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Multidimensional sequential compactness

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“War das das Leben?
Wohlan! Noch einmal!”

— Friedrich Nietzsche, *Also sprach Zarathustra*

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Resumen

Esta tesis explora generalizaciones de la compacidad secuencial en espacios topológicos. La primera parte avanza en la teoría de dimensión finita construyendo espacios n-secuencialmente compactos que no son $(n+1)$-secuencialmente compactos bajo las hipótesis $\mathfrak{b} = \mathfrak{c}$ o $\mathfrak{b} = \mathfrak{s}$. Entre los logros clave se incluye un ejemplo en ZFC de un espacio de Fréchet secuencialmente compacto que no es 2-secuencialmente compacto, respondiendo así a una pregunta abierta. Además, exhibimos un espacio secuencialmente compacto de carácter $\mathfrak{b}$ que no es 2-secuencialmente compacto, mostrando que $\mathfrak{b}$ es una cota óptima en [49, Teorema 3.1].

La segunda parte extiende el estudio a dimensiones infinitas vía barreras de Nash-Williams, donde la noción de "dimensión" es capturada por el rango de la barrera, el cual puede ser cualquier ordinal numerable. Un preorden $\preceq$ en las barreras facilita análogos de resultados finito-dimensionales, probando que si X es $\mathcal{B}$-secuencialmente compacto y $\mathcal{C} \preceq \mathcal{B}$, entonces X es $\mathcal{C}$-secuencialmente compacto. Bajo $\mathfrak{b} = \mathfrak{c}$, para $\alpha < \beta < \omega_1$, existen espacios α-secuencialmente compactos que no son β-secuencialmente compactos. Se muestra que ciertas clases bien estudiadas (como espacios importantes surgidos del análisis funcional) son α-secuencialmente compactas para todo $\alpha < \omega_1$. Finalmente, se estudian nuevos cardinales asociados a barreras, tanto en sí mismos como en el contexto de la compacidad secuencial con barreras.

Las contribuciones principales incluyen estas construcciones, caracterizaciones y extensiones, las cuales mejoran la comprensión de las jerarquías de compacidad.

Palabras clave: Compacidad $\mathcal{B}$-secuencial, orden de Katětov, barreras, invariantes cardinales del continuo, familias casi disjuntas.

Abstract

This thesis explores generalizations of sequential compactness in topological spaces. The first part advances finite-dimensional theory by constructing n-sequentially compact spaces that fail to be $(n+1)$-sequentially compact under the assumptions $\mathfrak{b} = \mathfrak{c}$ or $\mathfrak{b} = \mathfrak{s}$. Key achievements include a ZFC example of a Fréchet, sequentially compact space that is not 2-sequentially compact, answering an open question. Furthermore, we exhibit a sequentially compact space of character $\mathfrak{b}$ that is not 2-sequentially compact, showing that $\mathfrak{b}$ is an optimal bound in [49, Theorem 3.1].

The second part extends this analysis to infinite dimensions via Nash-Williams barriers, where the notion of "dimension" is captured by the rank of the barrier, which can be any countable ordinal. A preorder $\preceq$ on barriers facilitates analogues of finite-dimensional results, proving that if X is $\mathcal{B}$-sequentially compact and $\mathcal{C} \preceq \mathcal{B}$, then X is $\mathcal{C}$-sequentially compact. Under $\mathfrak{b} = \mathfrak{c}$, for $\alpha < \beta < \omega_1$, there exist α-sequentially compact spaces that are not β-sequentially compact. Well-studied classes (such as important spaces arising from functional analysis) are shown to be α-sequentially compact for all $\alpha < \omega_1$. Additionally, new cardinals associated with barriers are studied, both in themselves and in the context of barrier-sequential compactness.

The main contributions include these constructions, characterizations, and extensions, which improve the understanding of compactness hierarchies.

Keywords: $\mathcal{B}$-sequential compactness, Katětov order, barriers, cardinal invariants of the continuum, almost disjoint families.

Contents

Chapter 1

Introduction

A widely studied topological property is sequential compactness. Recall that a space X is *sequentially compact* if every sequence $(x_n)_{n\in\omega} \subseteq X$ has a convergent subsequence.

The concept originates in the study of sequences of real numbers. In 1817, Bernard Bolzano proved a lemma that later played a role in the proof of the intermediate value theorem; this result was eventually recognized as significant in its own right. In the late nineteenth century, Karl Weierstrass gave a fully rigorous proof of this result, which is now known as the Bolzano–Weierstrass theorem: every bounded sequence of real numbers has a convergent subsequence. A closely related late-nineteenth-century cornerstone that inspired the formal definition of sequential compactness is the Arzelà–Ascoli theorem, a fundamental tool for deciding when a family of real-valued continuous functions on a compact interval is compact.

Maurice Fréchet coined the term and formalized sequential compactness in his 1906 doctoral dissertation. He introduced the notion in the setting of metric spaces and certain more general classes and showed that, in metric spaces, sequential compactness is equivalent to compactness.

In the 1920s, Pavel Alexandroff and Pavel Urysohn developed the modern definition of compactness (with open covers) and emphasized that, in general topological spaces, sequential compactness does not necessarily coincide with compactness. For example, ω_1 with the order topology is sequentially compact but not compact, while $\beta\omega$, the Čech–Stone compactification of a countable discrete space, is compact but fails to be sequentially compact (see [64]).

In [49], Kubiś and Szeptycki introduced n-dimensional sequential compactness (for $n \geq 1$) and proved that compact metric spaces are sequentially compact in all such dimensions. The 2-dimensional case had previously been introduced and proved to hold in compact metric spaces in [9] by M. Bojańczyk, E. Kopczyński and S. Toruńczyk. The authors used this to obtain idempotents in compact semigroups.

To introduce the notion of n-sequential compactness we need the following notation: $[B]^n$ denotes the family of all subsets of B of cardinality n, $[B]^\omega$ denotes the family of all of its countably infinite subsets and $[B]^{<\omega} = \bigcup_{n\in\omega}[B]^n$. Recall that if X is a topological space, $p \in X$, C is a countable set and $f\colon C \to X$, then f *converges to* p if for every open neighborhood U of p there is a finite $F \subseteq C$ such that $f''(C \setminus F) \subseteq U$.[1] It is natural to generalize this definition to higher dimensions:

[1]Here, $f''A$ denotes the direct image of A under f, i.e., the set $\{f(x) \mid x \in A\}$. We use this notation throughout most of the thesis; however, in Chapter 4, we switch to the notation $f[A]$, which is more convenient when dealing with sets defined by lengthy expressions.

Definition 1.0.1. (See [49, Definition 2.1]) Let X be a topological space, $p \in X$, and $n \geq 1$.

(I) If B is countable and $f : [B]^n \to X$, then f *converges* to p if for every neighborhood U of p there is a finite set $F \subseteq B$ such that $f''[B \setminus F]^n \subseteq U$ and in this case we say that the function f is *convergent*.

(II) We say that X is an *n-sequentially compact space* if for every function $f : [\omega]^n \to X$ there exists $B \in [\omega]^\omega$ such that $f \restriction [B]^n$ is convergent. We say that $f \restriction [B]^n$ is a convergent subsequence of f.

Observe that if $B \in [\omega]^\omega$ then $C = [B]^n$ is countable, so for any function $f\colon [B]^n \to X$ we can consider two notions of convergence: the classical one (viewing f as a sequence indexed by the countable set C) and the n-dimensional notion from item (I) of Definition 1.0.1. It is important to realize that these two notions need not coincide. If f converges to $p \in X$ in the n-dimensional sense (Definition 1.0.1), then for every neighborhood U of p there is a finite $F \subseteq B$ such that $f''[B \setminus F]^n \subseteq U$. This does not rule out the possibility that the set $D = [B]^n \setminus ([B \setminus F]^n)$ satisfies $(f''D) \cap U = \emptyset$; if $F \neq \emptyset$ and $n \geq 2$, then D is infinite, so in this case f does not converge to p in the classical sense even though it converges to p in the n-dimensional sense.

On the other hand, classical convergence implies n-dimensional convergence. In fact, if f converges to p in the classical sense, then for every neighborhood U of p there is a finite $G \subseteq C$ with $f''(C \setminus G) \subseteq U$. Write each element of G as $a_i \in [B]^n$ for $i < k = |G|$, and set $F = \bigcup_{i<k} a_i$. Then $[B \setminus F]^n \subseteq C \setminus G$, so $f''[B \setminus F]^n \subseteq U$. Hence, classical convergence implies n-dimensional convergence. Thus, for maps $f\colon [B]^n \to X$, convergence in the n-dimensional sense is strictly weaker than classical convergence. To avoid confusion, from now on, whenever we talk about the convergence of $f : [B]^n \to X$, we refer to the sense of item (I) of Definition 1.0.1.

Observe that being 1-sequentially compact is equivalent to being sequentially compact. It is also not hard to see that Ramsey's theorem[2] is equivalent to the assertion that every finite space X is n-sequentially compact for all $n \geq 1$.

As noted earlier, in [49] it is proved that every compact metric space is n-sequentially compact for every $n \geq 1$, thus we cannot differentiate the class of n-sequentially compact spaces from the class of m-sequentially compact spaces with $n \neq m$ in the realm of metric spaces. So, the natural question is whether we can distinguish these properties in the wider category of general topological spaces.

It is easy to see that every $(n+1)$-sequentially compact space X is also n-sequentially compact [49, Proposition 2.2]. Indeed, given $f\colon [\omega]^n \to X$, define $\hat{f}\colon [\omega]^{n+1} \to X$ by

$$\hat{f}(\{a_0, \ldots, a_n\}) = f(\{a_0, \ldots, a_{n-1}\})$$

whenever $a_0 < \ldots < a_n$. Applying $(n+1)$-sequential compactness to $\hat{f}$ yields a convergent $\hat{f} \restriction [B]^{n+1}$. As $\hat{f}''[B \setminus F]^{n+1} = f''[B \setminus F]^n$ for every finite $F \subseteq B$, we get that $f \restriction [B]^n$ is convergent.

Therefore, the natural question is: for a fixed $n \geq 1$, does there exist a space X that is n-sequentially compact but fails to be $(n+1)$-sequentially compact?

The following theorem was the first partial answer to this question and was a major inspiration for my doctoral research.

[2]Ramsey's theorem states that if $n, m \geq 1$ and $f : [\mathbb{N}]^n \to m$, then there exists $Y \subseteq \mathbb{N}$ infinite such that $|f''[Y]^n| = 1$.

Theorem 1.0.2. *1.* [49, Example 4.1] *There exists a compact, sequentially compact space (that is,* 1*-sequentially compact) that is not* 2*-sequentially compact.*

2. [49, Theorem 4.7] *Under* CH*, for every $n \in \omega$ there exists an n-sequentially compact space that is not $(n+1)$-sequentially compact.*

The initial goal of my PhD research was to improve item 2 of Theorem 1.0.2 by constructing n-sequentially compact spaces that are not $(n+1)$-sequentially compact under various assumptions about cardinal invariants. We also provide a Fréchet example related to item (1), answering a question posed by the authors of [49]. In [49] the authors prove that a sequentially compact space of character[3] less than $\mathfrak{b}$ is n-sequentially compact for every $n \geq 1$; in my thesis I show that this bound is optimal by constructing a sequentially compact space of character exactly $\mathfrak{b}$ that fails to be 2-sequentially compact. In general, all of my results on the theory of n-sequentially compact spaces appear in Chapter 3, which is entirely based on the article [15], of which I am a coauthor.

The second part of my PhD work was devoted to extending the notion of sequential compactness to countably infinite dimensions and to obtaining analogues of finite-dimensional results in that setting. This is carried out using Nash–Williams barriers [55], where the notion of "dimension" is captured by the rank of a barrier (which may be any countable ordinal). Since the simplest examples of barriers are the families $[\omega]^n$, the notion of "barrier-sequential compactness" is a natural one.

Chapter 4 develops this new theory, which is entirely based on the article [16]. We begin with an analysis of barriers tailored to this topological setting and introduce a preorder $\preceq$ on the family of barriers that proves very useful for deriving analogues of the results from Chapter 3. For example, under $\mathfrak{b} = \mathfrak{c}$ we show that for every $\alpha < \beta < \omega_1$ there exists a space that is α-sequentially compact but not β-sequentially compact. We also extend several results from [49] to the infinite-dimensional setting.

Moreover, we prove that some well-studied classes of topological spaces—such as Rosenthal compacta—are α-sequentially compact for every $\alpha \in \omega_1$. Finally, we introduce certain cardinal invariants associated to barriers that are naturally related to the sequential compactness of various spaces.

More specifically, the main contributions of this thesis are the following:

Theorem. *Under $\mathfrak{b} = \mathfrak{c}$ or $\mathfrak{b} = \mathfrak{s}$, for every $n \geq 2$ there exists an n-sequentially compact space that is not $(n+1)$-sequentially compact.*

This result represents a significant advancement in distinguishing the classes of n-sequentially compact and m-sequentially compact spaces when $n \neq m$. It demonstrates that these properties are distinct under $\mathfrak{b} = \mathfrak{c}$ or $\mathfrak{b} = \mathfrak{s}$. This is particularly notable because these consistent assumptions hold in many standard models where CH fails—a setting where the previous results from [49] do not apply—thus establishing that these classes differ in a much wider context. It is also worth noting that the proofs for $\mathfrak{b} = \mathfrak{c}$ and $\mathfrak{b} = \mathfrak{s}$ utilize different approaches; notably, the latter employs Shelah's technique for constructing completely separable mad families. This highlights the depth and utility of this advanced method.

In [49], the authors asked whether every Fréchet sequentially compact space is necessarily 2-sequentially compact. We provide a negative answer in ZFC:

[3]For every $x \in X$, the *character of $x \in X$* is the minimum cardinality of a local base of x and it is denoted by $\chi(x, X)$. The character of X is $\chi(X) = \sup\{\chi(x, X) \mid x \in X\}$.

Theorem. *There exists a Fréchet sequentially compact space that is not 2-sequentially compact.*

A pivotal result, [49, Theorem 3.1], states that every sequentially compact space with character strictly less than $\mathfrak{b}$ is n-sequentially compact for all $n \geq 1$. We demonstrate the optimality of this bound with the following equality:

Theorem. $\mathfrak{b} = \min\{\chi(X) \mid X$ *is sequentially compact but not* 2*-sequentially compact*$\}$.

In the context of infinite dimensions, we show that the character bound identified in [49] actually yields a significantly stronger conclusion:

Theorem. *Suppose X is sequentially compact. If $\chi(X) < \mathfrak{b}$, then X is α-sequentially compact for every $\alpha \in \omega_1$.*

Leveraging the infinite-dimensional framework, we introduce a preorder relation $\preceq$ on the family of barriers. We prove that if $\mathcal{C} \preceq \mathcal{B}$ and a space X is $\mathcal{B}$-sequentially compact, then it is also $\mathcal{C}$-sequentially compact. This leads to the following main theorem concerning countable-dimensional sequential compactness:

Theorem. *Let X be a topological space and $\alpha \in \omega_1$, the following are equivalent:*

1. *X is α-sequentially compact;*
2. *X is $\mathcal{B}$-sequentially compact for every uniform barrier $\mathcal{B}$ of rank α;*
3. *X is $\mathcal{B}$-sequentially compact for some uniform barrier $\mathcal{B}$ of rank α.*

A desirable and expected consequence of this theorem and the properties of the preorder $\preceq$ is the following generalization of the finite-dimensional fact that n-sequential compactness implies m-sequential compactness whenever $m < n$:

Theorem. *If X is α-sequentially compact and $\beta < \alpha$, then X is β-sequentially compact. In particular, if α is infinite, then X is n-sequentially compact for every $n \in \omega$.*

We also establish that certain well-studied classes of spaces satisfy these properties for all $\alpha \in \omega_1$, illustrating the potential of this multidimensional theory to be applied to spaces within these classes and potentially obtain properties about them:

Theorem. *The following spaces are α-sequentially compact for every $\alpha \in \omega_1$:*

1. *Compact bisequential spaces (in particular, Rosenthal compacta).*
2. *Sequentially compact strongly monolithic spaces (in particular, Corson compacta and Eberlein compacta arising from adequate families).*

We successfully generalize the distinction between dimensions from the finite to the infinite setting. Specifically:

Theorem. *Assuming $\mathfrak{b} = \mathfrak{c}$, if $\{\alpha_n \mid n \in \omega\} \subseteq \alpha$, then there exists a space that is α_n-sequentially compact for all $n \in \omega$ but fails to be α-sequentially compact. In particular, for every $\alpha < \beta < \omega_1$ there is an α-sequentially compact space that is not β-sequentially compact.*

Regarding cardinal invariants associated with barriers, we provide a characterization of $\mathfrak{par}_2$, generalizing the classical result, [8, Theorem 3.5], that $\mathfrak{par}_2 = \mathfrak{par}_n$ for all $n \geq 2$:

Theorem. $\mathfrak{par}_{\mathcal{B}} = \mathfrak{par}_2$ *for every non-trivial barrier* $\mathcal{B}$*. In particular, if* $\kappa < \mathfrak{par}_2$ *and* $\mathcal{B}$ *is any barrier, then* 2^κ *is* $\mathcal{B}$*-sequentially compact. Moreover,*

$$\mathfrak{par}_2 = \min\{\kappa \mid 2^\kappa \textit{ is not } \mathcal{B}\textit{-sequentially compact}\}$$

for every non-trivial barrier $\mathcal{B}$*.*

Finally, in our specialized study of barriers, we obtain a characterization of the rank inequality in terms of the Katětov order, which reaffirms the importance of this order in the general structure of the combinatorics of ω:

Theorem. *Let* $\mathcal{B}$ *and* $\mathcal{C}$ *be uniform barriers such that* $\rho(\mathcal{B}) \leq \rho(\mathcal{C})$*. Then the following are equivalent:*

1. $\mathsf{FIN}^{\mathcal{C}} \not\leq_K \mathcal{G}_c(\mathcal{B})$.
2. $\rho(\mathcal{B}) < \rho(\mathcal{C})$.

Chapter 2

Preliminaries

Recall that for a set B, $[B]^n$ denotes the family of all subsets of B of cardinality n. Analogously, $[B]^\omega$ denotes the family of all of its countably infinite subsets and $[B]^{<\omega} = \bigcup_{n\in\omega}[B]^n$.

All spaces considered here are Hausdorff. Denote by ω the first infinite ordinal, by ω_1 the first uncountable ordinal, and by $\mathfrak{c}$ the cardinality of $\mathbb{R}$. Given two sets $A, B \subseteq \omega$, we write $A \subseteq^* B$ if $A \setminus B$ is finite, while we use $A =^* B$ if $A \subseteq^* B$ and $B \subseteq^* A$. Two sets A, B are *almost disjoint* if $A \cap B$ is finite. A family $\mathcal{A} \subseteq [\omega]^\omega$ is *almost disjoint* if its elements are pairwise almost disjoint, and it is *maximal almost disjoint (mad)* if it is almost disjoint and is not properly contained in any other almost disjoint family. Note that if we put $E = \{2n \mid n \in \omega\}$ and $O = \{2n+1 \mid n \in \omega\}$, then $\{E, O\}$ is a maximal almost disjoint family, so to avoid such trivial examples, whenever we speak of an almost disjoint family (and in particular of a mad family) we always assume the family is infinite.

We say that a set $S \in [\omega]^\omega$ *splits* $A \in [\omega]^\omega$ if $|S \cap A| = \omega = |A \setminus S|$. A family $\mathcal{S} \subseteq [\omega]^\omega$ is a *splitting family* if for every $X \in [\omega]^\omega$ there exists $S \in \mathcal{S}$ such that S splits X.

The set of functions from X to Y will be denoted by Y^X. For two functions $f, g \in \omega^\omega$, we write $f \leq^* g$ if $\{n \in \omega \mid f(n) > g(n)\}$ is finite. A family $\mathcal{B} \subseteq \omega^\omega$ is *unbounded* if for every $f \in \omega^\omega$ there exists $b \in \mathcal{B}$ such that $b \not\leq^* f$ and $\mathcal{B}$ is *dominating* if for every $f \in \omega^\omega$ there exists $d \in \mathcal{B}$ such that $f \leq^* d$. We also say that $\mathcal{B}$ is a *scale* if $\mathcal{B}$ is a dominating family and it is well ordered by $\leq^*$.

We define below the cardinal invariants used in this work for the ease of the reader.

- $\mathfrak{a}$ is the minimum size of a mad family,
- $\mathfrak{s}$ is the minimum size of a splitting family,
- $\mathfrak{b}$ is the minimum size of an unbounded family,
- $\mathfrak{d}$ is the minimum size of a dominating family.

Each of these cardinals is uncountable and is smaller than or equal to $\mathfrak{c}$. It is also well-known that $\mathfrak{b} \leq \mathfrak{d}$, $\mathfrak{b} \leq \mathfrak{a}$ and $\mathfrak{s} \leq \mathfrak{d}$ and that there are no other provable relations between them in ZFC. It is worth noting that scales exist if and only if $\mathfrak{b} = \mathfrak{d}$. A nice survey on cardinal characteristics and their properties is [8].

A family $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is an *ideal* if it is closed under finite unions and subsets, i.e., if $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$, and if $A \subseteq B$ and $B \in \mathcal{I}$, then $A \in \mathcal{I}$. We denote the family $\{X \subseteq \omega \mid X \notin \mathcal{I}\}$ by $\mathcal{I}^+$ and if $X \in \mathcal{I}^+$ we say that X *is positive for* $\mathcal{I}$ or X *is* $\mathcal{I}$*-positive*.

An ideal $\mathcal{I}$ is *tall* if for every $X \in [\omega]^\omega$ there exists $I \in \mathcal{I}$ such that $|X \cap I| = \omega$. Given a tall ideal $\mathcal{I}$, the cardinal $\mathsf{cov}^*(\mathcal{I})$ is defined as follows:

$$\mathsf{cov}^*(\mathcal{I}) = \min\{|\mathcal{H}| : \mathcal{H} \subseteq \mathcal{I} \ (\forall A \in [\omega]^\omega \exists B \in \mathcal{H}(|A \cap B| = \omega))\}.$$

We can also define $\mathsf{cof}(\mathcal{I})$, the cofinality of an ideal $\mathcal{I}$, as the minimum size of a subfamily of $\mathcal{I}$ that generates it, that is:

$$\mathsf{cof}(\mathcal{I}) = \min\{|\mathcal{J}| : \mathcal{J} \subseteq \mathcal{I} \wedge \forall\, I \in \mathcal{I}\ \exists J \in \mathcal{J}(I \subseteq J)\}.$$

For more information on ideals on countable sets and their cardinal invariants, the reader may consult [38]. We now summarize some definitions and results of almost disjoint families that will be helpful in our constructions.

Definition 2.0.1. If $\mathcal{A}$ is an almost disjoint family, we denote by $\mathcal{I}(\mathcal{A})$ the ideal generated by $\mathcal{A}$. Throughout this text, we assume that ideals contain all finite sets (i.e., $[\omega]^{<\omega} \subseteq \mathcal{I}$); thus, $\mathcal{I}(\mathcal{A}) = \langle \mathcal{A} \cup [\omega]^{<\omega} \rangle$.

Here, $\langle \mathcal{B} \rangle$ denotes the ideal generated by $\mathcal{B}$. Clearly, an element of $\mathcal{I}(\mathcal{A})$ is a set that can be almost covered by a finite subfamily of $\mathcal{A}$.

Definition 2.0.2. For an almost disjoint family $\mathcal{A}$ we define

$$\mathcal{I}^{++}(\mathcal{A}) := \{X \subseteq \omega : |\{A \in \mathcal{A} : |X \cap A| = \omega\}| \geq \omega\}.$$

Note that $\mathcal{I}^{++}(\mathcal{A}) \subseteq \mathcal{I}^{+}(\mathcal{A})$ and $\mathcal{A}$ is mad if and only if $\mathcal{I}^{++}(\mathcal{A}) = \mathcal{I}^{+}(\mathcal{A})$.

Let $\mathcal{A}$ be an almost disjoint family on a countable set N. The *Isbell–Mrówka space* $\Psi(\mathcal{A})$ has underlying set $N \cup \mathcal{A}$, where each $n \in N$ is isolated and a basic neighborhood of $A \in \mathcal{A}$ is of the form $\{A\} \cup A \setminus F$ for some $F \in [N]^{<\omega}$. Then $\Psi(\mathcal{A})$ is

(1) Hausdorff,

(2) zero-dimensional,

(3) separable and

(4) locally compact.

(1) Hausdorff. If $A, B \in \mathcal{A}$ with $A \neq B$, then

$$\{A\} \cup \Big(A \setminus (A \cap B)\Big) \quad \text{and} \quad \{B\} \cup \Big(B \setminus (A \cap B)\Big)$$

are disjoint neighborhoods of A and B, respectively. If $A \in \mathcal{A}$ and $n \in N$, then

$$\{A\} \cup (A \setminus \{n\}) \quad \text{and} \quad \{n\}$$

are disjoint neighborhoods of A and n, respectively.

(2) Zero-dimensional. Each basic neighborhood $\{A\} \cup (A \setminus F)$ is clopen. Indeed, if $B \in \mathcal{A}$ with $B \neq A$, then

$$\{B\} \cup \Big(B \setminus (A \cap B \cup F)\Big)$$

is open and disjoint from $\{A\} \cup (A \setminus F)$; if $n \in N \setminus (A \setminus F)$, then $\{n\}$ is open and disjoint from $\{A\} \cup (A \setminus F)$.

(3) Separable. The countable set N is dense in $\Psi(\mathcal{A})$.

(4) Locally compact. Each basic neighborhood $\{A\} \cup (A \setminus F)$ is compact. Given an open cover $\mathcal{U}$ of $\{A\} \cup (A \setminus F)$, choose $U_0 \in \mathcal{U}$ with $A \in U_0$. Then $U_0 \supseteq A \setminus G$ for some finite $G \subseteq N$, so

$$(A \setminus F) \setminus U_0 \ = \ G \setminus F$$

is finite. For each $x \in G \setminus F$, pick $U_x \in \mathcal{U}$ with $x \in U_x$. Then

$$\{U_0\} \cup \{U_x \mid x \in G \setminus F\}$$

is a finite subcover of $\mathcal{U}$.

Typically, the examples we construct are one-point compactifications of Isbell–Mrówka spaces, so it is convenient to recall the general definition of a one-point compactification. If X is a non-compact, locally compact, Hausdorff space, then its *one-point compactification* is the space

$$\overline{X} = X \cup \{\infty\},$$

where $\infty \notin X$. The open sets of X remain open in $\overline{X}$, and a basic neighborhood of ∞ is

$$\{\infty\} \cup \Big(X \setminus C\Big),$$

where $C \subseteq X$ is compact. It follows that $\overline{X}$ is compact and Hausdorff and X is a dense subspace of $\overline{X}$.

The *Franklin space* of an almost disjoint family $\mathcal{A}$ is the one-point compactification of the Isbell–Mrówka space $\Psi(\mathcal{A})$. Concretely,

$$\mathcal{F}(\mathcal{A}) = \Psi(\mathcal{A}) \cup \{\infty\},$$

and the basic neighborhoods of ∞ are

$$\{\infty\} \ \cup \ \Big(\mathcal{A} \setminus \{A_0, \dots, A_n\}\Big) \ \cup \ \Big(N \setminus \Big(\bigcup_{m \leq n}(A_m \cup F)\Big)\Big),$$

where $n \in \omega$, $A_0, \dots, A_n \in \mathcal{A}$, and $F \in [N]^{<\omega}$. For more on Isbell–Mrówka spaces, see the survey [35].

Since their early beginnings, the Isbell-Mrówka spaces $\Psi(\mathcal{A})$ have turned out to be a fruitful source of examples and counterexamples in topology, since combinatorial properties of $\mathcal{A}$ translate into topological properties of $\Psi(\mathcal{A})$. Several applications of Isbell-Mrówka spaces to topology and functional analysis can be found in the literature. However, constructing mad families with special combinatorial properties is usually a difficult task without extra assumptions beyond ZFC. An example of almost disjoint families with special combinatorial properties is the completely separable ones.

Definition 2.0.3. An almost disjoint family $\mathcal{A}$ is *completely separable* if for all $X \in \mathcal{I}^{++}(\mathcal{A})$ there exists $A \in \mathcal{A}$ such that $A \subseteq X$.

These families were introduced by S. H. Hechler in [34] and in [24]. P. Erdös and S. Shelah asked if there exist completely separable families that are mad. Most of the early work on this question was done by B. Balcar and P. Simon in [7], who proved that completely separable mad families exist under many assumptions. The next major advance on this was done by Shelah in [59], who introduced a technique that allowed him to prove that there are completely separable mad families under $\mathfrak{s} < \mathfrak{a}$ and also under $\mathfrak{s} \geq \mathfrak{a}$ + a certain "PCF hypothesis". The last big progress in the search of these families was done by H. Mildenberger, D. Raghavan and J. Steprāns in [54] and [57] who improved and extended Shelah's technique in order to get the existence of a completely separable mad family under $\mathfrak{s} \leq \mathfrak{a}$. Shelah's technique has turned out to be very powerful not only in the construction of completely separable mad families but also for other special mad families, e.g., weakly tight mad families (under $\mathfrak{s} \leq \mathfrak{b}$) and $+$-Ramsey mad families (see [57] and [29] respectively). This technique plays a crucial role in Section 3.5. The interested reader may consult [39] and [30].

An almost disjoint family $\mathcal{A}$ is *nowhere mad* if for every $X \in \mathcal{I}^+(\mathcal{A})$ there exists $B \in [X]^\omega$ that is almost disjoint with every element in $\mathcal{A}$. For example, every almost disjoint family of size less than $\mathfrak{a}$ is nowhere mad.

Finally, Simon proved the following theorem:

Theorem 2.0.4. (See [6, Corollary 2.3] and [26, Theorem 1]) *There is a completely separable nowhere mad family.*

So we can find a completely separable almost disjoint family in ZFC if we do not demand maximality of it. As we see in Section 3.2, this family leads to a sequentially compact space which is not 2-sequentially compact. The recursive construction can be carried out thanks to the following result:

Lemma 2.0.5. (See [26, Observation 1] and [30, Lemma 10]) *If $\mathcal{A}$ is a completely separable almost disjoint family and $X \in \mathcal{I}^{++}(\mathcal{A})$, then the set $\{A \in \mathcal{A} \mid A \subseteq X\}$ has cardinality $\mathfrak{c}$.*

Recall that a space X is *Fréchet* if whenever $x \in \overline{A}$, with $A \subseteq X$, there exists a sequence $\{x_n \mid n \in \omega\} \subseteq A$ that converges to x. Obviously, every first countable space is Fréchet. Indeed, let X be a first countable space, let $A \subseteq X$, and suppose $x \in \overline{A}$. Choose a decreasing local base $\{U_n \mid n \in \omega\}$ at x. For each $n \in \omega$, pick an element $x_n \in A \cap U_n$. Then the sequence $(x_n)_{n\in\omega}$ converges to x. On the other hand, there exist Fréchet spaces that are not first countable[1]. For example, the one-point compactification of a discrete space of cardinality $\mathfrak{c}$ is one such space.

Lemma 2.0.6. *Let $\mathcal{A}$ be an almost disjoint family and $X \subseteq \omega$. Then $X \in \mathcal{I}^+(\mathcal{A})$ if and only if $\infty \in \overline{X}$ (in $\mathcal{F}(\mathcal{A})$).*

Proof. If $A_0, \dots, A_n \in \mathcal{A}$ and $F \in [\omega]^{<\omega}$, then $X \not\subseteq \bigcup_{i=0}^n A_i \cup F$ if and only if

$$\left(\{\infty\} \cup (\mathcal{A} \setminus \{A_0, \dots, A_n\}) \cup \left(\omega \setminus \bigcup_{i=0}^{n}(A_i \cup F)\right)\right) \cap X \neq \emptyset.$$

This proves that $X \in \mathcal{I}^+(\mathcal{A})$ if and only if every basic neighborhood of ∞ meets X. □

Theorem 2.0.7. (See [39, Proposition 10.1]) *Let $\mathcal{A}$ be an almost disjoint family, then $\mathcal{A}$ is nowhere mad if and only if $\mathcal{F}(\mathcal{A})$ is a Fréchet space.*

Proof. We include the proof of this result for completeness, as it is often omitted in the literature due to its simplicity.

Assume $\mathcal{F}(\mathcal{A})$ is Fréchet and let $X \in \mathcal{I}^+(\mathcal{A})$. By Lemma 2.0.6, we have $\infty \in \overline{X}$. Since $\mathcal{F}(\mathcal{A})$ is Fréchet, there is $B \in [X]^\omega$ converging to ∞. In particular, for each $A \in \mathcal{A}$,

$$B \setminus \left(\{\infty\} \cup (\mathcal{A} \setminus \{A\}) \cup (\omega \setminus A)\right) = B \cap A$$

[1]In fact, as we see in Section 4.3, there is an important class strictly between first countable and Fréchet spaces (see Definition 4.3.3).

is finite. Hence, $\mathcal{A}$ is nowhere mad.

Now, suppose $\mathcal{A}$ is nowhere mad and let $Y \subseteq \mathcal{F}(\mathcal{A})$ and $x \in \overline{Y}$. We can assume, without loss of generality, that Y is infinite. If $Y \cap \mathcal{A}$ is infinite and $x = \infty$, choose any infinite sequence in $Y \cap \mathcal{A}$, which converges to ∞. If $x = A \in \mathcal{A}$, then $|Y \cap A| = \omega$, so any enumeration of $Y \cap A$ converges to A. Finally, if $x = \infty$ and $|Y \cap \mathcal{A}| < \omega$, then necessarily $|Y \cap \omega| = \omega$, then $X := Y \cap \omega \in \mathcal{I}^+(\mathcal{A})$, so by hypothesis there is $B \in [X]^\omega \cap \mathcal{A}^\perp$. Any enumeration of B converges to ∞ since, for every finite $F \subseteq \omega$ and $A_0, \dots, A_n \in \mathcal{A}$,

$$B \setminus \left(\{\infty\} \cup (\mathcal{A} \setminus \{A_0, \dots, A_n\}) \cup \left(\omega \setminus \bigcup_{i=0}^{n} (A_i \cup F) \right) \right)$$

is finite. □

We now recall some important results concerning n-sequential compactness proved in [49]. We include some proofs because we consider them important and because the techniques may appear later in subsequent sections; for example, the proof of the following lemma inspires a result that will later be crucial for determining when a space is not n-sequentially compact (see Lemma 3.2.2).

Proposition 2.0.8. (See [49, Example 4.1]) *The Franklin space of a maximal almost disjoint family is not* 2*-sequentially compact.*

Proof. Let $\mathcal{A}$ be an infinite almost disjoint family. Without loss of generality, we can assume that $\mathcal{A} \subseteq [\omega \times \omega]^\omega$ and that $A_n := \{n\} \times \omega \in \mathcal{A}$ for each $n \in \omega$. Then

$$\mathcal{F}(\mathcal{A}) = (\omega \times \omega) \cup \mathcal{A} \cup \{\infty\}.$$

Define

$$G \colon [\omega]^2 \to \mathcal{F}(\mathcal{A}) \qquad \text{by} \qquad G(\{n, m\}) = (n, m) \quad \text{for } n < m.$$

Fix any $B \in [\omega]^\omega$. We claim $G \restriction [B]^2$ is not convergent.

First, since $G \restriction [B]^2$ is injective, it cannot converge to any isolated point of $\omega \times \omega$.

Next, let $n \in \omega$ and $F \in [B]^{<\omega}$. Choose $l < k$ in $B \setminus F$ with $l > n$. Then

$$G(\{l, k\}) = (l, k) \in A_l,$$

so $G''[B \setminus F]^2 \not\subseteq \{A_n\} \cup A_n$. Hence, $G \restriction [B]^2$ cannot converge to any A_n.

Now, consider any $A \in \mathcal{A} \setminus \{A_n \mid n \in \omega\}$ and $F \in [B]^{<\omega}$. Let $m = \min(B \setminus F)$. Then $G''[B \setminus F]^2 \cap A_m$ is infinite, while $A \cap A_m$ is finite. Therefore

$$G''[B \setminus F]^2 \not\subseteq \{A\} \cup A,$$

so $G \restriction [B]^2$ does not converge to A.

Finally, we show that $G \restriction [B]^2$ does not converge to ∞. Enumerate $B = \{b_k \mid k \in \omega\}$ in increasing order. By maximality of $\mathcal{A}$, there is $A \in \mathcal{A} \setminus \{A_n \mid n \in \omega\}$ such that

$$A \cap \{(b_{2k}, b_{2k+1}) \mid k \in \omega\}$$

is infinite. Let

$$U := \mathcal{F}(\mathcal{A}) \setminus (\{A\} \cup A),$$

which is an open neighborhood of ∞. For any $F \in [B]^{<\omega}$, pick $k \in \omega$ with $b_{2k} > \max F$ and $(b_{2k}, b_{2k+1}) \in A$. Then

$$G(\{b_{2k}, b_{2k+1}\}) \notin U,$$

so $G''[B \setminus F]^2 \not\subseteq U$. This completes the proof. □

Note that if $\mathcal{A}$ is mad, then $\mathcal{F}(\mathcal{A})$ is sequentially compact. To see this, let $f\colon \omega \to \mathcal{F}(\mathcal{A})$. We must find $B \in [\omega]^\omega$ such that $f \restriction B$ converges. Define

$$c_f\colon [\omega]^2 \to \{0,1\}, \quad c_f(\{a,b\}) = 0 \iff f(a) = f(b).$$

By Ramsey's theorem, there is $B \subseteq [\omega]^\omega$ on which $c_f \restriction [B]^2$ is constant of color $i \in \{0,1\}$.
- If $i = 0$, then $f \restriction B$ is constant and hence converges.
- If $i = 1$, then there is $B' \in [B]^\omega$ such that either $f''B' \subseteq \omega$ or $f''B' \subseteq \mathcal{A}$. In the second case $f \restriction B'$ converges to ∞. In the first case, since $\mathcal{A}$ is maximal, pick $A \in \mathcal{A}$ such that $f''B' \cap A$ is infinite. Then the set

$$B'' = \{\, n \in B' \mid f(n) \in A\}$$

is infinite and $f \restriction B''$ converges to the point $A \in \mathcal{F}(\mathcal{A})$. Together with Proposition 2.0.8, this proves the following:

Corollary 2.0.9. *If $\mathcal{A}$ is a mad family, its Franklin space $\mathcal{F}(\mathcal{A})$ is a compact, sequentially compact space that is not 2-sequentially compact.*

Let $\mathcal{A}$ be an almost disjoint family on a countable set N, $f : [\omega]^n \to \mathcal{F}(\mathcal{A})$ and suppose that we want to see if we can find a convergent subsequence of f. As before, applying Ramsey's theorem and passing to an infinite subset of ω if needed, we may assume that one of the following cases holds:

1. f is constant ∞,
2. $f''[\omega]^n \subseteq \mathcal{A}$,
3. $f''[\omega]^n \subseteq N$.

The following lemma tells us that we only need to worry about the last case:

Lemma 2.0.10. (See [49, Lemma 4.2]) *If $\mathcal{A}$ is an almost disjoint family on a countable set N of isolated points and $n \geq 1$, $\mathcal{F}(\mathcal{A})$ is n-sequentially compact if and only if for all $f : [\omega]^n \to N$, there exists $B \in [\omega]^\omega$ such that $f \restriction [B]^n$ converges.*

Proof. Clearly, if $f\colon [\omega]^n \to N$, then $f\colon [\omega]^n \to \mathcal{F}(\mathcal{A})$, and since $\mathcal{F}(\mathcal{A})$ is n-sequentially compact, f has a convergent subsequence. To prove the converse, let $g\colon [\omega]^n \to \mathcal{F}(\mathcal{A})$. By Ramsey's theorem, there exists $B \in [\omega]^\omega$ such that

$$g''[B]^n \subseteq N, \quad g''[B]^n \subseteq \mathcal{A}, \quad \text{or} \quad g''[B]^n \subseteq \{\infty\}.$$

In the first case we are done by hypothesis, and in the last case $g \restriction [B]^n$ is constant and thus convergent. Hence, the only remaining case is $g''[B]^n \subseteq \mathcal{A}$. We claim that there is $C \in [B]^\omega$ such that one of the following holds:

(i) There exists $A \in \mathcal{A}$ with $g''[C]^n \subseteq \{A\}$.

(ii) For every $A \in \mathcal{A}$ there is $m \in \omega$ such that $g''[\,C \setminus m\,]^n \cap \{A\} = \emptyset$.

To see this, enumerate $g''[B]^n$ as $\{A_k \mid k \in \omega\}$ (if $g''[B]^n$ is finite, Case (i) follows immediately by Ramsey's theorem). Assume that Case (i) fails for every $C \in [B]^\omega$. Construct a sequence $(B_k)_{k\in\omega}$ such that:

1. $B_0 = B$,
2. $B_{k+1} \in [B_k]^\omega$ and
3. $g''[B_{k+1}]^n \cap \{A_k\} = \emptyset$.

For this, suppose that B_k has been constructed. Then, by Ramsey's theorem, there exists $B_{k+1} \in [B_k]^\omega$ so that either

$$g''[B_{k+1}]^n \subseteq \{A_k\} \quad \text{or} \quad g''[B_{k+1}]^n \cap \{A_k\} = \emptyset.$$

By assumption, the first option never occurs, so the second holds at each step. Let C be a pseudointersection[2] of the B_k's; then C satisfies condition (ii).

Finally, if Case (i) holds, then $g \restriction [C]^n$ is constant and thus convergent; if Case (ii) holds, then $g \restriction [C]^n$ converges to ∞. □

Recall that the *character of* $x \in X$ is the minimum cardinality of a local base of x and it is denoted by $\chi(x, X)$. The character of X is $\chi(X) = \sup\{\chi(x, X) \mid x \in X\}$.

Theorem 2.0.11. (See [49, Theorem 3.1]) *Every sequentially compact space with character less than $\mathfrak{b}$ is n-sequentially compact for all $n \geq 1$.*

In Theorem 3.6.3 we see that $\mathfrak{b}$ is optimal for the theorem above. A consequence of Theorem 2.0.11 and Lemma 2.0.10 is the following.

Corollary 2.0.12. *Let $\mathcal{A}$ be an almost disjoint family on a countable set N, such that for every $f : [\omega]^n \to N$ there exists $X \in [\omega]^\omega$ so that $f''[X]^n$ has closure of size less than $\mathfrak{b}$ in $\Psi(\mathcal{A})$. Then $\mathcal{F}(\mathcal{A})$ is n-sequentially compact.*

We also get:

Corollary 2.0.13. (See [49, Lemma 4.3]) *Let $\mathcal{A}$ be an almost disjoint family on a countable set N of isolated points and $n \geq 1$. If for all $f : [\omega]^n \to N$, there exists $B \in [\omega]^\omega$ such that*

$$\{A \in \mathcal{A} : |A \cap f''[B]^n| = \omega\}$$

has size less than $\mathfrak{b}$, then $\mathcal{F}(\mathcal{A})$ is n-sequentially compact.

One of the key features of the counterexamples given in [49] is the structure of the FIN^n ideal on ω^n. These ideals were introduced by M. Katětov in [45]; to define them, it is convenient to fix the following notation.

Notation 2.0.14. For finite sequences s and t, let $s^\frown t$ denote the concatenation of s and t. More formally, if $\mathrm{dom}(s) = n$ and $\mathrm{dom}(t) = m$, then $s^\frown t$ is defined by $\mathrm{dom}(s^\frown t) = n+m$ and:

$$(s^\frown t)(i) = \begin{cases} s(i) & \text{if } i < n, \\ t(i-n) & \text{if } n \leq i < n+m. \end{cases}$$

In the case where $\mathrm{dom}(t) = 1$ and $t(0) = k$, we simply write $s^\frown k$ or $s^\frown\{k\}$ to denote the finite sequence $s^\frown t$.

[2]For a family $\mathcal{F} \subseteq [\omega]^\omega$ and $X \in [\omega]^\omega$, we say that X is a *pseudointersection* of $\mathcal{F}$ if $X \setminus Y$ is finite for every $Y \in \mathcal{F}$, i.e., X is *almost contained* in Y.

Recall that $\mathsf{FIN} = [\omega]^{<\omega}$ denotes the ideal of finite sets of ω. The ideal FIN^n is defined recursively as the *Fubini product* of $\mathsf{FIN} \otimes \mathsf{FIN}^{n-1}$. That is, $X \in \mathsf{FIN}^n$ if

$$\{x \in \omega^{n-1} : |\{n \in \omega : x^\frown n \in X\}| = \omega\} \in \mathsf{FIN}^{n-1}.$$

The following combinatorial lemma is very useful for us further along.

Lemma 2.0.15. (See [49, Lemma 4.5]) *For every $f : [\omega]^n \to \omega^{n+1}$, there exists $X \in [\omega]^\omega$ such that $f''[X]^n \in \mathsf{FIN}^{n+1}$.*

To deal with these kinds of functions, we also need a useful tool known as the *canonical Ramsey theorem*. This very deep and powerful fact due to Erdös and R. Rado is that there is a finite, very easy to describe "base" for the equivalence relations on $[\omega]^n$.

For the rest of this paper, given $a \in [\omega]^n$, we write it as $\{a_i \mid i < n\}$, where $a_0 < \cdots < a_{n-1}$. If $I \subseteq n$, let $a \restriction I := \{a_i \mid i \in I\}$. For every $I \subseteq n$, we can define the equivalence relation E_I on $[\omega]^n$ as follows:

$$aE_Ib \iff a \restriction I = b \restriction I.$$

Note that:

- $aE_\emptyset b$ for every $a, b \in [\omega]^n$, and
- $aE_n b$ if and only if $a = b$.

Of course, not every equivalence relation is of this form. However, the surprising fact is that every equivalence relation is "locally" one of the ones we just described. Formally, we have the following:

Theorem 2.0.16. (See [23, Theorem 1]) *For every equivalence relation E on $[\omega]^k$ there are $M \in [\omega]^\omega$ and $I \subseteq k$ such that aEb if and only if $a \restriction I = b \restriction I$ for all $a, b \in [M]^k$.*

The reader can learn more about this topic from Todorcevic's book [62].

Applying Theorem 2.0.16 several times, we get the following:

Corollary 2.0.17. *Let $E_1, \ldots, E_l$ be equivalence relations on $[\omega]^n$. There exist $M \in [\omega]^\omega$ and $I_1, \ldots, I_l \subseteq n$ such that for all $a, b \in [M]^n$ and $i \leq l$, we have*

$$aE_ib \iff a \restriction I_i = b \restriction I_i.$$

We use Theorem 2.0.16 (and Corollary 2.0.17) in the context of n-sequentially compact spaces in the following uniform way:

Fix $f \colon [\omega]^n \to X$. Define an equivalence relation E_f on $[\omega]^n$ by

$$sE_ft \quad\iff\quad f(s) = f(t).$$

By Theorem 2.0.16, there are $I \subseteq n$ and $M \in [\omega]^\omega$ such that for all $s, t \in [M]^n$,

$$sE_ft \quad\iff\quad s \restriction I = t \restriction I.$$

Thus, on $[M]^n$, the value of f depends only on the projection to the coordinates in I. One then proceeds by a short case analysis on I:

- If $I = \emptyset$, then $f \restriction [M]^n$ is constant and thus convergent.

- If $n-1 \in I$, then $f \restriction [M]^n$ is *finite-to-one*[3] and, relying on the additional structure of X, this could be enough to prove that $f \restriction [M]^n$ is convergent.
- If $\emptyset \neq I \subseteq n-1$, then f factors through the projection $\pi\colon [M]^n \to [M]^{n-1}$, so the n-ary problem reduces to a lower-arity problem. Then one either repeats the same Erdös–Rado argument on the lower arity or applies sequential compactness of X (if possible) to extract a convergent subsequence.

In short: apply Erdös–Rado to obtain an infinite set M and an index set I, then handle the resulting projection-factorization of f by either reducing the arity or applying extra X-relying compactness/diagonal arguments to produce the desired convergent subsequence.

[3] $f : X \to Y$ is finite-to-one if $f^{-1}(\{y\})$ is finite for all $y \in Y$.

Chapter 3

Finite dimension

As noted in the introduction, this chapter is based entirely on the article [15] by Corral, Guzmán and the author of this thesis.

3.1 Examples of n-sequentially compact spaces

In this section we give some examples of n-sequentially compact spaces. By Theorem 2.0.11, we know that there are sequentially compact spaces that fail to even satisfy the 2-sequential compactness property and that, in general, the properties of n-sequential compactness and $(n+1)$-sequential compactness may be consistently different. To see how subtle it is to stratify the classes of n-sequentially compact spaces, we analyze some classical examples of sequentially compact spaces that are not first countable. These are necessary assumptions due to the following results:

Proposition 3.1.1. *(I)* (See [49, Proposition 2.2]) *If $n \geq 1$ and X is $(n+1)$-sequentially compact, then X is n-sequentially compact. In particular, if X is n-sequentially compact for some $n \geq 1$, then X is sequentially compact.*

(II) (See [49, Theorem 3.1]) *If X is sequentially compact and first countable, then X is n-sequentially compact for all $n \geq 1$.*

Indeed, to see *(I)*, it suffices to observe that given any function $f\colon [\omega]^n \to X$, one can define

$$\bar{f}\colon [\omega]^{n+1} \to X \quad \text{by} \quad \bar{f}(s) = f\Big(s \setminus \{\max s\}\Big).$$

Since X is $(n+1)$-sequentially compact, there exists $B \in [\omega]^\omega$ such that $\bar{f} \restriction [B]^{n+1}$ converges. But then $f \restriction [B]^n$ also converges. On the other hand, *(II)* follows directly from Theorem 2.0.11.

The following five examples are all the sequentially compact, non first countable spaces that appear in the book [60] (which can also be conveniently accessed on the website [14]):

1. The Fort space.

2. The closed long ray.

3. The altered long ray.

4. The Alexandroff square.

5. $\omega_1 + 1$.

We describe them in detail below and see that indeed all of them are n-sequentially compact for every $n \geq 1$. Note that as an easy consequence of Proposition 3.1.1 item (2) we have the following:

Corollary 3.1.2. (See. [49, Corollary 3.2]) *Let X be a sequentially compact space in which the closure of every countable set is first countable. Then X is n-sequentially compact for all $n \geq 1$.*

From the previous corollary we see that the one-point compactification of every discrete space is n-sequentially compact for all $n \geq 1$. In particular, the *Fort space*, which is the one-point compactification of the discrete space of size $\mathfrak{c}$, is n-sequentially compact for all $n \geq 1$. However, one can also give a more constructive, Ramsey-style proof of this fact. We include it here because it helps to familiarize the reader with arguments that are common in this area.

Let X be a discrete space, let $\bar{X} = X \cup \{\infty\}$ be its one-point compactification, $n \geq 1$ and let $f : [\omega]^n \to X$. Applying Ramsey's theorem, we can find $N \in [\omega]^\omega$ such that $f''[N]^2 \subseteq X$ or $f''[N]^2 \subseteq \{\infty\}$. Since in the latter case we are done, assume that we are in the former and let $N = \omega$. Now, we apply Theorem 2.0.16 to get $M \in [\omega]^\omega$ and $I \subseteq n$ such that for all $s, t \in [M]^n$, $f(s) = f(t)$ if and only if $s \restriction I = t \restriction I$.

If $I = \emptyset$, then $f \restriction [M]^n$ is constant, so it is convergent. If $n - 1 \in I$ then f is finite-to-one, so if $U = \bar{X} \setminus \{x_0, \ldots, x_l\}$ is a basic neighborhood of ∞ we find $m \in \omega$ such that $f''[M \setminus m]^n \cap \{x_0, \ldots, x_l\} = \emptyset$, i.e., $f \restriction M$ converges to ∞.

If $\emptyset \neq I \subseteq n - 1$, then let $k = \max I + 1$. Then we can define $g : [M]^k \to X$ by $g(s) = f(s \cup t)$ where $t \in [M \setminus (\max s + 1)]^{n-k}$. Note that the fact that $k = \max I + 1$ implies that $g(s)$ does not depend on the election of t. Note now that $g \restriction [M]^k$ is again finite-to-one and then convergent, but this implies that f is also convergent, since for all $F \in [\omega]^{<\omega}$ we have $f''[M \setminus F]^n = g''[M \setminus F]^{n-k}$. This finishes the proof.

Note that this argument does not work if the neighborhoods of ∞ are co-countable instead of cofinite i.e., if X is a discrete not countable space and we take $\tilde{X} = X \cup \{\infty\}$ for some $\infty \notin X$, where the basic neighborhoods of ∞ are the co-countable sets, then $\tilde{X}$ is not seq compact, since $\tilde{X}$ has no non-trivial convergent sequences.

Example 3.1.3. (**Closed long ray**) Consider $\mathbb{L} = \omega_1 \times [0, 1)$ with the lexicographic order and $\tilde{\mathbb{L}} = \mathbb{L} \cup \{\infty\}$ where ∞ is the maximum point. Then *the closed long ray* $\tilde{\mathbb{L}}$ with the order topology is n-sequentially compact for all $n \geq 1$.

Proof. Let $f : [\omega]^n \to \tilde{\mathbb{L}}$, applying Ramsey's theorem, we know that there exist $B \in [\omega]^\omega$ such that either $f''[B]^n \subseteq \mathbb{L}$ or $f''[B]^n = \{\infty\}$. In the latter case we are done and in the former note that $f''[B]^n \subseteq \mathbb{L}$ and that $\mathbb{L}$ is first countable and sequentially compact so we can apply Corollary 3.1.2. □

Note that $\tilde{\mathbb{L}}$ in Example 3.1.3 is sequentially compact but it is not first countable, since $\infty \in \tilde{\mathbb{L}}$ has no countable base.

Example 3.1.4. (**Altered long ray**)) Consider $\widehat{\mathbb{L}} = \omega_1 \times [0, 1) \cup \{\infty\}$ where $\omega_1 \times [0, 1)$ has the lexicographic order and the basic neighborhoods of ∞ are of the form:

$$N(\infty, \beta) = \{\infty\} \cup \left(\bigcup_{\alpha \geq \beta} (\{\alpha\} \times (0, 1)) \right) = \{\infty\} \cup ((\omega_1 \setminus \beta) \times (0, 1)).$$

Then $\widehat{\mathbb{L}}$ is n-sequentially compact for all $n \geq 1$.

Proof. Let $f : [\omega]^n \to \widehat{\mathbb{L}}$; by applying Ramsey's theorem, we know that there exist $B \in [\omega]^\omega$ such that either $f''[B]^n \subseteq \mathbb{L}$ or $f''[B]^n = \{\infty\}$. However, note that the topology of $\mathbb{L}$ as a subspace of $\widehat{\mathbb{L}}$ coincides with the topology of $\mathbb{L}$ as a subspace of $\tilde{\mathbb{L}}$. □

Let us denote by Δ the set of points of the form (x, x) for $x \in [0, 1]$. If $x \in [0, 1]$, then denote by V_x the vertical line given by x, i.e., $V_x = \{(x, y) \mid y \in [0, 1]\}$.

The following example illustrates the usual way in which we apply Theorem 2.0.16 to derive n-sequential compactness of certain spaces.

Example 3.1.5. (**Alexandroff square**) Let $\mathbb{A} = [0, 1] \times [0, 1]$ with the following topology: if $p = (x, x) \in \Delta$ then a basic neighborhood of p is of the form:

$$N(p, x_0, \dots, x_{n-1}, \epsilon) = ([0, 1] \times (x - \epsilon, x + \epsilon)) \setminus \left(\bigcup_{i \in n} V_{x_i} \right),$$

where $n \in \omega$, $\epsilon > 0$ and $x \notin \{x_i \mid i \in n\}$. If $p = (x, y) \notin \Delta$ then a basic neighborhood of p is of the form $N(p, \epsilon) = \{(x, t) : |y - t| < \epsilon\}$. Then $\mathbb{A}$ is n-sequentially compact for all $n \geq 1$.

Proof. Let $f : [\omega]^n \to \mathbb{A}$. Now, for every $i \in 2$ let $f_i : [\omega]^n \to [0, 1]$ be such that for all $s \in [\omega]^n$ we have $f(s) = (f_0(s), f_1(s))$. Note that:

(i) If there is $B \in [\omega]^\omega$ such that $f_0 \restriction [B]^n$ is constant, then $f \restriction [B]^n$ has a convergent subsequence[1].

Applying Theorem 2.0.16 to the function f_0, we can get $M \in [\omega]^\omega$ and $I \subseteq n$ such that for all $s, t \in [M]^n$, $f_0(s) = f_0(t)$ if and only if $s \restriction I = t \restriction I$.

If $I = \emptyset$ then we are done by (i). So assume that $I \neq \emptyset$ and consider $f_1 \restriction [M]^n$. As $[0, 1]$ is n-sequentially compact (with the usual topology), there exist $N \in [M]^\omega$ and $x \in [0, 1]$ such that $f_1 \restriction [N]^n$ converges to x. We claim that $f \restriction [N]^n$ converges to $p = (x, x)$. Thus, let $N(p, x_0, \dots, x_{n-1}, \epsilon)$ be a basic neighborhood of p. As $f_1 \restriction [N]^n$ converges to x, there exists $F \in [N]^{<\omega}$ such that $f_1''[N \setminus F]^n \subseteq (x - \epsilon, x + \epsilon)$.

Now, as $I \neq \emptyset$, we claim that there is $G \in [N]^{<\omega}$ such that $f_0''[N \setminus G]^n \cap \{x_i \mid i \in n\} = \emptyset$. Indeed, for every $i \in n$ there is $s \in [\omega]^{<\omega}$ non-empty such that $f_0(t) = x_i$ if and only if $t \restriction I = s_i$. Now, $G = \bigcup_{i \in n} s_i$ is as desired.

This shows that $f''[N \setminus (F \cup G)]^n \subseteq N(p, x_0, \dots, x_{n-1}, \epsilon)$ and then $f \restriction [N]^n$ converges to p. □

The fact that $\omega_1 + 1$ is n-sequentially compact for all $n \geq 1$ simply follows from Ramsey's theorem and Theorem 2.0.11:

Example 3.1.6. $\omega_1 + 1$ is n-sequentially compact for all $n \geq 1$.

Proof. Let $f : [\omega]^n \to \omega_1 + 1$. Applying Ramsey's theorem, we know that there is $B \in [\omega]^\omega$ such that $f''[B]^n = \{\omega_1\}$ or $f''[B]^n \subseteq \omega_1$. In the first case we are done, so we assume that $f''[B]^n \subseteq \omega_1$. As $f''[B]^n$ is countable, there exists $\gamma \in \omega_1$ such that $\overline{f''[B]^n} \subseteq \gamma + 1$, and $\gamma + 1$ is sequentially compact, has countable character and $\overline{f''[B]^n} \subseteq \gamma + 1$, so we can apply Theorem 2.0.11 to finish the proof. □

[1] In fact, more can be said here: if there is $p \in \mathbb{A} \setminus \Delta$ and $B \in [\omega]^\omega$ such that $f \restriction [B]^n$ converges to p, then f_0 is constant on $[B]^n$.

However, more can be said, as in the realm of limit ordinals, being sequentially compact, having uncountable cofinality and being n-sequentially compact for all $n \geq 1$ are equivalent. To see this, first let us prove the following simple fact:

Proposition 3.1.7. *If γ is an ordinal and $X \subseteq \gamma$ is countable, then $\overline{X}$ is countable.*

Proof. Suppose that $\overline{X}$ is not countable and let Y of size ω_1 be such that $X \subseteq Y \subseteq \overline{X}$. As Y is a set of ordinals, by going to a subset if necessary, we can enumerate Y in order type ω_1, i.e., $Y = \{x_\alpha \mid \alpha \in \omega_1\}$ where $x_\alpha < x_\beta$ if $\alpha < \beta$. Note that as X is countable, there exists $\alpha \in \omega_1$ such that $\{x_\beta \mid \alpha < \beta < \omega_1\} \cap X = \emptyset$, so in particular $(x_\alpha, x_\beta] \cap X = \emptyset$ for any $\beta > \alpha$, which shows that $x_\beta \notin \overline{X}$, a contradiction. □

Note that by Proposition 3.1.7, if γ is a limit ordinal with uncountable cofinality and $X \subseteq \gamma$ is countable, then the closure of X in γ is sequentially compact and first countable, so by Corollary 3.1.2 we have the following:

Corollary 3.1.8. *Let γ be a limit ordinal. Then $cof(\gamma) > \omega$ if and only if γ with the order topology is n-sequentially compact for all $n \geq 1$.*

Finally, we get a full characterization of those ordinals that are n-sequentially compact for all $n \in \omega$.

Corollary 3.1.9. *An ordinal γ is n-sequentially compact for all $n \geq 1$ if and only if $cof(\gamma) \neq \omega$.*

Proof. Let $f : [\omega]^n \to \gamma$ and assume that γ has uncountable cofinality. Let $X = f''[\omega]^n$ and define $\alpha = \sup(X)$. Since γ does not have countable cofinality, $\alpha \in \gamma$. Then it follows that $\alpha = \max(\overline{X})$. Take $\overline{X}$ with the inherited order, since the maximum of $\overline{X}$ exists, $\beta = ot(\overline{X})$ is a countable successor ordinal. Therefore, it is first countable and sequentially compact, which implies that β is n-sequentially compact by Proposition 3.1.1 item (2). Consequently, f has a convergent subsequence in $\overline{X} \subseteq \gamma$. □

With this we now know that all sequentially compact examples in [60] (and only those) are n-sequentially compact for all $n \geq 1$.

3.2 Sequentially compact Fréchet spaces that are not 2-sequentially compact

It is well-known that, in the case of Franklin spaces, being sequentially compact does not imply being Fréchet; however, Paul Szeptycki pointed out to us that a 2-sequentially compact Franklin space must be Fréchet.

Proposition 3.2.1. *If $\mathcal{A}$ is an almost disjoint family such that $\mathcal{F}(\mathcal{A})$ is 2-sequentially compact, then $\mathcal{F}(\mathcal{A})$ is Fréchet.*

Proof. Suppose that $\mathcal{F}(\mathcal{A})$ is not Fréchet, this means, by Theorem 2.0.7, that $\mathcal{A}$ is somewhere mad, i.e., there exists $X \in \mathcal{I}^+(\mathcal{A})$ such that $\mathcal{A} \restriction X := \{A \cap X \mid A \in \mathcal{A}(|A \cap X| = \omega)\}$ is an infinite mad family on X and then $\mathcal{F}(\mathcal{A} \restriction X)$ is not 2-sequentially compact by Proposition 2.0.8.

Now, let $Y := X \cup \{A \mid A \in \mathcal{A} \restriction X\} \cup \{\infty\}$ with the topology inherited by $\mathcal{F}(\mathcal{A})$. We can see that Y is a closed subset of $\mathcal{F}(\mathcal{A})$ and it is homeomorphic to $\mathcal{F}(\mathcal{A} \restriction X)$. As $\mathcal{F}(\mathcal{A})$ is 2-sequentially compact and $Y \subseteq \mathcal{F}(\mathcal{A})$ is closed, Y is 2-sequentially compact and consequently $\mathcal{F}(\mathcal{A} \restriction X)$ is 2-sequentially compact, which is a contradiction. □

In [49], the authors asked whether there is a ZFC example of a Fréchet sequentially compact space that is not 2-sequentially compact. They showed that a completely separable mad family suffices for the construction of such an example. As discussed earlier, the existence of a completely separable mad family holds under very weak assumptions like $\mathfrak{c} < \aleph_\omega$, however, as we stated in Theorem 2.0.4, completely separable almost disjoint families, possibly not maximal, do exist in ZFC. The existence of one of these families in ZFC is good enough for the construction sketched in [49], we give a full proof below.

If N is a countable set, $\mathcal{A} \subseteq \mathcal{P}(N)$ and $X \in [N]^\omega$, then $X \perp \mathcal{A}$ means that $X \cap A$ is finite for all $A \in \mathcal{A}$. The family $\{X \in [N]^\omega \mid X \perp \mathcal{A}\}$ is denoted by $\mathcal{A}^\perp$.

Lemma 3.2.2. *Let $\mathcal{A}$ be an almost disjoint family on a countable set N of isolated points. If there exist $\mathcal{A}_0 = \{A_m \mid m \in \omega\} \subseteq [N]^\omega$ and $f : [\omega]^n \to N$ such that:*

(I) Either, $\mathcal{A}_0 \subseteq \mathcal{A}$ or $\mathcal{A}_0 \subseteq \mathcal{A}^\perp$.

(II) $\mathcal{A}_0$ is a pairwise disjoint family.

(III) f is one-to-one.

(IV) If $\min(s) = m$ then $f(s) \in A_m$.

(V) For all $B \in [\omega]^\omega$ there exists $A \in \mathcal{A} \setminus \mathcal{A}_0$ such that $|f''[B]^n \cap A| = \omega$.

Then $\mathcal{F}(\mathcal{A})$ is not n-sequentially compact.

Proof. We show that $f \restriction [B]^n$ does not converge to any point in $\mathcal{F}(\mathcal{A})$ for all $B \in [\omega]^\omega$. Fix $B \in [\omega]^\omega$; then, as $f \restriction [B]^n$ is one-to-one, it does not converge to any isolated point. Let $F \in [B]^{<\omega}$ and $m \in \omega$. There are $m' > m$ and $a \in [B \setminus F]^n$ such that $\min(a) = m'$. Thus, $f(a) \in A_{m'}$. This shows that $f''[B \setminus F]^n \nsubseteq \{A_m\} \cup A_m$ for any F and thus $f \restriction [B]^n$ does not converge to A_m in the case where $\mathcal{A}_0 \subseteq \mathcal{A}$.

Now, let us see that $f \restriction [B]^n$ does not converge to any point in $\mathcal{A} \setminus \mathcal{A}_0$ either. Let $F \in [B]^{<\omega}$ and fix $A \in \mathcal{A} \setminus \mathcal{A}_0$. Take $m = \min(B \setminus F)$. Thus, by condition *(IV)*, $f''[B \setminus F]^n \cap A_m$ is infinite, and since $A \cap A_m$ is finite, it follows that $f''[B \setminus F]^n \nsubseteq \{A\} \cup A$.

The only thing that remains to be seen is that $f \restriction [B]^n$ does not converge to ∞. Let $A \in \mathcal{A} \setminus \mathcal{A}_0$ be given by item *(V)*. We show that $f''[B \setminus F]^n \nsubseteq U := \mathcal{F}(\mathcal{A}) \setminus (\{A\} \cup A)$ for all $F \in [B]^{<\omega}$. Since U is a basic neighborhood of ∞ in $\mathcal{F}(\mathcal{A})$, this suffices. Let $m = \max F$. Since $A \cap A_i$ is finite for every $i \leq m$ and $|f''[B]^n \cap A| = \omega$, there exists $a \in [B \setminus F]^n$ such that $f(a) \in A$ and we are done.

This shows that $\mathcal{F}(\mathcal{A})$ is not n-sequentially compact as witnessed by f. □

We can now prove the following, answering a question in [49].

Theorem 3.2.3. *There exists (in ZFC) a space which is Fréchet, sequentially compact and not 2-sequentially compact.*

Proof. Let $\mathcal{A}$ be an almost disjoint family on $\omega \times \omega$ which is completely separable, infinite and nowhere mad. We can assume that $A_n = \{n\} \times \omega \in \mathcal{A}$ for all $n \in \omega$. Now, let $G : [\omega]^2 \to \mathcal{F}(\mathcal{A})$ be given by $G(\{m, n\}) = (m, n)$, where $m < n$. We show that $\mathcal{A}$, $\mathcal{A}_0 = \{A_n \mid n \in \omega\}$ and $f = G$ satisfy the hypothesis of Lemma 3.2.2 for $n = 2$.

Conditions (II), (III) and (IV) are clear. For (V) note that (III) and (IV) imply that $|G''[B] \cap A_m| = \omega$ for every $m \in B$, in particular $G''[B]^2 \in \mathcal{I}^{++}(\mathcal{A})$. Since $\mathcal{A}$ is completely separable, using Lemma 2.0.5, there exists $A \in \mathcal{A} \setminus \mathcal{A}_0$ as required in (V). □

3.3 n-sequentially compact spaces from $\mathfrak{b} = \mathfrak{c}$

The relevance of the cardinals $\mathfrak{b}$ and $\mathfrak{d}$ in our examples is evident due to the following facts about $\mathsf{cov}^*(\mathsf{FIN}^n)$ and $\mathsf{cof}(\mathsf{FIN}^n)$. We first introduce the concept of Hechler trees in order to make the analysis of these results cleaner.

Given $n \in \omega$, we say that $T \subseteq \omega^{\leq n+1}$ is a *Hechler tree* if for every $s \in T \cap \omega^{\leq n}$, the set of successors of s is a final segment on ω, i.e., there exists $k_s \in \omega$ such that $succ_T(s) = \{i \in \omega \mid s^\frown i \in T\} = \omega \setminus k_s$. We can naturally associate a function $h_T : T \cap \omega^{\leq n} \to \omega$ to each Hechler tree $T \subseteq \omega^{\leq n+1}$ by defining $h(s) = k_s$.

For $s, t \in \omega^{\leq n}$, we say that $s \prec t$ if either $s \sqsubset t$ or $s = r^\frown l$ and $t = r^\frown m$ for some $r \in \omega^{\leq n}$ and $l < m$, where $s \sqsubset t$ means that s is a proper initial segment of t. Enumerate $\omega^{\leq n} = \{s_i \mid i \in \omega\}$ such that $s_i \prec s_j$ implies $i < j$. We fix this enumeration once and for all.

Definition 3.3.1. Let $f \in \omega^\omega$ and let $T \subseteq \omega^{n+1}$ be a Hechler tree. We define

- $T_f \subseteq \omega^{\leq n+1}$ where $\emptyset \in T_f$ and $succ(s_i) = \omega \setminus f(i)$ for every $s_i \in T_f \cap \omega^{\leq n}$.
- $f_T \in \omega^\omega$ where $f_T(n) = k$ if and only if $s_n \in T$ and $succ_T(s_n) = \omega \setminus k$ and $f_T(n) = 0$ otherwise.

Let $C_s = \{t \in \omega^{n+1} \mid s \sqsubseteq t\}$ for every $s \in \omega^{\leq n}$ and we write C_k instead of $C_{(k)}$. Also, for a tree $T \subseteq \omega^{\leq n}$ we denote its set of branches by $[T] = T \cap \omega^n$. It follows directly from the definition that for every $X \in \mathsf{FIN}^n$ there exists a Hechler tree $T \subseteq \omega^{\leq n}$ such that $X \cap [T] = \emptyset$.

Lemma 3.3.2. *For every $n \geq 1$, if $\mathcal{H} \subseteq \mathsf{FIN}^{n+1}$ is such that $|\mathcal{H}| < \mathfrak{b}$ and $B \in [\omega]^\omega$, then there exists a block sequence*[2] *$A \subseteq B^{n+1}$ such that $|A \cap X| < \omega$ for every $X \in \mathcal{H}$.*

Proof. For every $X \in \mathcal{H}$ let T_X be a Hechler tree disjoint from X. We write f_X instead of f_{T_X}.

Since $|\mathcal{H}| < \mathfrak{b}$, there is a function $f \in \omega^\omega$ such that $f >^* f_X$ for all $X \in \mathcal{H}$. We recursively define a block sequence $(a_m \mid m \in \omega) \subseteq \omega^{n+1}$. Let $(a_m^i : i \leq n)$ denote the sequence a_m. Assume that we have constructed a_k for every $k < m$. For a_m define $a_m^0 \in B \setminus (f(\emptyset) \cup \max(a_{m-1}))$ (where $a_{-1} = 0$). Now, for every $i < n$ pick

$$a_m^{i+1} \in B \setminus f((a_m^0, \ldots, a_m^i))$$

larger than a_m^i. These choices are possible since B is infinite. This finishes the construction of $A = (a_m \mid m \in \omega) \subseteq B^{n+1}$.

To see that it works, fix $X \in \mathcal{H}$. Since $f >^* f_X$ there exists $k \in \omega$ such that if $f_X(n) > f(n)$ then $s_n \in \bigcup_{i<k} C_i$. As $(a_m \mid m \in \omega)$ is a block sequence, $\{a_m^0 \mid m \in \omega\}$ is increasing and thus $a_m^0 > \max\{k, f_X(\emptyset)\}$ for all but finitely many $m \in \omega$. Given any a_m^0 as above, $a_m^0 > f_X(\emptyset)$ and as $(a_m^0, \ldots, a_m^i)$ extends a_m^0 in ω^{n+1} for any $i \leq n$, it follows from the assumption over k that $f((a_m^0, \ldots, a_m^i)) > f_X((a_m^0, \ldots, a_m^i))$. In particular, $(a_m^0, \ldots, a_m^n) \in T_X$, which implies that $X \cap A \subseteq \{a_m \mid a_m^0 < k\}$, with the latter set being finite. $\square$

[2]Recall that $(a_m \mid m \in \omega) \subseteq \omega^n$ is a block sequence if $\max a_m < \min a_{m+1}$ for all $m \in \omega$ and each a_m is increasing.

The previous lemma gives in particular that $\mathfrak{b} \leq \mathsf{cov}^*(\mathsf{FIN}^{n+1})$ for every $n \geq 1$. The other inequality can also be proved using similar ideas[3]:

Proposition 3.3.3. $\mathsf{cov}^*(\mathsf{FIN}^{n+1}) = \mathfrak{b}$ *for every* $n \geq 1$.

Proof. We only need to prove $\mathsf{cov}^*(\mathsf{FIN}^{n+1}) \leq \mathfrak{b}$. Suppose that $\kappa < \mathsf{cov}^*(\mathsf{FIN}^{n+1})$ and let $\{h_\beta \in \omega^\omega \mid \beta \in \kappa\}$. Recall that $\mathfrak{b}$ is the minimum size of a family $\mathcal{B} \subseteq \omega^\omega$ that is unbounded on each infinite subset of ω (see [20]), so it suffices to see that $\{h_\beta \in \omega^\omega \mid \beta \in \kappa\}$ is bounded on some $B \in [\omega]^\omega$. Enumerate ω^n as $\{t_l \mid l \in \omega\}$ such that if there is $r \in \omega^{n-1}$ such that $t_l = r^\frown i$ and $t_{l'} = r^\frown j$ and $i < j$ then $l < l'$.

For each $\beta \in \kappa$, consider the set $Y_\beta = \{t_l^\frown m \mid l \in \omega \wedge m \leq h_\beta(l)\}$. Clearly, each Y_β is in FIN^{n+1}. Now, consider $\mathcal{H} = \{Y_\beta \mid \beta \in \kappa\} \cup \{C_{t_l} \mid l \in \omega\}$. It is clear that $|\mathcal{H}| < \mathsf{cov}^*(\mathsf{FIN}^{n+1})$, so there exists $A \in [\omega^{n+1}]^\omega$ such that $A \cap C_{t_l} =^* \emptyset$ for all $l \in \omega$ and $A \cap Y_\beta =^* \emptyset$ for every $\beta \in \kappa$.

Now, define $f : \omega \to \omega$ as follows:

$$f(l) = \begin{cases} \max\{i \in \omega \mid t_l^\frown i \in A\} & \text{if } A \cap C_{t_l} \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Let $B = \{l \in \omega : A \cap C_{t_l} \neq \emptyset\} \in [\omega]^\omega$. Now, as $A \cap Y_\beta =^* \emptyset$, there exists $l \in \omega$ such that if $m > l$ then $C_{t_m} \cap (A \cap Y_\beta) = \emptyset$, which means that $f(m) > h_\beta(m)$ whenever $m \in B \setminus (l+1)$, i.e., f dominates h_β on B. □

Now, we are ready to prove that under $\mathfrak{b} = \mathfrak{c}$ we can distinguish the classes of n-sequentially compact and $(n+1)$-sequentially compact spaces for every $n \geq 1$.

Theorem 3.3.4. ($\mathfrak{b} = \mathfrak{c}$) *For all* $n \geq 2$ *there exists a* n*-sequentially compact space which is not* $(n+1)$*-sequentially compact.*

Proof. We construct an almost disjoint family $\mathcal{A}$ on ω^{n+1} such that its Franklin space is n-sequentially compact but not $(n+1)$-sequentially compact. Let $G : [\omega]^{n+1} \to \mathcal{F}(\mathcal{A})$ be given by $G(\{m_0, \ldots, m_n\}) = (m_0, \ldots, m_n)$, where $m_0 < \ldots < m_n$.

We enumerate $(\omega^{n+1})^{[\omega]^n} = \{f_\alpha \mid \omega \leq \alpha < \mathfrak{c}\}$ and $[\omega]^\omega = \{B_\alpha \mid \omega \leq \alpha < \mathfrak{c}\}$. We construct $\{A_\alpha \mid \alpha < \mathfrak{c}\} \subseteq \mathcal{P}(\omega^{n+1})$ and $\{X_\alpha \mid \omega \leq \alpha < \mathfrak{c}\} \subseteq \mathcal{P}(\omega)$ such that the following hold:

(1) For all $n \in \omega$, $A_n = C_n$ and A_α is a block sequence for every $\alpha \geq \omega$ (hence $A_\alpha \in \mathsf{FIN}^{n+1}$).

(2) $\forall \beta, \alpha \in \mathfrak{c}\ (\beta \neq \alpha \to A_\beta \cap A_\alpha =^* \emptyset)$.

(3) $f_\alpha''[X_\alpha]^n \in \mathsf{FIN}^{n+1}$.

(4) $\forall \beta, \alpha \in \mathfrak{c}\ (\omega \leq \beta < \alpha \to f_\beta''[X_\beta]^n \cap A_\alpha =^* \emptyset)$.

(5) $\forall \alpha \in [\omega, \mathfrak{c})\ (|A_\alpha \cap G''[B_\alpha]^{n+1}| = \omega)$.

Assume that we have already constructed $\{A_\beta \mid \beta < \alpha\}$ and $\{X_\beta \mid \beta < \alpha\}$ for some $\alpha \geq \omega$. We can apply Lemma 3.3.2 to the family $\mathcal{H}$ defined as follows:

$$\mathcal{H} = \{f_\beta''[X_\beta]^n \mid \omega \leq \beta < \alpha\} \cup \{A_\beta \mid \beta < \alpha\}$$

and the infinite set B_α. In this way, we get a block sequence $A_\alpha \subseteq B_\alpha^{n+1}$ such that:

[3] The following result is well-known, but we add a proof since we were unable to find a proof in the literature.

(a) $\forall \beta \in \alpha\ (|A_\beta \cap A_\alpha| < \omega)$.

(b) $\forall \beta \in \alpha\ (\omega \leq \beta < \alpha \to |f''_\beta[X_\beta]^n \cap A_\alpha| < \omega)$.

This ensures that conditions (2) and (4) hold; also, as A_α is a block sequence of elements of B_α, condition (5) holds. Now, we find $X_\alpha \in [\omega]^\omega$ such that $f''_\alpha[X_\alpha]^n \in \mathsf{FIN}^{n+1}$ (and this one exists by Lemma 2.0.15). This finishes the construction.

Consider $\mathcal{A} = \{A_\alpha \mid \alpha \in \mathfrak{c}\}$. Condition (4) implies that $f''_\beta[X_\beta]^n$ has closure of size less than $\mathfrak{c}$ in $\mathcal{F}(\mathcal{A})$ and, as $\mathfrak{b} = \mathfrak{c}$, it has closure of size less than $\mathfrak{b}$, so by Corollary 2.0.13, it has a convergent subsequence. On the other hand, condition (5) implies that condition *(V)* of Lemma 3.2.2 holds and the other three conditions when $f = G$ are clear, so $\mathcal{F}(\mathcal{A})$ is not $(n+1)$-sequentially compact. $\square$

3.4 High dimensional splitting-like cardinal invariants

We say that $\mathcal{P} = \{P_n \mid n \in \omega\} \subseteq [\omega]^{<\omega}$ is an *interval partition* if $\mathcal{P}$ is a partition of ω consisting of consecutive intervals.

Definition 3.4.1. (See [44, p. 266]) Let $\mathcal{P} = \{P_n \mid n \in \omega\}$ be an interval partition and $S \subseteq \omega$. We say that S *block-splits* $\mathcal{P}$ if

$$|\{n \in \omega \mid P_n \subseteq S\}| = \omega = |\{n \in \omega \mid P_n \cap S = \emptyset\}|.$$

A family $\mathcal{S} \subseteq [\omega]^\omega$ is a *block-splitting* family if for every interval partition $\mathcal{P}$, there exists $S \in \mathcal{S}$ such that S block-splits $\mathcal{P}$.
We also define

$$\mathfrak{bs} = \min\{|\mathcal{S}| : \mathcal{S} \text{ is a block splitting family}\}.$$

It is easy to see that every block-splitting family is a splitting family. It follows that $\mathfrak{s} \leq \mathfrak{bs}$. In fact, we have the following theorem due to A. Kamburelis and B. Węglorz.

Theorem 3.4.2. (See [44, Proposition 2.1]) $\mathfrak{bs} = \max\{\mathfrak{b}, \mathfrak{s}\}$.

Recall that for any $s \in \omega^{\leq n}$, we denote by C_s the set of $t \in \omega^n$ such that $t \restriction dom(s) = s$. If $n \geq 1$, we denote by $\mathcal{L}_n$ the set of functions $f : [\omega]^n \to \omega^{n+1}$ such that there is no $X \in [\omega]^\omega$ so that $f''[X]^n \subseteq C_s$ for some $s \in \omega^{<n+1} \setminus \{\emptyset\}$ (note that it is equivalent if we require this for $s \in \omega^1$).

Definition 3.4.3. Let $n \geq 1$, $f \in \mathcal{L}_n$ and $S \in [\omega]^\omega$. We say that S *splits* f (or more precisely S $\mathcal{L}_n$*-splits* f) if there are disjoint $X, Y \in [\omega]^\omega$ such that:

1. $f''[X]^n \subseteq S^{n+1}$, and
2. $f''[Y]^n \subseteq (\omega \setminus S)^{n+1}$.

We define μ_n as the minimum size of an $\mathcal{L}_n$-splitting family, i.e., the minimum size of a family $\mathcal{S} \subseteq [\omega]^\omega$ such that every $f \in \mathcal{L}_n$ is $\mathcal{L}_n$-splitted by an element in $\mathcal{S}$.

At the moment, it is not even clear that there are $\mathcal{L}_n$-splitting families, but we show later in this section that they do exist and we find an upper bound for μ_n.

We can consider ω^n as the set of increasing sequences of ω of length n through the computable bijection φ that maps $(i_0, \dots, i_{n-1})$ into

$$\left(i_0\ ,\ i_0 + i_1 + 1\ ,\ \dots, \sum_{j<n} i_j + (n-1)\right).$$

Notation 3.4.4. For a set $X \subseteq \omega$ and $n \in \omega$ let

- $nX = \{nx \mid x \in X\}$ and
- $\frac{X}{n} = \{\frac{x}{n} \mid x \in X\}$ if $n \neq 0$.

We start by finding a lower bound for these new invariants.

Proposition 3.4.5. $\mathfrak{s} \leq \mu_1$.

Proof. Let $\mathcal{S}$ be an $\mathcal{L}_1$-splitting family of size μ_1. With this, we define a splitting family of size at most μ_1. For every $X \in [\omega]^\omega$, define $f_X : \omega \to \omega^2$ such that if $X = \{x_n \mid n \in \omega\}$ is the increasing enumeration of X, then $f(n) = (2x_n, 2x_n + 1)$. Clearly, $f_X \in \mathcal{L}_1$.

There exists $S \in \mathcal{S}$ that $\mathcal{L}_1$-splits f_X. Hence, there are disjoint $Y, Z \in [\omega]^\omega$ such that $f''Y \subseteq S^2$ and $f''Z \subseteq (\omega \setminus S)^2$. In particular, for every $n \in Y$, we know that $f(n) \in S^2$, which implies that $2x_n \in S$. Similarly, $2x_m \in \omega \setminus S$ for every $m \in Z$.

Thus, we can define $S' = S \cap 2\omega$ and $\widehat{S} = \frac{S'}{2}$ for every $S \in \mathcal{S}$. It is now easy to see that $\{\widehat{S} \mid S \in \mathcal{S}\}$ is a splitting family of size at most μ_1. $\square$

Given $t \in \omega^n$, we write $t = (t_0, \dots, t_{n-1})$ and denote $im^*(t) = \{t_0, \dots, t_{n-1}\}$. Now, we see that the cardinals μ_n form a non-decreasing sequence.

Proposition 3.4.6. *Let $n \geq 1$. Every $\mathcal{L}_{n+1}$-splitting family is also an $\mathcal{L}_n$-splitting family. Hence, $\mu_n \leq \mu_{n+1}$ for every $n \geq 1$.*

Proof. Let $\mathcal{S}$ be an $\mathcal{L}_{n+1}$-splitting family and let $f \in \mathcal{L}_n$. Define $f' \in \mathcal{L}_{n+1}$ such that f' maps the cone C_s into $f(s)^\frown k$, where k is the minimum i that can be concatenated to the sequence $f(s)$. Formally,

$$f'(s) = f(s \setminus \max(s))^\frown(f(s \setminus \max(s))_n + 1)$$

for every $s \in [\omega]^{n+1}$. Let $S \in \mathcal{S}$ be such that it $\mathcal{L}_{n+1}$-splits f' and let $X, Y \in [\omega]^\omega$ be witnesses of this. Let $a \in [X]^n$ and pick $x \in X$ such that $x > \max(a)$. Since $f'[X]^{n+1} \subseteq S^{n+2}$, and $im^*(f(a)) \subseteq im^*(f'(a \cup \{x\}))$, it is now easy to see that $f(a) \in S^{n+1}$. Therefore, $f''[X]^n \subseteq S^{n+1}$. Similarly $f''[Y]^n \subseteq (\omega \setminus S)^{n+1}$. $\square$

Now, we give an upper bound for μ_n and show that $\mathcal{L}_n$-splitting families exist. In order to achieve this, we need to work with equivalence relations on $[\omega]^n$ and apply the Erdös–Rado theorem.

Theorem 3.4.7. *For every $n \in \omega$, $\mu_n \leq \mathfrak{bs}$.*

Proof. Fix $n \in \omega$. We show that every block splitting family is $\mathcal{L}_n$-splitting. Given $f \in \mathcal{L}_n$ and $a \in [\omega]^n$, we write $f(a)_i = b_i$ if $f(a) = (b_0, b_1, \ldots, b_n)$. Using Corollary 2.0.17, we find $M \in [\omega]^\omega$ and $I_j \subseteq n$ for every $j \leq n$, such that for every $a, a' \in [M]^n$, $f(a)_j = f(a')_j$ if and only if $a \restriction I_j = a' \restriction I_j$.

We claim that $I_j \neq \emptyset$ for every $j \leq n$. Indeed, if $I_j = \emptyset$ for some j, there exists $k \in \omega$ such that $f(a)_j = k$ for every $a \in [M]^n$. Since $f(a)$ is increasing for every $a \in [\omega]^n$, there is a finite set F such that $f(a)_0 \in F$ for every $a \in [M]^n$, thus applying Ramsey's theorem, we find $M' \in [M]^\omega$ and $l < k$ such that $f(a)_0 = l$ for every $a \in [M']^n$. That is, $f''[M']^n \subseteq C_l$, contradicting $f \in \mathcal{L}_n$.

For every $j \leq n$, let $m_j = \max(I_j)$. As none of the I_j is empty, the set $\{f(a)_j \mid a \in [M]^n\}$ is infinite. Moreover,

($*$) for every $j \leq n$ and $k \in \omega$, there exists $z \in M$ such that $f(a)_j > k$ for every $a \in [M]^n$ with $z \in a = \{a_0, \ldots, a_{n-1}\}$ and $z = a_i$ for some $i \leq m_j$.

To see this, fix $j \leq n$ and $k \in \omega$. For every $k' \leq k$ there exists $a(k') \in [M]^n \cup \{\emptyset\}$ such that $f(a)_j = k'$ if and only if $a \restriction I_j = a(k') \restriction I_j$ for every $a \in [M]^n$ (possibly $a(k') = \emptyset$ if $f(a)_j \neq k'$ for every $a \in [M]^n$). Now, let $F = \bigcup_{k' \leq k} a(k')$. Thus, any $z > \max(F)$ works.

It is also worth noting that given $j \leq n$, $z \in M$ and $t = (t_0, \ldots, t_{m_j - 1}) \in (M \cap z)^{m_j}$, there is $r(j, z, t) \in \omega$ such that

($**$) $f(t^\frown\{z\}^\frown s)_j = r(j, z, t)$

for every $s \in [M \setminus (z+1)]^{n - m_j}$ (recall that $t^\frown s$ is the concatenation of the sequences t and s).

We are going to define an interval partition $\mathcal{P} = \{P_l \mid l \in \omega\}$ such that any S, that block splits $\mathcal{P}$, also $\mathcal{L}_n$-splits f. It is clear that if we achieve this, then $\mu_n \leq \mathfrak{bs}$.

To define this partition $\mathcal{P}$, we also define an increasing sequence $\{x_l \mid l \in \omega\} \subseteq M$ and an increasing sequence $\{k_l \mid l \in \omega\}$.

Let $x_0 = \min(M)$ and $k_0 = 0$. For $l > 0$, by ($*$), we can find $x_l > \max\{x_i \mid i < l\}$ in M such that $f(a)_j > k_{l-1}$ for any $j \leq n$ and any $a \in [M]^n$ such that $a_i = x_l$ for some $i \leq m_j$. Given $j \leq n$ and $t = \{t_0, \ldots, t_{m_j - 1}\} \subseteq \{x_i \mid i < l\}$ let $r(j, x_l, t)$ be as in ($**$) and define

$$k_l > \max(\{r(j, x_l, t) \mid j \leq n \ \wedge \ t \in [\{x_i \mid i < l\}]^{m_j}\} \cup \{k_i \mid i < l\}).$$

Define $P_l = (k_{l-1}, k_l]$ for every $l \in \omega$ and let $\mathcal{P} = \{P_l \mid l \in \omega\}$.

We proceed to show that $\mathcal{P}$ is as required. Let $S \in [\omega]^\omega$ be such that block-splits $\mathcal{P}$ and define $X = \{x_l \mid P_l \subseteq S\}$ and $Y = \{x_l \mid P_l \cap S = \emptyset\}$. Let $a = \{x_{l_i} \mid i < n\} \subseteq X$ where $x_{l_0} < x_{l_1} < \cdots < x_{l_{n-1}}$. We prove $f(a) \in S^{n+1}$. Fix $j \leq n$ and let $i = m_j < n$. By the choice of l_i, it follows from the definition of x_{l_i} that $f(a)_j > k_{l_i - 1}$. On the other hand, we have

$$f(a)_j = f(t^\frown x_{l_i}^\frown s)_j = r(j, x_{l_i}, t) < k_{l_i},$$

with $t = \{x_{l_0}, \ldots, x_{l_{m_j - 1}}\}$ and $s = \{x_{l_{m_j + 1}}, \ldots, x_{l_{n-1}}\}$. Thus, $f(a)_j \in P_{l_i}$ and $P_{l_i} \subseteq S$ since $x_{l_i} \in X$. As this is true for every $j \leq n$, we have $f(a) \subseteq S^{n+1}$ and therefore $f''[X]^n \subseteq S^{n+1}$. A similar argument shows that $f''[Y]^n \subseteq (\omega \setminus S)^{n+1}$. $\square$

It follows from the results in this section that $\mathfrak{s} = \mu_n = \mathfrak{b}$ is consistent for every $n \geq 1$. We do not know whether any of these equalities holds in ZFC.

We ask the following natural questions:

Question 3.4.8. Given $n \geq 1$, is it consistent that $\mu_n < \mu_{n+1}$?

Question 3.4.9. Is it true that $\mu_n = \mathfrak{bs}$ for all $n \geq 1$?

Question 3.4.10. Is it consistent that $\mathfrak{s} < \mu_0 < \mu_1 < \ldots < \mu_n < \mu_{n+1} < \ldots < \mathfrak{b}$?

3.5 n-sequentially compact spaces from $\mathfrak{s} = \mathfrak{b}$.

In this section, we construct special almost disjoint families following the ideas and techniques developed by Shelah in [59] and improved by Mildenberger, Raghavan and Steprāns in [54] and [57]. For a survey of this method, see [43], [30]. We first need to prove that every $\mathcal{L}_n$-splitting family is *everywhere $\mathcal{L}_n$-splitting.*

Lemma 3.5.1. *Let $\mathcal{S}$ be a $\mathcal{L}_n$-splitting family, $A \in [\omega]^\omega$ and $f : [A]^n \to \omega^{n+1}$ such that there is no $X \in [A]^\omega$ so that $f''[X]^n \subseteq C_s$ for some $s \in \omega^{<n+1} \setminus \{\emptyset\}$. Then there are $X, Y \in [A]^\omega$ and $S \in \mathcal{S}$ such that $f''[X]^n \subseteq S^{n+1}$ and $f''[Y]^n \subseteq (\omega \setminus S)^{n+1}$.*

Proof. Let $g : \omega \to A$ be a bijection and consider the function $\bar{f} : [\omega]^n \to \omega^{n+1}$ given by $\bar{f}(s) = f(g''s)$. It is easy to see that $\bar{f} \in \mathcal{L}_n$, so there is $S \in \mathcal{S}$ and disjoint $X, Y \in [\omega]^\omega$ such that $\bar{f}''[X]^n \subseteq S^{n+1}$ and $\bar{f}''[Y]^n \subseteq (\omega \setminus S)^{n+1}$. Taking $\bar{X} = g''X$ and $\bar{Y} = g''Y$ we get that $\bar{f}''[X]^n = f''[\bar{X}^n]$ and $\bar{f}''[Y]^n = f''[\bar{Y}^n]$ so we get the result. $\square$

Recall that a set $Y \subseteq \omega$ is *almost monochromatic* (or *almost homogeneous*) for a partition $f : [\omega]^n \to 2$ if there is a finite set $F \subseteq Y$ such that f is constant on $[Y \setminus F]^n$. The cardinal $\mathfrak{par}_n$ denotes the smallest cardinal κ such that there is a family of partitions of $[\omega]^n$ of size κ such that no infinite set is almost monochromatic for all of them simultaneously.

Note that if we consider partitions into some finite number of pieces k, instead of 2, we obtain the same cardinal. Moreover, for all $n \geq 2$ we have $\mathfrak{par}_n = \mathfrak{par}_2$, in fact:

Theorem 3.5.2. (See [8, Theorem 3.5]) *For each $n \geq 2$, $\mathfrak{par}_n = \min\{\mathfrak{b}, \mathfrak{s}\}$.*

It is worth mentioning that Kubiś and Szeptycki characterized $\mathfrak{par}_2$ as the minimum κ such that 2^κ is not 2-sequentially compact [49].

Note that if $\mathfrak{s} = \mathfrak{b}$, then $\mathfrak{par}_2 = \mathfrak{par}_n = \mathfrak{s} = \mathfrak{b} = \mathfrak{bs} = \mu_n$ for every $n \geq 1$.

If $S \subseteq \omega$ then we follow the notation $S^0 := S$ and $S^1 := \omega \setminus S$. We are now ready to prove the main theorem in this section. It is interesting that the combinatorial characterizations of both $\max\{\mathfrak{b}, \mathfrak{s}\}$ and $\min\{\mathfrak{b}, \mathfrak{s}\}$ are used in the proof of this theorem.

Theorem 3.5.3. ($\mathfrak{s} = \mathfrak{b}$) *For every $n \geq 1$ there exists an almost disjoint family $\mathcal{A}$ on ω^{n+1} of size $\mathfrak{c}$ such that $\mathcal{F}(\mathcal{A})$ is n-sequentially compact but not $(n+1)$-sequentially compact.*

Proof. Fix a family $\mathcal{S} = \{S_\alpha \mid \alpha \in \mathfrak{b}\}$ that is splitting and $\mathcal{L}_n$-splitting and also fix the following enumerations:

- $[\omega]^\omega = \{B_\alpha \mid \omega \leq \alpha < \mathfrak{c}\}$.
- $\mathcal{L}_n = \{f_\alpha \mid \omega \leq \alpha < \mathfrak{c}\}$.

As usual, let $G : [\omega]^{n+1} \to \omega^{n+1}$ be the increasing enumeration.

We want to construct the following families:

- $\{A_\alpha \mid \alpha \in \mathfrak{c}\}$,

- $\{X_\alpha \mid \omega \leq \alpha < \mathfrak{c}\} \subseteq [\omega]^\omega$,
- $\{\tau_\alpha \mid \omega \leq \alpha < \mathfrak{c}\}, \{\sigma_\alpha \mid \omega \leq \alpha < \mathfrak{c}\} \subseteq 2^{<\mathfrak{b}}$,

with the following properties:

(1) $A_i = C_i$ for all $i \in \omega$.

(2) If $\beta < \alpha$, then $\tau_\alpha \nsubseteq \tau_\beta$, $\tau_\alpha \nsubseteq \sigma_\beta$, $\sigma_\alpha \nsubseteq \tau_\beta$ and $\sigma_\alpha \nsubseteq \sigma_\beta$.

(3) If $\beta \in dom(\tau_\alpha)$, then $A_\alpha \subseteq^* \left(S_\beta^{\tau_\alpha(\beta)}\right)^{n+1}$.

(4) $f''_\alpha[X_\alpha]^n \in \mathsf{FIN}^{n+1}$.

(5) If $\alpha \geq \omega$, then A_α is a block sequence.

(6) $\forall \alpha \in \mathfrak{c} \setminus \omega (A_\alpha \subseteq G''[B_\alpha]^{n+1})$.

(7) $\forall \beta, \alpha \in \mathfrak{c}\ (\omega \leq \beta < \alpha \implies \exists m_{\beta,\alpha} \in \omega (f''_\beta[X_\beta \setminus m_{\beta,\alpha}]^n \cap A_\alpha = \emptyset))$.

(8) If $\beta \in dom(\sigma_\alpha)$, then $\exists n_\beta \in \omega$ such that $f''_\alpha[X_\alpha \setminus n_\beta]^n \cap \left(S_\beta^{1-\sigma_\alpha(\beta)}\right)^{n+1} = \emptyset$.

(9) $\forall \beta, \alpha \in \mathfrak{c} (\beta \neq \alpha \implies A_\beta \cap A_\alpha =^* \emptyset)$.

Assume that the construction has been carried out for all $\beta < \alpha$. We proceed to define X_α and σ_α.

-X_α and σ_α construction:

We construct $\{\alpha_s \mid s \in 2^{<\omega}\} \subseteq \mathfrak{b}$, $\{\eta_s \mid s \in 2^{<\omega}\} \subseteq 2^{<\mathfrak{b}}$, $\{Y_s \mid s \in 2^{<\omega}\} \subseteq [\omega]^\omega$, $\{f_s \mid s \in 2^{<\omega}\}$ such that for all $s \in 2^{<\omega}$:

(a) $f_\emptyset = f_\alpha$, $Y_\emptyset = \omega$.

(b) $f_s : [Y_s]^n \to \omega^{n+1}$.

(c) $\alpha_s = dom(\eta_s)$.

(d) $Y_{s^\frown 0}, Y_{s^\frown 1} \in [Y_s]^\omega$ are disjoint.

(e) $f''_s[Y_{s^\frown i}]^n \subseteq (S^i_{\alpha_s})^{n+1}$ for every $i \in 2$.

(f) If $\beta \in dom(\eta_s)$ and $i \in 2$, then $f''_{s^\frown i}[Y_{s^\frown i} \setminus m]^n \cap \left(S_\beta^{1-\eta_s(\beta)}\right)^{n+1} = \emptyset$ for some $m \in \omega$.

(g) If $s \subseteq t$, then $\eta_s \subseteq \eta_t$.

(h) If $s \perp t$, then $\eta_s \perp \eta_t$.

(i) $f_{s^\frown i} = f_s \restriction Y_{s^\frown i}$.

(j) $\alpha_s < \alpha_{s^\frown i}$.

As $\mathcal{S}$ is $\mathcal{L}_n$-splitting, we can find $\alpha_\emptyset = \min\{\beta \in \mathfrak{b} \mid S_\beta \text{ splits } f_\emptyset\}$. Then there exist $Z_0, Z_1 \in [\omega]^\omega$ such that $f''_\emptyset[Z_i]^n \subseteq \left(S^i_{\alpha_\emptyset}\right)^{n+1}$ for $i \in 2$. Now, we consider the following family of colorings on $[Z_0 \cup Z_1]^n$: for every $\beta < \alpha_\emptyset$, let $g_\beta : [Z_0 \cup Z_1]^n \to 3$ as follows:

$$g_\beta(\{x_1, \dots, x_n\}) = \begin{cases} 0 & \text{if } f_\emptyset(\{x_1, \dots, x_n\}) \in (S^0_\beta)^{n+1}, \\ 1 & \text{if } f_\emptyset(\{x_1, \dots, x_n\}) \in (S^1_\beta)^{n+1}, \\ 2 & \text{otherwise.} \end{cases}$$

Note that for $i \in 2$, $|\{g_\beta \restriction [Z_i]^n \mid \beta < \alpha_\emptyset\}| \le |\alpha_\emptyset| < \mathfrak{b} = \mathfrak{par}_n$, then there exists $Y_i \in [Z_i]^\omega$ such that Y_i is almost monochromatic for g_β for every $\beta < \alpha_\emptyset$. It is worth to point out that we are defining $\eta_\emptyset$ in terms of $Y_0 = Y_{\emptyset^\frown 0}$ and $Y_1 = Y_{\emptyset^\frown 1}$.

Claim 3.5.4. For every $\beta < \alpha_\emptyset$, there exists $j \in 2$ such that for sufficiently large $m \in \omega$, we have $f''[Y_i \setminus m]^n \cap \left(S^{1-j}_\beta\right)^{n+1} = \emptyset$ for both $i \in 2$.

Proof of the claim. Consider Y_0 and Y_1 and the coloring g_β. If Y_i is almost 0-monochromatic for any $i \in 2$, then Y_{1-i} cannot be almost 1-monochromatic, since S_β does not split f. In this case, take $j = 0$. Hence, $1 - j = 1$ and regardless of which color Y_{1-i} takes, we know that there exists $n_\beta \in \omega$ such that $f_\emptyset([Y_i \setminus n_\beta]^n) \cap (S^1_\beta)^{n+1} = \emptyset$ for both i and $i - 1$. Analogously, if there is a $i < 2$ such that Y_i is 1-monochromatic, $j = 1$ works. In case both sets are 2-monochromatic, any $j \in 2$ satisfies the claim. $_{Claim}\square$

Now, define $\eta_\emptyset : \alpha_\emptyset \to 2$ so that for every $\beta < \alpha_\emptyset$ we have $\eta_\emptyset(\beta) = j$, where j is given by the claim. For $f_{(0)}$ and $f_{(1)}$ we just take the restriction $f_{(i)} = f \restriction [Y_i]^n$.

Now, suppose that we have already constructed f_s, Y_s, α_t and $\eta_t : \alpha_t \to 2$ for $s = t^\frown j$ where $j \in 2$ and $t \in 2^{<\omega}$. We are going to construct $Y_{s^\frown 0}, Y_{s^\frown 1}, \alpha_s$ and η_s.

As $f_s : [Y_s]^n \to \omega^{n+1}$, there exists a minimum $\alpha_s \in \mathfrak{b}$ such that S_{α_s} splits f_s. Note that $\alpha_s > \alpha_t$ since no $\beta \le \alpha_t$ splits f_s. Thus, there exist $Z_0, Z_1 \in [Y_s]^\omega$ disjoint with $f''_s[Z_i]^n \subseteq (S^i_{\alpha_s})^{n+1}$ for every $i \in 2$.

As before, consider the following family of colorings of $[Z_0 \cup Z_1]^n$: for every $\beta \in [\alpha_t, \alpha_s)$ let $g_\beta : [Z_0 \cup Z_1]^n \to 3$ be as follows:

$$g_\beta(\{x_1, \dots, x_n\}) = \begin{cases} 0 & \text{if } f_s(\{x_1, \dots, x_n\}) \in (S^0_\beta)^{n+1}, \\ 1 & \text{if } f_s(\{x_1, \dots, x_n\}) \in (S^1_\beta)^{n+1}, \\ 2 & \text{otherwise.} \end{cases}$$

Note that $|\{g_\beta \mid \beta \in [\alpha_t, \alpha_s)\}| \le |\alpha_s| < \mathfrak{b} = \mathfrak{par}_n$, so there exists $Y_{s^\frown i} \in [Z_i]^\omega$ such that $Y_{s^\frown i}$ is almost-homogeneous for every $\beta \in [\alpha_t, \alpha_s)$. At this point, we can define $\eta'_s : [\alpha_t, \alpha_s) \to 2$ with the proper adaptation of Claim 3.5.4. Now, let $\eta_s = \eta_t \cup \eta'_s$.

From the definition of η'_s, we know that for every $\beta \in [\alpha_t, \alpha_s)$,

$$f''[Y_{s^\frown i} \setminus m_\beta]^n \cap \left(S^{1-\eta_s(\beta)}_\beta\right)^{n+1} = \emptyset$$

for some $m_\beta \in \omega$ and every $i \in 2$.

On the other hand, we know that $Y_{s^\frown 0}, Y_{s^\frown 1} \subseteq Y_s$ and our induction hypothesis implies that if $\beta \in dom(\eta_t)$, then $f''[Y_s \setminus m]^n \cap \left(S^{1-\eta_t(\beta)}_\beta\right)^{n+1} = \emptyset$ for some $m \in \omega$, so this last condition is also true for $Y_{s^\frown 0}, Y_{s^\frown 1}$. Finally, let $f_{s^\frown i} = f_s \restriction [Y_{s^\frown i}]^n$.

This shows that condition (f) holds and all other conditions are clear from construction.

To finish the construction of X_α and σ_α, let $\eta_g = \bigcup_{n \in \omega} \eta_{g \restriction n}$ for every $g \in 2^\omega$. Notice that $\eta_g \in 2^{<\mathfrak{b}}$ as $\mathfrak{b}$ has uncountable cofinality (in fact, it is regular, see [8]). Furthermore, if $f \ne g$, then η_f and η_g are incompatible nodes of $2^{<\mathfrak{b}}$. Since $\alpha < \mathfrak{c}$, we can find $g \in 2^\omega$ that satisfies that there is no $\beta < \alpha$ such that σ_β or τ_β extends η_g. Let $\sigma_\alpha = \eta_g$ and X'_α be any pseudointersection of $\{Y_{g \restriction n} \mid n \in \omega\}$. It follows that X'_α satisfies (8). By Lemma 2.0.15, we can find $X_\alpha \in [X'_\alpha]^\omega$ which also works for (4).

-A_α and τ_α construction.

We are going to construct a collection $\{B_s \mid s \in 2^{<\omega}\} \subseteq [\omega]^\omega$, $\{\eta_s \mid s \in 2^{<\omega}\} \subseteq 2^{<\mathfrak{b}}$ and $\{\alpha_s \mid s \in 2^{<\omega}\}$ such that:

(i) $B_\emptyset = B_\alpha$.

(ii) If $s \subsetneq t$, then $\eta_s \subsetneq \eta_t$.

(iii) If $s \perp t$, then $\eta_s \perp \eta_t$.

(iv) $\alpha_s = dom(\eta_s)$.

(v) $B_{s^\frown 0}, B_{s^\frown 1} \in [B_s]^\omega$ are disjoint.

(vi) If $\beta < \alpha_s$, then $B_s \subseteq^* S_\beta^{\eta_s(\beta)}$.

Suppose that we have already constructed η_t and α_t for every $t \subsetneq s$ and B_s. Let $\alpha_s \in \mathfrak{b}$ be the minimum such that $|B_s \cap S^0_{\alpha_s}| = \omega = |B_s \cap S^1_{\alpha_s}|$, this is possible since $\mathcal{S}$ is splitting. Notice also that $\alpha_t < \alpha_s$ whenever $t \subsetneq s$ by (vi) and (v). Let $\eta_s : \alpha_s \to 2$ be such that for all $\beta < \alpha_s$, $|B_s \cap S_\beta^{\eta_s(\beta)}| = \omega$. Finally, for every $i \in 2$ let $B_{s^\frown i} = B_s \cap S^i_{\alpha_s}$. Note that for every $\beta \in \alpha_s$ and for $i \in \{0,1\}$, we have that $B_{s^\frown i}$ is almost contained in $S_\beta^{\eta_s(\beta)}$. The latter, the fact that S_{α_s} does not split $B_{s^\frown i}$ and the fact that $B_s \subseteq B_t$ for $t \subseteq s$, imply that $\eta_s \subsetneq \eta_{s^\frown i}$ for every $i \in 2$.

Now, for every $g \in 2^\omega$ let $\eta_g = \bigcup_{n\in\omega} \eta_{g\restriction n}$. As before $\eta_g \in 2^{<\mathfrak{b}}$ and if $f \neq g$, η_f and η_g are incompatible nodes of $2^{<\mathfrak{b}}$. As $\alpha < \mathfrak{c}$, we can find $g \in 2^\omega$ such that there is no $\beta < \alpha$ such that σ_β or τ_β extends η_g, then let $\tau_\alpha = \eta_g$. Consider Z a pseudointersection of $\{B_{g\restriction n} \mid n \in \omega\}$ contained in $B_\emptyset = B_\alpha$.

Note that if $\beta < \alpha$ and σ_α and τ_β are incompatible, then there exists $m \in \omega$ such that, letting $\xi = \sigma_\alpha \vartriangle \tau_\beta$,[4] we have $f''_\beta[X_\beta \setminus m]^n \cap \left(S_\xi^{\sigma_\alpha(\xi)}\right)^{n+1} = \emptyset$. In particular, as Z is almost contained in $S_\xi^{\sigma_\alpha(\xi)}$, $f''_\beta[X_\beta \setminus m]^n \cap (Z)^{n+1} =^* \emptyset$.

Consider the following families of elements of FIN^{n+1}:

$$\mathcal{H}_0 = \{f''_\beta[X_\beta]^n \mid \omega \leq \beta < \alpha \wedge (\tau_\alpha \supseteq \sigma_\beta)\},$$

$$\mathcal{H}_1 = \{A_\beta \mid \omega \leq \beta < \alpha(\tau_\alpha \supseteq \tau_\beta)\}$$

and

$$\mathcal{H}_2 = \{A_i \mid i \in \omega\}.$$

Define $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_1 \cup \mathcal{H}_2$. Note that $|\mathcal{H}| < \mathfrak{b}$, so we can apply Lemma 3.3.2 to the family $\mathcal{H}$ and the infinite set Z to obtain A_α a block sequence of elements of Z such that its intersection with every element of $\mathcal{H}$ is finite.

Given $\beta \in dom(\tau_\alpha)$, there is $s \in 2^{<\omega}$ such that $\beta \in dom(\eta_s)$ and $s \subseteq g$. Thus, $Z \subseteq^* B_s \subseteq^* S_\beta^{\eta_s(\beta)} = S_\beta^{\tau_\alpha(\beta)}$. Therefore, $A_\alpha \subseteq^* \left(S_\beta^{\tau_\alpha(\beta)}\right)^{n+1}$ as $A_\alpha \subseteq Z^{n+1}$ is a block sequence.

We verify that this construction satisfies conditions (1) to (9). Conditions (1), (2), (4), (5) are true by definition. Condition (6) holds since A_α is a block sequence of elements of $Z \subseteq B_\alpha$. Condition (8) follows from the choice of σ_α. We have already checked condition (3) at the end of the construction of A_α. It remains to show that (7) and (9) hold.

Condition (7): Let β and α in $\mathfrak{c}$ be such that $\omega \leq \beta < \alpha$. If $\sigma_\beta \subseteq \tau_\alpha$, then by the construction of A_α, we know that $f''_\beta[X_\beta]^n \cap A_\alpha =^* \emptyset$ and, as A_α is a block sequence, there exists $m_{\beta,\alpha} \in \omega$ such that $f''_\beta[X_\beta \setminus m_{\beta,\alpha}]^n \cap A_\alpha = \emptyset$.

[4] Recall that $\sigma \vartriangle \tau = \min\{\alpha \mid \sigma(\alpha) \neq \tau(\alpha)\}$.

On the other hand, if σ_β and τ_α are incompatible, then there exists $\xi \in dom(\tau_\alpha) \cap dom(\sigma_\beta)$ such that $\tau_\alpha(\xi) = i$ and $\sigma_\beta(\xi) = 1 - i$. By applying ((3)) and ((8)) to ξ, we get the result.

Condition (9): It is clear that if $i \neq j$ are finite ordinals, then $A_i \cap A_j = \emptyset$. Also, if $\omega \leq \alpha < \mathfrak{c}$, then A_α is a block sequence, so $A_\alpha \cap A_i$ is finite for every $i \in \omega$. Now, let $\omega \leq \beta < \alpha < \mathfrak{c}$. If $\tau_\alpha \supseteq \tau_\beta$, then $A_\alpha \cap A_\beta =^* \emptyset$ by the construction of A_α, since $A_\beta \in \mathcal{H}$. Otherwise, τ_α and τ_β are incompatible nodes in $2^{<\mathfrak{b}}$, so there exists $\xi \in dom(\tau_\alpha) \cap dom(\tau_\beta)$ such that $A_\alpha \subseteq^* (S_\xi^{\tau_\alpha(\xi)})^{n+1}$ and $A_\beta \subseteq^* (S_\xi^{1-\tau_\alpha(\xi)})^{n+1}$; in particular, $A_\alpha \cap A_\beta$ is finite. This finishes the construction.

We see that $\mathcal{F}(\mathcal{A})$ is the desired space. Conditions (3) and (6) imply that for every infinite $\alpha \in \mathfrak{c}$, the elements of the almost disjoint family in the closure of $f''_\alpha[X_\alpha]^n$, are at most those A_β such that $\tau_\beta \subsetneq \sigma_\alpha$. In particular, the closure of $f''_\alpha[X_\alpha]^n$ has size less than $\mathfrak{b}$. By Corollary 2.0.13 we get that $\mathcal{F}(\mathcal{A})$ is n-sequentially compact. On the other hand, condition (7) implies that condition *(V)* of Lemma 3.2.2 holds and the other three conditions when $f = G$ are clear, so $\mathcal{F}(\mathcal{A})$ is not $(n+1)$-sequentially compact. □

Most modifications of the technique developed by Shelah used previously in the literature use assumptions of the form $\iota \leq \theta$ where ι is a kind of splitting cardinal invariant and θ is another cardinal invariant imposed by the particularities of the problem. We do not know whether the previous result can be weakened to $\mathfrak{s} \leq \mathfrak{b}$.

Question 3.5.5. Does it follow from $\mathfrak{s} \leq \mathfrak{b}$ that there are n-sequentially compact spaces that fail to be $(n+1)$-sequentially compact?

Or even better:

Question 3.5.6. Is there in ZFC, for every $n \geq 2$, a space that is n-sequentially compact but not $(n+1)$-sequentially compact?

3.6 On the character of n-sequentially compact spaces

One of the key facts in the construction of our examples has been that any sequentially compact space of character less than $\mathfrak{b}$ is n-sequentially compact for every $n \geq 1$. It is then natural to ask what is the smallest possible character of an n-sequentially compact space that is not $(n+1)$-sequentially compact:

Definition 3.6.1. If $n \geq 2$ let λ_n be the minimum character of a space X such that X is $(n-1)$-sequentially compact but not n-sequentially compact.

We do not know if these cardinals are well defined, since we do not know if there are n-sequentially compact spaces that are not $(n+1)$-sequentially compact in ZFC (for $n > 1$). To avoid unnecessary complications, we agree that $\lambda_n = \mathfrak{c}$ when it is not well defined by the previous definition. Theorem 2.0.11 implies that $\mathfrak{b}$ is a lower bound for λ_n for every $n \geq 2$. Now, using the space introduced by D. K. Burke and E. K. van Douwen in [12] (see also [20, Example 7.3]), we can compute λ_2.

Let $\{f_\alpha \mid \alpha \in \mathfrak{b}\} \subseteq \omega^\omega$ be well ordered by $\leq^*$ consisting of strictly increasing functions and for each $\alpha \in \mathfrak{b}$, let $B_\alpha = \{(n, m) \mid m \leq f_\alpha(n)\}$. Now, let $X = (\omega \times \omega) \cup \mathfrak{b}$ be the topological space such that the elements of $\omega \times \omega$ are isolated and a basic neighborhood for $\beta \in \mathfrak{b}$ has the form

$$N(\alpha, \beta, F) = (\alpha, \beta] \cup ((B_\beta \setminus B_\alpha) \setminus F),$$

where $\alpha \in \beta \cup \{-1\}$, $B_{-1} = \emptyset$ and $F \in [\omega \times \omega]^{<\omega}$. It is easy to check that X is Hausdorff and that it is separable since $\omega \times \omega$ is a dense subset of X. In addition, it is not compact as $\mathfrak{b}$ is a closed subspace of it, and the topology of $\mathfrak{b}$ inherited by X coincides with the order topology of $\mathfrak{b}$ (which is not compact). We now see that X is locally compact; for this, call $\mathcal{B} := \{B_\alpha \mid \alpha \in \mathfrak{b}\}$ and note that $\subseteq^*$ well-orders $\mathcal{B} \cup \{\emptyset\}$.

Lemma 3.6.2. *X is locally compact.*

Proof. We prove this by showing that the basic open sets are compact. This is trivial for the singleton subsets of $\omega \times \omega$, so consider α and β such that $-1 \leq \alpha < \beta < \mathfrak{b}$ and $F \in [\omega \times \omega]^{<\omega}$.

Let $\mathcal{U}$ be any cover of $N(\alpha, \beta, F)$. As $\subseteq^*$ well-orders $\mathcal{B} \cup \{\emptyset\}$ then we can find $n \in \omega$ with $n \geq 1$ and sequences $(B_{\alpha_i})_{i \leq n} \subseteq \mathcal{B}$ and $(F_i)_{i \leq n} \subseteq [\omega \times \omega]^{<\omega}$ such that:

1. $B_{\alpha_0} = B_\beta$.
2. $\forall i \in n(B_{\alpha_{i+1}} \subset^* B_{\alpha_i})$.
3. $B_{\alpha_n} = B_\alpha$.
4. $\{N(\alpha_{i+1}, \alpha_i, F_i) \mid i \in n\}$ refines $\mathcal{U}$.

Clearly $N(\alpha, \beta, F) \cap \mathfrak{b} \subseteq \bigcup_{i \in n} N(\alpha_{i+1}, \alpha_i, F_i)$ and

$$\begin{aligned} N(\alpha, \beta, F) \cap (\omega \times \omega) = (B_\beta \setminus B_\alpha) \setminus F &= \bigcup_{i \in n} (B_{\alpha_{i+1}} \setminus B_{\alpha_i}) \setminus F \\ &\subseteq \bigcup_{i \in n} (N(\alpha_{i+1}, \alpha_i, F_i) \setminus F) \cup \bigcup_{i \in n} F_i. \end{aligned}$$

Since $\bigcup_{i \in n} F_i$ is finite, it follows that $\mathcal{U}$ has a finite subcover. $\square$

On the other hand, X is not sequentially compact since for all $n \in \omega$, the sequence $((n, m))_{m \in \omega}$ does not converge in X.

Since X is locally compact, non-compact and Hausdorff, it admits a one-point compactification, which is compact and Hausdorff, so let $\overline{X} = X \cup \{\infty\}$ be the one-point compactification of X.

Theorem 3.6.3. *The space $\overline{X}$ is a compact and sequentially compact space of character $\mathfrak{b}$ that is not 2-sequentially compact.*

Proof. We see that $\overline{X}$ is sequentially compact. Let $f : \omega \to \overline{X}$. If f takes the value ∞ infinitely many times, then we are done. Also, if f takes values in $\mathfrak{b}$ infinitely many times, then we are done since $\mathfrak{b}$ has uncountable cofinality (and then it is sequentially compact). Thus, we assume that $f : \omega \to \omega \times \omega$. Let $A_k = \{k\} \times \omega$ for every $k \in \omega$ and let $\mathcal{A} = \{A \in [\omega]^\omega \mid \forall n \in \omega\ (|A \cap A_n| < \omega)\}$. If $f''\omega \in \mathcal{A}$, then let $\beta \in \mathfrak{b}$ be the minimum such that $f''\omega \cap B_\beta$ is infinite. Taking $B \in [\omega]^\omega$ such that $f''B \subseteq B_\beta$ we have that $f \restriction B$ converges to β. If $f''\omega \notin \mathcal{A}$, then let $B \in [\omega]^\omega$ and $k \in \omega$ be such that $f''B \subseteq A_k$. It follows that $f \restriction B$ converges to ∞, since every basic neighborhood of ∞ contains all but finitely many elements of A_k for every $k \in \omega$.

We now show that $\overline{X}$ is not 2-sequentially compact . Consider $G : [\omega]^2 \to \overline{X}$ given by $G(\{n, m\}) = (n, m)$ where $n < m$. We see that for all $B \in [\omega]^\omega$, $G \restriction [B]^2$ does not converge in $\overline{X}$. Fix $B \in [\omega]^\omega$ and $\beta \in \mathfrak{b}$. Note that for every basic neighborhood N of β, $N \cap A_n$ is finite for every $n \in \omega$, but the intersection of $G''[B \setminus F]^2$ with A_n is infinite for every $n \in B \setminus F$. This shows that there is no $F \in [\omega]^{<\omega}$ such that $G''[B \setminus F]^2 \subseteq N$. On the other hand, as G is injective, $G \restriction [B]^2$ does not converge to any isolated point. Thus, it remains only to show that $G \restriction [B]^2$ does not converge to ∞.

Let $(m_k)_{k\in\omega}$ be the increasing enumeration of B and let $A = \{(m_k, m_{k+1}) \mid k \in \omega\}$. Note that:

1. $A \in \mathcal{A}$.

2. For all $F \in [\omega]^{<\omega}$ we have $A \subseteq^* G''[B \setminus F]^2$.

Now, let $\beta \in \mathfrak{b}$ be such that $|B_\beta \cap A| = \omega$ and β is the minimum with this property. Consider the following basic neighborhood of ∞:

$$N = \{\infty\} \cup (X \setminus N(-1, \beta, \emptyset)) = \{\infty\} \cup (\beta, \mathfrak{b}) \cup ((\omega \times \omega) \setminus B_\beta).$$

If there exists $F \in [\omega]^{<\omega}$ such that $G''[B \setminus F]^2 \subseteq N$, then in particular we have $G''[B \setminus F]^2 \cap B_\beta = \emptyset$, and as $A \subseteq^* G''[B \setminus F]^2$, then $A \cap B_\beta$ is finite, which is a contradiction. This implies that there is no such F and then $G \restriction [B]^2$ does not converge, proving that $\overline{X}$ is not 2-sequentially compact. □

With this last result and Theorem 2.0.11, we get the following nice topological characterization of $\mathfrak{b}$:

Corollary 3.6.4. $\mathfrak{b} = \lambda_2$, *i.e.,*

$$\mathfrak{b} = \min\{\chi(X) \mid X \textit{ is sequentially compact but not } 2\textit{-sequentially compact}\}.$$

We can still ask for the minimal size of an almost disjoint family $\mathcal{A}$ such that $\mathcal{F}(\mathcal{A})$ is not 2-sequentially compact, i.e., the minimal character of a space of the form $\mathcal{F}(\mathcal{A})$ that is not 2-sequentially compact. In general, we can ask this for every $n \geq 2$:

Definition 3.6.5. If $n \geq 2$ let $\mathfrak{a}_n^{sc}$ be the minimum size of an almost disjoint family such that $\mathcal{F}(\mathcal{A})$ is $(n-1)$-sequentially compact but not n-sequentially compact.

It follows from the results in [49] that $\mathfrak{b} \leq \mathfrak{a}_2^{sc} \leq \mathfrak{a}$. Also, it should be clear that $\lambda_n \leq \mathfrak{a}_n^{sc}$ for all $n \geq 2$ whenever they are well defined.

Definition 3.6.6. (See [59, Definition 1.4 (2)]) Let $\mathfrak{a}_*$ be the minimal size of an almost disjoint family $\mathcal{A}$, such that there are (disjoint) $C_0, C_1, C_2, \ldots \in \mathcal{A}^\perp$ so that for every $D \in [\omega]^\omega$, if $|C_n \cap D| = \omega$ for infinitely many $n \in \omega$, then there is $A \in \mathcal{A}$ for which $|A \cap D| = \omega$.

The following notions were introduced by A. V. Arhangel'skii [3]:

Definition 3.6.7. Let X be a topological space and $x \in X$. Then x is an α_i*-point* $(i = 1, 2, 3)$ if for every family $\{S_n \mid n \in \omega\}$ of sequences from X converging to x, there exists a sequence S that converges to x such that:

- (α_1) $S_n \setminus S$ is finite for all $n \in \omega$,

- (α_2) $S \cap S_n \neq \emptyset$ for all $n \in \omega$.

- (α_3) $|S \cap S_n| = \omega$ for infinitely many $n \in \omega$.

A space X is an α_i-*space* if every point $x \in X$ is an α_i-point.

Arhangel'skii introduced the α_i properties for $i \in \{1, \ldots, 5\}$ in [3]. It is easy to see that all α_1-points are α_2-points, and that all α_2-points are α_3-points. We refer the reader to [3] for the rest of the definitions and a deep study of their relationship.

Proposition 3.6.8. *$\mathfrak{a}_*$ is the minimum size of an almost disjoint family such that $\mathcal{F}(\mathcal{A})$ is not α_3.*

Proof. Let $\kappa < \mathfrak{a}_*$ and let $\mathcal{A}$ be an almost disjoint family of cardinality κ. As we want to prove that $\mathcal{F}(\mathcal{A})$ is α_3, it is enough to prove that ∞ is an α_3 point since any other point has countable character. Let $\{S_n \mid n \in \omega\}$ be a collection of convergent sequences to ∞. We can assume that S_n only takes values in ω for every $n \in \omega$, since any sequence in $\mathcal{A}$ converges to ∞. As S_n converges to ∞, $S_n \in \mathcal{A}^{\perp}$ for every $n \in \omega$. Since $|\mathcal{A}| < \mathfrak{a}_*$, there exists $B \in [\omega]^{\omega}$ such that $S_n \cap B$ is infinite for infinitely many S_n and $A \cap B$ is finite for every $A \in \mathcal{A}$. Then B is the required sequence converging to ∞.

Conversely, let $\mathcal{A}$ be an almost disjoint family such that $\mathcal{F}(\mathcal{A})$ is α_3. We want to prove that $\mathcal{A}$ is not a witness of $\mathfrak{a}_*$, so let $C_0, C_1, C_2, \ldots \in \mathcal{A}^{\perp}$. Note that every C_n is a convergent sequence to ∞ and as ∞ is an α_3-point, there exists a sequence $B \subseteq \Psi(\mathcal{A})$ that converges to ∞ and such that $|C_n \cap B| = \omega$ for infinitely many $n \in \omega$. We can assume that $B \subseteq \omega$, since otherwise $B' = B \cap (\bigcup_{i \in \omega} C_i)$ also witnesses the α_3-property for the family $\{C_i \mid i \in \omega\}$. As B converges to ∞, we also have that $B \cap A =^* \emptyset$ for all $A \in \mathcal{A}$, so $\mathcal{A}$ is not a witness for $\mathfrak{a}_*$ as we wanted to show. □

Following the idea of the previous proof, we can actually show that 2-sequentially compact ad families are α_3. Given a non α_3 almost disjoint family $\mathcal{A}$ and a family $\{S_n : n \in \omega\} \subseteq \mathcal{A}^{\perp}$ that witnesses this, we can define $f : [\omega]^2 \to \omega$ by $f(\{n, m\}) = S_n(m)$, where $S_n(m)$ is the m^{th} element of S_n. Thus, $\mathcal{A}$ and $\mathcal{A}_0 = \{S_n \mid n \in \omega\}$ satisfy the hypotheses of Lemma 3.2.2 for $n = 2$ and $\mathcal{F}(\mathcal{A})$ is not 2-sequentially compact.

In general, it is possible to prove that 2-sequentially compact spaces are α_3. This result was independently proved by Szeptycki, K. Almontashery and P. Memarpanahi. Moreover, they also proved that 2-sequentially compact spaces need not be α_2 and that α_1 spaces need not be 2-sequentially compact.

Theorem 3.6.9. *If X is 2-sequentially compact, then X is α_3.*

Proof. Let $x \in X$ and for every $n \in \omega$ let $Y_n = \{x(n, m) \mid m \in \omega\}$ be a sequence converging to x. Define $f : [\omega]^2 \to X$ by $f(n, m) = x(n, m)$, where $n < m$. Since X is 2-sequentially compact, there exists $B \in [\omega]^{\omega}$ such that $f \restriction [B]^2$ converges.

Let $Y := f''[B]^2 \in [X]^{\omega}$. It is clear that $Y \cap f''_n\omega$ is infinite for every $n \in B$, so we are done if we prove that Y is convergent to x. Let U be an open set such that $x \in U$. Since $f \restriction [B]^2$ converges to x, we know that there exists $m \in \omega$ such that $f''[B \setminus m]^2 \subseteq U$. It follows that $Y \setminus U \subseteq \bigcup_{n \in B \cap m} Y_n$. But $\bigcup_{n \in B \cap m} Y_n$ is a convergent sequence to x, so $\bigcup_{n \in B \cap m} Y_n \setminus U$ is finite; then

$$Y \setminus U \subseteq (\bigcup_{n \in B \cap m} Y_n) \setminus U$$

and the last is finite. We conclude that Y converges to x. □

It is also worth to mention that the previous proof follows the same idea as the proof that the spaces with the Ramsey property considered by Knaust [47] are α_3. Recall that a space X has the Ramsey property if whenever $\lim_{i \to \infty} \lim_{j \to \infty} x(i, j) = x$, there exists $B \in [\omega]^{\omega}$ such that $f \restriction [B]^2$ converges to x, where $f(\{i, j\}) = x(i, j)$.

Corollary 3.6.10. *If $\mathcal{A}$ is an almost disjoint family such that $\mathcal{F}(\mathcal{A})$ is 2-sequentially compact, then $\mathcal{F}(\mathcal{A})$ is α_3.*

In [32] it is proved that $\mathfrak{b} \leq \mathfrak{a}_* \leq \mathfrak{a}$ and Corollary 3.6.10 implies $\mathfrak{a}_2^{sc} \leq \mathfrak{a}_*$, so we have $\mathfrak{b} = \lambda_2 \leq \mathfrak{a}_2^{sc} \leq \mathfrak{a}_* \leq \mathfrak{a}$. Of course, the inequality $\lambda_n \leq \mathfrak{a}_n^{sc}$ is clear from the definition. This is pretty much what we know so far about these cardinals.

Question 3.6.11. Are there any other relations between the cardinals λ_n and $\mathfrak{a}_n^{sc}$ with $\mathfrak{b}$ and $\mathfrak{a}$?

Question 3.6.12. Is it consistent that $\mathfrak{a}_2^{sc} < \mathfrak{a}_*$?

Question 3.6.13. Are all the λ_n and $\mathfrak{a}_m^{sc}$ consistently different? In particular, is $\mathfrak{a}_2^{sc} < \cdots < \mathfrak{a}_n^{sc} < \cdots < \mathfrak{a}$ consistent?

Chapter 4

Infinite dimension

In the previous chapter, we analyzed the hierarchy of n-sequential compactness for finite n, showing that these properties can be consistently distinct. A pervasive theme in mathematics, particularly in set theory and topology, is the drive to extend finite phenomena into the countable realm. From this perspective, a natural question arises: can the notion of n-sequential compactness be meaningfully extended beyond finite dimensions?

To answer this, we must first identify the appropriate α-dimensional analogues of the sets $[\omega]^n$ for $\alpha \in \omega_1 \setminus \omega$. We accomplish this using the framework of Nash-Williams barriers. Once these objects are established, we can define what it means for a space to be $\mathcal{B}$-sequentially compact, where $\mathcal{B}$ is a barrier. It turns out that, just as $[\omega]^n$ is an n-dimensional object, the *dimension* of an arbitrary barrier is captured by its *rank*. This naturally leads to the notion of being α-sequentially compact, defined as being $\mathcal{B}$-sequentially compact for all barriers $\mathcal{B}$ of rank α.

However, a crucial distinction arises here: for $n \in \omega$, there is essentially only one barrier of rank n (namely, $[\omega]^n$). In contrast, for $\alpha \in \omega_1 \setminus \omega$, the situation is much more complex. Consequently, to make the notion of α-sequential compactness manageable, we must identify the *canonical* objects that play the role analogous to $[\omega]^n$; these are known as *uniform barriers*. This requires a focused, tailored study of the barriers themselves.

We begin this chapter by addressing this need. In Section 4.1, we develop the barrier theory necessary for the study of α-sequential compactness; specifically, we introduce a preorder on the family of barriers and show that *uniform barriers* play a crucial role in this preorder.

In Section 4.2, we introduce the notion of $\mathcal{B}$-sequential compactness. Here, the importance of the work done in Section 4.1 becomes apparent, as we prove, for instance, that for every $\alpha < \omega_1$ and every uniform barrier $\mathcal{B}$ of rank α, the notions of $\mathcal{B}$-sequential compactness and α-sequential compactness coincide (see Corollary 4.2.5). We also prove that if $\beta < \alpha$ and X is α-sequentially compact, then X is also β-sequentially compact (see Corollary 4.2.6). This demonstrates that the concept behaves as expected, confirming that this is indeed the correct generalization.

Following this, in Section 4.3, we show that the class of spaces that are α-sequentially compact for every $\alpha \in \omega_1$ is quite broad. We prove that this class includes *bisequential spaces* (see Definition 4.3.3), which in turn implies that important classes, such as Rosenthal compacta, are α-sequentially compact for all countable α. Moreover, we significantly strengthen Theorem 2.0.11 (originally from [49, Theorem 3.1]), which states that every sequentially compact space with character less than $\mathfrak{b}$ is n-sequentially compact for all n. We extend this to the infinite-dimensional setting, proving that such spaces are, in fact,

α-sequentially compact for every $\alpha \in \omega_1$.

Finally, in Section 4.4, our primary focus is to establish the distinctness of these new classes. We prove consistency results analogous to those in Section 3.3, demonstrating that under $\mathfrak{b} = \mathfrak{c}$, the classes of α-sequentially compact and β-sequentially compact spaces are distinct whenever $\alpha < \beta$. Additionally, we prove that being α-sequentially compact for every $\alpha \in \omega_1$ is strictly weaker than being bisequential. Section 4.5 concludes the chapter by exploring the cardinal invariants associated with these barriers.

As noted in the introduction, this chapter is based entirely on the article [16] by Corral, Guzmán, Memarpanahi, Szeptycki, Todorčević and the author of this thesis.

4.1 Barriers and fronts

We now introduce the basic concepts of barriers and fronts on $[\omega]^{<\omega}$. This section is mainly based on [2] and [62], although some results are new or adapted and will be applied in subsequent sections, where we extend the theory of n-sequentially compact spaces by defining the notion of barrier sequential compactness and ultimately α-sequential compactness for all $\alpha < \omega_1$.

For $\mathcal{F} \subseteq [\omega]^{<\omega}$ and $M \in [\omega]^\omega$, let

$$\mathcal{F}|M = \{s \in \mathcal{F} \mid s \subseteq M\}.$$

Recall that $s \sqsubseteq t$ means that t is an end-extension of s, i.e., every element of $t \setminus s$ is larger than every element of s. We write $s \sqsubset t$ to mean $s \sqsubseteq t$ and $s \neq t$.

Definition 4.1.1. A family $\mathcal{F} \subseteq [\omega]^{<\omega}$ is:

- *Ramsey*: if for every partition $\mathcal{F} = \mathcal{F}_0 \sqcup \cdots \sqcup \mathcal{F}_n$ and for every $N \in [\omega]^\omega$, there is $M \in [N]^\omega$ such that all but at most one of the restrictions $\mathcal{F}_i|M$ are empty.
- *Nash-Williams:* if $s \sqsubseteq t$ implies $s = t$ for $s, t \in \mathcal{F}$, i.e., $\mathcal{F}$ is an antichain respect to $\sqsubseteq$.
- *Sperner:* if $s \subseteq t$ implies $s = t$ for $s, t \in \mathcal{F}$. i.e., $\mathcal{F}$ is an antichain respect to $\subseteq$.

Given a Ramsey family $\mathcal{F}$, applying the Ramsey property to the partition $\mathcal{F} = \mathcal{F}_0 \cup (\mathcal{F} \setminus \mathcal{F}_0)$ where $\mathcal{F}_0$ is the set of $\subseteq$-minimal elements in $\mathcal{F}$ we get the following:

Proposition 4.1.2. *If $\mathcal{F}$ is Ramsey, there exists $M \in [\omega]^\omega$ such that $\mathcal{F}|M$ is Sperner.*

Theorem 4.1.3. (See [55, Theorem 1]) *Every Nash-Williams family is Ramsey.*

The previous two results show that (at least if one is willing to pass to an infinite subset) the three concepts of being Ramsey, Nash-Williams and Sperner are equivalent:

$$\text{Ramsey} \Rightarrow \text{(some restriction is) Sperner} \Rightarrow \text{Nash-Williams} \Rightarrow \text{Ramsey}.$$

Definition 4.1.4. A Nash-Williams family $\mathcal{F}$ such that for every $M \in [\omega]^\omega$ there is an initial segment $s \sqsubseteq M$ with $s \in \mathcal{F}$ is called a *front.* If moreover $\mathcal{F}$ is Sperner, we say that $\mathcal{F}$ is a *barrier* on ω.

The same argument for Proposition 4.1.2 can be used to show the following facts:

Fact 4.1.5. *If $\mathcal{F}$ is a front on a countable set M, there exists $N \in [M]^\omega$ such that $\mathcal{F}|N$ is a barrier.*

Fact 4.1.6. *For every barrier $\mathcal{B}$ on M, if we partition $\mathcal{B} = \mathcal{B}_0 \sqcup \cdots \sqcup \mathcal{B}_n$, there are $N \in [M]^\omega$ and $i \leq n$ such that $\mathcal{B}_i|N$ is a barrier.*

Given a front $\mathcal{F}$, let

$$T(\mathcal{F}) = \{s \in [\omega]^{<\omega} \mid \exists t \in \mathcal{F}\ (s \sqsubseteq t)\}.$$

We often view $T(\mathcal{F})$ as a subtree of $\omega^{<\omega}$ by the usual identification of each set $s \in [\omega]^{<\omega}$ with its increasing enumeration.

We also define the function $\rho_{T(\mathcal{F})} : T(\mathcal{F}) \to \omega_1$ by

$$\rho_{T(\mathcal{F})}(s) = \sup\{\rho_{T(\mathcal{F})}(t) + 1 \mid t \in T(\mathcal{F}) \wedge s \sqsubset t\},$$

where $\sup \emptyset = 0$. When referring to $\rho_{T(\mathcal{F})}$, we often omit the subscript $T(\mathcal{F})$ or replace it with $\mathcal{F}$ if no confusion arises.

Lemma 4.1.7. *Let $\mathcal{F}$ be a front. Then $(T(\mathcal{F}), \sqsupset)$ is well-founded*[1]*; in particular, $\rho_{T(\mathcal{F})}$ is well-defined.*

Proof. Suppose, towards a contradiction, that $(T(\mathcal{F}), \sqsupset)$ is not well-founded. Then there is a sequence $(s_n)_{n\in\omega}$ in $T(\mathcal{F})$ with $s_{n+1} \sqsupset s_n$ for every n. Set $S = \bigcup_{n\in\omega} s_n \in [\omega]^\omega$. Note that $s_n \sqsubset S$ for all n.

Since $\mathcal{F}$ is a front, pick $t \in \mathcal{B}$ with $t \sqsubseteq S$. Choose n sufficiently large so that $|s_n| > |t|$. Because both t and s_n are initial segments of S, we have $t \sqsubset s_n$. On the other hand, as $s_n \in T(\mathcal{F})$, there is $t' \in \mathcal{F}$ with $s_n \sqsubseteq t'$. Hence, $t \sqsubset t'$ and $t, t' \in \mathcal{F}$, contradicting that $\mathcal{F}$ is a Nash–Williams family (no two distinct members are comparable by $\sqsubseteq$).

This contradiction shows that $(T(\mathcal{F}), \sqsubseteq)$ is well-founded, so $\rho_{T(\mathcal{F})}$ is well-defined. □

The requirement that "for every $M \in [\omega]^\omega$ there is some $t \in \mathcal{B}$ with $t \sqsubseteq M$" is essential: the Nash-Williams property alone is insufficient to ensure that $(T(\mathcal{B}), \sqsupset)$ is well-founded, as the following example shows.

Example 4.1.8. Let

$$\mathcal{F} \ = \ \{\, s \in [\omega]^{<\omega} \setminus \{\emptyset\} \mid \max(s) \text{ is odd and every } n \in s \setminus \{\max(s)\} \text{ is even}\,\}.$$

Then $\mathcal{F}$ is a Nash–Williams family that cannot be extended to a front and $(T(\mathcal{F}), \sqsupset)$ is not well-founded.

Proof. Note that $s \in \mathcal{F}$ exactly when either $s = \{m\}$ for some odd m, or there exist integers $a_0 < \ldots < a_n \leq a_{n+1}$ such that $s = \{2a_0, \ldots, 2a_n,\ 2a_{n+1} + 1\}$.

It follows immediately that $\mathcal{F}$ is a Nash–Williams family. Indeed, every member of $\mathcal{F}$ contains exactly one odd element (its maximum). Hence, if $s \sqsubset t$ with $s, t \in \mathcal{F}$, then $\max(s) < \max(t)$ and both are odd elements of t, contradicting the fact that t has a unique odd element.

Next, for every $M \in [\omega]^\omega$ with $M \neq \{2n \mid n \in \omega\}$ there is $s \in \mathcal{F}$ with $s \sqsubseteq M$. Indeed, let a be the least odd element of M; then $M \cap (a + 1) \in \mathcal{F}$.

[1] A binary relation R on a set X, i.e., $R \subseteq X \times X$, is *well-founded* if there does not exist a sequence $(x_n)_{n\in\omega}$ of elements of X such that $x_{n+1} R x_n$ for all $n \in \omega$.

However, $E = \{2n \mid n \in \omega\}$ has no initial segment in $\mathcal{F}$. Moreover, $\mathcal{F}$ cannot be extended to a front $\widehat{\mathcal{F}}$. If such a front existed, there would be $s \in \widehat{\mathcal{F}}$ with $s \sqsubseteq E$. But then $s \sqsubset s \cup \{\max(s)+1\} \in \mathcal{F} \subseteq \widehat{\mathcal{F}}$, so both s and $s \cup \{\max(s)+1\}$ would belong to $\widehat{\mathcal{F}}$, contradicting the Nash–Williams property.

Finally, observe that $T(\mathcal{F})$ is not well-founded: for $n \in \omega$, set $s_n = \{0, 2, \ldots, 2n\}$. Each s_n belongs to $T(\mathcal{F})$ (witnessed by $s_n \cup \{2n+1\} \in \mathcal{F}$), and $s_{n+1} \sqsupset s_n$ for all n, so $(s_n)_{n\in\omega}$ is an infinite $\sqsupset$-descending chain in $T(\mathcal{F})$. $\square$

Definition 4.1.9. For a barrier $\mathcal{B}$ on a countable set M, its rank is defined by $\rho(\mathcal{B}) := \rho_{T(\mathcal{B})}(\emptyset)$.

Lemma 4.1.10. *$\rho([\omega]^n) = n$ for every $n \geq 1$.*

Proof. First, note that $T([\omega]^n) = [\omega]^{\leq n}$.

Claim 4.1.11. $\rho_{[\omega]^n}(s) = n - |s|$ for every $s \in T([\omega]^n)$.

Proof of the claim. We proceed by induction on $l := n - |s|$. If $l = 0$, i.e., $|s| = n$, then there is no $t \in T([\omega]^n)$ with $s \sqsubset t$, hence $\rho_{[\omega]^n}(s) = 0$. Now, suppose that the statement holds for all $k \leq l$ and let $s \in T([\omega]^n)$ with $n - |s| = l+1$ (so $|s| = n - (l+1)$). Note that

$$\begin{aligned}\{t \in T([\omega]^n) \mid s \sqsubset t\} &= \bigcup_{i=|s|+1}^{n} \{\, t \in T([\omega]^n) \mid s \sqsubset t,\ |t| = i \,\} \\ &= \bigcup_{k=0}^{l} \{\, t \in T([\omega]^n) \mid s \sqsubset t,\ |t| = n-k \,\}.\end{aligned}$$

By the induction hypothesis, if $t \in T([\omega]^n)$ satisfies $s \sqsubset t$ and $|t| = n - k$ for some $k \leq l$, then $\rho_{[\omega]^n}(t) = n - |t| = k$. Therefore,

$$\rho_{[\omega]^n}(s) = \sup\{\rho_{[\omega]^n}(t)+1 \mid t \in T([\omega]^n),\ s \sqsubset t\} = \sup\{k+1 \mid k \leq l\} = l+1 = n-|s|. {}_{Claim}\square$$

Applying Claim 4.1.11 to $s = \emptyset$, we get $\rho_{[\omega]^n}(\emptyset) = n$, and hence $\rho([\omega]^n) = n$. $\square$

It is worth to point out that in the definition of $T(\mathcal{B})$, we can replace the condition of being an initial segment of an element $t \in \mathcal{B}$, by being a subset of some $t' \in \mathcal{B}$ due to the following fact.

Lemma 4.1.12. *If $\mathcal{B}$ is a barrier on ω, then for every $s \in [\omega]^{<\omega}$, there exists $b \in \mathcal{B}$ such that $s \subseteq b$ if and only if there exists $b \in \mathcal{B}$ such that $s \sqsubseteq b$. In particular*

$$T(\mathcal{B}) = \{s \in [\omega]^{<\omega} \mid \exists b \in \mathcal{B}\ (s \subseteq b)\}.$$

Proof. Let $s \in [\omega]^{<\omega}$ and $b \in \mathcal{B}$ be such that $s \subseteq b$. Define $M = s \cup (\omega \setminus \max(b)+1)$. We can find $b' \in \mathcal{B}$ such that $b' \sqsubseteq M$ and hence $s \sqsubseteq b'$ since otherwise $b' \sqsubset s \subseteq b$ would contradict that $\mathcal{B}$ is a $\subseteq$-antichain. $\square$

Given $a = \{a_0, \ldots, a_n\}$ and $b = \{b_0, \ldots, b_k\}$ with $k < n$, denote by $a * b$ the *end replacement* of a with b, defined as follows:

$$a * b = \{a_0, \ldots, a_{n-k-1}, b_0, \ldots, b_k\}.$$

Lemma 4.1.13. *Let $\mathcal{B}$ be a barrier and $a \in \mathcal{B}$ be enumerated as $a = \{a_0, \ldots, a_{n-1}\}$. Then for every $b \in [\omega \setminus a_{n-1}]^{<n}$, there exists $s \in \mathcal{B}$ such that $a * b \sqsubseteq s$. In particular, $b \notin \mathcal{B}$.*

Proof. Fix $a = \{a_0, \dots, a_{n-1}\}$ and $b = \{b_0, \dots, b_{k-1}\}$ as stated in the hypothesis with $k < n$. We prove the lemma by induction on k.

If $k = 1$, then $b = \{b_0\}$ for some $b_0 \geq a_{n-1}$. Let $M \in [\omega]^\omega$ be such that $a * b \sqsubseteq M$. There is an $s \in \mathcal{B}$ such that $s \sqsubseteq M$. Notice that $|s| \geq n$, since otherwise $s \sqsubset a$ would contradict the fact that $\mathcal{B}$ is Sperner. Thus, $a * b \sqsubseteq s \in \mathcal{B}$.

Now, let $k > 1$ and assume the result is true for $k - 1$. Pick any $M \in [\omega]^\omega$such that $a * b \sqsubseteq M$ and let $s \in \mathcal{B}$ be an initial segment of M. If $|s| < n$, then

$$s \sqsubseteq \{a_0, \dots, a_{n-k-1}, b_0, \dots, b_{k-2}\} \subseteq \{a_0, \dots, a_{n-k}, b_0, \dots, b_{k-2}\} = a * b',$$

where $b' = \{b_0, \dots, b_{k-2}\}$. Since $|b'| = k - 1$, by the inductive hypothesis, we get that $s \subsetneq a * b' \sqsubseteq s'$ for some $s' \in \mathcal{B}$, contradicting that $\mathcal{B}$ is Sperner. Therefore, $|s| \geq n$ and thus $a * b \sqsubseteq s \in \mathcal{B}$. $\square$

Corollary 4.1.14. *If $\mathcal{B}$ is a barrier on ω of rank $k \in \mathbb{N}$, there exists $m \in \omega$ such that $\mathcal{B}|(\omega \setminus m) = [\omega \setminus m]^k$.*

Proof. Let $\mathcal{B}$ be a barrier of rank k. If there is $a \in \mathcal{B}$ of size l, then $\rho(\mathcal{B}) \geq l$. Thus, $\mathcal{B} \subseteq [\omega]^{\leq k}$. It is also easy to see that $\mathcal{B} \cap [\omega]^k \neq \emptyset$ since otherwise $\rho(\mathcal{B}) < k$.

Take any $a \in \mathcal{B} \cap [\omega]^k$ and let $m = \max(a) + 1$. By Lemma 4.1.13, if $b \in \mathcal{B}|(\omega \setminus m)$ has size less than k, we can find $s \in \mathcal{B}$ such that $b' = a * b \sqsubseteq s$. As $|b| < k$, we have $b \subsetneq b' \sqsubseteq s$, which contradicts that $\mathcal{B}$ is Sperner. Henceforth, $\mathcal{B}|(\omega \setminus m) = [\omega \setminus m]^k$. $\square$

Notation 4.1.15. Given a family $\mathcal{B} \subseteq [\omega]^{<\omega}$, $s \in [\omega]^{<\omega}$ and $n \in \omega$ set

- $\mathcal{B}(s) = \{t \in \mathcal{B} \mid s \sqsubseteq t\}$,
- $\mathcal{B}[s] = \{t \setminus s \mid t \in \mathcal{B}(s)\}$,
- $s^\frown\mathcal{B} = \{s \cup t \mid t \in \mathcal{B} \ \wedge \ \min(t) > \max(s)\}$,
- $\mathcal{B} + n = \{s + n \mid s \in \mathcal{B}\}$, where $s + n = \{m + n \mid m \in s\}$.

We write $\mathcal{B}(n)$, $\mathcal{B}[n]$ and $n^\frown\mathcal{B}$ instead of $\mathcal{B}(\{n\})$, $\mathcal{B}[\{n\}]$ and $\{n\}^\frown\mathcal{B}$, respectively. Notice that some confusion may arise as $n = \{0, \dots, n-1\}$, but this notation should not lead to any confusion, as $\mathcal{B}(n)$ is always to be interpreted as $\mathcal{B}(\{n\})$, and the same applies for $\mathcal{B}[n]$ and $n^\frown\mathcal{B}$. We see that if $\mathcal{B}$ is a barrier (respectively, a front) on M, then $\mathcal{B}[n]$ is a barrier (respectively, a front) on $M \setminus (n+1)$. Conversely, if $\mathcal{B}_n$ is a barrier on $M \setminus (n+1)$ for every $n \in M$, then $\bigcup_{n \in M} n^\frown\mathcal{B}_n$ is a front on M and there is an infinite set on which its restriction is a barrier. We can now describe a canonical barrier of rank ω: The *Schreier barrier* $\mathcal{S}$ is the barrier of rank ω such that $\{0\} \in \mathcal{S}$ and $\mathcal{S}[n] = [\omega \setminus (n+1)]^n$ for every $n > 0$. In other words, $s \in \mathcal{S}$ if and only if $|s| = \min(s) + 1$.

Definition 4.1.16. Let $\mathcal{B}$ be a barrier on $M \in [\omega]^\omega$ with $\rho(\mathcal{B}) = \alpha$. If $\alpha > 1$, we say that $\mathcal{B}$ is a *uniform* barrier if each $\mathcal{B}[n]$ is a uniform barrier (on $M \setminus (n+1)$) and

- $\rho_{T(\mathcal{B})}(\{n\}) = \beta$ for every $n \in \omega$ if $\alpha = \beta + 1$, or
- $\{\rho_{T(\mathcal{B})}(\{n\}) \mid n \in \omega\}$ is an increasing sequence with limit α if α is a limit ordinal.

For $\alpha = 1$, we declare the unique barrier $\mathcal{B} = [\omega]^1$ as a uniform barrier.

The following result, together with Corollary 4.1.14, shows that all barriers of rank ω are somewhere uniform and preserve their rank under this restriction.

Proposition 4.1.17. *If $\mathcal{B}$ is a barrier of rank ω, there exists $M \in [\omega]^\omega$ such that $\mathcal{B}|M$ is uniform and has rank ω.*

Proof. By definition, we can find m_i such that $\rho(\{m_i\}) \geq i$ for every $i \in \omega$. By Corollary 4.1.14, we can also find k_i for every $i \in \omega$, such that $\mathcal{B}[m_i] = [\omega \setminus k_i]^{n_i}$, where $n_i = \rho(\{m_i\})$. Notice that we can pick $\{m_i \mid i \in \omega\}$ listed in increasing order such that $m_{i+1} > k_i$. We can now see that $M = \{m_i \mid i \in \omega\}$ works. □

A partial analogue of Proposition 4.1.17 for any barrier, independent of its rank, is also true.

Proposition 4.1.18. (See [2, Corollary II.3.10]) *For every barrier $\mathcal{B}$ on ω there exists an infinite set $M \in [\omega]^\omega$ such that $\mathcal{B}|M$ is uniform.* □

By Corollary 4.1.14, the only uniform barrier of rank $k \in \omega$ is $[\omega]^k$. Thus, Lemma 4.1.13 states that the sequence of ranks $\{\rho(\{n\}) \mid n \in \omega\}$ is non-decreasing and the sequence $\{m_i \mid i \in \omega\}$ can be taken to be the identity in the previous theorem. This shows that the only way to get a non-uniform barrier of rank ω is by embedding a non-uniform barrier $\mathcal{B}[n]$ on top of $\{n\}$, so basically every barrier of rank ω is uniform. This is as far as we can go since there is a non-uniform barrier $\mathcal{B}$ of rank $\omega + 1$ such that $\mathcal{B}[n]$ is uniform for every $n \in \omega$.

Example 4.1.19. There is a non-uniform barrier of rank $\omega + 1$.

We define a barrier $\mathcal{B}$ by describing $\mathcal{B}(n)$ for every $n \in \omega$. For every $k \in \omega$, let $\mathcal{S}_k = \{s \in [\omega]^{<\omega} : |s| = \min(s) + k\}$, (hence, the Schreier barrier is $\mathcal{S}_1$). If n is even, define $\mathcal{B}(n) = n^\frown(\mathcal{S}_n + n + 1)$. For n odd, let $\mathcal{B}(n) = n^\frown[\omega \setminus (n+1)]^n$.

It is clear that every infinite subset of ω has an initial segment in $\mathcal{B}$ and that each $\mathcal{B}(n)$ is a $\subseteq$-antichain. It remains to show that if $s = \{s_0, \ldots, s_i\} \in \mathcal{B}(n)$, $t = \{t_0, \ldots, t_j\} \in \mathcal{B}(m)$ and $n < m$, then $t \nsubseteq s$ (the other contention is impossible as $n \in s \setminus t$).

Let us first compute the size of an element $b \in \mathcal{B}$. Let $b = \{b_0, \ldots, b_k\}$. If b_0 is odd, then $b \in \mathcal{B}(b_0)$ and has size $b_0 + 1$. Otherwise, $b \in b_0^\frown(\mathcal{S}_{b_0} + b_0 + 1)$. Let $b' = b \setminus \{b_0\} \in \mathcal{S}_{b_0} + b_0 + 1$. It is clear that $|b'| = |b' - b_0 - 1|$ and since $(b' - b_0 - 1) \in \mathcal{S}_{b_0}$ and $\min(b' - b_0 - 1) = b_1 - b_0 - 1$, we conclude that $|b'| = |b' - b_0 - 1| = (b_1 - b_0 - 1) + b_0 = b_1 - 1$. Therefore, $|b| = b_1$.

We are now ready to show that s and t are $\subseteq$-incomparable. If both n and m are odd, we have $|t| = m + 1 > n + 1 = |s|$ and we are done. If both n and m are even, then $|s| = s_1$ and $|t| = t_1$ but $s_0 = n \notin t$ implies that if $t \subseteq s$, hence $|s| = s_1 \leq t_0 < t_1 = |t|$, which is a contradiction.
Similarly if n is odd and m is even, assuming that $t \subseteq s$ yields that $|t| = t_1 > t_0 \geq s_1 \geq s_0 + 1 = |s|$, a contradiction. Finally, if n is even and m is odd, and if we assume that $t \subseteq s$, we reach a contradiction as $|t| = t_0 + 1 > t_0 \geq s_1 = |s|$. □

The persistence of the rank under restriction for a uniform barrier is the key property of the notion of uniformity. We can imitate this behaviour by considering a lower bound for the ranks of uniform restrictions of a given barrier $\mathcal{B}$.

Definition 4.1.20. Given a barrier $\mathcal{B}$, let

$$spec(\mathcal{B}) = \{\alpha < \omega_1 \mid \exists M \in [\omega]^\omega \ (\mathcal{B}|M \text{ is uniform with rank } \alpha)\}.$$

We define the *uniform rank of $\mathcal{B}$* as $\rho_u(\mathcal{B}) = \min(spec(\mathcal{B}))$.

In [49], it is inductively proved that sequentially compact spaces of character less that $\mathfrak{b}$ are n-sequentially compact for every n. To perform the induction on $[\omega]^{n+1}$, it suffices to notice that any element of $[\omega]^{n+1}$ has a unique initial segment in $[\omega]^n$. It is also true that any element in $[\omega]^n$ is end-extended by at least one element in $[\omega]^{n+1}$. This fact is used to analyze a splitting-like cardinal invariant in [15] that helps with the construction of n-sequentially compact spaces that fail to be $(n+1)$-sequentially compact under some assumptions involving this cardinal. Other examples of results that use this fact are Ramsey's theorem itself and the proof that $\mathfrak{par}_n = \mathfrak{par}_2 = \max\{\mathfrak{b}, \mathfrak{s}\}$ in [8]. In order to obtain analogous results in our framework, we need a generalization of this fact for barriers of any rank. The statement of Lemma 4.1.22 below and its proof appear in [2] in a slightly different way; we add here a proof for completeness.

Notation 4.1.21. Given two families $\mathcal{B}, \mathcal{C} \subseteq [\omega]^{<\omega}$ we denote by $\mathcal{B} \sqsubseteq \mathcal{C}$ if

- $\forall s \in \mathcal{B}\ \exists t \in \mathcal{C}\ (s \sqsubseteq t)$, and
- $\forall t \in \mathcal{C}\ \exists s \in \mathcal{B}\ (s \sqsubseteq t)$.

Lemma 4.1.22. *Given two barriers $\mathcal{B}$ and $\mathcal{C}$ on a countable set $N \in [\omega]^\omega$, there is an infinite set $M \in [N]^\omega$ such that either $\mathcal{B}|M \sqsubseteq \mathcal{C}|M$ or $\mathcal{C}|M \sqsubseteq \mathcal{B}|M$. Moreover, if $\rho(\mathcal{B}) < \rho(\mathcal{C})$ and $\mathcal{C}$ is uniform, then $\mathcal{B}|M \sqsubseteq \mathcal{C}|M$ necessarily holds.*

Proof. Define $\mathcal{B}_0 = \{b \in \mathcal{B} \mid \exists c \in \mathcal{C}(b \subseteq c)\}$. Since $\mathcal{B} = \mathcal{B}_0 \cup (\mathcal{B} \setminus \mathcal{B}_0)$, we can find, by the Nash-Williams theorem, an infinite set $M \in [N]^\omega$ such that either $(\mathcal{B} \setminus \mathcal{B}_0)|M = \emptyset$ or $\mathcal{B}_0|M = \emptyset$. If $(\mathcal{B} \setminus \mathcal{B}_0)|M = \emptyset$, then we are done, as this implies that $\mathcal{B}|M \sqsubseteq \mathcal{C}|M$. To see this, note that for every $b \in \mathcal{B}|M$ (and hence $b \in \mathcal{B}_0$), there is $c \in \mathcal{C}$ such that $b \subseteq c$, but if we define $X = b \cup (M \setminus \max(b))$, there is also $c' \in \mathcal{C}$ such that $c' \sqsubseteq X$. As $b \subseteq c \in \mathcal{C}$, it happens that $b \sqsubseteq c'$ and $c' \in \mathcal{C}|M$. For $c \in \mathcal{C}|M$, we can also find $b \in \mathcal{B}|M$ such that $b \sqsubseteq c \cup (M \setminus (\max(c) + 1))$. The case where $c \sqsubset b$ is not possible since $b \in \mathcal{B}_0$ and thus there is another $c' \in \mathcal{C}|M$ such that $b \subseteq c'$.

Assume otherwise $\mathcal{B}_0|M = \emptyset$ and define $\mathcal{C}_0 = \{c \in \mathcal{C}|M \mid \exists b \in \mathcal{B}|M(c \subseteq b)\}$. Again, by the Nash-Williams theorem, we can find $M' \in [M]^\omega$ such that either $\mathcal{C}_0|M' = \emptyset$ or $(\mathcal{C} \setminus \mathcal{C}_0)|M' = \emptyset$. The case $\mathcal{C}_0|M' = \emptyset$ is impossible since we can find $b \in \mathcal{B}|M' \subseteq \mathcal{B}|M$ and $c \in \mathcal{C}|M'$ such that $b, c \sqsubseteq M'$ and then either $b \subseteq c$ or $c \subseteq b$ contradicting the choices of M and M'. Thus, $(\mathcal{C} \setminus \mathcal{C}_0)|M' = \emptyset$ and as in the previous case we get that $\mathcal{C}|M' \sqsubseteq \mathcal{B}|M'$.

For the last assertion of the statement, notice that if $\mathcal{C}|M \sqsubseteq \mathcal{B}|M$, then $T(\mathcal{C}|M) \subseteq T(\mathcal{B}|M)$. In particular,

$$\rho(\mathcal{C}) = \rho(\mathcal{C}|M) = \rho_{T(\mathcal{C}|M)}(\emptyset) \leq \rho_{T(\mathcal{B}|M)}(\emptyset) = \rho(\mathcal{B}|M) \leq \rho(\mathcal{B}). \qquad \square$$

The previous lemma tells us that we can define a preorder on the family of all barriers on ω by defining $\mathcal{B} \leq \mathcal{C}$ if there is an infinite set M such that $\mathcal{B}|M \sqsubseteq \mathcal{C}|M$. One may be tempted to say that the partial order defined by identifying $\mathcal{B}$ and $\mathcal{C}$ if $\mathcal{B} \leq \mathcal{C}$ and $\mathcal{C} \leq \mathcal{B}$, collapses to ω_1 when restricted to the family of uniform barriers, that is, $\mathcal{B} \leq \mathcal{C}$ if and only if $\rho(\mathcal{B}) \leq \rho(\mathcal{C})$. However, the unpleasant fact about this is that for two barriers $\mathcal{B}$ and $\mathcal{C}$ of the same rank, it is not always true that $\mathcal{B} \leq \mathcal{C}$ and $\mathcal{C} \leq \mathcal{B}$. For example, if $\mathcal{B}$ is the Schreier barrier and $\mathcal{C}$ is any uniform barrier defined such that $\rho_{T(\mathcal{C})}(\{n\}) = f(n)$ for an increasing function $f \in \omega^\omega$ that is strictly bigger than the identity, then $\mathcal{C} \not\leq \mathcal{B}$. Notice that any uniform barrier $\mathcal{B}$ of rank ω is completely determined by a function $f \in \omega^\omega$ that encodes the ranks of the first level on $T(\mathcal{B})$. A more complex but naturally defined coding of the ranks of the successors of any element $s \in T(\mathcal{B})$ also completely determines the structure of any barrier $\mathcal{B}$.

We see now that this function is the only obstruction and that if we are willing to compress some finite intervals of ω into points, we can define a weaker relation between barriers that depends only on their ranks and extends $\leq$. This allows us to establish, in some instances, that satisfying a property for a single uniform barrier of rank α is equivalent to satisfying it for every barrier of that rank (e.g., regarding "$\mathcal{B}$-sequentially compact" in Corollary 4.2.5).

Definition 4.1.23. Given two barriers $\mathcal{B}$ and $\mathcal{C}$ on a countable set M, we write $\mathcal{C} \preceq \mathcal{B}$ if there is a finite-to-one, non-decreasing function $f \in \omega^\omega$ such that for every infinite subset $M' \in [M]^\omega$ there exists $N \in [M']^\omega$ so that $f \restriction N$ is one-to-one and

- $\forall b \in (\mathcal{B}|N)\ \exists c \in \mathcal{C}\ (c \sqsubseteq f[b])$.

Notice that if $\mathcal{B}$ is uniform and $\rho(\mathcal{C}) < \rho(\mathcal{B})$, then $\mathcal{C} \preceq \mathcal{B}$ with f being the identity map, since $\mathcal{C}|M \leq \mathcal{B}|M$ for every restriction to an infinite set M. The condition that $f \restriction N$ is one-to-one is superfluous, as we can always shrink N in order to obtain this property; however, it is convenient to add this to the definition to avoid stating it explicitly each time we use the preorder $\preceq$.

Having said this, it is clear that the relation $\preceq$ is transitive, that is, if $\mathcal{C} \preceq \mathcal{B}$ and $\mathcal{D} \preceq \mathcal{C}$ is witnessed by f and g, respectively, then $\mathcal{D} \preceq \mathcal{B}$ and it is witnessed by $g \circ f$.

Proposition 4.1.24. *If $\mathcal{B}$ and $\mathcal{C}$ are two barriers in ω, $\mathcal{B}$ is uniform and $\rho(\mathcal{B}) \geq \rho(\mathcal{C})$, then $\mathcal{C} \preceq \mathcal{B}$.*

Proof. We alternately define two sequences $\{a_i \mid i \in \omega\}$ and $\{b_i \mid i \in \omega\}$ such that $a_i < b_i < a_{i+1}$ for every $i \in \omega$. We also denote by $\{k_i \mid i \in \omega\}$ the increasing sequence defined by these two sequences, that is, $k_{2i} = a_i$ and $k_{2i+1} = b_i$. We also define simultaneously $f \in \omega^\omega$ which is completely determined by the sequence $\{k_i \mid i \in \omega\}$ as follows: $f \restriction [0, k_0) = 0$ and $f(n) = i$ if and only if $n \in [k_i, k_{i+1})$. We show that f satisfies the definition of $\mathcal{C} \preceq \mathcal{B}$. To save notation, we write $\rho_{\mathcal{B}}$ and $\rho_{\mathcal{C}}$ instead of $\rho_{T(\mathcal{B})}$ and $\rho_{T(\mathcal{C})}$, respectively. Also, given $s \in [\omega]^\omega \setminus T(\mathcal{C})$ we define $\rho_{\mathcal{C}}(s) = -1$ so that we can talk about the rank of s even if s is not in $T(\mathcal{C})$.

Step 0:

Since $\rho_{\mathcal{B}}(\emptyset) \geq \rho_{\mathcal{C}}(\emptyset)$, there are $a_0 < b_0 < \omega$ such that $\rho_{\mathcal{B}}(\{a_0\}) \geq \rho_{\mathcal{C}}(\{0\})$ and $\rho_{\mathcal{B}}(\{b_0\}) \geq \rho_{\mathcal{C}}(\{1\})$.

Now, assume we have defined $\{a_i \mid i \leq n\}$ and $\{b_i \mid i \leq n\}$.

Step a:

Inductive hypothesis at n:

Let us call

$$S_a(n) = \Big\{ s \in T(\mathcal{B}) \mid s \subseteq \bigcup_{i<n} [a_i, b_i) \ \wedge\ \forall i < n(|s \cap [a_i, b_i)| \leq 1) \Big\}.$$

We ensure that the following inductive hypothesis holds throughout the construction: Given $s \in S_a(n)$

- $\phi_a(n, s)$ denotes the formula $\rho_{\mathcal{B}}(s) \geq \rho_{\mathcal{C}}(f[s])$,
- $\psi_a(n, s)$ denotes the formula: if $j \geq a_n$, then $\rho_{\mathcal{B}}(s \cup \{j\}) \geq \rho_{\mathcal{C}}(f[s \cup \{2n\}])$.

We also write $\phi_a(n)$ as a shorthand for $\forall s \in S_a(n)(\phi_a(n,s))$ and $\psi_a(n)$ as a shorthand for $\forall s \in S_a(n)(\psi_a(n,s))$. It is worth pointing out that $S_a(n)$, $\phi_a(n,s)$ and $\psi_a(n,s)$ do not mention b_n. Note that if $j \geq a_0$, we have $\rho_{\mathcal{B}}(\{j\}) \geq \rho_{\mathcal{B}}(\{a_0\}) \geq \rho_{\mathcal{C}}(\{0\})$ since $\mathcal{B}$ is uniform. Thus, $\psi_a(0)$ is satisfied and $\phi_a(0)$ is also vacuously satisfied (here $S_a(0) = \{\emptyset\}$).

Construction of a_{n+1}:

We need to take care of all $s \in S_a(n+1)$ (note that this set is well defined since it only depends on $\{a_i, b_i \mid i \leq n\}$).

First, consider $s \in S_a(n)$. By $\phi_a(n,s)$, we have $\rho_{\mathcal{B}}(s) \geq \rho_{\mathcal{C}}(f[s])$ and thus we can find m_s^a such that

$(\star_1^a)$ $\rho_{\mathcal{B}}(s \cup \{m\}) \geq \rho_{\mathcal{C}}(f[s] \cup \{2n+2\})$ for every $m \geq m_s^a$.

Otherwise, $s \in S_a(n+1) \setminus S_a(n)$ and we can write $s' = s \setminus \{j\}$ where $j = \max(s) \geq a_n$ and $s' \in S_a(n)$. Then, we know that $\rho_{\mathcal{B}}(s' \cup \{j\}) \geq \rho_{\mathcal{C}}(f[s'] \cup \{2n\})$ by $\psi_a(n,s)$. Since $j \in [a_n, b_n)$, we have $f(j) = 2n$. Thus

$(*^a)$ $\rho_{\mathcal{B}}(s) = \rho_{\mathcal{B}}(s' \cup \{j\}) \geq \rho_{\mathcal{C}}(f[s'] \cup \{f(j)\}) = \rho_{\mathcal{C}}(f[s' \cup \{j\}]) = \rho_{\mathcal{C}}(f[s])$.

Then there is $m_s^a \in \omega$ such that $\rho_{\mathcal{B}}(s \cup \{m_s^a\}) \geq \rho_{\mathcal{C}}(f[s] \cup \{2n+2\})$. Since $\mathcal{B}$ is uniform,

$(\star_2^a)$ $\rho_{\mathcal{B}}(s \cup \{m\}) \geq \rho_{\mathcal{C}}(f[s] \cup \{2n+2\})$ for every $m \geq m_s^a$.

Finally, define

$$a_{n+1} = \max\{m_s^a \mid s \in S_a(n+1)\}.$$

Inductive hypothesis for $n+1$:

We want to see that $\phi_a(n+1)$ and $\psi_a(n+1)$ hold. Let $s \in S_a(n+1)$. If $s \in S_a(n)$ then $\phi_a(n+1,s)$ follows from $\phi_a(n,s)$. Consequently, we can assume that $s \in S_a(n+1) \setminus S_a(n)$ and let $s' = s \setminus \{j\}$ where $j = \max(s)$. Hence, by $(*^a)$, we have $\rho_{\mathcal{B}}(s) \geq \rho_{\mathcal{C}}(f[s])$, which means that $\phi_n(n+1,s)$ holds and, consequently, $\phi_a(n+1)$ does.

Now, let us see that $\psi_a(n+1)$ is also true. Let $s \in S_a(n+1)$ and $j \geq a_{n+1}$. If $s \in S_a(n)$ then m_a^s satisfies $(\star_1^a)$ and if $s \in S_a(n+1) \setminus S_a(n)$ we see that m_a^s satisfies $(\star_2^a)$, either case, as $a_n + 1 \geq m_a^s$ we have that $\psi_a(n+1,s)$ holds and so does $\psi_a(n+1)$.

Inductive hypothesis at n:

The construction is dual to that of step a. We start by calling

$$S_b(n) = \left\{ s \in T(\mathcal{B}) \;\middle|\; s \subseteq \bigcup_{i<n} [b_i, a_{i+1}) \;\wedge\; \forall i < n(|s \cap [b_i, a_{i+1})| \leq 1) \right\}.$$

The corresponding inductive formulas for $s \in S_b(n)$ are now:

- $\phi_b(n,s)$ denotes the formula $\rho_{\mathcal{B}}(s) \geq \rho_{\mathcal{C}}(f[s])$.
- $\psi_b(n,s)$ denotes the formula: if $j \geq b_n$, then $\rho_{\mathcal{B}}(s \cup \{j\}) \geq \rho_{\mathcal{C}}(f[s \cup \{2n+1\}])$.

We write again $\phi_b(n)$ and $\psi_b(n)$ as shorthands for $\forall s \in S_b(n)(\phi_a(n,s))$ and $\forall s \in S_b(n)(\psi_a(n,s))$, respectively. In the particular case of ϕ_b, we have that it is the same formula as ϕ_a with a different domain. Since a_{n+1} has already been defined, the interval $[b_n, a_{n+1})$ considered in $S_b(n+1)$ makes complete sense.

Construction of b_{n+1}:

Given $s \in S_b(n)$, from $\phi_b(n,s)$ it follows that $\rho_{\mathcal{B}}(s) \geq \rho_{\mathcal{C}}(f[s])$ and thus we can find m_s^b such that

$(\star_1^b)$ $\rho_{\mathcal{B}}(s \cup \{m\}) \geq \rho_{\mathcal{C}}(f[s] \cup \{2n+3\})$ for every $m \geq m_s^b$.

Otherwise, $s \in S_b(n+1) \setminus S_b(n)$ and we can write $s' = s \setminus \{j\}$ where $j = \max(s) \geq b_n$ and $s' \in S_b(n)$. Thus, $\rho_{\mathcal{B}}(s' \cup \{j\}) \geq \rho_{\mathcal{C}}(f[s'] \cup \{2n+1\})$ by $\psi_b(n, s)$. Since $j \in [b_n, a_{n+1})$, we have $f(j) = 2n+1$. Then

$(*^b)$ $\rho_{\mathcal{B}}(s) = \rho_{\mathcal{B}}(s' \cup \{j\}) \geq \rho_{\mathcal{C}}(f[s'] \cup \{f(j)\}) = \rho_{\mathcal{C}}(f[s' \cup \{j\}]) = \rho_{\mathcal{C}}(f[s])$.

We can again find $m_s^b \in \omega$ such that $\rho_{\mathcal{B}}(s \cup \{m_s^b\}) \geq \rho_{\mathcal{C}}(f[s] \cup \{2n+3\})$ and moreover,

$(\star_2^b)$ $\rho_{\mathcal{B}}(s \cup \{m\}) \geq \rho_{\mathcal{C}}(f[s] \cup \{2n+3\})$ for every $m \geq m_s^b$.

We then define

$$b_{n+1} = \max\{m_s^b \mid s \in S_b(n+1)\}.$$

Inductive hypothesis for $n+1$:

If $s \in S_b(n)$ then $\phi_b(n+1, s)$ follows from $\phi_b(n, s)$ and otherwise $s \in S_b(n+1) \setminus S_b(n)$ can be written as $s' = s \setminus \{j\}$ where $j = \max(s)$. By $(*^b)$, we have $\rho_{\mathcal{B}}(s) \geq \rho_{\mathcal{C}}(f[s])$ and then $\phi_b(n+1)$ holds.

To see that $\psi_b(n+1)$ is true, let $s \in S_b(n+1)$ and $j \geq b_{n+1}$. Then $b_{n+1} \geq m_s^b$ and either by $(\star_1^b)$ or $(\star_2^b)$ we have that $\phi_b(n+1, s)$ holds.

This finishes the construction and it remains to prove that f works.

Clearly f is finite-to-one and non-decreasing. Given $M \in [\omega]^\omega$, we can find an infinite subset $N \in [M]^\omega$ such that $|N \cap [k_i, k_{i+1}]| \leq 1$ for every $i \in \omega$ and either $N \cap \bigcup_{i \in \omega}[k_{2i}, k_{2i+1}) = \emptyset$ or $N \cap \bigcup_{i \in \omega}[k_{2i+1}, k_{2i+2}) = \emptyset$. Without loss of generality, assume that $N \subseteq \bigcup_{i \in \omega}[k_{2i}, k_{2i+1}) = \bigcup_{i \in \omega}[a_i, b_i)$.

Now, let $b \in \mathcal{B}|N$. It follows from the choice of N that there is an $n \in \omega$ such that $b \in S_a(n)$. By $\phi_a(n, b)$ we know that $0 = \rho_{\mathcal{B}}(b) \geq \rho_{\mathcal{C}}(f[b])$. Necessarily $\rho_{\mathcal{C}}(f[b]) \in \{0, -1\}$, but in either case we can conclude that there is $c \in \mathcal{C}$ such that $c \sqsubseteq f[b]$. This finishes the proof. □

The previous lemma is new and constitutes the final component of the theory of barriers needed to prove that the general notion of barrier sequential compactness, which extends n-sequential compactness, depends only on the rank of the associated uniform barrier. In the case where $\mathcal{B}$ has rank ω, we can eliminate the requirement that the barrier is uniform.

Corollary 4.1.25. *Let $\mathcal{B}$ and $\mathcal{C}$ be barriers on ω such that $\rho(\mathcal{C}) \leq \rho(\mathcal{B}) = \omega$, then $\mathcal{C} \preceq \mathcal{B}$.*

Proof. By the fact that the preorder $\preceq$ is transitive and the previous proposition, it suffices to show that $\mathcal{S} \preceq \mathcal{B}$, where $\mathcal{S}$ is the Schreier barrier.

Since $\mathcal{B}$ has rank ω, it follows from Lemma 4.1.13 that for every $i \in \omega$, we can find m_i such that $\rho_{\mathcal{B}}(m) \geq i+1$ for every $m \geq m_i$. Let $f \in \omega^\omega$ be given by $f(n) = i$ if and only if $n \in [m_i, m_{i+1})$ (and we define $f \restriction m_0$ arbitrarily). We claim that f witnesses $\mathcal{S} \preceq \mathcal{B}$.

Indeed, given $M \in [\omega]^\omega$, we can find $N_0 \in [M]^\omega$ such that $N_0 \cap [0, m_0) = \emptyset$ and $|N_0 \cap [m_i, m_{i+1})| \leq 1$ for every $i \in \omega$. Let $n_0 = \min(N_0)$ and for every $i \in \omega$ define n_{i+1} as follows: for each $j < i+1$ we can apply Corollary 4.1.14 and find k_j such that $\mathcal{B}[n_j] = [\omega \setminus k_j]^r$ for some $r \geq l$ where l is the unique natural number such that $n_j \in [m_l, m_{l+1})$, pick $n_{i+1} > k_j$ for every $j < i+1$. Thus, $N = \{n_i \mid i \in \omega\} \in [M]^\omega$.

Fix $b \in (\mathcal{B}|N)$ and let $n_i = \min(b)$. It follows from the definitions that $b \setminus \{n_i\} \in [\omega \setminus k_j]^r$ for some $r \geq l$ where $n_i \in [m_l, m_{l+1})$. Then $|f[b]| \geq l+1$ as f is one to one on N but $f(n_i) = l$ and $\mathcal{S}(l) = \{s \in [\omega]^{<\omega} \mid \min(s) = l \wedge |s| = l+1\}$. Thus, we can conclude that $c \sqsubseteq f[b]$ defined as the first $l+1$ elements of $f[b]$ is an element of $\mathcal{S}$. □

4.2 $\mathcal{B}$-sequential compactness

We now define the notion of $\mathcal{B}$-sequential compactness for barriers $\mathcal{B}$ and prove the necessary results to formulate the natural notion of α-sequential compactness, where $\alpha \in \omega_1$ corresponds to the rank of the barrier. All the results of this section are natural generalizations of the results presented in [49] for finite n.

Definition 4.2.1. Let $\mathcal{B}$ be a barrier on $M \in [\omega]^\omega$, let X be a topological space, $f : \mathcal{B} \to X$ and $x \in X$. We say that f *converges* to x if for all $U \in \mathcal{N}(x)$ there is $n \in \omega$ such that $f[(\mathcal{B}|(M \setminus n))] \subseteq U$.[2]

The term "converges" might seem ambiguous since f has a countable domain; however, its meaning will be clear from the context throughout this thesis. The function f should be viewed as a $\mathcal{B}$-dimensional sequence (or an α-dimensional sequence, where $\alpha = \rho(\mathcal{B})$), rather than a classical 1-dimensional sequence indexed by the countable set $\mathcal{B}$.

Definition 4.2.2. Let $\mathcal{B}$ be a barrier on ω and let X be a topological space. We say that X is *$\mathcal{B}$-sequentially compact* if for all $f : \mathcal{B} \to X$ there is $M \in [\omega]^\omega$ such that $f \restriction (\mathcal{B}|M)$ converges.

The natural stratification of barriers given by their ranks gives us a natural classification of $\mathcal{B}$-sequentially compact spaces grouping them and associating them to a countable ordinal attending the following definition:

Definition 4.2.3. Let $\alpha < \omega_1$. We say that X is *α-sequentially compact* if X is $\mathcal{B}$-sequentially compact for all barriers $\mathcal{B}$ of rank α.

The first application of the theory developed in Section 4.1 and the main reason for the introduction of the order $\preceq$ is due to the following theorem:

Theorem 4.2.4. *If $\mathcal{C} \preceq \mathcal{B}$ and X is $\mathcal{B}$-sequentially compact, then X is also $\mathcal{C}$-sequentially compact.*

Proof. Assume X is $\mathcal{B}$ sequentially compact and let $f \in \omega^\omega$ be as in the definition of $\mathcal{C} \preceq \mathcal{B}$. Let $h : \mathcal{C} \to X$ and define $\hat{h} : \mathcal{B} \to X$ by $\hat{h}(b) = h(c)$ for any $c \in \mathcal{C}$ that is $\sqsubseteq$-compatible with $f[b]$. Notice that since $\mathcal{C}$ is a barrier, there is at least one of such c and then $\hat{h}$ is well defined.

Since X is $\mathcal{B}$-sequentially compact, there is $M \in [\omega]^\omega$ such that $\hat{h} \restriction (\mathcal{B}|M)$ converges to some $x \in X$. By the definition of $\mathcal{C} \preceq \mathcal{B}$, take $N \in [M]^\omega$ such that

$$\forall b \in (\mathcal{B}|N)\ \exists c \in \mathcal{C}\ (c \sqsubseteq f[b]),$$

and define $M_0 = f[N]$. It turns out that $\hat{h} \restriction (\mathcal{B}|N)$ also converges to x. We prove that $h \restriction (\mathcal{C}|M_0)$ converges to the same $x \in X$.

To see this, fix an open neighbourhood $U \in \mathcal{N}(x)$. We can find $n \in \omega$ such that $\hat{h}(b) \in U$ whenever $b \in \mathcal{B}|(N \setminus n)$. Take $c \in \mathcal{C}|(M_0 \setminus f(n))$. Thus, $c = \{f(m_0), \ldots, f(m_j)\}$ for some set $a = \{m_i \mid i \leq j\} \subseteq N \setminus n$. Take any infinite set $N' \subseteq N$ such that $a \sqsubseteq N'$. We can find $b \in \mathcal{B}|N$ such that $b \sqsubseteq N'$. It follows that $a \sqsubseteq b$ since otherwise $f[b] \sqsubset f[a] = c$ would contradict $b \subseteq N$. Therefore, $h(c) = \hat{h}(b) \in U$ as desired. □

[2] As mentioned in the introduction, throughout this chapter we use the notation $f[A]$ to denote the set $\{f(x) \mid x \in A\}$, instead of the notation $f''A$ used previously. We adopt this change because the bracket notation is more convenient when dealing with images of sets defined by long expressions, which occurs frequently in this chapter.

Corollary 4.2.5. *Let X be a topological space and $\alpha \in \omega_1$, the following are equivalent:*

(I) X is α-sequentially compact,

(II) X is $\mathcal{B}$-sequentially compact for every uniform barrier of rank α,

(III) X is $\mathcal{B}$-sequentially compact for some uniform barrier $\mathcal{B}$ of rank α.

Proof. That *(III)* implies *(I)* follows directly from Proposition 4.1.24 and Theorem 4.2.4. The other implications are clear from the definitions. □

From Lemma 4.1.24 and Theorem 4.2.4, we can derive the following result.

Corollary 4.2.6. *If X is α-sequentially compact and $\beta < \alpha$ then X is β-sequentially compact. In particular, if α is infinite, X is n-sequentially compact for every $n \in \omega$.*

The stronger result for barriers of rank ω given in Corollary 4.1.25 yields the following result:

Corollary 4.2.7. *If X is $\mathcal{B}$-sequentially compact for some barrier of rank ω (not necessarily uniform), then X is ω-sequentially compact.*

In general, this is not true for every countable ordinal. A trivial example is a barrier with a single node in $T(\mathcal{B})$ of rank ω (say $\{0\}$) and such that $\mathcal{B}|(\omega \setminus 1)$ is isomorphic to $\mathcal{S}_2$. In this case, every infinite restriction has rank ω, but the barrier itself has rank $\omega + 1$. It is not hard to modify our constructions in Section 4.4 to show that, consistently, there is a $\mathcal{B}$-sequentially compact space that is not $(\omega+1)$-sequentially compact. We do not know whether the requirement that $\mathcal{B}$ has an infinite restriction of rank α suffices to show that $\mathcal{B}$-sequentially compact spaces are α-sequentially compact. We believe that the answer is "no", but we conjecture that a weaker result is true:

Conjecture 4.2.8. *If X is $\mathcal{B}$-sequentially compact, then it is $\rho_u(\mathcal{B})$-sequentially compact.*

The following easy fact is a partial answer to Conjecture 4.2.8.

Proposition 4.2.9. *Let $\mathcal{B}$ be a barrier on ω and X a $\mathcal{B}$-sequentially compact space. If there is $M \in [\omega]^\omega$ such that $\mathcal{B}|M$ is uniform and either X is not $(\mathcal{B}|(\omega \setminus M))$–sequentially compact or $\omega \setminus M$ is finite, then X is $\rho(\mathcal{B}|M)$-sequentially compact. In particular X is $\rho_u(\mathcal{B})$-sequentially compact.*

Proof. By Corollary 4.2.5 it is enough to prove that X is $(\mathcal{B}|M)$-sequentially compact. Let $F := \omega \setminus M$. Regardless of whether F is finite or X is not $(\mathcal{B}|F)$-sequentially compact, there exists a function $g : \mathcal{B}|F \to X$ without infinite convergence subsequences (here convergence means convergence with respect to barriers). Now, let $f : \mathcal{B}|M \to X$. We want to prove that f admits an infinite convergent subsequence $f \restriction (\mathcal{B}|M')$ for some $M' \in [\omega]^\omega$. For this, let $\hat{f} : \mathcal{B} \to X$ be any common extension of both f and g. As X is $\mathcal{B}$-sequentially compact, there is $N \in [\omega]^\omega$ such that $\hat{f} \restriction N$ is convergent. Now, $N \cap F$ should be finite, since otherwise $\hat{f} \restriction (N \cap F)$ would also be an infinite convergent subsequence for g. In this way, $N' := N \setminus F$ is infinite and thus $\hat{f} \restriction N' = f \restriction N'$ is an infinite convergent subsequence for f. □

4.3 Some classes of ω_1-sequentially compact spaces

In [49], it is proved that compact metric spaces are n-sequentially compact for every $n \in \omega$. This result extends to any $\alpha < \omega_1$. We start this section by giving two classes of spaces, each containing the class of compact metric spaces, that sit at the top of the hierarchy of α-sequentially compact spaces.

Definition 4.3.1. We say that X is ω_1-*sequentially compact*, if it is α-sequentially compact for every $\alpha < \omega_1$.

The first family of spaces we prove to be ω_1-sequentially compact is the class of sequentially compact spaces whose character is less than $\mathfrak{b}$. Theorem 2.0.11 (proved in [49]) asserts that every sequentially compact space of character $< \mathfrak{b}$ is n-sequentially compact for every $n \in \omega$. We now state an infinite-dimensional generalization.

Theorem 4.3.2. *Suppose X is sequentially compact. If $\chi(X) < \mathfrak{b}$, then X is ω_1-sequentially compact.*

Proof. We show by induction that X is α-sequentially compact for every $\alpha < \omega_1$. If $\alpha = 1$, then X being sequentially compact is equivalent to X being 1-sequentially compact, since the only uniform barrier of rank 1 is $[\omega]^1$.

We prove the inductive step at β, treating the successor and limit cases simultaneously. If β is a successor ordinal, let $\beta = \alpha+1$; if it is a limit ordinal, let $(\alpha_n)_{n\in\omega}$ be an increasing sequence with supremum β.

Let $f\colon \mathcal{B} \to X$ be any function, where $\mathcal{B}$ is uniform of rank β. Note that for all $n \in \omega$, $\mathcal{B}[n]$ is a uniform barrier of rank α (in the successor case) or of rank α_n (in the limit case). For all $n \in \omega$, define $f_n\colon \mathcal{B}[n] \to X$ by $s \mapsto f(\{n\} \cup s)$. By the inductive hypothesis, there exists an infinite subset N_0 and a point $x_0 \in X$ such that $f_0 \restriction (\mathcal{B}[0]|N_0) \to x_0$.

Let $n_0 = 0$ and $n_1 = \min N_0 \setminus \{0\}$. Note that for any infinite subset A of ω and any $n \in \omega$, $\mathcal{B}[n]|A$ is also uniform of rank α if $\beta = \alpha + 1$ or of rank α_n if β were a limit ordinal. Now, suppose that we have defined a decreasing sequence of infinite subsets $N_0 \supseteq N_1 \supseteq \cdots \supseteq N_{i-1}$, and an increasing sequence $n_0 < n_1 < \ldots < n_i$ in addition to a subset $\{x_0, \ldots, x_{i-1}\}$ of X. Consider the function $f_{n_i} : (\mathcal{B}[n_i]|N_{i-1}) \to X$, applying our inductive hypothesis yields an infinite subset $N_i \subseteq N_{i-1}$ such that $f_{n_i} \restriction (\mathcal{B}[n_i]|N_i) \to x_i$, for some point $x_i \in X$. We also define $n_{i+1} = \min(N_i \setminus \{n_0, \ldots, n_i\})$. Thereby, we obtain a decreasing sequence of infinite subsets $N_0 \supseteq N_1 \supseteq N_2 \supseteq \ldots$, a subset $\{x_i \mid i \in \omega\}$ of the space X and a strictly increasing sequence $\langle n_i \mid i \in \omega\rangle$ such that $f_{n_i} \restriction (\mathcal{B}[n_i]|N_i) \to x_i$ where $n_{i+1} = \min N_i \setminus \{n_0, \ldots, n_i\}$ for all $i \in \omega$.

Since $\{x_i \mid i \in \omega\} \subseteq X$ and X is sequentially compact, there is a convergent subsequence, $Y = \{x_{i_j} \mid j \in \omega\}$, with limit point x. Re-numerating the indices, we can view $Y = \{x_i \mid i \in \omega\}$ as the convergent subsequence. Also, let $N = \{n_i \mid i \in \omega\}$.

Take any open set U in X that contains the point x. Since $x_i \to x$, there is an integer $m_U \in \omega$ such that $x_i \in U$ for all $i > m_U$. Also, for all such i there is another integer $\phi_U(n_i)$ such that

$$f_{n_i}[\mathcal{B}[n_i]|(N_i \setminus \phi_U(n_i))] \subseteq U. \tag{$*$}$$

For any $j \leq m_U$ and any $j \notin \{n_i \mid i \in \omega\}$, let $\phi_U(j) = 0$. Thus, any open subset U of X containing x induces a function $\phi_U : \omega \to \omega$.

Let $\eta(x)$ be a local neighborhood base of x of minimum size. Since $\chi(X) < \mathfrak{b}$ the set of functions $\{\phi_U \mid U \in \eta(x)\}$ is bounded by some increasing function $\psi : \omega \to \omega$, that is, $\phi_U <^* \psi$ for every $U \in \eta(x)$.

We can find an increasing subsequence $\langle m_i \mid i \in \omega\rangle$ of $\langle n_i \mid i \in \omega\rangle$ such that $m_{i+1} > \psi(m_i)$ for every $i \in \omega$. We claim that $f \restriction (\mathcal{B}|M) \to x$ where $M = \{m_i \mid i \in \omega\}$.

Fix an open set $U \in \eta(x)$ and let $k \in \omega$ be such that $x_j \in U$ and $\psi(j) > \phi_U(j)$ for every $j \geq k$. Let $s \in \mathcal{B}|(M \setminus \psi(k))$ and write s as $\{m_{k_0}, \ldots, m_{k_n}\}$ and $s' = s \setminus \{m_{k_0}\}$. Note that for every $i > 0$, we have

$$m_{k_i} > \psi(m_{k_0}) > \phi(m_{k_0}),$$

as $m_{k_0} \geq \psi(k) \geq k$. In particular, $s' \cap \phi(m_{k_0}) = \emptyset$.

Now, let $j \in \omega$ be such that $m_{k_0} = n_j$. Note that if $j' > j$, then $n_{j'} \in N_j$. As any $m_{k_i} > m_{k_0}$ for $i > 0$ and this implies that $m_{k_i} = n_{j'}$ for some $j' > j$, we can conclude that $s' \subseteq N_j$.

Combining the previous arguments, we get that $s' \subseteq N_j \setminus \phi_U(n_j)$ and we already know $s = \{n_j\} \cup s' \in \mathcal{B}$, which implies $s' \in \mathcal{B}[n_j]$. Therefore, $f(s) = f_{n_j}(s') \in U$ by $(*)$ as desired. $\square$

The second class of spaces that we show as ω_1-sequentially compact is the class of compact bisequential spaces. Recall that $\mathcal{U} \subseteq \mathcal{P}(X)$ clusters at x (often written as $x \in \overline{\mathcal{U}}$) if $x \in \overline{U}$ for every $U \in \mathcal{U}$. Of course, if $\mathcal{U}$ is an ultrafilter, then $\mathcal{U}$ clusters at x is equivalent to $\mathcal{U}$ converging to x.

Definition 4.3.3. [53, Definition 3.D.1] Let X be a topological space and $x \in X$. The space X is said to be *bisequential at x* if, for every ultrafilter $\mathcal{U}$ on X that converges to x, there exists a countable subfamily $\{A_n \mid n \in \omega\} \subseteq \mathcal{U}$ such that $\{A_n \mid n \in \omega\}$ converges to x (i.e., every neighborhood of x contains some A_n). The space X is *bisequential* if it is bisequential at every point $x \in X$.

The concept of bisequentiality serves as a robust generalization of first countability. While first countable spaces require a fixed countable local base at each point, bisequential spaces allow for a more flexible condition dependent on the converging ultrafilter. Furthermore, it is straightforward to see that bisequentiality implies the Fréchet property. Thus, we have the following chain of implications:

$$\text{First Countable} \implies \text{Bisequential} \implies \text{Fréchet}. \tag{4.1}$$

To demonstrate that the class of bisequential spaces is distinct from the two established classes (first countable and Fréchet), we present the following examples which show that none of the implications are reversible.

1. [53, Example 10.4] If Z is a compact, first countable space and one collapses a closed subset C to a single point, the resulting quotient space Z/C is a bisequential space that fails to be first countable at the new point, provided C is not a G_δ set. A specific instance is the *Alexandroff Duplicate* of the unit interval $I = [0, 1]$, denoted by $A(I)$ (see [22, Exercise 3.1.G]). The underlying set is $I \times \{0, 1\}$, where the points in $I \times \{1\}$ are isolated, while a basic neighborhood of $(x, 0)$ is of the form $(U \times \{0, 1\}) \setminus F$, where $U \subseteq I$ open, $x \in U$ and $F \subseteq I \times \{1\}$ is finite. It is straightforward to verify that $Z = A(I)$ satisfies the aforementioned properties by taking $C = I \times \{0\}$, as the latter is a closed subset that is not a G_δ set.

2. The theory of Eberlein compacta (see Definition 4.3.11 (1)) provides a natural source of non first countable bisequential spaces. Nyikos established that every uniform Eberlein compact space of weight less than the first measurable cardinal[3] is bisequential [13, Theorem 5.2]. A canonical example is $B_{\ell_2(\Gamma)}$, the unit ball of $\ell_2(\Gamma)$ endowed with the weak topology. Since $B_{\ell_2(\Gamma)}$ is a uniform Eberlein compact space of weight $|\Gamma|$ (see [5]), choosing an uncountable Γ below the first measurable cardinal, yields a bisequential space that is not first countable.

3. [51, Proposition 2.2] Let $S_\omega = (\omega \times \omega) \cup \{\infty\}$ be the *sequential fan*. Points in $\omega \times \omega$ are isolated, and a set $U \ni \infty$ is open if and only if U contains all but finitely many points of each spine $\mathrm{Sp}_n = \{n\} \times \omega$. The space S_ω is Fréchet but not bisequential.

Before stating the main result connecting this to the α-sequential compactness, we introduce necessary terminology regarding trees and barriers.

Definition 4.3.4. 1. Given a tree $T \subseteq \omega^{<\omega}$ and $s \in T$, let

$$Succ_T(s) = \{n \in \omega \mid s^\frown n \in T\}.$$

For an ultrafilter $\mathcal{U} \subseteq \mathcal{P}(\omega)$, T is called *$\mathcal{U}$-branching* if $Succ_T(s) \in \mathcal{U}$ for every $s \in T$.

2. If $\mathcal{B}$ is a barrier on ω, we define the ultrafilter $\mathcal{U}^{\mathcal{B}} \subseteq \mathcal{P}(\mathcal{B})$ by declaring $Y \in \mathcal{U}^{\mathcal{B}}$ if and only if there exists a $\mathcal{U}$-branching tree $S \subseteq T(\mathcal{B})$ such that $Y = S \cap \mathcal{B}$ (identifying finite subsets with increasing sequences).

Theorem 4.3.5. *Every compact bisequential space is ω_1-sequentially compact.*

Proof. Let X be a compact bisequential space, let $\mathcal{B}$ be a barrier and let $f : \mathcal{B} \to X$. Fix an ultrafilter $\mathcal{U} \subseteq \mathcal{P}(\omega)$ and define an ultrafilter $\mathcal{V} \subseteq \mathcal{P}(f[\mathcal{B}]) \subseteq \mathcal{P}(X)$ by $V \in \mathcal{V}$ if and only if $f^{-1}(V) \in \mathcal{U}^{\mathcal{B}}$.

As X is compact, we can find $x \in X$ such that $x \in \overline{\mathcal{V}}$ and by bisequentiality, there exists a decreasing family $\{V_n \mid n \in \omega\} \subseteq \mathcal{V}$ such that for any open neighborhood $W \in \mathcal{N}(x)$, there is $n \in \omega$ such that $V_n \subseteq W$. Define $U_n = f^{-1}(V_n)$ for every $n \in \omega$ and let $T_n = \{s \in T(\mathcal{B}) \mid \exists b \in U_n (s \subseteq b)\}$. Then each T_n is a $\mathcal{U}$-branching tree.

We now recursively define an infinite set $M = \{m_i \mid i \in \omega\}$. Let $m_0 = \min(succ_{T_0}(\emptyset))$. If we have already defined $\{m_i \mid i < n\}$, define m_n such that for every $s \subseteq \{m_i \mid i < n\}$ with $s \in T(\mathcal{B}) \setminus \mathcal{B}$, we have $s^\frown m_n \in T_n$. The choice of m_n is possible since every relevant s is an element of T_0 and the trees T_n are $\mathcal{U}$-branching. Moreover, if $s \subseteq M$, $s \in T(\mathcal{B}) \setminus \mathcal{B}$ and $\min(s) \geq m_n$, then $s \in T_k$ for every $k \geq n$.

It remains to show that $f \restriction (\mathcal{B}|M)$ converges to x. Fix an open set $W \in \mathcal{N}(x)$ and find $n \in \omega$ such that $V_n \subseteq W$. Given $s \in \mathcal{B}|(M \setminus m_n)$, let $m_k = \max(s)$ and $s' = s \setminus m_k$. By the choice of m_k, we have $s = s'^\frown m_k \in T_n \cap \mathcal{B} = U_n$ as $k \geq n$, thus $f(s) \in V_n \subseteq W$. Therefore, $f[\mathcal{B}|(M \setminus m_n)] \subseteq W$. □

Recall that a function is of *Baire-class-1* if it is the pointwise limit of a sequence of continuous functions. A *Rosenthal compactum* is a compact subset $K \subseteq B_1(X)$, where $B_1(X)$ denotes the set of all Baire-class-1 functions from X to $\mathbb{R}$ endowed with the pointwise topology, for some Polish space X. In [56], R. Pol proved that Rosenthal compacta are bisequential. It can also be deduced from the arguments of G. Debs in [18] (see [62] and Lemma 6 in [63]). Therefore, we have the following:

[3] An infinite cardinal κ is *measurable* if there exists a non-principal κ-complete ultrafilter on κ (i.e., the intersection of any family of strictly less than κ-many sets in the ultrafilter is also in the ultrafilter).

Corollary 4.3.6. *Every Rosenthal compact space is ω_1-sequentially compact.*

A space X is *angelic* if the relatively compact subsets of X are relatively compact, and for every relatively compact subset $A \subseteq X$ and for every $x \in \overline{A}$, there is a sequence $\{x_n \mid n \in \omega\} \subseteq A$ that converges to x. It is a result of J. Bourgain, D. H. Fremlin and M. Talagrand [11] that Rosenthal compacta are angelic spaces. Another closely investigated property is the following.

Definition 4.3.7. A space X has *the Ramsey property* if for every function $f : [\omega]^2 \to X$ such that $\lim_{i\to\infty} \lim_{j\to\infty} f(\{i, j\}) = x$, there is $M \in [\omega]^\omega$ such that $f \restriction [M]^2 \to x$.

It was shown by Knaust in [48] that Rosenthal compacta have the Ramsey property, as any 2-sequentially compact space has the Ramsey property, Corollary 4.3.6 is a strengthening of Knaust's result.

The list of angelic spaces with the Ramsey property was expanded in [47], however, the question of whether every angelic space has the Ramsey property was left open. We show now that this is not the case by pointing out that the example of a sequentially compact space that is not 2-sequentially compact considered in [15] is such a counterexample.

Theorem 4.3.8. *There is an angelic space without the Ramsey property.*

Proof. Note that a compact space is angelic if and only if it is Fréchet. In [15], an almost disjoint family $\mathcal{A} \subseteq [\omega \times \omega]^\omega$ is constructed such that $\mathcal{F}(\mathcal{A})$ is Fréchet and sequentially compact, but fails to be 2-sequentially compact. Hence, $\mathcal{F}(\mathcal{A})$ is an angelic space.

In order to prove that $\mathcal{F}(\mathcal{A})$ is not 2-sequentially compact, $\mathcal{A}$ was constructed so that $A_n := \{(n, m) \mid m \in \omega\} \in \mathcal{A}$ for every $n \in \omega$, and so that the function $G : [\omega]^2 \to \omega \times \omega$, given by $G(n, m) = (n, m)$ for $n < m$, has no convergent subsequence in $\mathcal{F}(\mathcal{A})$.

To avoid confusion, denote by $*$ the point at infinity on $\mathcal{F}(\mathcal{A})$. Since $\lim_{m\to\infty}(n, m) = A_n$ and any infinite subset of $\mathcal{A}$ converges to $*$ in $\mathcal{F}(\mathcal{A})$, we conclude that

$$\lim_{n\to\infty} \lim_{m\to\infty} (n, m) = *.$$

Then, moreover, G is a witness that $\mathcal{F}(\mathcal{A})$ does not have the Ramsey property. $\square$

The notion of n-sequential compactness in [49] was motivated by the 2-dimensional version introduced in [9], where the main application was to show that compact metric semigroups have idempotents naturally defined as the limit of a 2-dimensional sequence. From Theorem 4.3.5 and the results in [9], we can conclude the following:

Corollary 4.3.9. *Every compact bisequential semigroup K has an idempotent naturally representable as the Ramsey limit of a restriction of any given $f : [\omega]^2 \to K$.*

The examples of n-sequentially compact spaces that are not $(n+1)$-sequentially compact that were constructed in [49] and [15] were all of the form $\mathcal{F}(\mathcal{A})$, i.e., the one point compactifications of Ψ-spaces over the almost disjoint family $\mathcal{A}$. An important class of almost disjoint families consists of those whose Ψ-space embeds into $\mathbb{R}$. An almost disjoint family $\mathcal{A}$ on ω is $\mathbb{R}$-*embeddable* if there is a one-to-one function $f : \omega \to \mathbb{Q}$ that can be continuously extended to a function $\widehat{f} : \Psi(\mathcal{A}) \to \mathbb{R}$ (see [31] and [36]). We now note that the $\mathbb{R}$-embeddable families all give rise to ω_1-sequentially compact spaces $\mathcal{F}(\mathcal{A})$. As a Ψ-space is first countable and hence bisequential, when we say that an almost disjoint family is bisequential, we mean that $\mathcal{F}(\mathcal{A})$ is bisequential. By Proposition 3.2 in [28], every $\mathbb{R}$-embeddable family is bisequential; thus, we have:

Corollary 4.3.10. *Every $\mathbb{R}$-embeddable almost disjoint family is ω_1-sequentially compact, i.e., if $\mathcal{A}$ is $\mathbb{R}$-embeddable then $\mathcal{F}(\mathcal{A})$ is ω_1-sequentially compact.*

Definition 4.3.11. (1) A compact space is said to be *(uniform) Eberlein compact* if it embeds homeomorphically into a Banach (Hilbert) space with its weak topology.

(2) A compact space is *Corson compact* if it homeomorphically embeds into a Σ-product, $\Sigma(\kappa)$, where

$$\Sigma(\kappa) = \{f \in \mathbb{R}^\kappa : |supp(f)| \leq \omega\}$$

and $supp(f) = \{\alpha \in \kappa \mid f(\alpha) \neq 0\}$.

A nice way to obtain Eberlein and Corson compacta is through adequate families of subsets introduced by Talagrand [61]. A family $\mathcal{D} \subseteq \mathcal{P}(X)$ is *adequate* if it is closed under subsets and $D \in \mathcal{D}$ if and only if every finite subset of D is an element of $\mathcal{D}$. Every adequate family of subsets of X defines a compact subset of 2^X via characteristic functions. We call the set $K_{\mathcal{D}} = \{1_D \in 2^X \mid D \in \mathcal{D}\}$ an *adequate compact*, where 1_D is the characteristic function of D. It is clear that $K_{\mathcal{D}}$ is Corson if and only if $\mathcal{D} \subseteq [X]^{\leq\omega}$. A characterization of (uniform) Eberlein adequate compacts is due to A. G. Leiderman and G. A. Sokolov.

Theorem 4.3.12. (See [50, Theorems 4.8 and 4.9]) *Let $\mathcal{D} \subseteq \mathcal{P}(X)$ be an adequate family, then:*

(I) $K_{\mathcal{D}}$ is Eberlein compact if and only if $X = \bigcup_{n\in\omega} X_n$ and $supp(f) \cap X_n$ is finite for every $f \in K_{\mathcal{D}}$ and $n \in \omega$.

(II) $K_{\mathcal{D}}$ is uniform Eberlein compact if and only if $X = \bigcup_{n\in\omega} X_n$ and there is a function $g \in \omega^\omega$ such that $|supp(f) \cap X_n| < g(n)$ for every $f \in K_{\mathcal{D}}$ and $n \in \omega$.

In order to see that Eberlein compacta defined from adequate families are ω_1-sequentially compact, we need the following definition.

Definition 4.3.13. (See [4, pp. 83]) A space X is *strongly monolithic* if the weight of $\overline{A}$ does not exceed $|A|$ for every $A \subseteq X$.

We are only interested in the case where $|A| = \omega$. Note that (uniform) Eberlein compacta of the form $K_{\mathcal{D}}$ for an adequate family $\mathcal{D}$ and Corson compacta are strongly monolithic. Hence, the next result shows that they are also ω_1-sequentially compact.

Theorem 4.3.14. *Every sequentially compact strongly monolithic space is ω_1-sequentially compact. In particular, Corson compacta and any Eberlein compacta defined from adequate families are ω_1-sequentially compact.*

Proof. As the closure of any countable set on a strongly monolithic space X is first countable, we can apply Theorem 4.3.2 to $f[\mathcal{B}]$ for every barrier $\mathcal{B}$ in order to find a convergent subsequence. Hence, X is ω_1-sequentially compact. As Corson and Eberlein compacta are sequentially compact [21], the second part of the result follows. $\square$

4.4 Some examples delineating the classes of α-sequentially compact spaces

We construct counterexamples that show that the classes of α-sequentially compact spaces and related spaces do not coincide (at least consistently, in some cases) giving analogous results to those presented in [49] and [15].

We start this section by pointing out that neither Theorem 4.3.2 nor Theorem 4.3.5 supersedes the other. On the one hand, if $X \subseteq 2^\omega$ has size $\mathfrak{c}$, then $\mathcal{F}(\mathcal{A}_X)$ is a compact bisequential space, but its character is $|\mathcal{A}_X| = \mathfrak{c} \geq \mathfrak{b}$ (so it satisfies the hypotheses of 4.3.2 but not 4.3.5). On the other hand, ω_1 is a first countable sequentially compact space, but it is non-compact (so it satisfies 4.3.5 but not 4.3.2). However, this leaves the question of whether there exists an ω_1-sequentially compact, compact space that is not bisequential, or even one of character less than $\mathfrak{b}$ (of course, assuming $\mathfrak{b} > \omega_1$ since first countable spaces are bisequential).

In order to find a compact and ω_1-sequentially compact space that is not bisequential, we use the following space defined by T. Cieśla in [13]: consider the set

$$\mathcal{D} = \{A \in [\omega_1 \times \omega_1]^{<\omega} \mid A \text{ is the graph of a partial strictly decreasing function}^4\}.$$

Clearly, $\mathcal{D}$ is an adequate family, and then $\mathcal{Z} = K_{\mathcal{D}}$ is a subset of the Tychonoff product $2^{\omega_1 \times \omega_1}$ by identifying elements of $\mathcal{D}$ with their characteristic functions. Then $\mathcal{Z}$ is an Eberlein compactum of weight ω_1 given by an adequate family.

Theorem 4.4.1. (See [13, Theorem 1.2]) *The space $\mathcal{Z}$ is not bisequential.*

With the previous result and Theorem 4.3.14, we get the following:

Theorem 4.4.2. *There exists a compact and ω_1-sequentially compact space that is not bisequential.* □

If we moreover assume that $\mathfrak{b} > \omega_1$, then the example has character less than $\mathfrak{b}$.

A more general version of $\mathcal{Z}$ was considered by C. Agostini and J. Somaglia in [1]. Given a tree T, we say that a finite partial function $f; T \to T$ is *decreasing* if $s < t$ implies $f(s) > f(t)$. It is *strongly decreasing* if f is decreasing and $dom(f)$ is a totally ordered subset of T. Hence, $\mathcal{Z}_T \subseteq 2^{T \times T}$ is the space of strongly decreasing finite partial functions on T and $\mathcal{Y}_T \subseteq 2^{T \times T}$ is the space of finite decreasing functions on T.

Theorem 4.4.3. *If T is a tree with no uncountable antichains and has an uncountable branch, then $\mathcal{Z}_T$ and $\mathcal{Y}_T$ are compact and ω_1-sequentially compact spaces but fail to be bisequential.*

Proof. The bisequentiality of $\mathcal{Z}_T$ and $\mathcal{Y}_T$ was characterized in terms of the combinatorial properties of T: the space $\mathcal{Z}_T$ is bisequential if and only if T has size less than the first measurable cardinal and T has no uncountable branches, while $\mathcal{Y}_T$ is bisequential only in the case where T is countable [1]. It is also easy to see that $\mathcal{Z}_T$ is always Eberlein compact (defined from an adequate family) and that $\mathcal{Y}_T$ is Corson compact if and only if every antichain in T is countable. □

[4] That is, there exist $n \in \omega$, $a_0 < \cdots < a_n < \omega_1$ and $b_n < \cdots < b_0 < \omega_1$ such that $A = \{(a_0, b_0), \ldots, (a_n, b_n)\}$.

By Proposition 2.0.8, if $\mathcal{F}(\mathcal{A})$ is 2-sequentially compact for an almost disjoint family $\mathcal{A}$, then $\mathcal{A}$ is nowhere mad. Indeed, otherwise choose $X \in \mathcal{I}(\mathcal{A})^+$ with $\mathcal{A} \upharpoonright X$ mad. By Proposition 2.0.8 the Franklin space $\mathcal{F}(\mathcal{A} \upharpoonright X) = X \cup (\mathcal{A} \upharpoonright X) \cup \{\infty\}$ is not 2-sequentially compact, but $\mathcal{F}(\mathcal{A} \upharpoonright X)$ is a closed subspace of $\mathcal{F}(\mathcal{A})$, and 2-sequential compactness is inherited by closed subspaces. This contradiction shows that $\mathcal{A}$ is nowhere mad.

Since an almost disjoint family is Fréchet if and only if it is nowhere mad, all counterexamples constructed from almost disjoint families that are at least 2-sequentially compact in [49] and [15] are also Fréchet. It was also shown in [15] that 2-sequentially compact spaces are α_3. This gives more examples of non-bisequential almost disjoint families that are Fréchet and α_3. The existence of these kind of almost disjoint families is not known in ZFC. For more consistent examples and for the definition of α_3, the reader may consult [17].

In [49] it was shown that the classes of n-sequentially compact spaces and $(n+1)$-sequentially compact spaces do not coincide (assuming CH for $n > 1$). This result was later strengthened by the authors in [15], who showed that the same separation holds under weaker hypotheses, e.g. when $\mathfrak{b} = \mathfrak{c}$. We show that it is also consistent to separate α-sequential compactness from β-sequential compactness for any $\alpha \neq \beta$ by extending the combinatorial analysis of [15] to barriers. To that end, we introduce some concepts and carry out constructions analogous to those in Section 3.3, beginning with a slightly more general notion of a Hechler tree. We say that $T \subseteq \omega^{<\omega}$ is a *Hechler tree* if, for every $s \in T$, the set of successors of s is a final segment of ω, i.e., there exists $k_s \in \omega$ such that $succ_T(s) = \{i \in \omega \mid s^\frown i \in T\} = \omega \setminus k_s$. We associate to each Hechler tree T the function $h_T \colon T \to \omega$ given by $h_T(s) = k_s$.

Remark 4.4.4. This is a slight abuse of terminology compared to the notion used in Section 3.3. There, a "Hechler tree" of height $\leq n$ was understood as $S \cap \omega^{\leq n}$ for some $S \subseteq \omega^{<\omega}$ of the present kind. To avoid ambiguity, from now on we adopt the present (full) definition: whenever we say "Hechler tree", we mean a subtree $T \subseteq \omega^{<\omega}$ satisfying the condition above.

For $s, t \in \omega^{<\omega}$, we say that $s \prec t$ if either $s \sqsubset t$ or $s = r^\frown l$, $t = r^\frown m$ for some $r \in \omega^{<\omega}$ and $l < m$. Here $s \sqsubset t$ means that s is a proper initial segment of t. Enumerate $\omega^{<\omega} = \{s_i \mid i \in \omega\}$ so that $s_i \prec s_j$ implies $i < j$. We fix this enumeration for the rest of the thesis.

Definition 4.4.5. Let $f \in \omega^\omega$ and let $T \subseteq \omega^{<\omega}$ be a Hechler tree. We define:

- $T_f \subseteq \omega^{<\omega}$ where $\emptyset \in T_f$ and $succ(s_i) = \omega \setminus f(i)$ for every $s_i \in T_f$.
- $f_T \in \omega^\omega$ where $f_T(n) = k$ if and only if $s_n \in T$ and $succ_T(s_n) = \omega \setminus k$ and $f_T(n) = 0$ otherwise.

Recall that $\mathsf{FIN} \subseteq \mathcal{P}(\omega)$ is the ideal of finite sets of ω.

Definition 4.4.6. Given a barrier $\mathcal{B}$, define the ideal $\mathsf{FIN}^{\mathcal{B}}$ as the set of all $X \subseteq [\omega]^{<\omega}$ such that

$$\{n \in \omega \mid \{s \setminus \{n\} \mid s \in X \cap \mathcal{B}(n)\} \notin \mathsf{FIN}^{\mathcal{B}[n]}\} \in \mathsf{FIN}.$$

Notice that in the case of the barrier $\mathcal{B} = [\omega]^n$, the ideal $\mathsf{FIN}^{\mathcal{B}}$ coincides with the well-known Fubini product FIN^n. In general, the Fubini product FIN^α for a countable limit ordinal α is not uniquely determined and depends on the choice of an increasing sequence $\{\alpha_n \mid n \in \omega\}$ converging to α, the corresponding previous choices for each α_n and so on. For a barrier $\mathcal{B}$ of rank α such that $\alpha_n = \rho(\mathcal{B}[n])$ forms an increasing sequence, the ideal $\mathsf{FIN}^{\mathcal{B}}$ also coincides with a Fubini product FIN^{α} where the sequence $\{\alpha_n \mid n \in \omega\}$ is precisely the sequence converging to α used in the Fubini product. For more on these ideals and their presentations, the reader may consult Dobrinen's paper [19].

It follows directly from the definition that we can characterize the ideals $\mathsf{FIN}^{\mathcal{B}}$ via Hechler trees.

Lemma 4.4.7. *Let $\mathcal{B}$ be a barrier and $X \in \mathsf{FIN}^{\mathcal{B}}$, then there exists a Hechler tree $H \subseteq \omega^{<\omega}$ such that $X \cap H \cap \mathcal{B} = \emptyset$.*

Proof. We prove it by induction on $\rho(\mathcal{B})$. If $\mathcal{B} = [\omega]^1$ and $X \in \mathsf{FIN}^{\mathcal{B}} = \mathsf{FIN}$, define $H \subseteq \omega^{<\omega}$ such that $\emptyset \in H$, $succ_H(\emptyset) = \omega \setminus k$ where $X \subseteq k$ and $succ_H(s) = \omega$ for any $s \in H \setminus \{\emptyset\}$. This H clearly works.

Now, let $\mathcal{B}$ be an arbitrary barrier with $\rho(\mathcal{B}) > 1$. We may assume that $\{k\} \notin \mathcal{B}$ for every $k \in \omega$. Fix $X \in \mathsf{FIN}^{\mathcal{B}}$. Thus, $\{n \in \omega \mid X \cap \mathcal{B}(n) \notin \mathsf{FIN}^{\mathcal{B}[n]}\}$ is finite and we can find $k \in \omega$ that contains this set. Hence, for every $n \in \omega \setminus k$, there is a Hechler tree H_n with root $\{n\}$ such that $X \cap H_n \cap \mathcal{B}(n) = \emptyset$. Define $H = \{\emptyset\} \cup \bigcup_{n \geq k} H_n$. It is clear that H is a Hechler tree and $X \cap H \cap \mathcal{B} = \emptyset$. □

Lemma 4.4.8. *For every $\mathcal{X} \subseteq \mathsf{FIN}^{\mathcal{B}}$ with $|\mathcal{X}| < \mathfrak{b}$, there is a Hechler tree $H \subseteq \omega^{<\omega}$ such that for every $X \in \mathcal{X}$, there exists $n \in \omega$ such that*

$$X \cap H \cap \mathcal{B} \subseteq \bigcup_{i \leq n} \mathcal{B}(i).$$

Proof. Let $\mathcal{X} \subseteq \mathsf{FIN}^{\mathcal{B}}$ where $|\mathcal{X}| < \mathfrak{b}$. By Lemma 4.4.7, for every $X \in \mathcal{X}$ there is a Hechler tree H_X such that $X \cap H_X \cap \mathcal{B} = \emptyset$. Let $f_X = f_{H_X}$ for every $X \in \mathcal{X}$. Since $|\mathcal{X}| < \mathfrak{b}$ we can find $f \in \omega^\omega$ such that $f >^* f_X$ for every $X \in \mathcal{X}$. It follows from our enumeration of $\omega^{<\omega}$ that if $f >^* f_X$, then there exists $n_X \in \omega$ such that if $f_X(s) > f(s)$ for some $s \neq \emptyset$ then $s \in \mathcal{B}(i)$ for some $i \leq n_X$. Thus, letting $H = H_f$ we have $H \setminus H_X \subseteq \bigcup_{i \leq n_X} \mathcal{B}(i)$ and this implies that

$$X \cap H \cap \mathcal{B} \subseteq \bigcup_{i \leq n_X} \mathcal{B}(i).$$

□

The following proposition appears in [49] for the particular case of barriers of the form $[\omega]^n$. The same argument shows that this is true for every barrier, so we have the following:

Proposition 4.4.9. *Let $\mathcal{A}$ be an almost disjoint family on a countable set N and let $\mathcal{B}$ be a barrier. If for every function $f : \mathcal{B} \to N$ there is an infinite set $X \in [\omega]^\omega$ such that*

$$|\{A \in \mathcal{A} : |f[\mathcal{B}|X] \cap A| = \omega\}| < \mathfrak{b}$$

then $\mathcal{F}(\mathcal{A})$ is $\mathcal{B}$-sequentially compact. □

The following lemma will also be helpful to prove that a space is not $\mathcal{B}$-sequentially compact for some barrier $\mathcal{B}$. We say that a barrier $\mathcal{C}$ is non-trivial if $\mathcal{C} \neq [\omega]^1$.

Lemma 4.4.10. *Let $\mathcal{A}$ be an almost disjoint family on a non-trivial barrier $\mathcal{C}$, such that $|A \cap \mathcal{C}(n)| \leq 1$ for every $A \in \mathcal{A}$ and every $n \in \omega$. If for every $E \in [\omega]^\omega$ there exists $A \in \mathcal{A}$ such that $A \subseteq \mathcal{C}|E$, then $\mathcal{F}(\mathcal{A})$ is not $\mathcal{C}$-sequentially compact.*

Proof. Since $\mathcal{C}$ is non-trivial, we may assume without loss of generality that every element of the $\mathcal{C}$ has size at least 2, in particular, $\mathcal{C}(m)$ is infinite for every $m \in \omega$.

Consider $i : \mathcal{C} \to \mathcal{C}$ to be the identity map and $E \in [\omega]^\omega$. We shall show that $i \restriction (\mathcal{C}|E)$ does not converge in $\mathcal{F}(\mathcal{A})$. Since $i[\mathcal{C}|E]$ is infinite, it does not converge to any isolated point in $\mathcal{C}$.

On the other hand, if $A \in \mathcal{A}$ and $n \in \omega$, we have $|\mathcal{C}(m) \cap i[\mathcal{C}|E \setminus n]| = \omega$ for every $m \in E \setminus n$, but $|A \cap \mathcal{C}(m)| \leq 1$ implies that $i[\mathcal{C}|(E \setminus n)] \nsubseteq A$ and hence does not converge to A. It remains to show that it does not converge to ∞.

Fix the neighborhood $U = \mathcal{F}(\mathcal{A}) \setminus (\{A\} \cup A)$ of ∞ for some $A \subseteq \mathcal{C}|E$ and let $n \in \omega$. As $|A \cap \mathcal{C}(i)| \leq 1$ for every $i \in \omega$, we can find $c \in A \subseteq \mathcal{C}|E$ such that $\min(c) > n$. Thus, $c \in \mathcal{C}|(E \setminus n) \setminus U$ and, consequently, $i[\mathcal{C}|(E \setminus n)] \nsubseteq U$. Since this is true for any $n \in \omega$, we conclude that $i \restriction (\mathcal{C}|E)$ does not converge to ∞. $\square$

The previous lemmas allow us to show that there are spaces that are $\mathcal{B}$-sequentially compact but fail to be $\mathcal{C}$-sequentially compact whenever their associated ideals satisfy a suitable relation. For this reason, it is useful to introduce the terminology of the Katětov order.

Definition 4.4.11. (See [46, Definitions 1.4 and 1.6]) Let X and Y be countable sets, $\mathcal{I}$ and $\mathcal{J}$ ideals on X and Y respectively and $f : Y \to X$.

1. f is a *Katětov* function from $(Y, \mathcal{J})$ to $(X, \mathcal{I})$ if $f^{-1}(A) \in \mathcal{J}$ for all $A \in \mathcal{I}$.

2. $\mathcal{I} \leq_K \mathcal{J}$ ($\mathcal{I}$ *is Katětov below* $\mathcal{J}$) if there exists a Katětov function from $(Y, \mathcal{J})$ to $(X, \mathcal{I})$.

Although this order was introduced by Katětov more than fifty years ago, its systematic study in recent decades has established it as a central tool in modern set theory. This framework has provided a unified approach for addressing problems in topological convergence, ultrafilter classification, and forcing destructibility. Fundamental references for this order include Hrušák's paper [38] and Meza-Alcántara's thesis [52].

Applications of the order include the characterization of Hausdorff ultrafilters via critical ideals (see [42, Theorem 2.3]) and the analysis of the destructibility of mad families and ideals by forcing notions (see [41]). Moreover, the order is closely related to cardinal invariants of the continuum and Cichoń's diagram (see [37]).

Regarding the structural properties of the order itself, fundamental dichotomies have been identified that organize Borel ideals into a rigorous hierarchy [40]. Furthermore, the global structure of the Katětov order is known to be complex: there are no minimal tall Borel ideals (see [27]), maximal ideals are cofinal, ideals generated by mad families are coinitial among tall ideals, and every tall ideal bounds large antichains and long decreasing chains (see [38, Proposition 4.1 and Theorem 4.2]).

Now, an easy consequence of Definition 4.4.11 is the following:

Lemma 4.4.12. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on countable sets X and Y, respectively. Then the following are equivalent:*

(I) $\mathcal{I} \nleq_K \mathcal{J}$.

(II) For every $f : Y \to X$, there exists $B \in \mathcal{J}^+$ such that $f[B] \in \mathcal{I}$.

Proof. To see that *(I)* implies *(II)*, let $f : Y \to X$. As $\mathcal{I} \not\leq_K \mathcal{J}$, f is not a Katětov function; therefore, there exists $A \in \mathcal{I}$ such that $B := f^{-1}(A) \in \mathcal{J}^+$. Now, $f[B] \in \mathcal{I}$ since $f[B] \subseteq A \in \mathcal{I}$. Conversely, to see that $\mathcal{I} \not\leq_K \mathcal{J}$, let $f : Y \to X$; we want to prove that f is not a Katětov function. Thus, let $B \in \mathcal{J}^+$ be such that $A := f[B] \in \mathcal{I}$. Now, $f^{-1}(A) \in \mathcal{J}^+$ since $B \subseteq f^{-1}(A)$. $\square$

In addition to the ideal $\mathsf{FIN}^{\mathcal{B}}$, another ideal that is useful in the construction of counterexamples is the following:

Definition 4.4.13. If $\mathcal{B}$ is a barrier on ω, then $\mathcal{G}_c(\mathcal{B})$ is the ideal on $\mathcal{B}$ such that $(\mathcal{G}_c(\mathcal{B}))^+ = \langle\{\mathcal{B}|X \mid X \in [\omega]^\omega\}\rangle$, i.e., $S \subseteq \mathcal{B}$ is an element of $\mathcal{G}_c(\mathcal{B})$ if and only if there is no $X \in [\omega]^\omega$ such that $\mathcal{B}|X \subseteq S$.

Note that $(\mathcal{G}_c([\omega]^2))^+$ is the collection of all subsets of $[\omega]^2$ that contain an infinite complete subgraph, hence the notation $\mathcal{G}_c$. Also note that the fact that $\mathcal{G}_c(\mathcal{B})$ is closed under finite unions follows from the Nash-Williams theorem.

Theorem 4.4.14. ($\mathfrak{b} = \mathfrak{c}$) *Let $\mathcal{B}$ and $\mathcal{C}$ be two barriers such that $\mathsf{FIN}^{\mathcal{C}} \not\leq_K \mathcal{G}_c(\mathcal{B})$. Then there is a space that is $\mathcal{B}$-sequentially compact, but not $\mathcal{C}$-sequentially compact.*

Proof. Enumerate $\mathcal{C}^{\mathcal{B}} = \{f_\alpha \mid \alpha < \mathfrak{c}\}$ and $[\omega]^\omega = \{E_\alpha \mid \alpha < \mathfrak{c}\}$. For $E \in [\omega]^\omega$, let $E^\uparrow = T(\mathcal{C}|E)$. Notice that for any such E, the tree $E^\uparrow$ is everywhere ω-splitting and hence it has infinite intersection with any Hechler tree H inside $\mathcal{C}$; that is, $|E^\uparrow \cap H \cap \mathcal{C}| = \omega$. Moreover, for any $n \in E$, we have

$$|E^\uparrow \cap H \cap \mathcal{C}(n)| = \omega.$$

Recursively construct $\{A_\alpha \mid \alpha < \mathfrak{c}\} \subseteq [\mathcal{C}]^\omega$ and $\{X_\alpha \mid \alpha < \mathfrak{c}\} \subseteq [\omega]^\omega$ such that for all $\beta < \alpha < \mathfrak{c}$:

(1) $f_\alpha[\mathcal{B}|X_\alpha] \in \mathsf{FIN}^{\mathcal{C}}$,

(2) $|A_\alpha \cap A_\beta| < \omega$,

(3) $|A_\alpha \cap f_\beta[\mathcal{B}|X_\beta]| < \omega$,

(4) $|A_\alpha \cap \mathcal{C}(n)| \leq 1$ for all $n \in \omega$,

(5) $A_\alpha \subseteq E_\alpha^\uparrow$.

At step $\alpha < \mathfrak{c}$, let $X_\alpha \in [\omega]^\omega$ be such that $f_\alpha[\mathcal{B}|X_\alpha] \in \mathsf{FIN}^{\mathcal{C}}$ (this set exists by Lemma 4.4.12). Note also that every A_β with $\beta < \alpha$ is an element of $\mathsf{FIN}^{\mathcal{C}}$, then we can find a Hechler tree H as in Lemma 4.4.8 such that

$$X \cap H \cap \mathcal{C} \subseteq \bigcup_{i \leq n_X} \mathcal{C}(i)$$

for either $X = f_\beta[\mathcal{B}|X_\beta]$ or $X = A_\beta$ for some $\beta < \alpha$. Let $A_\alpha \in [\mathcal{C}]^\omega$ be such that $A_\alpha \subseteq H \cap \mathcal{C} \cap E_\alpha^\uparrow$ and $|A_\alpha \cap \mathcal{C}(i)| \leq 1$ for every $i \in \omega$. It follows that $A_\alpha \cap X \subseteq \bigcup_{i \leq n_X} \mathcal{C}(i)$ for some $n_X \in \omega$ and as $|A_\alpha \cap \mathcal{C}(i)| \leq 1$ for every $i \leq n_X$, this intersection is finite and (2) and (3) hold. The remaining three properties follow from the construction.

Since $\{A \in \mathcal{A} : |f_\alpha[\mathcal{B}|X_\alpha] \cap A| = \omega\} \subseteq \{A_\beta \mid \beta \leq \alpha\}$ and this set has size less than $\mathfrak{b}$, by Proposition 4.4.9, we get that $\mathcal{F}(\mathcal{A})$ is $\mathcal{B}$-sequentially compact.

On the other hand, item (5) implies that $\mathcal{F}(\mathcal{A})$ is not $\mathcal{C}$-sequentially compact by Lemma 4.4.10, since $\mathcal{C}$ has rank greater than 1. $\square$

The natural question now is when the barriers $\mathcal{B}$ and $\mathcal{C}$ satisfy $\mathsf{FIN}^{\mathcal{C}} \not\leq_K \mathcal{G}_c(\mathcal{B})$. We see in the following that this depends only on the rank of the barriers, assuming that at least the one with the higher rank is uniform. See Theorem 4.4.15 and Corollary 4.4.17 below.

The relevance of the Katětov order in the study of subclasses of sequentially compact spaces is also demonstrated in the work of R. Filipów, K. Kowitz and A. Kwela [25], where the authors study many subclasses of sequentially compact spaces, prove some inclusions among them and find counterexamples by using the Katětov order relationship between some ideals naturally associated with these classes.

Recall that if $\mathcal{B}$ is a barrier, then $\mathcal{B}[s]$ is a barrier on $\omega \setminus (\max(s)+1)$. By abuse of notation, for any function $g : \mathcal{B}(s) \to \mathcal{C}(t)$, we will use the same symbol g to denote its naturally associated function from $\mathcal{B}[s]$ to $\mathcal{C}[t]$. Also, for a Hechler tree H, we denote by $H[n]$ its copy above $\{n\}$, that is, a tree T with root $\{n\}$ and such that $succ_T(s) = succ_H(s \setminus \{n\})$. In particular, $\omega^{<\omega}[n] = \{s \in \omega^{<\omega} \mid n = \min(s)\}$.

Theorem 4.4.15. *Let $\mathcal{B}, \mathcal{C}$ be barriers with $\mathcal{C}$ uniform and $\rho(\mathcal{B}) < \rho(\mathcal{C})$. Then,*

$$\mathsf{FIN}^{\mathcal{C}} \not\leq_K \mathcal{G}_c(\mathcal{B}).$$

Proof. By Lemma 4.4.12, it is enough to prove that for every $f : \mathcal{B} \to \mathcal{C}$ there is $X \in [\omega]^\omega$ such that $f[\mathcal{B}|X] \in \mathsf{FIN}^{\mathcal{C}}$. The proof is by induction on $\rho(\mathcal{C})$. If $\rho(\mathcal{C}) = 1$ there is nothing to do, so we can assume that $\rho(\mathcal{C}) > 1$ and the result is true for $\mathcal{C}'$ whenever $\rho(\mathcal{C}') < \rho(\mathcal{C})$. We now prove this by induction on $\rho(\mathcal{B})$. If $\rho(\mathcal{B}) = 1$, we can find an infinite set $X \in [\omega]^\omega$ such that either $|f[\mathcal{B}|X] \cap \mathcal{C}(i)| \leq 1$ for every $i \in \omega$ or $f[\mathcal{B}|X] \subseteq \mathcal{C}(n)$ for some $n \in \omega$. Hence, assume that for any barriers $\mathcal{B}'$ and $\mathcal{C}'$ with $\rho(\mathcal{B}') < \rho(\mathcal{B})$, $\rho(\mathcal{C}') \leq \rho(\mathcal{C})$ and $\rho(\mathcal{B}') < \rho(\mathcal{C}')$ the result holds. We can also think in $\{\emptyset\}$ as a barrier of rank 0. In this case the result is trivial, but it is worth to mention since $\mathcal{B}[s]$ is a barrier of rank 0 whenever $s \in \mathcal{B}$ and we also consider these kind of barriers as part of our inductive hypothesis. We can also assume, by possibly passing to a cofinite set, that $\rho(\mathcal{C}(n)) \geq \rho(\mathcal{B})$ for every $n \in \omega$.

Define a function $\pi_n : \mathcal{B} \to 2$ for every $n \in \omega$ given by $\pi_n(s) = 0$ if and only if $f(s) \in \bigcup_{i \leq n} \mathcal{C}_i$. By the Nash-Williams theorem, we can find an infinite set $Z_n \in [\omega]^\omega$ such that $\pi \restriction (\mathcal{B}|Z_n)$ is monochromatic. If for some $n \in \omega$ we can find a monochromatic set Z_n with color 0, then the Hechler tree H defined by $\emptyset \in H$, $succ_H(\emptyset) = \omega \setminus (n+1)$ and $succ_H(s) = \omega$ for all $s \in H \setminus \{\emptyset\}$ works for the infinite set $X = Z_n$. Hence, we can assume that there is no 0-monochromatic set for every $n \in \omega$.

Let us recursively define $\{X_n \mid n \in \omega\}$, $\{m_i \mid i \in \omega\}$, $\{H_n \mid n \in \omega\}$ and $\{g(s,n) \mid s \subseteq \{m_i \mid i < n\} \wedge n \in \omega\}$ such that:

(1) $m_i = \min(X_i)$,

(2) X_n is monochromatic for π_n,

(3) $X_{n+1} \subseteq X_n$,

(4) $g(s,n) : \mathcal{B}(s) \to \mathcal{C}(n)$ is given by $g(s,n)(t) = f(t)$ if and only if $f(s) \in \mathcal{C}(n)$ and otherwise let $g(s,n)(t) \in \mathcal{C}(n)$ be arbitrary.

(5) $H_n \subseteq \omega^{<\omega}[n]$ is a Hechler tree.

(6) $g(s,n)[\mathcal{B}(s)|X_n] \cap H_n = \emptyset$ for every $s \subseteq \{m_i \mid i \leq n\}$ with $s \neq \emptyset$.

Since m_i and $g(s,n)$ are defined from X_n, we only need to define X_n and H_n. Note that if in point (6) we have $s \notin T(\mathcal{B})$, then $\mathcal{B}(s) = \emptyset$ and then the choices of X_n and H_n are trivial, so we assume in the rest of the proof that $s \in T(\mathcal{B})$.

Let X_0 be π_n-monochromatic and let $H_0 = \omega^{<\omega}[0]$, i.e., H_0 is a copy of $\omega^{<\omega}$ on top of $\{0\}$. Assume that we have defined X_i and H_i for $i \leq n$. We can find a π_{n+1} monochromatic set (with color 1) $X' \subseteq X_n$. Enumerate the set $\{s \subseteq \{m_i \mid i \leq n\} \mid s \neq \emptyset\}$ as $\{s_i \mid i \leq k\}$ for some $k \in \omega$, We can recursively define $X^0_{n+1} \subseteq X'$, $X^{i+1}_{n+1} \subseteq X^i_{n+1}$ for $i < k$ and H^i_{n+1} for $i \leq k$ such that:

(∗) $g(s,n)[\mathcal{B}[s]|X^i_{n+1}] \cap H^s_{n+1} = \emptyset$.

To see that the choices of X^i_{n+1} and H^i_{n+1} are possible, note that $g(s,n) : \mathcal{B}[s] \to \mathcal{C}[n]$ and $\rho(\mathcal{B}[s]) < \rho(\mathcal{B}) \leq \rho(\mathcal{C}[n])$, and then we can apply the inductive hypothesis. Without loss of generality, we can assume that $\min(X^i_{n+1}) > m_n$ for every $i \leq k$, so that no confusion arises in (∗) with the identification of $\mathcal{B}(s)$ and $\mathcal{B}[s]$. Also, since each H^i_{n+1} is a Hechler tree for every $i \leq k$, we can define $H_{n+1} = \bigcap_{i \leq k} H^i_{n+1}[n+1]$ which is a Hechler tree on top of $\{n\}$. Define $X_{n+1} = X^k_{n+1}$. This finishes the construction.

Let $X = \{x_i \mid i \in \omega\}$ and $H = \{\emptyset\} \cup \bigcup_{n \in \omega} H_n$. We show that $f[\mathcal{B}|X] \cap H = \emptyset$. Fix $s \in \mathcal{B}|X$, then $s = (x_{i_0}, \dots, x_{i_k})$ for some $k \in \omega$ and some sequence $i_0 < \cdots < i_k$. As $\{x_{i_0}, \dots, x_{i_k}\} \subseteq X_{i_0}$ and X_{i_0} is π_{i_0} monochromatic with color 1, we have $f(s) \in C(n)$ for some $n > i_0$. Let $s' = \{x_i \in s \mid i < n\}$ and notice that $s \setminus s' \subseteq X_n$ and $s' \neq \emptyset$. Hence, $g(s', n)$ was considered at step n in (6) and s' was indexed as s_i for some i. Since we have $H \cap \omega^{<\omega}[n] \subseteq H^i_n$, $s \setminus s' \in \mathcal{B}[s']|X_n$ and $X_n \subseteq X^i_n$, we can conclude, by (∗), that $f(s) = g(s',n)(s) \notin H$. Therefore, $f[\mathcal{B}|X] \cap H = \emptyset$, so $f[\mathcal{B}|X] \in \mathsf{FIN}^{\mathcal{C}}$. □

It is worth noting that the previous result gives us that for two barriers $\mathcal{B}$ and $\mathcal{C}$, the classes of $\mathcal{B}$-sequentially compact spaces and $\mathcal{C}$-sequentially compact spaces only depend on the ranks of $\mathcal{B}$ and $\mathcal{C}$ whenever the largest is uniform. If $\rho(\mathcal{B}) < \rho(\mathcal{C})$ and $\mathcal{C}$ is uniform, every $\mathcal{C}$-sequentially compact space is $\mathcal{B}$-sequentially compact by Corollaries 4.2.5 and 4.2.6. On the other hand, there are (at least consistently) examples of $\mathcal{B}$-sequentially compact spaces that are not $\mathcal{C}$-sequentially compact by Theorem 4.4.15. If otherwise $\rho(\mathcal{B}) = \rho(\mathcal{C})$ and both of them are uniform, then the classes of $\mathcal{B}$-sequentially compact and $\mathcal{C}$-sequentially compact spaces coincide by Corollary 4.2.5.

A more subtle analysis can be done by considering $\rho_u(\mathcal{B})$ instead of $\rho(\mathcal{B})$, but what we have done suffices to show that (consistently) the classes of α-sequentially compact and β-sequentially compact spaces do not coincide if $\alpha \neq \beta$.

Corollary 4.4.16. ($\mathfrak{b} = \mathfrak{c}$) *If $\{\alpha_n \mid n \in \omega\} \subseteq \alpha$, there exists a space that is α_n-sequentially compact for all $n \in \omega$ but fails to be α-sequentially compact. In particular, for every $\alpha < \beta < \omega_1$, there exists an α-sequentially compact space that is not β-sequentially compact.*

Proof. Fix a uniform barrier $\mathcal{B}_n$ of rank α_n for every $n \in \omega$ and a barrier $\mathcal{C}$ of rank α. Repeat the proof of Theorem 4.4.14 by enumerating

$$\bigcup_{n \in \omega} \mathcal{C}^{\mathcal{B}_n} = \{f_\alpha \mid \alpha < \mathfrak{c}\}.$$

The hypothesis of Theorem 4.4.14 holds for every barrier $\mathcal{B}_n$ by Theorem 4.4.15. Finally, by Corollary 4.2.5, we get the result. □

If $\mathcal{B}$ and $\mathcal{C}$ are barriers of the same rank and $\mathcal{B}$ is uniform, then $\mathsf{FIN}^{\mathcal{C}} \leq_K \mathcal{G}_c(\mathcal{B})$. To see this, note that otherwise, by Theorem 4.4.14, there is a space X that is $\mathcal{B}$-sequentially compact but not $\mathcal{C}$-sequentially compact under $\mathfrak{b} = \mathfrak{c}$, but this is a contradiction to Corollary 4.2.5, since the Katětov order is absolute among Borel ideals due to Shoenfield's absoluteness theorem (see [40]). We then get the following corollary:

Corollary 4.4.17. *Let $\mathcal{B}$ and $\mathcal{C}$ be uniform barriers such that $\rho(\mathcal{B}) \leq \rho(\mathcal{C})$. Then the following are equivalent:*

(I) $\mathsf{FIN}^{\mathcal{C}} \not\leq_K \mathcal{G}_c(\mathcal{B})$.

(II) $\rho(\mathcal{B}) < \rho(\mathcal{C})$.

It is worth pointing out that for the implication *(I)* $\Rightarrow$ *(II)*, we use the uniformity of $\mathcal{B}$, while for *(II)* $\Rightarrow$ *(I)*, we use the uniformity of $\mathcal{C}$.

4.5 Cardinal invariants associated to barriers

In this section, we define and analyze some cardinal invariants associated to barriers that play an important role in the structure of α-sequentially compact spaces. In particular, we extend a result in [49] by showing that for every $\alpha > 1$, the Cantor cube 2^κ is α-sequentially compact if and only if $\kappa < \min\{\mathfrak{b}, \mathfrak{s}\}$. When $\alpha = 1$, it is known that 2^κ is sequentially compact if and only if $\kappa < \mathfrak{s}$ (see [20]).

It is also shown in [15], that the cardinal $\mathfrak{par}$ is closely related to the constructions of examples of spaces that are n-sequentially compact but fail to be $(n+1)$-sequentially compact under $\mathfrak{s} = \mathfrak{b}$. This suggests that if we aim to do similar constructions for arbitrary barriers, we should start analyzing and computing the analogous cardinal invariants for $\mathfrak{par}$ in arbitrary barriers. If $\mathcal{B}$ is a barrier, $\pi : \mathcal{B} \to 2$ and $H \in [\omega]^\omega$, then we say that H is *monochromatic for* π if π is constant on $\mathcal{B}|H$. We say that H is *almost monochromatic* for π if there is $F \in [H]^{<\omega}$ such that $H \setminus F$ is monochromatic for π. If Π is a family of colorings on $\mathcal{B}$, then we say that H is *almost monochromatic for* Π if it is almost monochromatic for every $\pi \in \Pi$. We refer to $[\omega]^1$ as the trivial barrier.

Definition 4.5.1. If $\mathcal{B}$ is a barrier, then $\mathfrak{par}_{\mathcal{B}}$ is the minimum size of a collection of colorings on $\mathcal{B}$ without an almost monochromatic infinite set.

If $t \in [\omega]^{<\omega} \setminus \{\emptyset\}$, recall that we write t as $\{t_0, \dots, t_{n-1}\}$ in the increasing way and if $I \subseteq n$, then $t \restriction I = \{t_i \mid i \in I\}$. Also note that $t \restriction 2 \sqsubseteq t$ for $n \geq 2$. Given a barrier $\mathcal{B}$ such that each of its elements has size at least 2, if we let $\mathcal{B} \restriction 2 := \{t \restriction 2 : t \in \mathcal{B}\}$, then $\mathcal{B} \restriction 2 = [\omega]^2$.

Theorem 4.5.2. *If $\mathcal{B}$ is a non-trivial barrier, then* $\mathfrak{par}_{\mathcal{B}} \leq \mathfrak{par}_2$.

Proof. As $\mathcal{B}$ is a non-trivial barrier, there is $n \in \omega$ such that every element of $\mathcal{B}|(\omega \setminus n)$ has size at least two. Without loss of generality, we may assume that $n = 0$. Let $\Pi = \{\pi_\alpha \mid \alpha \in \mathfrak{par}_2\}$ be a family of colorings on $[\omega]^2$ without almost monochromatic infinite sets. Define $\widehat{\pi}_\alpha : \mathcal{B} \to 2$ for every $\alpha \in \mathfrak{par}_2$, given by $\widehat{\pi}_\alpha(t) = \pi_\alpha(t \restriction 2)$. Note that if H is an almost monochromatic infinite set for $\widehat{\Pi} = \{\widehat{\pi}_\alpha \mid \alpha \in \mathfrak{par}_2\}$, then H is also an almost monochromatic infinite set for Π. Hence, $\widehat{\Pi}$ has no infinite almost monochromatic sets. $\square$

Recall that if $\mathcal{B}$ is a barrier, the rank of $\mathcal{B}[n]$ is strictly smaller than the rank of $\mathcal{B}$ for every $n \in \omega$. This allows us to show that the same idea that proves that $\mathfrak{par}_n = \min\{\mathfrak{b}, \mathfrak{s}\}$ for every $n \in \omega$ in [8] also works for arbitrary barriers.

Theorem 4.5.3. *If $\mathcal{B}$ is a barrier such that each of its elements has size at least 2, then $\mathfrak{par}_{\mathcal{B}} \geq \mathfrak{par}_2$.*

Proof. By induction on the rank of $\mathcal{B}$. Suppose that $\mathcal{B}$ has rank γ and the result is true for every barrier of rank strictly smaller than γ. Let $\kappa < \mathfrak{par}_2$ and let $\{\pi_\alpha : \alpha \in \kappa\}$ be a set of colorings on $\mathcal{B}$. We show that there exists an almost monochromatic infinite set for this family.

For every $\alpha \in \kappa$ and every $n \in \omega$ let $\pi_\alpha^n : \mathcal{B}[n] \to 2$ be given by $\pi_\alpha^n(s) = \pi_\alpha(\{n\} \cup s)$. We recursively construct a collection $\{H_n : n \in \omega\} \subseteq [\omega]^\omega$ such that:

1. $H_{n+1} \in [H_n]^\omega$ and

2. H_{n+1} is almost monochromatic for $\{\pi_\alpha^n \restriction (\mathcal{B}[n]|H_n) : \alpha \in \kappa\}$.

We start by defining $H_0 = \omega \setminus 1$ and as $B[0]$ has rank smaller than γ, by our inductive hypothesis we know that there is $H_1 \in [\omega \setminus 1]^\omega$ such that H_1 is almost monochromatic for $\{\pi_\alpha^0 \mid \alpha \in \kappa\}$. Now, suppose that we have already defined H_n and consider the family $\{\pi_\alpha^n \restriction (\mathcal{B}[n]|H_n) : \alpha \in \kappa\}$. As $\mathcal{B}[n]|H_n$ also has a lower rank than γ, there is $H_{n+1} \in [H_n]^\omega$ such that H_{n+1} is almost monochromatic for this family.

Take H an infinite pseudointersection of $\{H_n : n \in \omega\}$. Note that for every $n \in H$ and every $\alpha \in \kappa$ we have H_n is almost monochromatic for π_α^n with color $j(\alpha, n) \in \{0, 1\}$. For every $\alpha \in \kappa$, define an increasing $f_\alpha \in H^\omega$ such that $H_n \setminus f_\alpha(n)$ is monochromatic for π_α^n. Also, let $g \in \omega^\omega$ increasing such that $H \setminus g(n) \subseteq H_n$. As $\mathcal{F} = \{f_\alpha : \alpha \in \kappa\} \cup \{g\} \subseteq \omega^\omega$ has size less than $\mathfrak{par}_2 \leq \mathfrak{b}$, there exists $f \in \omega^\omega$ strictly increasing that dominates $\mathcal{F}$.

For every $i \in 2$, let $X_\alpha^i := \{n \in H : j(\alpha, n) = i\}$. Note that $\{X_\alpha^0 \mid \alpha \in \kappa\}$ is a collection of less than $\mathfrak{par}_2 \leq \mathfrak{s}$ subsets of H, so there is $X \in [H]^\omega$ such that for every $\alpha \in \kappa$ there is $j(\alpha) \in \{0, 1\}$ such that $X \subseteq^* X_\alpha^{j(\alpha)}$.

Define $J := \{x_n \mid n \in \omega\} \subseteq X \subseteq H$ such that $x_{n+1} > f(x_n)$ for every $n \in \omega$.

Claim 4.5.4. J is almost monochromatic for $\{\pi_\alpha \mid \alpha \in \kappa\}$.

Proof of the claim: Let $\alpha \in \kappa$. We know that $X \subseteq^* X_\alpha^{j(\alpha)}$, $f_\alpha <^* f$ and $g <^* f$, so there exists $k \in \omega$ such that:

1. $X \setminus k \subseteq X_\alpha^{j(\alpha)}$,

2. $f_\alpha(n) < f(n)$ for every $n > k$ and

3. $g(n) < f(n)$ for every $n > k$.

Fix $s \in \mathcal{B}|(J \setminus (k+1))$ and let $n = \min(s)$ and $t = s \setminus \{n\}$. Then we have $n \in X_\alpha^{j(\alpha)}$ and

$$t \subseteq J \setminus f(n) \subseteq X \setminus f(n) \subseteq H \setminus f(n) \subseteq H \setminus g(n) \subseteq H_n.$$

Notice that $\min(t) > f(n) > f_\alpha(n)$, hence $t \subseteq H_n \setminus f_\alpha(n)$. But then $\pi_\alpha(s) = \pi_\alpha^n(t) = j(\alpha, n)$ and $j(\alpha, n) = j(\alpha)$ since $n \in X_\alpha^{j(\alpha)}$. This proofs that π_α is constant on $\mathcal{B}|(J \setminus (k+1))$ with color $j(\alpha)$. $_{Claim}\square$

The proof of the previous claim finishes the proof of the theorem. $\square$

Corollary 4.5.5. $\mathfrak{par}_{\mathcal{B}} = \mathfrak{par}_2$ *for every non-trivial barrier $\mathcal{B}$ on ω and* $\mathfrak{par}_{[\omega]^1} = \mathfrak{s}$.

Corollary 4.5.6. *If $\kappa < \mathfrak{par}_2$ and $\mathcal{B}$ is a barrier, then 2^κ is $\mathcal{B}$-sequentially compact. Moreover,*

$$\mathfrak{par}_2 = \min\{\kappa \mid 2^\kappa \text{ is not } \mathcal{B}\text{-sequentially compact}\}$$

for every non-trivial barrier $\mathcal{B}$.

Proof. Let $f : \mathcal{B} \to 2^\kappa$ for some $\kappa < \mathfrak{par}_2$ and for every $\alpha \in \kappa$ let $\pi_\alpha : \mathcal{B} \to 2$ be the α-th projection of f. Thus, there is $M \in [\omega]^\omega$ that is almost monochromatic with color $j(\alpha)$ for each $\alpha \in \kappa$. It follows that $f \restriction (\mathcal{B}|M)$ converges to $j = \langle j(\alpha) : \alpha \in \kappa\rangle \in 2^\kappa$. Conversely, let $\{\pi_\alpha \mid \alpha < \mathfrak{par}_2\}$ be a family of colorings on $\mathcal{B}$ with no infinite almost monochromatic set, and define $f : \mathcal{B} \to 2^{\mathfrak{par}_2}$ by $f(b)(\alpha) = \pi_\alpha(b)$. Then, any infinite set $M \in [\omega]^\omega$ such that $f \restriction (\mathcal{B}|M)$ converges to a point $\langle j(\alpha) \mid \alpha < \mathfrak{par}_2\rangle$ is easily seen to be an almost monochromatic set for $\{\pi_\alpha \mid \alpha < \mathfrak{par}_2\}$. Thus, no such infinite set M exists. □

The dual cardinal to $\mathfrak{par}$ is the cardinal $\mathfrak{hom}$. The characterization of $\mathfrak{par}$ in terms of $\mathfrak{b}$ and $\mathfrak{s}$ also has a dual version using $\mathfrak{d}$ and a variant of $\mathfrak{r}$, the dual cardinals of $\mathfrak{b}$ and $\mathfrak{s}$ respectively (see [8]).

Notation 4.5.7. Let N be a countable set, $\mathcal{I}$ an ideal on N and $X, Y \subseteq N$. Then $X \subseteq_{\mathcal{I}} Y$ means that $X \setminus Y \in \mathcal{I}$.

Definition 4.5.8. Given a barrier $\mathcal{B}$ let,

$$\mathfrak{hom}_{\mathcal{B}} = \min\{|\mathcal{H}| : \mathcal{H} \subseteq [\omega]^\omega \ \wedge \ \forall S \subseteq \mathcal{B}\ \exists M \in \mathcal{H}\ \exists i \in 2\ (\mathcal{B}|M \subseteq S^i)\}$$

and

$$\mathfrak{r}_{\mathcal{B}} = \min\{|\mathcal{R}| : \mathcal{R} \subseteq [\omega]^\omega\ \forall S \subseteq \mathcal{B}\ \exists R \in \mathcal{R}\ \exists i \in 2\ (\mathcal{B}|R \subseteq_{\mathsf{FIN}^{\mathcal{B}}} S^i)\}.$$

A family, as defined in $\mathfrak{hom}_{\mathcal{B}}$, is called a *homogeneous family for $\mathcal{B}$* or a *$\mathcal{B}$-homogeneous family.* Analogously, a family, as defined in $\mathfrak{r}_{\mathcal{B}}$, is called a *reaping family for $\mathcal{B}$* or a *$\mathcal{B}$-reaping family.*

Recall that by the Nash-Williams theorem, for every $\pi : \mathcal{B} \to 2$ there is $M \in [\omega]^\omega$ such that $\pi \restriction (\mathcal{B}|M)$ is constant. In this way, $\mathfrak{hom}_{\mathcal{B}}$ is well defined and $\mathfrak{r}_{\mathcal{B}} \leq \mathfrak{hom}_{\mathcal{B}}$ since every $\mathcal{B}$-homogeneous family is a $\mathcal{B}$-reaping family. In particular, $\mathfrak{r}_{\mathcal{B}}$ is well defined.

Note that if $\mathcal{B}$ is the trivial barrier, i.e., $\mathcal{B} = [\omega]^1$, then $\mathfrak{r}_{\mathcal{B}} = \mathfrak{hom}_{\mathcal{B}} = \mathfrak{r}$. On the other hand, if $\mathcal{B}$ is a non-trivial barrier, then $\mathfrak{hom}_2 \leq \mathfrak{hom}_{\mathcal{B}}$. To see this, we can suppose that every element of $\mathcal{B}$ has size at least 2 and thus every coloring $\pi : [\omega]^2 \to 2$ induces a coloring of $\hat{\pi} : \mathcal{B} \to 2$ given by $\hat{\pi}(s) = \pi(s \restriction 2)$. Now, an infinite homogeneous set for $\hat{\pi}$ is also homogeneous for π, which in turn gives that any homogeneous family for $\mathcal{B}$ is an homogeneous family for $[\omega]^2$ and then $\max\{\mathfrak{d}, \mathfrak{r}_\sigma\} = \mathfrak{hom}_2 \leq \mathfrak{hom}_{\mathcal{B}}$. In particular $\mathfrak{d} \leq \mathfrak{hom}_{\mathcal{B}}$. Ramsey ultrafilters are helpful in our study of the cardinal $\mathfrak{hom}_{\mathcal{B}}$.

Recall that a non-principal ultrafilter $\mathcal{U}$ on ω is *Ramsey* if for all $\pi : [\omega]^2 \to 2$, there is $U \in \mathcal{U}$ such that $\pi \restriction [U]^2$ is constant. A family $\mathcal{F} \subseteq [\omega]^\omega$ is *selective* if for every decreasing sequence $\{Y_n \mid n \in \omega\} \subseteq \mathcal{F}$, there is $f : \omega \to \omega$ such that $f[\omega] \in \mathcal{F}$, $f(0) \in Y_0$ and $f(n+1) \in Y_{f(n)}$ for all $n \in \omega$. The following result, which according to D. Booth is mostly due to K. Kunen, relates Ramsey ultrafilters and selective families.

Theorem 4.5.9. (See [10, Theorem 4.9, items (i) and (v)]) *If $\mathcal{U}$ is a non-principal ultrafilter, then $\mathcal{U}$ is Ramsey if and only if it is selective.*

We show now that a base for a Ramsey ultrafilter is an homogeneous family for every barrier $\mathcal{B}$. Hence, if we define $\mathfrak{u}_R$ as the minimum character of a Ramsey ultrafilter[5] we get $\mathfrak{hom}_{\mathcal{B}} \leq \mathfrak{u}_R$.

The next theorem is a standard result of Local Ramsey theory (see [62] Chapter 7). It follows immediately from Theorem 7.42 in the same way that the Nash-Williams theorem follows from the Ellentuck theorem. For the convenience of the reader, we give the argument.

Theorem 4.5.10. *If $\mathcal{U}$ is a Ramsey ultrafilter, $\mathcal{F}$ is a front on ω and $\pi : \mathcal{F} \to 2$, then there is $U \in \mathcal{U}$ such that $\pi \restriction (\mathcal{F}|U)$ is constant.*

Proof. By induction on the rank of the front. If $\mathcal{F}$ has rank zero, i.e, $\mathcal{F} = \{\emptyset\}$, then the result is trivial.

Suppose that $\mathcal{F}$ has rank γ and the result is true for every front of rank smaller than γ. As $\mathcal{F}[n]$ is a front on $\omega \setminus (n+1)$ of rank less than γ, there is $V_n \in \mathcal{U}$ such that the coloring $\pi_n : \mathcal{F}[n] \to 2$ is constant on $\mathcal{F}[n]|V_n$, where $\pi_n(s) = \pi(\{n\} \cup s)$.

For every $n \in \omega$ we know that $\pi_n \restriction (\mathcal{F}[n]|V_n)$ is constant in color $j(n) \in 2$. So take $V \in \mathcal{U}$ and $i \in 2$ such that $j(n) = i$ for all $n \in V$.

Now, call $U_n = (\bigcap_{m \leq n} V_n) \cap V \cap (\omega \setminus n+1)$ and note that $\{U_n \mid n \in \omega\}$ is a decreasing sequence of elements of $\mathcal{U}$, so applying that $\mathcal{U}$ is selective, there is $f \in \omega^\omega$ such that $f[\omega] \in \mathcal{U}$, $f(0) \in U_0$ and $f(n+1) \in U_{f(n)}$ for all $n \in \omega$. Define $U := f[\omega]$.

It remains to show that $\pi \restriction (\mathcal{F}|U)$ is constant with value i. Indeed, let $s = \{a_0, \ldots, a_k\} \in \mathcal{F}|U$. Then $a_0 \in V$ and

$$\{a_1, \ldots, a_k\} \subseteq U_{f(a_0)} \subseteq U_{a_0} \subseteq V_{a_0},$$

so that $\{a_1, \ldots, a_k\} \in \mathcal{B}[a_0]|V_{a_0},$. Then $\pi(s) = \pi_{a_0}(\{a_1, \ldots, a_k\}) = j(a_0) = i$. □

It is well-known that the forcing notion $\mathbb{U} = (\mathcal{P}(\omega)/\mathsf{FIN}, \subseteq^*)$ adds a Ramsey ultrafilter $\mathcal{U}$ to the ground model without adding reals (see for example [33, pp. 358]), thus Theorem 4.5.10 implies Theorem 4.1.3 in the case where the Nash-Williams family is a front. To see this, let $\mathcal{F}$ be a front and $\pi : \mathcal{F} \to 2$. Now, if we force with $\mathbb{U}$ then in $V[G]$ there is $\mathcal{U}$ a Ramsey ultrafilter and $\pi \in V[G]$. Now, by Theorem 4.5.10, there is $X \in \mathcal{U} \cap V[G]$ such that $\pi \restriction (\mathcal{F}|X)$ is constant, but as $\mathbb{U}$ does not add reals, $X \in V$ and we are done.

Thus, we obtain an alternative proof to the one given in [55] that Nash–Williams families are Ramsey, but only for families that are fronts — in other words, our proof is less general. One might be tempted to argue that, if every Nash–Williams family could be extended to a front, the general Nash–Williams theorem would follow from the front case. However, Example 4.1.8 shows that not every Nash–Williams family can be extended to a front.

Notation 4.5.11. If $R \in [\omega]^\omega$ then $\mathbb{H}(R)$ is the collection of all non-empty trees $T \subseteq R^{<\omega}$ such that $succ_T(s)$ is cofinite in R for all $s \in T$.

We need the following folklore facts.

Proposition 4.5.12. *If $R \in [\omega]^\omega$ and $T \in \mathbb{H}(R)$, then there is $A \in [R]^\omega$ such that $[A]^\omega \subseteq [T]$.*

Proposition 4.5.13. $\mathsf{cof}(\mathsf{FIN}^{\mathcal{B}}) = \mathfrak{d}$ *for every non-trivial barrier $\mathcal{B}$.*

[5]Let $\mathfrak{u}_R = \mathfrak{c}$ if there is no such ultrafilter.

An explicit proof for the previous result on $\mathsf{cof}(\mathsf{FIN}^{\mathcal{B}})$ where $\mathcal{B}$ is a barrier of finite rank appears in [15]. The same argument works for general barriers.

Theorem 4.5.14. $\mathfrak{hom}_{\mathcal{B}} = \max\{\mathfrak{d}, \mathfrak{r}_{\mathcal{B}}\}$ *for every non-trivial barrier* $\mathcal{B}$ *on* ω.

Proof. We already noted that $\max\{\mathfrak{d}, \mathfrak{r}_{\mathcal{B}}\} \leq \mathfrak{hom}_{\mathcal{B}}$, so it is enough to prove the other inequality. For this, let $\mathcal{R} = \{R_\beta \mid \beta \in \mathfrak{r}_{\mathcal{B}}\}$ be a reaping family for $\mathcal{B}$ and for every $\beta \in \mathfrak{r}_{\mathcal{B}}$ let $\{X^\beta_\alpha \mid \alpha \in \mathfrak{d}\}$ cofinal in $\mathsf{FIN}^{\mathcal{B}|R_\beta}$. Now, by the definition of $\mathsf{FIN}^{\mathcal{B}|R_\beta}$ and Proposition 4.5.12, for every $\beta \in \mathfrak{r}_{\mathcal{B}}$ and every $\alpha \in \mathfrak{d}$ there are $T^\beta_\alpha \in \mathbb{H}(R_\beta)$ and $A^\beta_\alpha \in [R_\beta]^\omega$ such that:

1. $T^\beta_\alpha \cap X^\beta_\alpha = \emptyset$,
2. $[A^\beta_\alpha]^\omega \subseteq [T^\beta_\alpha]$.

Now, let $\mathcal{H} = \{A^\beta_\alpha \mid \beta \in \mathfrak{r}_\beta \wedge \alpha \in \mathfrak{d}\}$. We claim that $\mathcal{H}$ is a homogeneous family for $\mathcal{B}$. To see this, let $S \subseteq \mathcal{B}$. As $\mathcal{R}$ is a reaping family for $\mathcal{B}$, there is $\beta \in \mathfrak{r}_\beta$ and $i \in 2$ such that $\mathcal{B}|R_\beta \subseteq_{\mathsf{FIN}^{\mathcal{B}}} S^i$. Now, call $X := (\mathcal{B}|R_\beta) \setminus S^i \in \mathsf{FIN}^{\mathcal{B}}$. Moreover, as $X \subseteq \mathcal{B}|R_\beta$, then $X \in \mathsf{FIN}^{\mathcal{B}|R_\beta}$. Thus, there is $\alpha \in \mathfrak{d}$ such that $X \subseteq X^\beta_\alpha$ and consequently $X \cap T^\beta_\alpha = \emptyset$. Also, we know that $[A^\beta_\alpha]^\omega \subseteq [T^\beta_\alpha]$, so $X \cap (\mathcal{B}|A^\beta_\alpha) = \emptyset$. Indeed, if $b \in X \cap (\mathcal{B}|A^\beta_\alpha)$, then there is $Z \in [A^\beta_\alpha]^\omega$ such that $b \sqsubseteq Z$ and, as $Z \in [T^\beta_\alpha]$, then $b \in T^\beta_\alpha \cap X$, which is a contradiction. In this way, we have $\mathcal{B}|A^\beta_\alpha \subseteq \mathcal{B}|R_\beta \subseteq S^i \cup X$ and $(\mathcal{B}|A^\beta_\alpha) \cap X = \emptyset$, so $\mathcal{B}|A^\beta_\alpha \subseteq S^i$ and we are done. $\square$

Question 4.5.15. Is $\mathfrak{hom}_{\mathcal{B}} = \mathfrak{hom}_2$ for every non-trivial barrier $\mathcal{B}$?

Chapter 5

Open questions

Of course, our biggest unresolved question is:

Question 5.0.1. Is it true in ZFC that for every $n \geq 2$ there exists a space that is n-sequentially compact but not $(n+1)$-sequentially compact?

Because most of our partial solutions to Question 5.0.1 are Franklin spaces of almost disjoint families, a natural strengthening of Question 5.0.1 is the following:

Question 5.0.2. Is it true in ZFC that for every $n \geq 2$ there is an almost disjoint family $\mathcal{A}$ such that $\mathcal{F}(\mathcal{A})$ is n-sequentially compact but not $(n+1)$-sequentially compact?

Although constructing mad families with rich combinatorial or topological features is notoriously difficult, it is equally challenging to build models of ZFC in which certain classes of mad families do not exist. Hence, Question 5.0.2 may be very hard.

As we mentioned, most adaptations of Shelah's technique for constructing completely separable mad families that have appeared in the literature use hypotheses of the form $\iota \leq \theta$, where ι is some splitting-type cardinal invariant and θ is another invariant tailored to the problem. Thus, the following is natural:

Question 5.0.3. Does $\mathfrak{s} \leq \mathfrak{b}$ imply the existence of n-sequentially compact spaces that fail to be $(n+1)$-sequentially compact?

However, our current proof relies heavily on the inequality $\mathfrak{par}_2 \geq \mathfrak{b}$, which is equivalent to asking $\mathfrak{s} \geq \mathfrak{b}$ (and the reverse inequality $\mathfrak{s} \leq \mathfrak{b}$ is also used in the proof to obtain $\mu_n = \mathfrak{b}$). In other words, our argument does not appear susceptible to weakening to only $\mathfrak{s} \leq \mathfrak{b}$. To settle Question 5.0.3 one likely needs a much more delicate analysis of the cardinals μ_n or a proof that depends less on the precise values of those cardinals.

On the other hand, there are models where $\mathfrak{b} < \mathfrak{s}$; the earliest and simplest such model was constructed by Shelah (see [58]). A related question is:

Question 5.0.4. In Shelah's model with $\mathfrak{b} < \mathfrak{s}$, does there exist, for every $n \geq 2$, an n-sequentially compact space that is not $(n+1)$-sequentially compact?

Concerning the cardinals μ_n, the following questions seem natural.

Question 5.0.5. Given $n \geq 1$, is it consistent that $\mu_n < \mu_{n+1}$?

Question 5.0.6. Is it true that $\mu_n = \mathfrak{bs}$ for all $n \geq 1$?

Question 5.0.7. Is it consistent that $\mathfrak{s} < \mu_0 < \mu_1 < \ldots < \mu_n < \mu_{n+1} < \ldots < \mathfrak{b}$?

Regarding the cardinals λ_n and $\mathfrak{a}_n^{\mathrm{sc}}$, the natural question—typical when a new cardinal invariant is introduced—is how they relate to the classical invariants. Of particular interest to us is the comparison of them with the cardinals that are most relevant for the study of n-sequentially compact spaces: $\mathfrak{a}$ and $\mathfrak{b}$.

Question 5.0.8. Besides the relations $\mathfrak{b} = \lambda_2 \leq \mathfrak{a}_2^{sc} \leq \mathfrak{a}_* \leq \mathfrak{a}$ and $\lambda_n \leq \mathfrak{a}_n^{sc}$, are there other ZFC-provable relationships between the cardinals λ_n and $\mathfrak{a}_n^{sc}$ and the classical invariants $\mathfrak{b}$ and $\mathfrak{a}$?

Question 5.0.9. Are all the cardinals λ_n and $\mathfrak{a}_m^{sc}$ consistently pairwise different? In particular, is it consistent that $\mathfrak{a}_2^{sc} < \cdots < \mathfrak{a}_n^{sc} < \cdots < \mathfrak{a}$?

We know that 2-sequentially compact spaces are $\mathfrak{a}_3$; for Franklin spaces, one can ask whether these two properties can be separated:

Question 5.0.10. Is it consistent that there is an almost disjoint family $\mathcal{A}$ such that $\mathcal{F}(\mathcal{A})$ is $\mathfrak{a}_3$ but not 2-sequentially compact? Equivalently, is it consistent that $\mathfrak{a}_2^{sc} < \mathfrak{a}_*$?

In the infinite-dimensional setting, the most important generalization of Question 5.0.1 is:

Question 5.0.11. For every $\alpha < \beta < \omega_1$, does ZFC prove the existence of a space X that is α-sequentially compact but not β-sequentially compact?

We were unable to answer Question 5.0.11 positively from the hypothesis $\mathfrak{s} = \mathfrak{b}$.

Question 5.0.12. Does $\mathfrak{b} = \mathfrak{s}$ imply that for every $\alpha < \beta < \omega_1$ there exists a space that is α-sequentially compact but not β-sequentially compact?

The difficulty in generalizing the proof of Theorem 3.5.3 to the infinite-dimensional case lies in computing the analogues of the cardinals μ_n. In other words, for arbitrary uniform barriers $\mathcal{B}$ and $\mathcal{C}$ one can define analogues of the cardinals μ_n, but it seems hard to establish general inequalities comparing those cardinals with $\mathfrak{b}$ and $\mathfrak{s}$. Concretely, we make the following definition.

Definition 5.0.13. Let $\mathcal{B}$ and $\mathcal{C}$ be barriers.

1. $\mathcal{L}_{\mathcal{B},\mathcal{C}}$ is the family of all functions $f : \mathcal{B} \to \mathcal{C}$ such that there is no $n \in \omega$ and $X \in [\omega]^\omega$ with $f[\mathcal{B} \restriction X] \subseteq C_n$, where $C_n = \{t \in [\omega]^{<\omega} \mid \min(t) = n\}$.

2. If $f \in \mathcal{L}_{\mathcal{B},\mathcal{C}}$ and $S \in [\omega]^\omega$, we say that S $(\mathcal{B},\mathcal{C})$-*splits* f if there exist $X, Y \in [\omega]^\omega$ such that $f[\mathcal{B} \restriction X] \subseteq \mathcal{C} \restriction S$ and $f[\mathcal{B} \restriction Y] \subseteq \mathcal{C} \restriction (\omega \setminus S)$.

3. A family $\mathcal{S} \subseteq [\omega]^\omega$ is $(\mathcal{B},\mathcal{C})$-*splitting* if for every $f \in \mathcal{L}_{\mathcal{B},\mathcal{C}}$ there is $S \in \mathcal{S}$ that splits f.

4. $\mu(\mathcal{B},\mathcal{C})$ is the minimum size of a $(\mathcal{B},\mathcal{C})$-splitting family.

Note that with this general definition one recovers $\mu_n = \mu\Big([\omega]^n, [\omega]^{n+1}\Big)$. The main case of interest is when $\mathcal{B}$ and $\mathcal{C}$ are uniform barriers with $\rho(\mathcal{B}) < \rho(\mathcal{C})$, but it can be hard even to show that $(\mathcal{B},\mathcal{C})$-splitting families exist. Thus, a route toward Question 5.0.12 is to start with the following:

Question 5.0.14. Let $\mathcal{B}$ and $\mathcal{C}$ be uniform barriers with $\rho(\mathcal{B}) < \rho(\mathcal{C})$ and fix $f \in \mathcal{L}_{\mathcal{B},\mathcal{C}}$. Does there exist $S \in [\omega]^\omega$ that $(\mathcal{B},\mathcal{C})$-splits f? Equivalently, is $[\omega]^\omega$ itself a $(\mathcal{B},\mathcal{C})$-splitting family? Equivalently, do $(\mathcal{B},\mathcal{C})$-splitting families exist, i.e., is $\mu(\mathcal{B},\mathcal{C})$ well-defined?

To make progress on Question 5.0.12, the crucial step would be to show something like:

Question 5.0.15. Let $\mathcal{B}$ and $\mathcal{C}$ be uniform barriers. Is it true that $\mu(\mathcal{B},\mathcal{C}) \leq \mathfrak{bs}$? Or, weaker but sufficient: does $\mathfrak{b} = \mathfrak{s}$ imply that $\mu(\mathcal{B},\mathcal{C})$ exists and equals $\mathfrak{bs}$?

Finally, in contrast with $\mathfrak{par}_\mathcal{B}$, the following remains open:

Question 5.0.16. Is $\mathfrak{hom}_\mathcal{B} = \mathfrak{hom}_2$ for every nontrivial barrier $\mathcal{B}$?

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