s)<\infty. \tag{2.4.22}$$

From (2.4.22) and (2.4.18), it follows that $f_n(z)$ is continuous on $|z| = 1$ and, by (2.4.19), admits an analytic extension inside $|z| < 1$. Clearly,

$$\left|\sum_{m=1}^{\infty} \frac{K_{n,n+m}}{K_{nn}} z^m\right| \leq \sum_{m=1}^{\infty} \left|\frac{K_{n,n+m}}{K_{nn}}\right| |z^m| = \sum_{m=1}^{\infty} \left|\frac{K_{n,n+m}}{K_{nn}}\right| |z|^m \leq |z| \sum_{m=1}^{\infty} \left|\frac{K_{n,n+m}}{K_{nn}}\right|; \quad (2.4.23)$$

therefore, the asymptotic relation (2.4.20) follows from (2.4.19) and (2.4.23). From (2.4.19),

$$f_n(z) = z^n \left[1 + (K_{nn} - 1) + K_{nn} \sum_{m=1}^{\infty} \frac{K_{n,n+m}}{K_{nn}} z^m\right]. \quad (2.4.24)$$

Because of (2.4.17), $K_{nn} \to 1$ as $n \to \infty$ and for $|z| \leq 1$

$$\left|\sum_{m=1}^{\infty} \frac{K_{n,n+m}}{K_{nn}} z^m\right| \leq \sum_{m=1}^{\infty} \left|\frac{K_{n,n+m}}{K_{nn}}\right| \leq c \sum_{s=2n+1}^{\infty} \eta(s) \to 0$$

as $n \to \infty$. The asymptotic relation (2.4.21) follows from (2.4.24). ■

2.5 Derivation of the equation for the Jost solution

Through the fundamental equation (2.2.4), the lemma below can be proven.

Lemma 2.5.1. *The following equality holds:*

$$\varrho_n \frac{K_{n+1,n+m+1}}{K_{n+1,n+1}} - \frac{K_{n,n+m}}{K_{nn}} + b_n \frac{K_{n,n+m+1}}{K_{nn}} + \frac{K_{n-1,n+m+1}}{K_{n-1,n-1}} - \frac{K_{n,n+m+2}}{K_{nn}} = 0, \qquad n, m \geq 1, \quad (2.5.1)$$

where ϱ_n and b_n are defined by (2.4.13) and (2.4.12), respectively.

Theorem 2.5.1. *The function $f_n(z)$ defined by (2.4.18) satisfies the following equality:*

$$a_{n+1}\, f_{n+1}(z) + b_n\, f_n(z) + a_n\, f_{n-1}(z) = (z + z^{-1})\, f_n(z), \qquad n \geq 1. \quad (2.5.2)$$

Furthermore, the equation

$$\varrho_n = \frac{K_{n+1,n+1}^2}{K_{nn}^2}, \qquad n \geq 1 \quad (2.5.3)$$

holds, where K_{nn}, a_n, b_n, and ϱ_n are defined by (2.4.10)-(2.4.13), respectively.

Proof.

By using (2.5.1) with a direct substitution, it is verified that $\{f_n(z)\}_{n=0}^{\infty}$ defined by formula (2.4.18) satisfies the equality

$$\frac{K_{nn}}{K_{n-1,n-1}}\, f_{n-1}(z) + b_n\, f_n(z) + \varrho_n \frac{K_{nn}}{K_{n+1,n+1}}\, f_{n+1}(z) = (z + z^{-1})\, f_n(z), \qquad n \geq 1. \quad (2.5.4)$$

Moreover, by substituting (2.2.1)-(2.2.3) into (2.2.4), we obtain

$$F_{2n+m}^{(1)} + \frac{K_{n,n+m}}{K_{nn}} + \sum_{k=1}^{\infty} \frac{K_{n,n+k}}{K_{nn}}\, F_{k+m+2n}^{(1)} + \sum_{j=1}^{N} M_j^2\, z_j^{2n+m} \left(1 + \sum_{k=1}^{\infty} \frac{K_{n,n+k}}{K_{nn}}\, z_j^k\right) = 0, \quad n \geq 0, \quad m \geq 1, \quad (2.5.5)$$

where

$$F_m^{(1)} = \frac{1}{2i\pi} \oint_{\mathbf{T}} [1 - S(z)]\, z^{m-1}\, d\, z. \tag{2.5.6}$$

By taking into account (2.4.19), Equation (2.5.5) can be rewritten as the formula

$$F_{2n+m}^{(1)} + \frac{K_{n,n+m}}{K_{nn}} + \sum_{k=1}^{\infty} \frac{K_{n,n+k}}{K_{nn}}\, F_{k+m+2n}^{(1)} + \sum_{j=1}^{N} \frac{1}{K_{nn}} M_j^2\, z_j^n\, z_j^m\, f_n(z_j) = 0, \quad n \geq 0, \quad m \geq 1. \tag{2.5.7}$$

Let us define

$$G_{nm} := F_{2n+m}^{(1)} + \frac{K_{n,n+m}}{K_{nn}} + \sum_{k=1}^{\infty} \frac{K_{n,n+k}}{K_{nn}}\, F_{k+m+2n}^{(1)}, \qquad m \in \mathbb{Z}, \quad n \geq 0. \tag{2.5.8}$$

By comparing (2.5.7) and (2.5.8) and realizing that $K_{n,n+m}/K_{nn} = 0$ for $m \leq 0$, we obtain

$$G_{nm} = \begin{cases} -\sum_{j=1}^{N} \frac{1}{K_{nn}}\, M_j^2\, z_j^n\, z_j^m\, f_n(z_j) & \text{for} \quad m \geq 1, \\ F_{2n+m}^{(1)} + \sum_{k=1}^{\infty} \frac{K_{n,n+k}}{K_{nn}}\, F_{k+m+2n}^{(1)} & \text{for} \quad m \leq 0, \quad n \geq 0, \end{cases} \tag{2.5.9}$$

from which we can obtain

$$1 + G_{n_0} + \sum_{j=1}^{N} \frac{1}{K_{nn}}\, M_j^2\, z_j^n\, f_n(z_j) = 1 + F_{2n} + \sum_{k=1}^{\infty} \frac{K_{n,n+k}}{K_{nn}}\, F_{k+2n} = \frac{1}{K_{nn}^2}. \tag{2.5.10}$$

Additionally, a function of the form

$$\psi_n(z) = \frac{1}{K_{nn}}\, z^{-n} \left(1 + \sum_{-\infty}^{m=-1} A_{nm}\, z^{-m} \right), \quad \{A_{nm}\}_{-\infty}^{m=-1} \in \ell^2(-\infty, -1) \qquad n \geq 0 \tag{2.5.11}$$

can be constructed to satisfy (2.5.4):

$$\frac{K_{nn}}{K_{n-1,n-1}}\, \psi_{n-1}(z) + b_n\, \psi_n(z) + \varrho_n\, \frac{K_{nn}}{K_{n+1,n+1}}\, \psi_{n+1}(z) = (z + z^{-1})\, \psi_n(z), \qquad n \geq 1. \tag{2.5.12}$$

By substituting (2.5.11) into (2.5.12) and comparing the coefficients of z on both sides of the equation, we obtain

$$\varrho_n\, \frac{K_{nn}}{K_{n+1,n+1}^2} = \frac{1}{K_{nn}}, \qquad n \geq 1,$$

which leads to

$$\varrho_n = \frac{K_{n+1,n+1}^2}{K_{nn}^2}, \qquad n \geq 1. \tag{2.5.13}$$

Thus, the Equality (2.5.3) holds. By substituting (2.5.13) into (2.5.4) and taking into account (2.4.11), we obtain the equality (2.5.2). ■

Corollary 2.5.1. *From* (2.4.14), (2.5.3), *and* (2.5.4), *it follows that*

$$\sum_{n=1}^{\infty} n\, (|1 - a_n| + |b_n|) < \infty.$$

2.6 Derivation of the boundary condition

Theorem 2.6.1. *The following equalities hold:*

$$f_0(z_j) = 0, \qquad j = 1, \dots, N, \tag{2.6.1}$$

$$f_0(z^{-1}) - S(z) f_0(z) = 0, \qquad |z| = 1. \tag{2.6.2}$$

Proof.

1. By the fundamental equation (2.2.4) for $n = 0$, we have

$$F_m + \frac{K_{0m}}{K_{00}} + \sum_{k=1}^{\infty} \frac{K_{0k}}{K_{00}} F_{k+m} = 0, \qquad m \geq 1.$$

By substituting F_m by using its expression from Equations (2.2.1)-(2.2.3) and by applying Equation (2.4.19), we obtain

$$\sum_{j=1}^{N} \frac{1}{K_{00}} M_j^2 \, z_j^m \, f_0(z_j) + F_m^{(1)} + \frac{K_{0m}}{K_{00}} + \sum_{k=1}^{\infty} \frac{K_{0k}}{K_{00}} F_{k+m}^{(1)} = 0, \qquad m \geq 1. \tag{2.6.3}$$

By Condition III, the equation

$$h_m + \sum_{k=1}^{\infty} h_k \, F_{k+m}^{(1)} = 0, \qquad m \geq 1 \tag{2.6.4}$$

in $\ell^2(1, \infty)$ has exactly N linearly independent solutions $h^{(p)} = \{h_m^{(p)}\}_{m=1}^{\infty}$, $p = 1, \dots, N$. By extending these vectors to non-positive indices with zeros, we have

$$H_p(z) := \sum_{m=1}^{\infty} h_m^{(p)} \, z^m, \qquad p = 1, \dots, N. \tag{2.6.5}$$

The function $H_p(z)$ allows an analytic extension to $|z| < 1$. By multiplying both sides of Equation (2.6.3) by $h_m^{(p)}$, summing over m from 1 to ∞, and using Equation (2.6.5), we get

$$\sum_{j=1}^{N} \frac{1}{K_{00}} M_j^2 \, f_0(z_j) \, H_p(z_j) + \sum_{m=1}^{\infty} F_m^{(1)} \, h_m^{(p)} + \sum_{m=1}^{\infty} \frac{K_{0m}}{K_{00}} \left(h_m^{(p)} + \sum_{k=1}^{\infty} h_k^{(p)} \, F_{k+m}^{(1)} \right) = 0.$$

Since $\{h_m^{(p)}\}_{m=1}^{\infty}$ is a solution of Equation (2.6.4), from the last equality we obtain

$$\sum_{j=1}^{N} \frac{1}{K_{00}} M_j^2 \, f_0(z_j) \, H_p(z_j) + \sum_{m=1}^{\infty} F_m^{(1)} \, h_m^{(p)} = 0, \qquad p = 1, \dots, N. \tag{2.6.6}$$

Furthermore, since $\{h_m^{(p)}\}_{m=1}^{\infty}$ is a solution of Equation (2.6.4), we have

$$H_p(z^{-1}) - S(z) \, H_p(z) = 0, \qquad |z| = 1, \qquad p = 1, \dots, N.$$

Thus, by considering Equation (2.2.3), we obtain

$$\begin{aligned}\sum_{m=1}^{\infty} F_m^{(1)}\, h_m^{(p)} &= \sum_{m=1}^{\infty} \left(\frac{1}{2i\pi} \oint_{\mathbf{T}} [1 - S(z)]\; z^m\, \frac{d\,z}{z} \right) h_m^{(p)} \\ &= \frac{1}{2i\pi} \oint_{\mathbf{T}} [1 - S(z)]\; H_p(z)\, \frac{d\,z}{z} \\ &= \frac{1}{2i\pi} \oint_{\mathbf{T}} \left(H_p(z) - H_p(z^{-1}) \right) \frac{d\,z}{z} = 0, \qquad p = 1, \ldots, N.\end{aligned}$$

Consequently, from Equation (2.6.6),

$$\sum_{j=1}^{N} M_j^2\, f_0(z_j)\, H_p(z_j) = 0, \qquad p = 1, \ldots, N.$$

or, in matrix form,

$$\underbrace{\begin{pmatrix} H_1(z_1) & H_1(z_2) & H_1(z_3) & \cdots \\ H_2(z_1) & H_2(z_2) & H_2(z_3) & \cdots \\ H_3(z_1) & H_3(z_2) & H_3(z_3) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}}_{R} \begin{pmatrix} M_1^2 f_0(z_1) \\ M_2^2 f_0(z_2) \\ M_3^2 f_0(z_3) \\ \vdots \end{pmatrix} = 0.$$

To complete the proof, it remains to verify that the determinant of the matrix R is non-zero. If this determinant were zero, then there would exist constants c_p, $p = 1, \ldots, N$, not all zeros, such that

$$\sum_{p=1}^{N} c_p\, H_p(z_j) = 0, \qquad j = 1, \ldots, N.$$

Then function

$$H(z) := \sum_{p=1}^{N} c_p\, H_p(z)$$

would then satisfy the equalities

$$\begin{aligned} &H(z_j) = 0, && j = 1, \ldots, N, \\ &H(z^{-1}) - S(z)H(z) = 0, && |z| = 1, \end{aligned}$$

which means the vector $\{h_m\}_{m=1}^{\infty} = \{\sum_{p=1}^{N} c_p h_m^{(p)}\}_{m=1}^{\infty}$ would be a non-zero solution of the homogeneous Equation (2.2.14) for $n = 0$. By Theorem 2.2.1, however, this is not possible; therefore,

$$\det R \neq 0.$$

Thus, Equation (2.6.1) holds.

2. By setting

$$A(z) := \sum_{m=1}^{\infty} \frac{K_{0m}}{K_{00}}\, z^m, \tag{2.6.7}$$

we have

$$f_0(z) = K_{00}\left(1 + \sum_{m=1}^{\infty} \frac{K_{0m}}{K_{00}}\, z^m\right) = K_{00}\left[1 + A(z)\right].$$

Therefore,

$$f_0(z^{-1}) - S(z) f_0(z) = K_{00}\left(1 - S(z) + A(z^{-1}) - A(z) + A(z)\left[1 - S(z)\right]\right),$$

and thus, Equation (2.6.2) is equivalent to

$$1 - S(z) + A(z^{-1}) - A(z) + A(z)\left[1 - S(z)\right] = 0, \qquad |z| = 1. \tag{2.6.8}$$

By (2.2.3), we have

$$1 - S(z) = \sum_{m=-\infty}^{\infty} F_m^{(1)}\, z^{-m}. \tag{2.6.9}$$

By substituting (2.6.9) and (2.6.7) into (2.6.8) and considering that $K_{0m}/K_{00} = 0$ for $m \le 0$, we obtain

$$\sum_{m=-\infty}^{\infty} \left(F_m^{(1)} + \frac{K_{0m}}{K_{00}} - \frac{K_{0,-m}}{K_{00}} + \sum_{k=1}^{\infty} \frac{K_{0k}}{K_{00}}\, F_{k+m}^{(1)}\right) z^{-m} = 0.$$

Thus, by setting

$$g_m := F_m^{(1)} + \frac{K_{0m}}{K_{00}} - \frac{K_{0,-m}}{K_{00}} + \sum_{k=1}^{\infty} \frac{K_{0k}}{K_{00}}\, F_{k+m}^{(1)}, \qquad m \in \mathbb{Z}, \tag{2.6.10}$$

we see that Equation (2.6.2) is equivalent to the equality

$$g_m \equiv 0.$$

From the fundamental equation (2.2.4), by setting $n = 0$, and by using $f_0(z_j) = 0$ for $j = 1, \ldots, N$, we obtain (as seen in (2.6.3)) for $m \ge 1$

$$F_m^{(1)} + \frac{K_{0m}}{K_{00}} + \sum_{k=1}^{\infty} \frac{K_{0k}}{K_{00}}\, F_{k+m}^{(1)} = 0,$$

while $K_{0,-m}/K_{00} = 0$ for $m \ge 1$. Hence,

$$g_m = 0, \qquad 1 \le m < \infty. \tag{2.6.11}$$

It remains to prove that $g_m = 0$ for $-\infty < m \le 0$. We multiply both sides of (2.6.10) by $F_{m+n}^{(1)}$ and sum with respect to m over the limits from $-\infty$ to ∞, and, by considering (2.6.11), we obtain

$$\begin{aligned}\sum_{m=-\infty}^{0} g_m\, F_{m+n}^{(1)} = \sum_{m=-\infty}^{\infty} F_m^{(1)}\, F_{m+n}^{(1)} + \sum_{m=1}^{\infty} \frac{K_{0m}}{K_{00}}\, F_{m+n}^{(1)} - \sum_{m=-\infty}^{-1} \frac{K_{0,-m}}{K_{00}}\, F_{m+n}^{(1)} \\ + \sum_{k=1}^{\infty} \frac{K_{0k}}{K_{00}} \sum_{m=-\infty}^{\infty} F_{k+m}^{(1)}\, F_{m+n}^{(1)}.\end{aligned} \tag{2.6.12}$$

We will show that

$$\sum_{m=-\infty}^{\infty} F_m^{(1)}\, F_{m+n}^{(1)} = F_{-n}^{(1)} + F_n^{(1)}. \tag{2.6.13}$$

In fact, by denoting

$$G_n := \sum_{m=-\infty}^{\infty} F_m^{(1)}\, F_{m+n}^{(1)}$$

and by taking into account (2.6.9) and (2.1.1), we have

$$\begin{aligned}\sum_{n=-\infty}^{\infty} G_n\, z^n &= \sum_{n=-\infty}^{\infty}\left[\sum_{m=-\infty}^{\infty} F_m^{(1)} F_{m+n}^{(1)}\right] z^n = \sum_{p=-\infty}^{\infty} F_p^{(1)}\, z^p \sum_{m=-\infty}^{\infty} F_m^{(1)}\, z^{-m} \\ &= [1 - S(z^{-1})]\cdot[1 - S(z)] = [1 - S(z)] + [1 - S(z^{-1})].\end{aligned} \tag{2.6.14}$$

By multiplying both sides of (2.6.14) by $\frac{1}{2i\pi}\, z^{-n-1}$ and then by integrating over $\mathbf{T}$ and considering (2.2.3), we obtain (2.6.13). In view of (2.6.13), we have

$$\sum_{m=-\infty}^{\infty} F_{k+m}^{(1)}\, F_{m+n}^{(1)} = \sum_{p=-\infty}^{\infty} F_p^{(1)}\, F_{p-k+n}^{(1)} = F_{k-n}^{(1)} + F_{n-k}^{(1)}. \tag{2.6.15}$$

By substituting (2.6.13) and (2.6.15) into (2.6.12), we get

$$\sum_{m=-\infty}^{0} g_m\, F_{m+n}^{(1)} = F_{-n}^{(1)} + F_n^{(1)} + \sum_{m=1}^{\infty} \frac{K_{0m}}{K_{00}}\, F_{m+n}^{(1)} + \sum_{m=1}^{\infty} \frac{K_{0m}}{K_{00}}\, F_{m-n}^{(1)}.$$

By adding and subtracting $K_{0,-n}/K_{00} - K_{0n}/K_{00}$ on the right-hand side of the last equality and by considering (2.6.10), we obtain

$$\sum_{m=-\infty}^{0} g_m\, F_{m+n}^{(1)} = g_{-n} + g_n, \qquad n \in \mathbb{Z}. \tag{2.6.16}$$

In view of (2.6.11), $g_{-n} = 0$ for $n \leq -1$. Therefore, by (2.6.16), we have

$$\sum_{m=-\infty}^{0} g_m\, F_m^{(1)} = 2g_0, \qquad \sum_{m=-\infty}^{0} g_m\, F_{m+n}^{(1)} = g_n, \qquad -\infty < n \leq -1. \tag{2.6.17}$$

It remains to show that $\{g_m\}_{-\infty}^{m=0} \in \ell^2(-\infty, 0)$. By Condition IV of (2.6.17), we then have $g_m = 0$ for $-\infty < m \leq 0$. From (2.6.10) and by noting that $K_{0m}/K_{00} = 0$ for $m \leq 0$, we have

$$g_m = F_m^{(1)} - \frac{K_{0,-m}}{K_{00}} + \sum_{k=1}^{\infty} \frac{K_{0k}}{K_{00}}\, F_{k+m}^{(1)}, \quad -\infty < m \leq -1. \tag{2.6.18}$$

From (2.3.1),

$$\left|\frac{K_{0,-m}}{K_{00}}\right| \leq c\,\eta(-m), \qquad -\infty < m \leq -1,$$

where, by virtue of (2.2.7), it follows that $\{K_{0,-m}/K_{00}\}_{-\infty}^{-1} \in \ell^2(-\infty, -1)$. Additionally,

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{K_{0k}}{K_{00}}\, F_{k+m}^{(1)} &= \frac{1}{2i\pi}\oint_{\mathbf{T}} [1 - S(z)]\, z^m \left(\sum_{k=1}^{\infty} \frac{K_{0k}}{K_{00}}\, z^k\right) \frac{d\,z}{z} \\ &= \frac{1}{2i\pi}\oint_{\mathbf{T}} [1 - S(z)]\, A(z)\, z^m\, \frac{d\,z}{z}.\end{aligned}$$

Therefore, by Parseval's equality, it follows that $\{\sum_{k=1}^{\infty}(K_{0k}/K_{00})\,F_{k+m}^{(1)}\}_{-\infty}^{\infty} \in \ell^2(-\infty,\infty)$ and in particular, $\{\sum_{k=1}^{\infty}(K_{0k}/K_{00})\,F_{k+m}^{(1)}\}_{-\infty}^{m=-1} \in \ell^2(-\infty,-1)$. By virtue of (2.6.9), we have that $\{F_m^{(1)}\}_{-\infty}^{\infty} \in \ell^2(-\infty,\infty)$. Consequently, from (2.6.18), it follows that $\{g_m\}_{-\infty}^{0} \in \ell^2(-\infty,0)$.

■

2.7 Derivation of Parseval's equality

Let us denote

$$u_n(z) = f_n(z^{-1}) - S(z)\,f_n(z), \quad |z|=1, \quad n\geq 0 \tag{2.7.1}$$

$$u_n(z_j) = M_j\,f_n(z_j), \quad z\in(-1,0)\cup(0,1), \quad j=1,\dots,N, \quad n\geq 0. \tag{2.7.2}$$

By the results of previous sections, it follows that these sequences are solutions to the boundary problem:

$$a_{n+1}\,y_{n+1} + b_n\,y_n + a_n\,y_{n-1} = \lambda\,y_n, \qquad \lambda = z+z^{-1}, \quad n\geq 1, \tag{2.7.3}$$

$$y_0 = 0, \tag{2.7.4}$$

where a_n and b_n are defined by formulas (2.4.11) and (2.4.12), respectively, with

$$a_n>0, \quad \operatorname{Im} b_n = 0, \quad n=1,2,3,\dots, \quad \sum_{n=1}^{\infty} n\,(|1-a_n|+|b_n|)<\infty.$$

Furthermore, they have the asymptotic behavior

$$u_n(z) = z^{-n} - S(z)z^n + o(1), \qquad n\to\infty, \quad |z|=1 \tag{2.7.5}$$

$$u_n(z_j) = M_j\,z_j^n\,[1+o(1)], \qquad n\to\infty, \qquad j=1,\dots,N, \quad z_j\in(-1,0)\cup(0,1). \tag{2.7.6}$$

To assert that the sequence $\{S(z);\ z_1,z_2,\dots,z_N;\ M_1,M_2,\dots,M_N\}$ constitutes the scattering data for the boundary problem (2.7.3)+(2.7.4), we must show that $\{u_n(z)\}_{n=1}^{\infty}$ and $\{u_n(z_j)\}_{n=1}^{\infty}$ for $j=1,\dots,N$ fulfil a Parseval's equality.

Theorem 2.7.1. *Parseval's equality*

$$\delta_{nm} = \sum_{j=1}^{N} u_n(z_j)\,\overline{u_m(z_j)} + \frac{1}{2\pi}\int_0^{\pi} u_n(e^{i\theta})\,\overline{u_m(e^{i\theta})}\,d\theta, \qquad n,m\geq 1$$

holds.

Proof.

By using the fundamental equation, we can obtain

$$f_n(z^{-1}) + [1-S(z)]\,f_n(z) = K_{nn}\,z^{-n}\left(1+\sum_{-\infty}^{m=0} G_{nm}\,z^{-m} - \sum_{j=1}^{N}\frac{1}{K_{nn}}M_j^2\,z_j^n\,f_n(z_j)\,\frac{z_j\,z^{-1}}{1-z_j\,z^{-1}}\right), \tag{2.7.7}$$

where G_{nm} is given by (2.5.9). By multiplying both sides of (2.7.7) by $(1/2i)f_m(z)\,z^{-1}$, $m \geq n$, by integrating over the closed simple curve $\mathbf{T}$ in a counterclockwise direction and considering the notations (2.7.1), (2.7.2), and the easily verified formulas

$$\oint_{\mathbf{T}} f_n(z)\, f_m(z)\, \frac{d\,z}{z} = 0, \qquad n, m \geq 1,$$

$$K_{nn}\, \frac{1}{2i\pi} \oint_{\mathbf{T}} f_m(z)\, z^{-n}\, \frac{d\,z}{z} = \begin{cases} 0, & \text{if} \quad m > n \\ K_{nn}^2 & \text{if} \quad m = n, \end{cases}$$

$$\oint_{\mathbf{T}} \left(f_n(z^{-1})\, f_m(z) - S(z)\, f_n(z)\, f_m(z) \right) \frac{d\,z}{z} = \frac{1}{2} \oint_{\mathbf{T}} u_n(z)\, u_m(z^{-1}) \frac{d\,z}{z}$$

$$= \frac{1}{2} \int_0^{2\pi} u_n(e^{i\theta})\, \overline{u_m(e^{i\theta})}\, i\, d\,\theta = \int_0^{\pi} u_n(e^{i\theta})\, \overline{u_m(e^{i\theta})}\, i\, d\,\theta,$$

$$K_{nn} \sum_{-\infty}^{k=0} G_{nk}\, \frac{1}{2i\pi} \oint_{\mathbf{T}} f_m(z)\, z^{-(n+k)}\, \frac{d\,z}{z} = \begin{cases} 0, & \text{if} \quad m > n \\ K_{nn}^2\, G_{n0} & \text{if} \quad m = n, \end{cases}$$

$$\frac{1}{2i\pi} \oint_{\mathbf{T}} \frac{z^{-n}\, z_j\, z^{-1}}{1 - z_j\, z^{-1}} f_m(z)\, \frac{d\,z}{z} = \begin{cases} z_j^{-n}\, f_m(z_j), & \text{if} \quad m > n \\ -K_{nn} + z_j^{-n}\, f_n(z_j) & \text{if} \quad m = n, \end{cases}$$

we obtain

$$\sum_{j=1}^{N} u_n(z_j)\, u_m(z_j) + \frac{1}{2\pi} \int_0^{\pi} u_n(e^{i\theta})\, \overline{u_m(e^{i\theta})} d\,\theta$$

$$= \begin{cases} 0, & \text{if} \quad m > n \\ K_{nn} \sum_{j=1}^{N} M_j\, z_j^n\, f_n(z_j) + K_{nn}^2 + K_{nn}^2\, G_{n0} & \text{if} \quad m = n, \end{cases} \tag{2.7.8}$$

By considering (2.5.10) and (2.4.10), we have

$$K_{nn} \sum_{j=1}^{N} M_j\, z_j^n\, f_n(z_j) + K_{nn}^2 + K_{nn}^2\, G_{n0} = 1.$$

Consequently, from (2.7.8) and by taking into account that $u_n(z_j)$ are real, we obtain

$$\sum_{j=1}^{N} u_n(z_j)\, \overline{u_m(z_j)} + \frac{1}{2\pi} \int_0^{\pi} u_n(e^{i\theta})\, \overline{u_m(e^{i\theta})} d\,\theta = \delta_{nm}, \quad m \geq n \geq 1. \tag{2.7.9}$$

By conjugating (2.7.9) and noting that u_n and u_m are equal, we find that the equation holds for $m < n$. ■

Thus, we have completely proven the fundamental theorem 2.1.1. From the unique solvability of the fundamental equation, it follows that the inverse problem is uniquely solvable from the scattering data.

Example 1. *Consider the set of magnitudes*

$$\left\{ S(z) = \frac{1 - a\, z^{-1}}{1 - a\, z}; \quad z_1 = \frac{1}{a}, \quad a > 1; \quad M > 0 \right\}.$$

It is clear that this set satisfies condition I of the fundamental theorem 2.1.1. Additionally, by applying the contour integration method and using residue formulas, it is easy to calculate that

$$F_1 = \frac{M^2}{a} + \frac{1}{a}, \qquad F_m = \frac{M^2}{a^m} + \frac{1}{a^m} - a^m, \qquad m \geq 2,$$

where

$$F_m = M^2\, z^m - \frac{1}{2i\pi} \oint_{\mathbf{T}} \frac{1 - a\, z^{-1}}{1 - a\, z} z^{m-1}\, d\, z.$$

It is evident that condition II is also satisfied. The fundamental equation (2.2.4) is solved explicitly, yielding

$$\frac{K_{01}}{K_{00}} = -a, \quad \frac{K_{0m}}{K_{00}} = 0, \quad m \geq 2, \qquad \frac{K_{n,n+m}}{K_{nn}} = -\frac{(a^2-1)(M^2-a^2+1)a^{-(2n+m)}}{a^2 - 1 + (M^2-a^2+1)a^{-2n}}, \quad m, n \geq 1,$$

$$\begin{aligned} \frac{1}{K_{nn}^2} &= 1 + F_{2n} + \sum_{k=1}^{\infty} \frac{K_{n,n+k}}{K_{nn}} F_{k+2n} \\ &= 1 + (M^2 - a^2 + 1)a^{-2n} - \frac{(a^2-1)(M^2-a^2+1)a^{-(2n+m)}}{(1-a^{-2})[a^2 - 1 + (M^2-a^2+1)a^{-2n}]}, \quad n \geq 1. \end{aligned}$$

After this, the coefficients a_n and b_n are determined by the formulas

$$a_n = \frac{K_{nn}}{K_{n-1,n-1}}, \qquad b_n = \frac{K_{n,n+1}}{K_{nn}} - \frac{K_{n-1,n}}{K_{n-1,n-1}}, \qquad n \geq 1.$$

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