Light by light scattering at one-loop order in scalar QED

THESIS
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PRESENTS

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Para Jaqueline.
Gracias por hacerme sentir vivo de nuevo, hacerme creer que vale la pena seguir adelante y por amarme de esa forma tan única como solo tú pudiste hacerlo.

Contents

Resumen

El fenómeno de la luz se ha estudiado durante muchos años, desde la dispersión Compton hasta la dispersión luz por luz en la Teoría Cuántica de Campos. En el formalismo estándar de la Teoría Cuántica de Campos, con el uso de las reglas de Feynman, podemos calcular varias de estas interacciones, como la polarización del vacío, que implica dos fotones unidos a un lazo fermiónico, e incluso trabajar con la dispersión de cuatro fotones. En particular, este último proceso se ha estudiado en el formalismo estándar en varios límites. El objetivo de esta tesis es calcular la amplitud a un lazo en el vacío bajo el marco del formalismo línea de mundo para la QED escalar. Como la amplitud es finita, debido a la invariancia de norma, todas las divergencias involucradas deben desaparecer. Los resultados mostrados están escritos en términos de funciones como polilogaritmo y para el resultado final, hemos realizado integración numérica. Como algunas de las ventajas de utilizar el formalismo línea de mundo podemos mencionar que el resultado no depende del ordenamiento elegido para las piernas externas, ni de la suma de diagramas de Feynman, también se eliminan algunas divergencias UV, que en el formalismo estándar deben cancelarse entre diferentes diagramas. La perspectiva para el futuro es pasar a la QED de espinores y generalizar el resultado a un orden de bucle superior.

Palabras clave: Formalismo Worldline, Invarianza de Norma, Fórmula Maestra de Bern-Kosower, Teoría Cuántica de Campos, Amplitud de Dispersión.

Abstract

The phenomenon of light has been studied for many years, from Compton scattering to light-by-light scattering in Quantum Field Theory. In the standard QFT formalism, with the use of Feynman's rules, we can calculate several of these interactions, such as vacuum polarisation, which involves two photons attached to a fermionic loop, and even work with four-photon scattering. Particularly the latter process has been studied in the standard formalism in various limits. The aim of this thesis is to calculate the amplitude for one-loop order in vacuum under the framework of worldline formalism for scalar QED. Since the amplitude is finite, due to gauge invariance, all divergences involved must vanish. The results shown are written in terms of functions as polylogarithm and for the final result, we have performed numerical integration. As some of the advantages of using the worldline formalism we can mention that the result does not depend on the ordering chosen for the external legs, nor the sum of Feynman diagrams, also some UV divergences, which in the standard formalism must cancel between different diagrams, are eliminated. The perspective for the future is to pass to spinor QED and generalise the result to higher loop order.

Introduction

From the beginning, mankind has had an intrinsic curiosity about its environment and how it works. As a consequence of this curiosity, people tried to explain in different ways, how nature works and its rules, from magic and esoteric ideas until the most important development; science, and more specifically, physics. As physicists, our task is to observe, describe, model, make predictions, test them and write these rules of nature with only the purpose of having a better understanding of the universe in all the possible ways, from the macroscopic scale to the smallest one. As the first areas of physics, also called Classical Physics, we can find Mechanics, Thermodynamics, Electrodynamics, and Optics, these areas were able to explain several phenomena; just to mention some examples, Mechanics explains how the earth moves around the sun, Thermodynamics explains the transformation of energy into heat and vice versa with the postulation of the three laws of Thermodynamics, Electromagnetism and Optics into Maxwell's equations encrypt how light has a relation with electromagnetic fields.

In classical physics we have different approaches to the theory, for example, we have Newtonian mechanics, Lagrangian mechanics, Hamiltonian mechanics, etc. These approaches are equivalent to each other, the difference is in how they do it.

Newton's mechanics by characterizing physical systems by their position, velocity and acceleration as time-dependent functions and with the help of Newton's three laws, allow us to find the equations of motion and integrate them, particularly, the second Newton's law:

$$\mathbf{F} = m\mathbf{a} = \frac{d\mathbf{p}}{dt} \tag{1}$$

On the other hand, Lagrangian mechanics is founded on the stationary action principle, defining a function called the Lagrangian:

$$L = L(q_i, \dot{q}_i, t) = T - V, \tag{2}$$

where $q_i, \dot{q}_i$ are the generalised coordinates, and the velocities respectively for the i-particle, T is the kinetic energy and V the potential energy of the system, we can define a functional called the *action functional*:

$$S[q] = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)dt. \tag{3}$$

Now, imposing that S be stationary:

$$\delta S = 0, \tag{4}$$

where the symbol *delta* is the virtual displacement of the system, we can find that the equations of motion for the Lagrangian function are given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \tag{5}$$

which are called *Euler-Lagrange equations of motion*. Knowing the Lagrangian function, the associated equations of motion can be found and eventually integrated to explicitly find the solution to our problem. For Hamiltonian mechanics, we can pass from Lagrangian mechanics by doing a Legendre transformation:

$$H = H(q_i, p_i, t) = \sum_i p_i q_i - L = T + V, \tag{6}$$

where p_i is the canonical momentum of the i-particle. What we can do with these approaches is to analyze different theories, for example for a free particle, we have the lagrangian:

$$L = \frac{1}{2} m \dot{q}^2, \tag{7}$$

which leads to the equation of motion:

$$\ddot{q} = 0 \tag{8}$$

Another example, now involving special relativity, is the relativistic free particle:

$$L = -m \sqrt{-\dot{x}^\mu \dot{x}_\mu} \tag{9}$$

which leads to the equations of motion:

$$\ddot{x}^\mu = 0 \tag{10}$$

And as a last example, classical electrodynamics can be described with a lagrangian density function as follows:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + j_\mu A_\mu, \tag{11}$$

where $F_{\mu\nu}$ are the electromagnetic tensor components, A_μ the components of the vector potential and j_μ the four current components. Since this is a lagrangian density, we have to integrate over the coordinates:

$$L = \int d^3 x \mathcal{L}. \tag{12}$$

From equation 11, we can obtain Gauss's Law and Ampere Law for classical electrodynamics, and the rest from Bianchi's identity, which can be written as:

$$\partial_\sigma F^{\mu\nu} + \partial_\mu F^{\nu\sigma} + \partial_\nu F^{\sigma\mu} = 0 \tag{13}$$

We can write Maxwell equations in terms of the electromagnetic tensor $F_{\mu\nu}$:

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= \mu_0 J^\nu \\ \partial_\mu \left(\frac{1}{2} \varepsilon^{\mu\nu\gamma\delta} F_{\gamma\delta} \right) &= 0 \end{aligned} \tag{14}$$

We can notice that they are linear in fields and sources:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \end{aligned} \tag{15}$$

But even with these powerful tools and their accomplishments, Classical Physics areas have a lot of limitations. For example, they cannot explain the movement of Mercury around the sun or the UV catastrophe in the black body radiation. For years, scientists were looking for different ways to solve these limitations of the theory, for example redoing Classical Physics and the partial solution to the problem. Resulting from of these attempts we get two of the most important areas for modern physics, *General Relativity* and *Quantum Mechanics*. The first is in charge of the geometry and dynamics of spacetime, meanwhile the second is in charge of the microscopic world and its probabilistic behavior.

For these approaches, we pass from classical ideas to more general ideas, for example, in General Relativity now the space and the time are a single element of the universe and we pass from 3-vector and time representation to 4-vector including the time as another coordinate. For Quantum Mechanics due to the nature and scales, now we cannot think in a deterministic approach. We pass from absolute measurements to probabilities, and the coordinates now pass to linear operators and states for our systems:

$$x(t) \rightarrow \hat{x}, |x\rangle, \tag{16}$$

$$p(t) \rightarrow \hat{p}, |p\rangle. \tag{17}$$

The operators and states are defined in a linear space with a positive norm, called *Hilbert space* $\mathscr{H}$. The operators are maps into our Hilbert space, *i.e.* they map from $\mathscr{H} \rightarrow \mathscr{H}$.

Their action on the states return to us eigenvalues as follows:

$$\hat{A}|A\rangle = a|A\rangle \tag{18}$$

The operators that give us physical information of the system have a particular property, their eigenvalues are always real numbers and they are called hermitian operators. There are commutation relations between operators, for example between position and momentum operators:

$$[\hat{x}, \hat{p}] = i\hbar, \quad [\hat{x}, \hat{x}] = 0, \quad [\hat{p}, \hat{p}] = 0. \tag{19}$$

This process of changing our classical variables to operators and states into Hilbert space is known as canonical quantization. With the operators, we can now define other physical quantities, for example, the hamiltonian:

$$\hat{H}(\hat{x}, \hat{p}) = \frac{\hat{p}^2}{2m} + \hat{V}(\hat{x}). \tag{20}$$

Now the time evolution of the quantum system is given through the Schröedinger equation:

$$\hat{H}|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle = E|\psi\rangle, \tag{21}$$

where we can identify the energy of the system. Also, we have the uncertainty principle wich express that now we have a fundamental limit to the precision of measurements to certain set of variables, for example in the case of position and momentum:

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}, \tag{22}$$

where σ_x, σ_p are the standard deviation of position and momentum respectively.

With the passage of time, Quantum Mechanics opened the way to new theories as *Quantum Field Theory* which involves the relativistic behaviour for particles that Quantum Mechanics do not take into account, for example, introducing the idea of path integral for a relativistic particle. Another important contribution from QFT is *Quantum Electrodynamics*, which predicts some phenomena that Classical Physics were not able to predict, for example, the principal topic of this thesis, light by light scattering.

Beyond the appearance of QFT and its advantages, we can suggest new ways to present old ideas, in Richard Feynman's words, "There are, therefore, no fundamentally new results. However, there is a pleasure in recognizing old things from a new point of view" [1]. Since the microscopic world is non-deterministic, we cannot know precisely how the processes are taking place; What we can do, is to suppose about how they are and start the job with reasonable ideas. This fact allows us to imagine and propose all kinds of processes that can occur virtually, and that new theories and formulations allow us to describe that in turn coincide with observations and experiments.

Light by Light Scattering

Light was a mystery for physicists for centuries, its nature, its properties, its interactions and particularly, the idea of light as a wave or a particle. This mystery evolves at the same time as science was, and now we know that actually light has a duality between both behaviours. Before starting with the light by light scattering, we need to talk about one of the more incredible works involving the light phenomena, the Compton scattering [2], observed for the first time in 1923 by Arthur H. Compton. The nature of this process was not understood, but it was studied with the mathematical tools of the era by Compton. The experiment consists of X-rays with energy around $17keV$, striking a graphite target, there occurs Compton's scattering which is the interaction of an incoming photon and an electron, taking as result the scattering of another photon and another electron. After that, the scattered X-ray photons pass through a slit, this only avoids certain photons scattered at a selected angle and sends them to an ionization chamber. In the chamber, it is possible to measure the total energy over time and not just the energy of a single scattered photon.

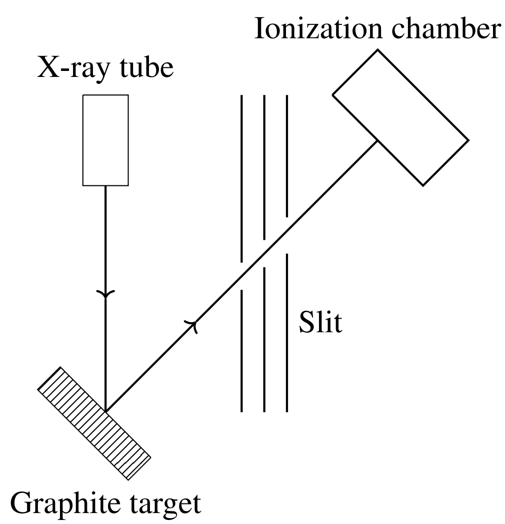


Figure 1: Compton's experiment

As result of this experiment, Compton finds that the wavelenght shift of scattered photons is:

$$\Delta\lambda = \frac{h}{m_e c}(1 - \cos\theta), \tag{23}$$

where θ is the scattering angle, m_e is the electron rest mass, h is the Planck constant, and c is the speed of the light.

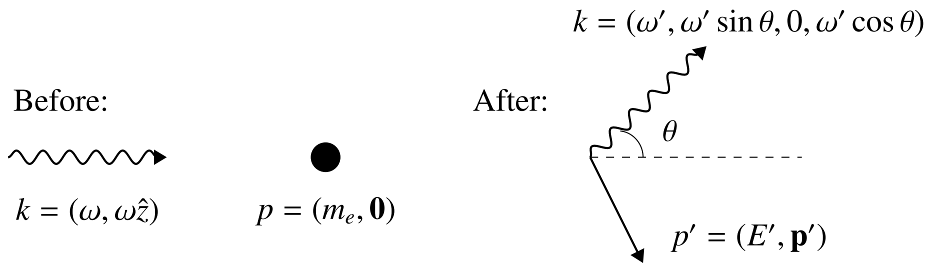


Figure 2: Compton scattering in the lab frame

Equation 23 can be derived from relativistic kinematics, working in the lab frame and finding the energy of the final photon:

$$m_e^2 = p'^2 = (p + k - k')^2 = p^2 + 2p \cdot (k - k') - 2k \cdot k' \quad (24)$$
$$= m_e^2 + 2m_e(\omega - \omega') - 2\omega\omega'(1 - \cos\theta), \quad (25)$$

thus, it is easy to find that:

$$\frac{1}{\omega'} - \frac{1}{\omega} = \frac{1}{m_e}(1 - \cos\theta), \quad (26)$$

and this is equivalent to equation 23, changing ω with the relation for photon's energy $\omega = \frac{c}{\lambda}$, we can check that they are the same.

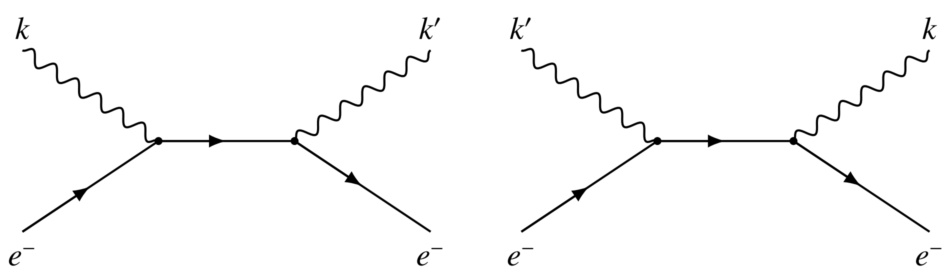


Figure 3: Feynman Diagrams for Compton Scattering

Eventually with the introduction of QFT methods and Feynman diagrams, it was possible to re-calculate with the new technology. One of this contributions is the calculation of the scattering matrix elements $i\mathcal{M}$ defined by:

$$i\mathcal{M} = \text{sum of all connected, amputed diagrams} \quad (27)$$

We can work out $i\mathcal{M}$ at tree level from diagrams in figure 3 for Compton scattering:

$$i\mathcal{M} = -ie^2\varepsilon^*_\mu(k')\varepsilon_\nu(k)\bar{u}(p')\left[\frac{\gamma^\mu \not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{-\gamma^\nu \not{k}'\gamma^\mu + 2\gamma^\nu p^\mu}{-2p\cdot k'}\right]u(p). \tag{28}$$

The rest of the computation for the Compton scattering is straightfoward. Furthermore, summing over spins and photon porlarizations, we can compute the cross section in the lab frame:

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m_e^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right]. \tag{29}$$

The last equation is the spin-averaged *Klein-Nishina formula* [3]. As we can notice, the application of QFT to well-studied processes brings us new information and allows us to explore other interesting cases, for example, high-energy behaviour or the opposite case, the low-energy limit. This limit give us the Thompson cross section which is essentially what Compton was sensitive to see in his experiment.

But this is just the beginning of the story, after that, several photon-photon processes were studied. There is no light by light scattering in free Maxwell theory for electromagnetic fields, this is because Maxwell's equations are linear in the fields and the sources, as we can notice from equations 15, thus the principle of superposition holds. On the other hand using Quantum Mechanics and QFT, introducing the creation of virtual particles pairs we can allow these kinds of effects. This means that photons interact with virtual particles in a vertex with a fermionic loop, following the process, the virtual particles annihilate one to each other in another vertex creating a photon, we can see figure 4 where this picture is illustrated.

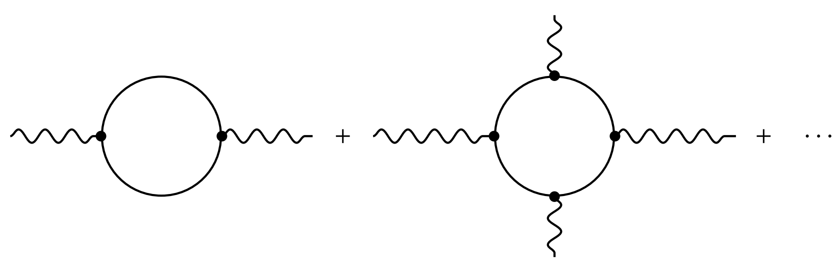


Figure 4: Perturbative expansion in the low-energy limit of the unormalized EH lagrangian. In equation 30, the second and third factors remove the vacuum expectation of the field and the vacuum polarization diagrams.

This interaction was calculated for the first time by W. Heisenberg and H. Euler in 1936[4], from the one-loop effective lagrangian for spinor QED in a constant classical electromagnetic background:

$$\mathcal{L}_{EH} = -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3}e^{-m^2T}\left[\frac{(eaT)(ebT)}{\tan(eaT)\tanh(ebT)} - \frac{2}{3}(eT)^2\mathcal{F} - 1\right]. \tag{30}$$

Here a and b are given by:

$$a^2 = \sqrt{\mathcal{F}^2 + \mathcal{G}^2} - \mathcal{F}, \quad b^2 = \sqrt{\mathcal{F}^2 + \mathcal{G}^2} + \mathcal{F}. \tag{31}$$

And the electromagnetic Lorentz invariants quantities:

$$\mathcal{F} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2), \quad \mathcal{G} = \frac{1}{4} F_{\mu\nu} \bar{F}^{\mu\nu} = -\mathbf{E} \cdot \mathbf{B} \tag{32}$$

Later in the same year, Weisskopf found an analog expression for scalar QED[5]:

$$\mathcal{L}_W = \frac{1}{16\pi^2} \int_0^\infty \frac{dT}{T^3} e^{-m^2 T} \left[\frac{(eaT)(ebT)}{\sin(eaT)\sinh(ebT)} + \frac{1}{6}(eT)^2 \mathcal{F} - 1 \right]. \tag{33}$$

In this case for the expansion, we shall have extra diagrams, due to the existence of the *seagull vertex*, we shall discuss this in chapter 2. We can expand this lagrangian in the low-energy limit obtaining:

$$\int_0^\infty dT (4\pi)^{-2} e^{-m^2 T} \left(\frac{1}{T^3} + \frac{e^2}{12} \frac{tr[F^2]}{T} + \frac{e^4}{36} \left[\frac{1}{10} tr[F^4] + \frac{1}{8} (tr[F^2])^2 \right] T \right). \tag{34}$$

We shall derive this formula eventually. We can notice the appearence of powers of traces of F, this implies the existence of light by light interaction. About these lagrangians, Gerald V. Dunne conducts a thorough analysis including further applications[6].

Going ahead into history, the limit of low-energy photons calculation was performed by H. Euler and B. Kockel around 1935-1936[7, 8]. They found the cross section in relation to the wavelenght:

$$\sigma \sim \left(\frac{\hbar^2 \alpha}{m^2 c^2} \right)^4 \frac{1}{\lambda^6}. \tag{35}$$

where $\alpha = \frac{e^2}{\hbar c}$. In the same year, the opposite case, the high-energy limit, was computed by A. Akhiezer et al[9]. They found that the integral cross section is:

$$\sigma \sim a\alpha^4 \left(\frac{c}{\omega} \right)^2, \tag{36}$$

here a is a constant difficult to calculate. On the other hand, they also show the differential cross section in the case for small angles:

$$d\sigma = 8\pi\alpha^4 \left(\frac{c}{\omega} \right)^2 (\log\theta)^4 d\Omega, \tag{37}$$

with θ the scattering angle and $d\Omega$ the solid angle.

Eventually, in 1950, Karplus and Neuman carry on with the treatment for the light by light scattering amplitude for arbitrary on-shell kinematics.This means that their results are valid for both high and low energy cases. Also, they analyze the vacuum polarization case[10, 11], focusing in the pair creation, writing their results in terms of all the polarizations possibilities. In the same way, they analyze some particular cases, as low energy limit and right angle scattering. This process is called vacuum polarization because, the fermionic loop divides the incoming photon into an electron-positron pair, and they act as an electric dipole that polarizes the vacuum just as a capacitor.

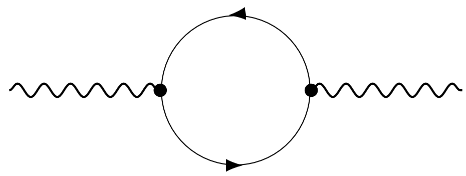
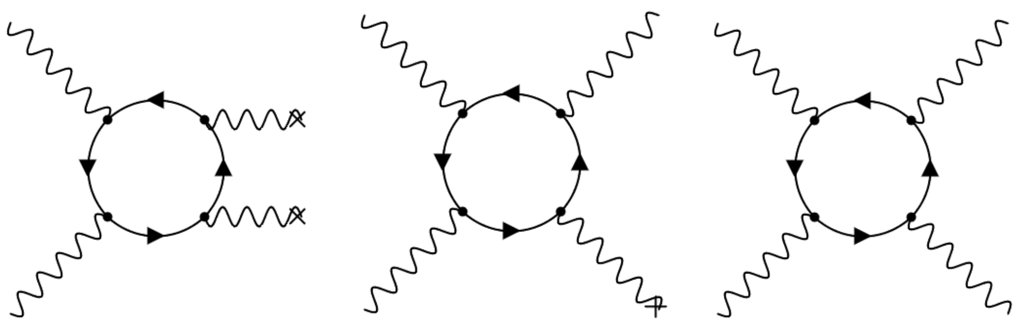


Figure 5: Vacuum polarization diagram

As a result of this polarization, we can appreciate is a partial screening effect of the electric field, in other words, the electric field will be weaker than would be expected if the vacuum were empty. Around 1964 De Tollis using dispersion relations calculates the amplitude obtaining a more compact result[12], however, those two results are equivalent.



(a) Delbrück scattering (b) Photon splitting (c) 4-photon scattering

Figure 6: Light by light processes

Light by light scattering studies increase in number and complexity; a process called Delbrück scattering appears, this involves four photons interacting, in figure 6 a) is represented in a diagram. There, photons with the cross are photons produced by an external field that is treated semi-clasically*, whereas the photons without the cross are true quantized photons that we take as the scattering ones. In the Delbrück scattering case, the external field is the Coulomb field. It was predicted in 1933[13] and observed for the first

*It is not quantized.

time in 1953 [14], and measured under different conditions in 1973 [15] and recently in 2017 with the ATLAS detector at the LHC was found evidence of the light by light scattering in heavy ion collisions[16]. For more details about measurement of those phenomena, we can mention the work of Scharnhorts [17], where he compiles in detail the experiments made over the years and also includes different perspectives of the theory behind. But this is not the only possible four-photon process, also we have the photon splitting and the four-photon scattering as we show in figure 6, for the first process we have only one interacting photon, and for the second it occurs totally in vacuum.

For this thesis, we shall focus on the four-photon scattering problem in the framework of worldline formalism, particularly in scalar QED, this means that we are going to work with a scalar loop instead of a fermionic loop.

The general structure of this thesis is:

- In Chapter 1, we study the standard Quantum Field Theory, the Feynman rules, and how to apply them to the light by light scattering in the framework of scalar QED.
- In Chapter 2, afterward, we shall approach the formalism where we shall work out our problem; the worldline formalism.
- In Chapter 3, we show the full amplitude for the four-photon scattering in the worldline formalism and we carry on with the calculations.
- In Chapter 4, we present our conclusions and perspectives for future work.

Chapter 1

Standard QFT and Scalar QED

The aim of this chapter is to show the standard formalism methods to calculate scattering amplitudes. This includes a quick review of Feynman diagrams and their rules. Succeeding, we shall be able to work out special cases for light by light scattering, such as *tadpoles*, vacuum polarization, and the three-photon scattering, introducing here an important and general result; Furry's theorem.

1.1 Quantization of the Klein-Gordon Field

1.1.1 Classical Field Theory

First, we consider a system that requires a scalar field $\phi(x)$ to describe it, this field $\phi(x)$ may be a real or complex field. In the case of a complex field, $\phi(x)$ and $\phi^*(x)$ are treated as independent fields, or we can treat a complex field as a pair of real fields, thus for the following calculations, we shall consider $\phi(x)$ as a real field. Now we have to introduce a Lagrangian density:

$$\mathcal{L}(\phi, \partial_\mu \phi). \tag{1.1}$$

This density has a relation with the Lagrangian:

$$L(t) \equiv \int d^3x \mathcal{L}(\phi, \partial_\mu \phi). \tag{1.2}$$

With this density, we can define the action functional $S[\phi|\Omega]$:

$$S[\phi|\Omega] = \int d^4x \mathcal{L}(\phi, \partial\phi), \tag{1.3}$$

where Ω is an arbitrary region of the four-dimensional space-time continuum. Just as in classical mechanics, to get the equations of motion, we need to do a local variation on the field:

$$\phi(x) \rightarrow \phi(x) + \delta\phi(x), \tag{1.4}$$

This variations $\delta\phi(x)$ vanish on the surface boundary of the region Γ, *i.e* Ω. Now we are going to demand that the variation on the action has a stationary value:

$$\delta S\,[\phi|\Omega] = 0. \tag{1.5}$$

By doing this, we will get the Euler-Lagrange equations of motion for a scalar field:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0. \tag{1.6}$$

In the case that we have N scalar fields, we shall have N equations of motion. Next we can do is to define a momentum density conjugate to $\phi(x)$:

$$\pi(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}}. \tag{1.7}$$

With this now we are able to define our Hamiltonian and its respective density:

$$H \equiv \int d^4x \mathcal{H}(x) = \int d^3x[\pi(x)\dot{\phi}(x) - \mathcal{L}(\phi, \partial_\mu \phi)]. \tag{1.8}$$

Thinking in the phase space formalism we can define Poisson brackets from the symplectic form:

$$\{f, g\}_{P.B.} = \int d^3x \left[\frac{\delta f}{\delta\phi(x)} \frac{\delta g}{\delta\pi(x)} - \frac{\delta g}{\delta\phi(x)} \frac{\delta f}{\delta\pi(x)} \right], \tag{1.9}$$

where the simbol δ refers to functional differentiation. With the relations:

$$\frac{\delta\phi(x')}{\delta\phi(x)} = \delta^3(x - x'), \quad \frac{\delta\pi(x')}{\delta\pi(x)} = \delta^3(x - x'), \tag{1.10}$$

we can find the fundamental Poisson brackets:

$$\begin{aligned} \{\phi(x), \pi(x')\}_{P.B.} &= \delta^3(x - x') \\ \{\phi(x), \phi(x')\}_{P.B.} = \{\pi(x), \pi(x')\}_{P.B.} &= 0 \end{aligned} \tag{1.11}$$

Defining the phase-space variables:

$$\eta(x) = (\phi(x), \pi(x)), \tag{1.12}$$

the fundamental Poisson brackets can be represented as:

$$\{\eta, \eta\}_{P.B.} = \Omega. \tag{1.13}$$

With Ω the symplectic form matrix:

$$\Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}. \tag{1.14}$$

1.1.2 The Klein-Gordon Equation

Considering the Lagrangian density:

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}\left((\partial_\mu \phi)^2 - m^2\phi^2\right), \tag{1.15}$$

and using 1.6, we will get:

$$(\Box + m^2)\phi(x) = 0, \tag{1.16}$$

where $\Box = \partial_\mu \partial^\mu$ is the d'Alembert operator and m should be interpreted as a mass, this can be notice from introducing the plane wave decomposition for the Klein-Gordon field into the equations of motion. Equation 1.16 is the Klein-Gordon equation. We can notice easily that the conjugate momentum is $\pi(x) = \dot{\phi}(x)$ and the Hamiltonian:

$$H = \int d^3x \mathcal{H} = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right]. \tag{1.17}$$

1.1.3 Canonical Quantization

The solutions to the Klein-Gordon equation are plane waves, and we can expand them into Fourier space:

$$\phi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \phi(\mathbf{p}, t). \tag{1.18}$$

By replacing this field in equation 1.16, we get:

$$\left[\partial_t^2 + (|\mathbf{p}|^2 + m^2)\right] \phi(\mathbf{p}, t) = 0. \tag{1.19}$$

We can notice that this last equation is the same for a harmonic oscillator, with frequency:

$$\omega_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}. \tag{1.20}$$

Taking advantage, we can model our Klein-Gordon field as a harmonic oscillator; writing the Hamiltonian for a single field mode:

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{\phi}^2. \tag{1.21}$$

As in quantum mechanics, we can write ϕ and p in terms of ladder operators:

$$\hat{\phi} = \frac{1}{\sqrt{2\omega}}(\hat{a} + \hat{a}^\dagger), \tag{1.22}$$

$$\hat{p} = -i\sqrt{\frac{\omega}{2}}(\hat{a} - \hat{a}^\dagger). \tag{1.23}$$

Now we can re-write the Hamiltonian 1.21 in terms of the ladder operators with the help of equations 1.22 and 1.23:

$$\hat{H} = \omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right). \tag{1.24}$$

As we know, the eigen-states of H are $|n\rangle$, with $|0\rangle$ as the ground state with energy $\frac{1}{2}\omega$. It is easy to find that the eigen-states $|n\rangle = (a^\dagger)^n|0\rangle$ have eigen-values $\omega\left(n + \frac{1}{2}\right)$. Also we have the commutation relations:

$$[\hat{\phi}, \hat{p}] = i, \quad [\hat{H}, \hat{a}^\dagger] = \omega\hat{a}^\dagger, \quad [\hat{H}, \hat{a}] = -\omega\hat{a}. \tag{1.25}$$

Now we are ready to go ahead with second quantization for the Klein-Gordon field.

1.1.4 Field Operators and Commutators

Next, we want to promote the Klein-Gordon field to an operator. The Klein-Gordon field operator is analogous to equation 1.22 but now we are going to split it into two contributions, the positive and negative frequency parts as:

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right). \tag{1.26}$$

And for equation 1.23 we have:

$$\pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right). \tag{1.27}$$

The decomposition into plane waves is because they satisfy the Klein-Gordon motion equation 1.19. Now we can re-write our Hamiltonian with the equations 1.17, 1.26 and 1.27 :

$$H = \int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + \frac{1}{2}\left[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}}\right]\right). \tag{1.28}$$

The second term is actually, proportional to $\delta(0)$, this is due to the sum of all zero-point energies. This term is expected and we can ignore it, because in the measurements, we are only working with energy differences from the ground state. It can be demostrated that now, in analogy to equation 1.11, we have the following commutation relations:

$$\begin{aligned}[\phi(\mathbf{x}), \pi(\mathbf{x}')] &= i\delta(\mathbf{x}-\mathbf{x}'), \\ [\phi(\mathbf{x}), \phi(\mathbf{x}')] &= [\pi(\mathbf{x}), \pi(\mathbf{x}')] = 0.\end{aligned} \tag{1.29}$$

And for the Hamiltonian, we have the next commutation relations:

$$[H, a_{\mathbf{p}}] = -\omega_{\mathbf{p}}a_{\mathbf{p}}, \quad \left[H, a^\dagger_{\mathbf{p}}\right] = \omega_{\mathbf{p}}a^\dagger_{\mathbf{p}}, \quad \left[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}\right] = (2\pi)^3\delta(\mathbf{p}-\mathbf{p}'). \tag{1.30}$$

In order to give an interpretation for the field operator in equation 1.26, we define the state $|\mathbf{p}\rangle = \sqrt{2\omega_{\mathbf{p}}}a^\dagger_{\mathbf{p}}|0\rangle$ as a momentum state with the following properties:

$$\hat{H}|\mathbf{p}\rangle = \omega_{\mathbf{p}}|\mathbf{p}\rangle = \sqrt{|\mathbf{p}|^2 + m^2}|\mathbf{p}\rangle, \tag{1.31}$$

$$\hat{\mathbf{P}}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle, \tag{1.32}$$

where $\hat{\mathbf{P}}$ is the total momentum operator defined by:

$$\hat{\mathbf{P}} = -\int d^3x\pi(\mathbf{x})\nabla\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\mathbf{p}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}. \tag{1.33}$$

Taking the inner product $\langle 0|\phi(x)|\mathbf{p}\rangle$:

$$\langle 0|\phi(\mathbf{x})|\mathbf{p}\rangle = \langle 0|\int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}'}}}\left(a_{\mathbf{p}'}e^{i\mathbf{p}'\cdot\mathbf{x}} + a^\dagger_{\mathbf{p}'}e^{-i\mathbf{p}'\cdot\mathbf{x}}\right)\sqrt{2\omega_{\mathbf{p}'}}a^\dagger_{\mathbf{p}'}|0\rangle = e^{i\mathbf{p}\cdot\mathbf{x}}. \tag{1.34}$$

We notice that analogous to the non-relativistic quantum mechanics, equation 1.34 is the wavefunction of the state $|\mathbf{p}\rangle$ for a single particle. Following with important calculations, we can make our operators time depending in the Heisenberg picture as usual:

$$\phi(x) = \phi(\mathbf{x}, t) = e^{iHt}\phi(\mathbf{x})e^{-iHt}. \tag{1.35}$$

$$\pi(x) = \pi(\mathbf{x}, t) = e^{iHt}\pi(\mathbf{x})e^{-iHt}. \tag{1.36}$$

Now with equation 1.35 we can compute an important commutator that we shall use in the next section:

$$
\begin{aligned}
[\phi(x), \phi(x')] &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{q}}}}\left[\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + a^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right), \left(a_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}'} + a^\dagger_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{x}'}\right)\right] \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}\left(e^{ip\cdot(x-x')} - e^{ip\cdot(x-x')}\right). \qquad (1.37)
\end{aligned}
$$

This equation is called the *Klein-Gordon function* or *Pauli-Jordan function*, and is a commutator, not a propagator. We shall discuss the propagators in the next section.

1.1.5 Green's Functions and Propagators

To calculate the Green's function for the Klein-Gordon field, we shall work with the inhomogeneous Klein-Gordon equation:

$$(\Box + m^2)G(x - x') = -i\delta^4(x - x'), \qquad (1.38)$$

where $G(x - x')$ is the Green's function. The Fourier transform of $G(x - x')$ is given by:

$$G(x - x') = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-x')}\tilde{G}(p). \qquad (1.39)$$

Recalling a definition of the delta function:

$$\delta(x - x') = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-x')}. \qquad (1.40)$$

By substituting those functions into equation 1.38 we obtain:

$$(-p^2 + m^2)\tilde{G}(p) = -i. \qquad (1.41)$$

Solving for $\tilde{G}(p)$:

$$\tilde{G}(p) = \frac{i}{p^2 - m^2}. \qquad (1.42)$$

Going back to equation 1.38 and replacing $\tilde{G}(p)$:

$$G(x - x') = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m^2} e^{-ip\cdot(x-x')}. \qquad (1.43)$$

If we assume that $x_0 - x'_0 > 0$, we can split the integrals into:

$$G(x - x') = \int \frac{d^3p}{(2\pi)^4} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} \int_{-\infty}^{\infty} dp_0 \frac{i}{(p_0)^2 - \omega_{\mathbf{p}}^2} e^{-ip_0(x^0 - x'^0)}. \tag{1.44}$$

The last integral has poles at $p_0 = \pm\omega_{\mathbf{p}}$, using the *Feynman prescription* we can shift the poles by a factor of $-i\epsilon$, focusing only in the last integral:

$$\lim_{\epsilon\to 0} \int_{-\infty}^{\infty} dp_0 \frac{i}{(p_0)^2 - (\omega_{\mathbf{p}} - i\epsilon)^2} e^{-ip_0(x^0 - x'^0)}. \tag{1.45}$$

We notice that:

$$(p_0)^2 - (\omega_{\mathbf{p}} - i\epsilon)^2 \approx p_0^2 - \omega^2 + i\epsilon + O(\epsilon^2). \tag{1.46}$$

Integrating by the residue theorem:

$$\lim_{\epsilon\to 0} \int_{-\infty}^{\infty} dk_0 \frac{i}{(p_0)^2 - (\omega_{\mathbf{p}} - i\epsilon)^2} e^{-ip_0(x^0 - x'^0)} = 2\pi \frac{e^{-ip_0(x^0 - x'^0)}}{2p_0}. \tag{1.47}$$

Returning to equation 1.45:

$$G(x - x') = \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip\cdot(x-x')}}{2p_0} \equiv \langle 0|\phi(x)\phi(x')|0\rangle \quad \text{for } x_0 - x'_0 > 0. \tag{1.48}$$

For $x_0 - x'_0 < 0$, similarly we have:

$$G(x - x') = \int \frac{d^3p}{(2\pi)^3} \frac{e^{ik\cdot(x-x')}}{2p_0} \equiv \langle 0|\phi(x')\phi(x)|0\rangle \quad \text{for } x_0 - x'_0 < 0. \tag{1.49}$$

Now we define the Klein-Gordon Feynman propagator as:

$$\begin{aligned} \Delta_F(x - x') &= \theta(x^0 - x'^0)\langle 0|\phi(x)\phi(x')|0\rangle + \theta(x'^0 - x^0)\langle 0|\phi(x')\phi(x)|0\rangle \\ &\equiv \langle 0|T(\phi(x)\phi(x'))|0\rangle. \end{aligned} \tag{1.50}$$

Equation 1.50 defines the *time ordering* symbol. This symbol acts moving the object with the latest time to the left and so on, with no penalty for commutation. Thereupon, with equations 1.48, 1.49 and 1.46 we can express the Klein-Gordon Feynman propagator as:

$$\Delta_F(x - x') = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-x')}, \tag{1.51}$$

and we can recognize the Klein-Gordon propagator in momentum space:

$$\Delta_F(p) = \frac{i}{p^2 - m^2 + i\epsilon}, \tag{1.52}$$

this last equation is the expression we shall use for the propagation of virtual particles, in other words, that expression is for the internal lines in our Feynman's diagrams.

1.2 Functional Formulation of QFT

It is well known that one of the important applications of QFT is the calculation of scattering amplitudes for several processes. To perform these calculations, we need to learn Feynman's diagrams and their rules. Because of that, we shall present the path integral and functional methods to derive Feynman rules. It is true that there are different theories where we can find Feynman's rules, as ϕ^4 or Yukawa's theory. In this work, our interest is to present the Feynman rules in scalar QED.

1.2.1 Path Integrals

From quantum mechanics we know the form of the propagator function:

$$A(x_i, x_f; T) = \langle x_f | e^{-i\hat{H}T} | x_i \rangle, \tag{1.53}$$

where $T = t_f - t_i$ is the total propagation time. Considering Hamiltonians of the simple form:

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \hat{V}(\hat{x}), \tag{1.54}$$

we can split the equation 1.53 into a product of N factors and inserting $N-1$ completeness relations:

$$\begin{aligned} A(x_i, x_f; T) &= \langle x_f | e^{-i\hat{H}T} | x_i \rangle = \langle x_f | (e^{-i\hat{H}\frac{T}{N}})^N | x_i \rangle = \langle x_f | e^{-i\hat{H}\frac{T}{N}} \mathbb{1} e^{-i\hat{H}\frac{T}{N}} \mathbb{1} \cdots e^{-i\hat{H}\frac{T}{N}} | x_i \rangle \\ &= \int \left(\prod_{k=1}^{N-1} dx_k \right) \prod_{k=1}^{N} \langle x_k | e^{-i\epsilon \hat{H}} | x_{k-1} \rangle, \end{aligned} \tag{1.55}$$

where $x_0 = x_i$, $x_N = x_f$ and $\epsilon = \frac{T}{N}$. Now we introduce the identity for momentum eigenstates:

$$\mathbb{1} = \int \frac{dp}{2\pi} |p\rangle\langle p|, \tag{1.56}$$

and we obtain:

$$A = \int \left(\prod_{k=1}^{N-1} dx_k \right) \left(\prod_{k=1}^{N} \frac{dp_k}{2\pi} \right) \prod_{k=1}^{N} \langle x_k | p_k \rangle \langle p_k | e^{-i\epsilon \hat{H}} | x_{k-1} \rangle \tag{1.57}$$

Evaluating the matrix elements by expanding the exponential into Taylor series:

$$\langle p_k | e^{-i\epsilon \hat{H}(\hat{x},\hat{p})} | x_{k-1} \rangle = \langle p_k | (\mathbb{1} - i\epsilon \hat{H}(\hat{x}, \hat{p}) + \cdots) | x_{k-1} \rangle. \tag{1.58}$$

And now assuming the simple form 1.54 for the Hamiltonian up to order ϵ, we compute:

$$\langle p_k | e^{-i\epsilon \hat{H}(\hat{x},\hat{p})} | x_{k-1} \rangle = \langle p_k | x_{k-1} \rangle e^{-i\epsilon H(p_k, x_{k-1})} = e^{-i p_k x_{k-1}} e^{-i\epsilon H(p_k, x_{k-1})}. \tag{1.59}$$

Going back to equation 1.57:

$$\begin{aligned} A &= \lim_{N\to\infty} \int \left(\prod_{k=1}^{N-1} dx_k \right) \left(\prod_{k=1}^{N} \frac{dp_k}{2\pi} \right) e^{i\epsilon \sum_{k=1}^{N} \left[p_k \frac{(x_k - x_{k-1})}{\epsilon} - H(x_{k-1}, p_k) \right]} \\ &= \int Dx(t) Dp(t) e^{iS[x,p]}, \end{aligned} \tag{1.60}$$

with $S[x, p]$ the discretized classical action in phase space:

$$S[x, p] = \int_{t_i}^{t_f} dt (p\dot{x} - H(x, p)). \tag{1.61}$$

The integral over p can be performed if we assume Hamiltonians as 1.54, completing the square in equation 1.60 and integrating we get:

$$\begin{aligned} A &= \lim_{N\to\infty} \int \left(\prod_{k=1}^{N-1} dx_k \right) \left(\frac{m}{2\pi i \epsilon} \right)^{\frac{N}{2}} e^{i\epsilon \sum_{k=1}^{N} \left[\frac{m}{2} \frac{(x_k - x_{k-1})^2}{\epsilon^2} - V(x_{k-1}) \right]} \\ &= \int Dx(t) e^{iS[x]}. \end{aligned} \tag{1.62}$$

Equation 1.62 is the path integral in configuration space. In the case of a scalar field, we need to remember equation 1.17 and then we shall have:

$$\langle \phi_b(\mathbf{x}) | e^{-i\hat{H}T} | \phi_a(\mathbf{x}) \rangle = \int D\phi D\pi e^{i \int_{t_i}^{t_f} d^4x (\pi\dot{\phi} - \frac{1}{2}\pi^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi))}. \tag{1.63}$$

With restrictions for $\phi_a(\mathbf{x})$ at $x^0 = t_i$ and $\phi_b(\mathbf{x})$ at $x^0 = t_f$. We can compute the integral over π by completing the square obtaining:

$$\langle\phi_b(\mathbf{x})|e^{-iHT}|\phi_a(\mathbf{x})\rangle = \mathcal{N}\int D\phi e^{iS[\phi,\partial_\mu\phi]}, \tag{1.64}$$

where $\mathcal{N}$ is a normalization factor and $S[\phi, \partial_\mu\phi]$ is given by:

$$S[\phi, \partial_\mu\phi] = \int_{t_i}^{t_f} d^4x\mathcal{L}(\phi) = \int_{t_i}^{t_f} d^4x\left(\frac{1}{2}(\partial_\mu\phi)^2 - V(\phi)\right) \tag{1.65}$$

Equation 1.64 tell us, we need to integrate over all the possible intermediate configurations for the scalar field between the time interval from t_i until t_f.

1.2.2 Correlation Functions and Generating Functionals

We define a correlation function for n scalar fields as:

$$G_n = \langle\Omega|T\{\phi_H(x_1)\phi_H(x_2)\cdots\phi_H(x_n)\}|\Omega\rangle. \tag{1.66}$$

Where we projected the states $|\phi_a\rangle$ and $\langle\phi_b|$ into the vacuum state $|\Omega\rangle$ by taking the limit $T \to \infty(1 - i\epsilon)$. Explicitly in terms of the path integral:

$$\langle\Omega|T\{\phi_H(x_1)\phi_H(x_2)\cdots\phi_H(x_n)\}|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int_{\phi(a)}^{\phi(b)} D\phi\phi(x_1)\phi(x_2)\cdots\phi(x_n)e^{iS[\phi,\partial_\mu\phi]}}{\int_{\phi(a)}^{\phi(b)} D\phi e^{iS[\phi,\partial_\mu\phi]}}. \tag{1.67}$$

We would like to have an easier way to evaluate this integral, because of that we define a generating functional $Z[J]$:

$$Z[J] \equiv \int D\phi e^{i\int d^4x\left[\mathcal{L}(\phi)+J(x)\phi(x)\right]}, \tag{1.68}$$

where J is an arbitrary source. Now we are able to re-write equation 1.67 in terms of $Z[J]$ as follows:

$$\int D\phi\phi(x_1)\phi(x_2)\cdots\phi(x_n)e^{iS[\phi,\partial_\mu\phi]} = (-i)^n \frac{\delta^n}{\delta J(x_1)\cdots\delta J(x_n)}Z[J]\bigg|_{J=0}. \tag{1.69}$$

Here the δ refers to functional diferentiation, and after that we need to remove the source J evaluating it at zero. The functional differentiation is defined by:

$$\frac{\delta J(x')}{\delta J(x)} = \delta^4(x - x') \quad \text{Equivalent to} \quad \frac{\delta}{\delta J(x)}\int d^4x' J(x')\phi(x') = \phi(x) \tag{1.70}$$

With equation 1.69 we can express the correlation fuction in terms of the generating functional:

$$\langle\Omega|T\{\phi_H(x_1)\phi_H(x_2)\cdots\phi_H(x_n)\}|\Omega\rangle = \frac{(-i)^n}{Z_0}\frac{\delta^n}{\delta J(x_1)\cdots\delta J(x_n)}Z[J]\bigg|_{J=0}. \tag{1.71}$$

Where $Z_0 = Z[J = 0]$. With this last result we can perform more complex correlation functions only taking derivatives of the source J.

1.3 Scalar QED

Previously we studied how to quantize a free scalar field, the Klein-Gordon field. In this section, we shall analize the structure of the scalar field theory attached to an electromagnetic field. This theory is also known as scalar QED and it is the main theory for this work.

1.3.1 Lagrangian for Scalar QED

For scalar QED, we have the Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - m^2\phi^*\phi, \tag{1.72}$$

where D_μ is called covariant derivative and it is defined as:

$$D_\mu\phi = \partial_\mu\phi + ieA_\mu\phi. \tag{1.73}$$

We can notice that now we have added a electromagnetic term to the scalar theory studied before. This lagrangian can be expanded as:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \phi^*(\Box + m^2)\phi - ieA_\mu\left[\phi^*(\partial^\mu\phi) - (\partial^\mu\phi^*)\phi\right] + e^2A_\mu A^\mu|\phi|^2 \tag{1.74}$$

Its equations of motion are:

$$(\Box + m^2)\phi = -2ieA_\mu\partial^\mu\phi + e^2A_\mu A^\mu\phi, \tag{1.75}$$

$$(\Box + m^2)\phi^* = 2ieA_\mu\partial^\mu\phi^* + e^2A_\mu A^\mu\phi^*. \tag{1.76}$$

We can realize from the linear term in e that the charge of ϕ^* is the opposite of ϕ, but they have the same mass.

1.3.2 Symmetries of Scalar QED

We have introduced the Lagrangian for QED and now we are going to analyze its symmetries. There is a global symmetry in the scalar field:

$$\phi \to e^{-i\alpha}\phi, \tag{1.77}$$

where α is a constant parameter. This symmetry is a rotation in the complex scalar field and it is called gauge invariance. This field theory admits a conserved current, that we can compute with Noether's theorem:

$$j_\mu = \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \frac{\delta \phi_n}{\delta \alpha} = -i(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi) - 2eA_\mu\phi^*\phi, \tag{1.78}$$

here n represents each of the two fields ϕ and ϕ^*, also $\frac{\delta\phi}{\delta\alpha} = -i\phi$ and $\frac{\delta\phi^*}{\delta\alpha} = i\phi^*$. This current is conserved because of the global phase symmetry. Here we can notice that in the lagrangian 1.74, the linear term in A_μ is just $-eA_\mu j^\mu$:

$$-ieA_\mu\left[\phi^*(\partial^\mu\phi) - (\partial^\mu\phi^*)\phi\right] + e^2 A_\mu A^\mu \phi^*\phi = -eA_\mu j^\mu. \tag{1.79}$$

Now we can promote this global symmetry to a local symmetry. In order to do it, α will be a function of the coordinates, *i.e.* $\alpha(x)$:

$$\phi \to e^{-i\alpha(x)}\phi, \tag{1.80}$$

but this change does not leave the derivatives invariant:

$$\partial_\mu\phi \to \partial_\mu(e^{-i\alpha(x)}\phi) = e^{-i\alpha(x)}\partial_\mu\phi - ie^{-i\alpha(x)}\phi\partial_\mu\alpha(x). \tag{1.81}$$

Then, it is necessary to define the covariant derivative as we showed before:

$$D_\mu\phi = \partial_\mu\phi + ieA_\mu\phi. \tag{1.82}$$

That derivative demands a change into the vector field A_μ:

$$A_\mu \to A_\mu + \partial_\mu\alpha(x). \tag{1.83}$$

With this, now the covariant derivative transforms just as the field does:

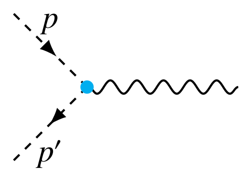
$$D_\mu\phi \to e^{-i\alpha(x)}D_\mu\phi. \tag{1.84}$$

All these transformations together let the lagrangian invariant. As such, the lagrangian for scalar QED in equation 1.72, is gauge invariant under both global and local symmetries showed above.

1.3.3 Feynman Rules in Scalar QED

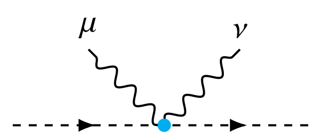
Inserting equation 1.74 into the generating functional 1.68, splitting into free part and interaction part, expanding into series in powers of e and taking the Fourier transform for the interaction vertices, we get two different vertices:

- From the term $-ieA_\mu\left[\phi^*(\partial^\mu\phi) - (\partial^\mu\phi^*)\phi\right]$, we can find the vertex with incoming scalar particle, incoming scalar anti-particle, outgoing photon



$$= -ie(p_\mu - p'_\nu)$$

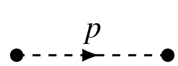
- And from the quadratic term $e^2 A_\mu A^\mu |\phi|^2$, we can find the vertex with incoming scalar particle, outgoing scalar anti-particle, two outgoing photons, also called *seagull vertex*



$$= 2ie^2\eta_{\mu\nu}$$

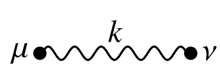
And we have the propagators for scalar particles and photons, choosing Feynman's gauge we get:

- Propagator for scalar particle



$$= \frac{i}{p^2 - m^2 + i\varepsilon}$$

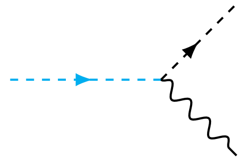
- Propagator for photon



$$= \frac{-i\eta_{\mu\nu}}{k^2 + i\varepsilon}$$

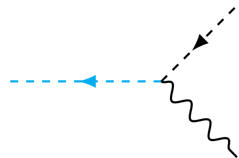
Just as a reminder, it is necessary to impose momentum conservation in each vertex, and integrate over each undeterminated momentum. Now, for external legs:

- Ingoing scalar particle



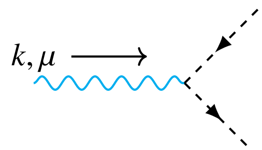
$= 1$

- Ingoing scalar anti-particle



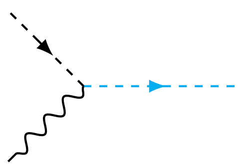
$= 1$

- Ingoing photon



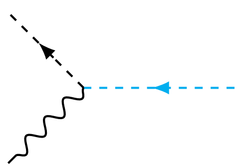
$= \varepsilon_\mu(k)$

- Outgoing scalar particle



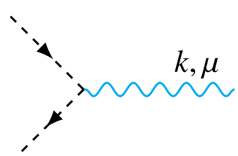
$= 1$

- Outgoing scalar anti-particle



$= 1$

- Outgoing photon



$= \varepsilon^*_\mu(k)$

1.3.4 Ward Identity for Scalar QED

First, we have to recall the matrix elements $\mathcal{M}$, which depends on each of the polarizations as follows:

$$\mathcal{M} = \varepsilon_\mu M^\mu. \tag{1.85}$$

Where M^μ transforms as a 4-vector. As we see above, there are polarizations for the incoming photons, represented by ε_μ and the conjugate for the outgoing photons. We can analyze how the matrix elements change by applying a gauge transformation of the form:

$$\varepsilon_\mu \to \varepsilon_\mu + ck_\mu, \tag{1.86}$$

with c a constant and k_μ the momentum of the photon. Doing a gauge transformation to the polarizations and matrix elements we get:

$$\varepsilon_\mu M^\mu \to (\varepsilon_\mu + ck_\mu)M'^\mu. \tag{1.87}$$

Where $M^\mu \to M'^\mu = \Lambda^\mu{}_\nu M^\nu$. This last equation requires that the product $k_\mu M'^\mu = 0$, because the matrix elements are invariant under a gauge transformations. The expression:

$$k_\mu M^\mu = 0, \tag{1.88}$$

is called *the Ward Identity* on-shell and it tell us that we can change the polarization vector by its momentum vector and then the result must be zero. This identity is a manifestation of the gauge invariance. As we see previously, the scalar QED lagrangian is invariant under the change $A_\mu \to A_\mu + \partial_\mu \alpha(x)$, which in the momentum space directly imply that $\varepsilon_\mu \to \varepsilon_\mu + ck_\mu$ is also a symmetry for scalar QED.

1.4 Scattering Amplitudes For Light-by-light Processes In QFT

In this section we are going to present easier cases for light by light processes, as one-photon scattering*, vacuum polarization and the three-photon scattering, in scalar QED for standard QFT.

1.4.1 Tadpoles

The one-photon diagram is also known as the *tadpole*, its representation is as we show in the figure 1.1.

*This is the vacuum expectation of the field, but here we interpret as the self-scattering of one photon.

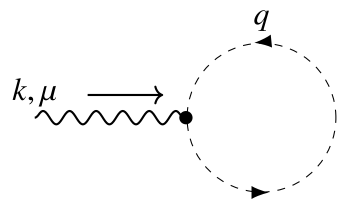


Figure 1.1: Tadpole diagram

Applying the Feynman rules for scalar QED at one-loop level, we get:

$$i\mathcal{M} = -i^2(2\pi)^D\delta^D(k)e\varepsilon_\mu(k)\int\frac{d^Dq}{(2\pi)^D}\frac{(2q+k)^\mu}{q^2-m^2+i\epsilon}. \tag{1.89}$$

The linear term in q integrates to zero, since the integrand is odd under a parity transformation. Thus we only have the integral:

$$-i^2(2\pi)^D\delta^D(k)e\varepsilon_\mu(k)\int\frac{d^Dq}{(2\pi)^D}\frac{k^\mu}{q^2-m^2+i\epsilon}. \tag{1.90}$$

From the application of momentum conservation at the vertex, we get the $\delta^D(k)$, this tell us that all the momentum components of the incoming photon are zero. Thus, the integral of equation 1.90 is zero, in other words, the tadpole diagram in vacuum gives no contribution.

1.4.2 Vacuum Polarization

The two-photon case is a special process, it is called *vacuum polarization*. The figure 1.2 represents the process into the two possible diagrams in scalar QED.

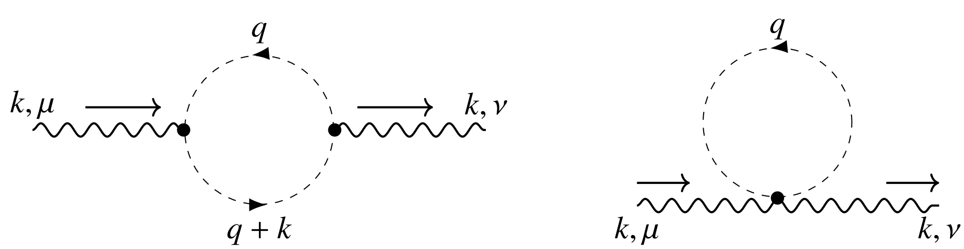


Figure 1.2: Vacuum polarization diagrams

Applying the scalar QED Feynman rules for both diagrams, we get:

$$i\mathcal{M} = e^2\varepsilon_\mu(k)\varepsilon^*_\nu(k)\int\frac{d^Dq}{(2\pi)^D}\left[\frac{(2q+k)^\mu(2q+k)^\nu}{((q+k)^2-m^2+i\epsilon)(p^2-m^2+i\epsilon)}-\frac{2\eta^{\mu\nu}}{p^2-m^2+i\epsilon}\right]. \tag{1.91}$$

The rest of the calculation consists of introducing Feynman parameters and performing a Wick rotation, afterwards we integrate over all the variables†. The result is:

$$i\mathcal{M} = \frac{ie^2}{(4\pi)^2}\varepsilon_\mu(k)\varepsilon^*_\nu(k)\left[\eta^{\mu\nu}k^2 - k^\mu k^\nu\right]\int_0^1 dx(2x-1)^2\left(\ln\left(\frac{m^2 - x(1-x)k^2}{4\pi}\right) + \gamma_E - \frac{1}{\epsilon}\right), \tag{1.92}$$

where γ_E is the Euler constant and we can notice that there is a divergent part equal to:

$$-\frac{ie^2}{(4\pi)^2}\left[\eta^{\mu\nu}k^2 - k^\mu k^\nu\right]\frac{1}{3\epsilon}. \tag{1.93}$$

We can appreciate the appearence of the transversal projector. This process is not considered light-by-light scattering, because is more like a correction to the photon propagator, or in other words, it is the self-energy process for the photon mediated by a virtual pair of particle-antiparticle. About the divergence in equation 1.93, we can handle it by field strength renormalisation.

1.4.3 Three-photon Scattering and Furry's Theorem

Now going ahead, we treat the case of three-photon scattering. Figure 1.3 represents the possible diagrams for this process with the first kind of vertex in scalar QED.

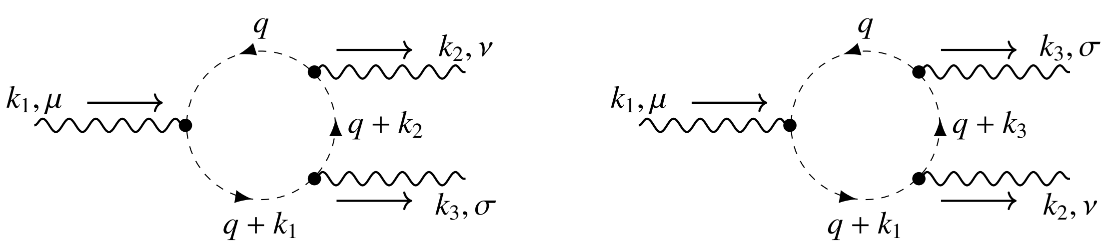


Figure 1.3: Feynman diagrams for three photon scattering with the first kind of vertex

Working out the first diagram we get:

$$i\mathcal{M} = (-ie)^3\varepsilon_\mu(k_1)\varepsilon^*_\nu(k_2)\varepsilon^*_\sigma(k_3)\int\frac{d^Dq}{(2\pi)^D}\frac{(2q+k_1)^\mu}{q^2 - m^2 + i\epsilon}\frac{(2q+k_2)^\nu}{(q+k_2)^2 - m^2 + i\epsilon}\frac{(2q+k_1+k_2)^\sigma}{(q+k_1)^2 - m^2 + i\epsilon} \tag{1.94}$$

†To see how the process can be done, we recommend to check Peskin and Schroeder Chapter 6.

And for the second diagram:

$$i\mathcal{M} = (-ie)^3 \varepsilon_\mu(k_1)\varepsilon^*_\nu(k_2)\varepsilon^*_\sigma(k_3) \int \frac{d^D q}{(2\pi)^D} \frac{(2q+k_1)^\mu}{q^2 - m^2 + i\epsilon} \frac{(2q+k_3)^\sigma}{(q+k_3)^2 - m^2 + i\epsilon} \frac{(2q+k_1+k_3)^\nu}{(q+k_1)^2 - m^2 + i\epsilon} \tag{1.95}$$

Doing a change of variables $q = -(q' + k_1)$ for the second diagram, we are changing the flux of momentum in the loop. As result of this, we transform the integrand of equation 1.94 into minus the integrand of the equation 1.95, and the total sum is zero. We have no contribution from these diagrams.

Now we have represented the diagrams of the seagull vertex in figure 1.4. Each diagram

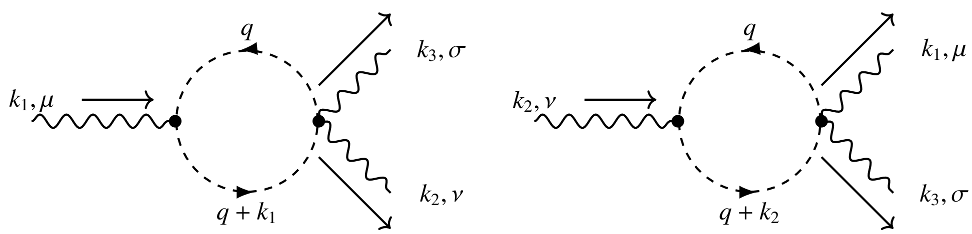


Figure 1.4: Feynman diagrams for three photon scattering with the seagull vertex

is zero by itself, for example, if we compute the first diagram we get:

$$i\mathcal{M} = (-ie)\varepsilon_\mu(k_1)\varepsilon^*_\nu(k_2)\varepsilon^*_\sigma(k_3) \int \frac{d^D q}{(2\pi)^D} \frac{(2q+k_1)^\mu}{q^2 - m^2 + i\epsilon} \frac{2ie^2\eta^{\sigma\nu}}{(q+k_1)^2 - m^2 + i\epsilon} \tag{1.96}$$

Changing the flux of momentum again with the same change of variables, we shall obtain minus the integrand of equation 1.96, but we are still working the same diagram, this means the diagram must be zero. Now it is clear that we have no contribution from three photon scattering, and this can be generalized to a theorem called *Furry's theorem*. This theorem tell us that from any $N-$photon scattering, with N an odd number, we shall get zero as result for the scattering amplitude. We shall prove this in the worldline formalism in next chapter.

1.4.4 Four-photon Scattering

At this point, we are not going to compute the integrals, we are just going to show how obtain the integral representation for the different diagrams of the four-photon scattering processes.

In the figure 1.5, we can see represented the diagrams plus permutations[‡] of the external

[‡]There are 6 permutations for this kind of diagram.

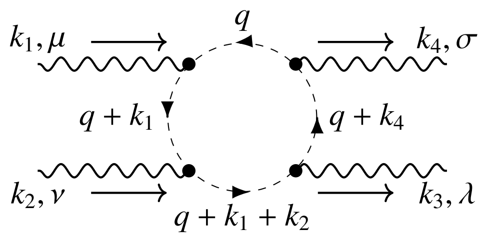


Figure 1.5: Four photon scattering diagrams with the first kind of vertex

legs, working out the first diagram we get:

$$
\begin{aligned}
i\mathcal{M}^1 &= (-ie)^4 \varepsilon_\mu(k_1)\varepsilon_\nu(k_2)\varepsilon^*_\lambda(k_3)\varepsilon^*_\sigma(k_4) \int \frac{d^Dq}{(2\pi)^D} \frac{(2q+k_1)^\mu}{q^2-m^2+i\epsilon} \times \\
&\times \frac{(2q+k_4)^\sigma}{(q+k_4)^2-m^2+i\epsilon} \frac{(2q+2k_4+k_3)^\lambda}{(q+k_1+k_2)^2-m^2+i\epsilon} \frac{(2q+2k_1+k_2)^\nu}{(q+k_1)^2-m^2+i\epsilon}. \qquad (1.97)
\end{aligned}
$$

Where the index 1, is a label for the diagram and not a tensor index. As we can see, this integral is not trivial to do, and we are not computing it. For the seagull vertex we have represented in figure 1.6 the diagrams plus permutations[§].

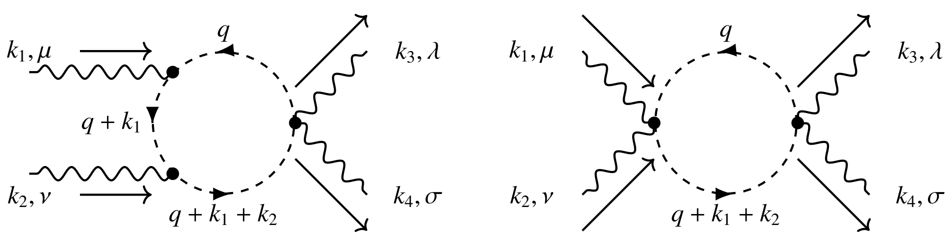


Figure 1.6: Four photon scattering diagrams with the seagull vertex

Applying Feynman's rules for the first diagram:

$$
\begin{aligned}
i\mathcal{M}^2 &= (-ie)^2 \varepsilon_\mu(k_1)\varepsilon_\nu(k_2)\varepsilon^*_\lambda(k_3)\varepsilon^*_\sigma(k_4) \int \frac{d^Dq}{(2\pi)^D} \frac{(2q+k_1)_\mu}{q^2-m^2+i\epsilon} \times \\
&\times \frac{2ie^2\eta^{\sigma\lambda}}{(q+k_1+k_2)^2-m^2+i\epsilon} \frac{(2q+2k_1+k_2)_\nu}{(q+k_1)^2-m^2+i\epsilon} \qquad (1.98)
\end{aligned}
$$

For the second diagram:

$$
i\mathcal{M}^3 = \varepsilon_\mu(k_1)\varepsilon_\nu(k_2)\varepsilon^*_\lambda(k_3)\varepsilon^*_\sigma(k_4) \int \frac{d^Dq}{(2\pi)^D} \frac{2ie^2\eta^{\mu\nu}}{q^2-m^2+i\epsilon} \frac{2ie^2\eta^{\sigma\lambda}}{(q+k_1+k_2)^2-m^2+i\epsilon} \qquad (1.99)
$$

[§]There are 3 permutations for each kind of diagram.

Again, we are not going to calculate the integrals, this is just to ilustrate how Feynman's rules works for these diagrams. What we can do is the analysis for the superficial degree of divergence. We can do this by counting powers of momentum in the integral expressions for the amplitude. For example, if we take equation 1.97 and we count the different combinations of powers in q, we will obtain zero as the highest power in q, this result let us know that perhaps there are divergences of kind $\ln \Lambda$, with Λ a cutoff in the momentum space. The remaing combinations, will give us negative numbers and that means the finiteness of the rest of integrals. The same argument holds for equations 1.98 and 1.99.

Now the question in the air is: *How do we know that the complete amplitude is finite?*. The answer is quite simple, and it is due to gauge invariance and Ward identity. Now in our case for the four photon scattering the identity takes the form:

$$k_\mu^{(1)} k_\nu^{(2)} k_\sigma^{(3)} k_\lambda^{(4)} \mathcal{M}_{\mu\nu\sigma\lambda} = 0, \tag{1.100}$$

since we have four polarizations, for each of our four photons, here each supra-index is to indicate the number of the photon. Moreover, Ward's identity is held for any of the polarizations.

$$k_\mu^{(i)} \mathcal{M}_{\mu\nu\sigma\lambda} = 0, \tag{1.101}$$

Now we can change any photon polarization by its momentum vector and sum all the contributions for the complete amplitude:

$$k_\mu^{(i)} (\mathcal{M}^1_{\mu\nu\sigma\lambda} + \mathcal{M}^2_{\mu\nu\sigma\lambda} + \mathcal{M}^3_{\mu\nu\sigma\lambda} + \text{permutations}) = 0 \tag{1.102}$$

after that, the amplitude must vanishes. It can be demostrated that summing over all diagrams, we can cancel the divergent part of each diagram by applying the Ward identity[¶].

[¶]We recommend to check Peskin and Schoesder, Chapter 10, problem 10.1.

Chapter 2

Worldline Formalism

The beginnings of path integrals were born as an alternative way to represent non-relativistic quantum mechanics by Richard Feynman[1, 18]. This formulation is completely equivalent to those offered by Dirac and Schröedinger, but from a functional point of view. As time went by, it was by analysing this new representation for quantum mechanics that different ways of encompassing QFT were obtained, and this allowed new formulations to be explored. The worldline formalism was born precisely as a first quantized approach to QFT, using the formulation of path integrals over relativistic particles trajectories, and proposing a different way of performing the calculations offered by Feynman diagrams.

Its history begins with Feynman and his path integral representation, suggesting a first quantised particle path integral representation of the propagator in scalar and spinor QED, but it is not really developed the idea of worldline formalism[1, 18, 19]. Eventually, the use of this representation brought about other approaches to interesting problems, such as the work of Affleck et al[20], in which they study the asymptotics of perturbation theory in electromagnetic backgrounds using a semi classical approach. Later, Bern and Kosower use string theories to derive a master formula for field theory amplitudes from the infinite tension limit[21, 22, 23]. They note that the final results do not depend on string theory details. Moving forward in time, we come across a paper published by Strassler[24], in which we can see how he developed Feynman's work into what is now called the worldline formalism, recovering the Bern-Kosower formulae from a point particle path integral representation within field theory. These methods were shown to be more efficient for the calculation of one-loop amplitudes in gauge theories, showing several advantages over the standard formalism and quickly there were articles where we can find a broad of applications of it[25, 26, 27, 28, 29, 30, 31]. Currently, most of the details about modern notation, examples and applications are into Christian Schubert report[32] and the update to that report[33].

Now, our task is to show how this formalism works and how to apply it to our calculations for light by light scattering processes.

2.1 The One-loop N-photons Amplitudes

The one-loop effective action, in scalar QED, is given by the Det of the kinetic operator as follows:

$$\Gamma[\phi, A] = \ln \mathrm{Det}^{-1}[-(\partial + ieA)^2 + m^2] = -\mathrm{Tr}\ln[-(\partial + ieA)^2 + m^2]. \tag{2.1}$$

Now, recalling the integral representation of ln:

$$\ln(A/B) = \int_0^\infty \frac{dT}{T}(e^{-BT} - e^{-AT}), \tag{2.2}$$

and representing the Tr in x-space, we get:

$$\Gamma[\phi, A] = \int_0^\infty \frac{dT}{T} \int d^D x \langle x| \exp\left\{-T[m^2 - (\partial + ieA)^2]\right\} |x\rangle. \tag{2.3}$$

Notice that, in the loop, our boundary conditions allow us to write:

$$\int d^D x \int_{x(0)=x}^{x(T)=x} Dx = \int_{x(0)=x(T)} Dx. \tag{2.4}$$

Thus, the worldline respresentation of the one-loop effective action is given by:

$$\Gamma_{\mathrm{scal}}[A] = \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_{x(T)=x(0)} Dx e^{-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + ie\dot{x}\cdot A(x)\right)}. \tag{2.5}$$

Choosing our background Maxwell field as the sum of N plane waves:

$$A_\mu(x) = \sum_{i=1}^N \varepsilon_{i\mu} e^{ik_i \cdot x}, \tag{2.6}$$

this defines a polarization ε_i and momentum k_i for each photon. Just taking the terms linear in each of the N plane waves, we obtain the representation of the N - photon amplitude in terms of photon vertex operators:

$$\begin{aligned} \Gamma_{\mathrm{scal}}[k_1, \varepsilon_1; \ldots; k_N, \varepsilon_N] &= (-ie)^N \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_{x(0)=x(T)} Dx e^{-\int_0^T d\tau \frac{1}{4}\dot{x}^2} \\ &\times V^\gamma_{\mathrm{scal}}[k_1, \varepsilon_1] \cdots V^\gamma_{\mathrm{scal}}[k_N, \varepsilon_N], \end{aligned} \tag{2.7}$$

we can notice that the expansion of the interaction term left the free kinetic term in the worldline action. Here the photon vertex operator is given by:

$$V^{\gamma}_{\text{scal}}[k,\varepsilon] = \int_0^T d\tau\varepsilon \cdot \dot{x}(\tau)e^{ik\cdot x(\tau)}. \tag{2.8}$$

At this point, we have to be careful about the constant functions of the the bosonic kinetic operator, also called zero modes. In these terms, the kinetic term vanishes, corresponding to a zero eigenvalue, having as a consequence that we cannot invert the kinetic operator. We can treat them by choosing *String Inspired* boundary conditions, by doing this, we change to an orthogonal Hilbert space to the zero mode and there the kinetic term is invertible. First, we change the trajectories by a fluctuation $q(\tau)$:

$$x^\mu = x_0^\mu + q^\mu(\tau), \tag{2.9}$$

and now we have to integrate over x_0^μ and the fluctuation variable:

$$\int Dx = \int d^D x_0 \int Dq(\tau). \tag{2.10}$$

For these conditions, we have fixed a point in the center of mass of the loop as we show in figure 2.1, defined by:

$$x_0^\mu \equiv \frac{1}{T}\int_0^T d\tau x^\mu(\tau). \tag{2.11}$$

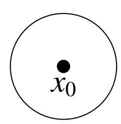


Figure 2.1: String Inspired zero mode fixed point

Integrating over τ the new trajectories, we shall find the conditions for $q(\tau)$:

$$\int_0^T d\tau q^\mu(\tau) = 0. \tag{2.12}$$

These boundary conditions are called *String Inspired*. Performing the integral over x_0^μ produces the energy-momentum conservation factor and the remaining integral over Dq is gaussian form, integrating it, we obtain the *Bern-Kosower master formula* for scalar QED:

$$\begin{aligned}\Gamma_{\text{scal}}[k_1,\varepsilon_1;\ldots;k_N,\varepsilon_N] &= (-ie)^N(2\pi)^D\delta(\sum k_i)\int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}}e^{-m^2T}\prod_{i=1}^N\int_0^T d\tau_i \\ &\times \exp\left\{\sum_{i,j=1}^N\left[\frac{1}{2}G_{ij}k_i\cdot k_j - i\dot{G}_{ij}\varepsilon_i\cdot k_j + \frac{1}{2}\ddot{G}_{ij}\varepsilon_i\cdot\varepsilon_j\right]\right\}\Bigg|_{\varepsilon_1\varepsilon_2\cdots\varepsilon_N}.\end{aligned} \tag{2.13}$$

Where $G_{ij} = G(\tau_i,\tau_j)$ is called *worldline Green's function* for SI boundary conditions:

$$G(\tau_i,\tau_j) = |\tau_i-\tau_j| - \frac{(\tau_i-\tau_j)^2}{T}, \tag{2.14}$$

$$\dot{G}(\tau_i,\tau_j) = \text{sgn}(\tau_i-\tau_j) - 2\frac{\tau_i-\tau_j}{T}, \tag{2.15}$$

$$\ddot{G}(\tau_i,\tau_j) = 2\delta(\tau_i-\tau_j) - \frac{2}{T}, \tag{2.16}$$

and the dot is the derivative of G_{ij} respect to the first variable. We can notice a factor of $-\frac{2}{T}$, this factor is due to the transformation we did to treat the zero mode, which modifies our Hilbert space and it has no effect because of equation 2.12 in the Hilbert space of functions orthogonal to the zero mode.

Usually, in our calculations we shall do a rescaling for the worldline Green's function $\tau_i = u_iT$, with these changes the functions take the form:

$$G_{ij} = |\tau_i-\tau_j| - \frac{(\tau_i-\tau_j)^2}{T} = T[|u_i-u_j| - (u_i-u_j)^2]. \tag{2.17}$$

And $\dot{G}_{ij}$:

$$\dot{G}_{ij} = \text{sgn}(\tau_i-\tau_j) - 2\frac{(\tau_i-\tau_j)}{T} = \text{sgn}(u_i-u_j) - 2(u_i-u_j). \tag{2.18}$$

We are not treating $\ddot{G}_{ij}$, because in all our calculations, we shall integrate by parts to replace them for $\dot{G}_{ij}$. Equation 2.13 represents the full N-photon amplitude without the need of summing over permutations of the photon legs, we only need to choose a value for N and perform the integrals, in other words, these integrals automatically include that sum over the permutations of the external legs. We can expand the exponential in equation 2.13 in certain specific way to define the polynomials P_N as follows:

$$\exp\{\cdot\}\Big|_{\varepsilon_1\varepsilon_2\cdots\varepsilon_N} = (-i)^N P_N(\dot{G}_{ij},\ddot{G}_{ij})\exp\left[\frac{1}{2}\sum_{i,j=1}^N G_{ij}k_i\cdot k_j\right]. \tag{2.19}$$

Integrating by parts to eliminate the contributions involving $\ddot{G}_{ij}$ we obtain new polynomials called Q_N. This Q representation brings to us some advantages over the P and the standard QFT representations:

- It is a more compact representation. As we see in the last chapter, the calculation for four photon scattering is not a small equation, also we have to sum over all the permutations for the external legs. Here, we only have to use the master formula with $N = 4$.

- The integrand is homogeneous. The P representation has $\ddot{G}_{ij}$ terms, now with the Q representation, we get rid of all the second derivatives terms and instead we have N factors of $\dot{G}_{ij}$ and N factors of external momentum.

- We can pass to spinor QED with the application of the replacement rule:

$$\dot{G}_{i_1 i_2}\dot{G}_{i_2 i_3}\cdots\dot{G}_{i_N i_1} \rightarrow \dot{G}_{i_1 i_2}\cdots\dot{G}_{i_N i_1} - G_{Fi_1 i_2}\cdots G_{Fi_N i_1}, \tag{2.20}$$

  where G_{Fij} is the *fermionic worldline Green's function*, and it is given by:

$$G_{Fij} = \text{sgn}(\tau_i - \tau_j) \tag{2.21}$$

These Q_N, let us define new structures that we will recognize, for example the photon field-strength tensor associated to the photon with momentum k_i and polarization vector ε_i:

$$f_i^{\mu\nu} = k_i^\mu \varepsilon_i^\nu - \varepsilon_i^\mu k_i^\nu. \tag{2.22}$$

With the trace of products of this tensor, we can define another structure called a *Lorentz-cycle* $Z_n(i_1 i_2 \ldots i_n)$:

$$Z_2(ij) = \frac{1}{2}\text{tr}(f_i f_j), \tag{2.23}$$

$$Z_n(i_1 i_2 \ldots i_n) = \text{tr}\left(\prod_{j=1}^{n} f_{i_j}\right). \tag{2.24}$$

Here, Z_2 is the transversal projector and Z_n is the generalization to n-points. Another important structure is called a *one-tail* defined as:

$$T(a) = \dot{G}_{ai}\varepsilon_a \cdot k_i. \tag{2.25}$$

At this point, it is important to mention why we expect to write the amplitude in terms of all this previous structures. The reason is coming from Ward identity and gauge invariance. As we saw in the last chapter, we are looking for structures invariant under Lorentz transformations in the polarizations. In this case, the photon field strength tensor, the Lorentz cycles and the terms involving tails are invariants under these transformations, for example, taking the strength field tensor:

$$f_i^{\mu\nu} = k_i^\mu \varepsilon_i^\nu - \varepsilon_i^\mu k_i^\nu = k_i^\mu(\varepsilon_i^\nu + k_i^\nu) - (\varepsilon_i^\mu + k_i^\mu)k_i^\nu = k_i^\mu \varepsilon_i^\nu + k_i^\mu k_i^\nu - \varepsilon_i^\mu k_i^\nu - k_i^\mu k_i^\nu = f_i^{\mu\nu} \quad (2.26)$$

This last equation makes clear that the Lorentz cycles are invariant too, because they are written in terms of the strength tensor. On the other hand, the terms with tails are also invariant under the integral; the integrand transforms as a total derivative which vanishes around the loop. To see this fact, we need to take the complete term and apply the transformation as we did in the last equation.

In order to show how this formalism works, we are going to calculate the particular cases we have studied before for standard QFT.

2.2 Advantages of Worldline Formalism

Between the several advantages of worldline formalism over standard QFT and Feynman diagrams representation, we can mention:

- It is explicitly gauge invariant.
- The effective action is written as a one-dimensional path integral.
- The master formula represents the complete amplitude, we don't need to add permutations as in Feynman diagramatic representation.
- The integral can be performed to any order in the gauge coupling*.
- Avoid the usual algebra required from Feynman diagrams.
- It has been extended to a number of other field theories.
- In the framework of scalar QED, we don't need to worry about the existence of the seagull vertex. The diagrams involving it, are included into the master formula.

2.3 Scattering Amplitudes For Light-by-light Processes In Worldline Formalism

For this section, we are going to present the results in a new way. We shall begin with the vacuum polarization and afterwards, we present Furry's theorem in this formalism. To

*The meaning of this statement, is just that in principle, we can choose an arbitrary number of external photons.

finish, we present the complete four-photon scattering amplitude and in the next chapter we shall show how to treat this problem.

2.3.1 Vacuum Polarization

For $N = 2$ in the master formula and expanding the exponential, we get:

$$\begin{aligned}\Gamma_{\text{scal}}[k_a, \varepsilon_1; k_2, \varepsilon_2] &= (-ie)^2(2\pi)^D\delta(k_1 + k_2)\int_0^\infty \frac{dT}{T}(4\pi T)^{-D/2}e^{-m^2T} \\ &\times \int_0^T d\tau_1 \int_0^T d\tau_2(-i)^2 P_2 e^{G_{12}k_1\cdot k_2}. \end{aligned} \tag{2.27}$$

Where P_2 is given by:

$$P_2 = \dot{G}_{12}\varepsilon_1 \cdot k_2\dot{G}_{21}\varepsilon_2 \cdot k_1 - \ddot{G}_{12}\varepsilon_1 \cdot \varepsilon_2. \tag{2.28}$$

Doing an integration by parts we can obtain Q_2

$$Q_2 = \dot{G}_{12}\dot{G}_{21}(\varepsilon_1 \cdot k_2\varepsilon_2 \cdot k_1 - \varepsilon_1 \cdot \varepsilon_2 k_1 \cdot k_2). \tag{2.29}$$

Imposing momentum conservation, we can set a unique momentum $k := k_1 = -k_2$, and introducing Q_2 into equation 2.27 we get:

$$\Gamma_{\text{scal}}[k, \varepsilon_1; \varepsilon_2] = (2\pi)^D\varepsilon_1 \cdot \Pi_{\text{scal}} \cdot \varepsilon_2, \tag{2.30}$$

where $\Pi^{\mu\nu}_{\text{scal}}$ is given by:

$$\begin{aligned}\Pi^{\mu\nu}_{\text{scal}}(k) &= e^2(\delta^{\mu\nu}k^2 - k^\mu k^\nu)\int_0^\infty \frac{dT}{T}(4\pi T)^{-D/2}e^{-m^2T} \\ &\times \int_0^T d\tau_1 \int_0^T d\tau_2\dot{G}_{12}\dot{G}_{21}e^{-G_{12}k^2}. \end{aligned} \tag{2.31}$$

Re-scaling to the unitary circle by introducing an dimensionless parameter $\tau_i = Tu_i$, and fixing the zero in the second vertex operator, due to the translation invariance in the loop, leading us to $u_2 = 0$, $u_1 = u$:

$$G(\tau_1, \tau_2) = Tu(1 - u), \quad \dot{G}(\tau_1, \tau_2) = 1 - 2u. \tag{2.32}$$

Now we can re-write equation 2.31 as:

$$\Pi^{\mu\nu}_{\text{scal}} = -\frac{e^2}{(4\pi)^{D/2}}(\delta^{\mu\nu}k^2 - k^\mu k^\nu)\int_0^\infty \frac{dT}{T}e^{-m^2T}T^{2-D/2}\int_0^1 du(1 - 2u)^2e^{-Tu(1-u)k^2}. \tag{2.33}$$

Performing the integral over T:

$$\Pi^{\mu\nu}_{\text{scal}} = -\frac{e^2}{(4\pi)^{D/2}}(\delta^{\mu\nu}k^2 - k^\mu k^\nu)\Gamma\left(2-\frac{D}{2}\right)\int_0^1 du(1-2u)^2[m^2+(1-u)uk^2]^{\frac{D}{2}-2}. \quad (2.34)$$

It is important to notice that if we try to substitute $D = 4$, equation 2.34 is divergent. To solve this problem, we are going to introduce a small parameter ϵ into the dimension $D = 4 - 2\epsilon$. Doing this and expanding around $\epsilon = 0$, we pass to the result showed in equation 1.92.

2.3.2 Furry's Theorem

To show how Furry's theorem works in the worldline fomalism we need to remember the *Bern-Kosower master formula* for $N = 3$ where Q_3 appears:

$$\Gamma_{scal}[k_1,\varepsilon_1;k_2,\varepsilon_2,k_3,\varepsilon_3] = (-ie)^3(2\pi)^D\delta(\Sigma k_i)\int_0^\infty\frac{dT}{T}(4\pi T)^{-\frac{D}{2}}e^{-m^2T}\times$$
$$\times\int_0^T d\tau_1\int_0^T d\tau_2\int_0^T d\tau_3(-i)^3 Q_3(\dot{G}_{ij})\exp\left(G_{12}k_1\cdot k_2 + G_{13}k_1\cdot k_3 + G_{23}k_2\cdot k_3\right) \quad (2.35)$$

We calculate Q_3 using the expansion in equation 2.19 and integrating by parts, splitting it into two contributions:

$$Q_3^3 = \dot{G}_{12}\dot{G}_{23}\dot{G}_{31}\text{tr}(f_1f_2f_3),$$
$$Q_3^2 = \dot{G}_{12}\dot{G}_{21}\frac{1}{2}\text{tr}(f_1f_2)T(3) + \dot{G}_{13}\dot{G}_{31}\frac{1}{2}\text{tr}(f_1f_3)T(2) + \dot{G}_{23}\dot{G}_{32}\frac{1}{2}\text{tr}(f_2f_3)T(1). \quad (2.36)$$

Expanding the tails in Q_3^2:

$$\begin{aligned} Q_3^2 &= \dot{G}_{12}\dot{G}_{21}(\dot{G}_{31}\varepsilon_3\cdot k_1 + \dot{G}_{32}\varepsilon_3\cdot k_2)\frac{1}{2}\text{tr}(f_1f_2) + \\ &+ \dot{G}_{13}\dot{G}_{31}(\dot{G}_{21}\varepsilon_2\cdot k_1 + \dot{G}_{23}\varepsilon_2\cdot k_3)\frac{1}{2}\text{tr}(f_1f_3) + \\ &+ \dot{G}_{23}\dot{G}_{32}(\dot{G}_{12}\varepsilon_1\cdot k_2 + \dot{G}_{13}\varepsilon_1\cdot k_3)\frac{1}{2}\text{tr}(f_2f_3). \end{aligned} \quad (2.37)$$

Let's see the structure of G_{ij} with the rescaling $\tau_i = u_iT$:

$$G_{ij} = |\tau_i - \tau_j| - \frac{(\tau_i-\tau_j)^2}{T} = T[|u_i - u_j| - (u_i-u_j)^2]. \quad (2.38)$$

And $\dot{G}_{ij}$:

$$\dot{G}_{ij} = \text{sgn}(\tau_i - \tau_j) - 2\frac{(\tau_i - \tau_j)}{T} = \text{sgn}(u_i - u_j) - 2(u_i - u_j). \tag{2.39}$$

Notice that, there are three $\dot{G}$ in each term of Q_3, we need to exploit this property to calculate the integral. We know G_{ij} is a periodic and even function in the interval $[0, 1]$, this means it will be invariant under the transformation $u_k \to (1 - u_k)$. The consequence is in $\dot{G}_{ij}$, if we perform this transfomation, it brings us an extra minus sign, for example:

$$\dot{G}_{12} = \text{sgn}(u_1 - u_2) - 2(u_1 - u_2) \to \text{sgn}(-u_1 + u_2) - 2(-u_1 + u_2) = -\dot{G}_{12}. \tag{2.40}$$

Taking the first term Q_3^3 we can see its transformation:

$$Q_3^3 = \dot{G}_{12}\dot{G}_{23}\dot{G}_{31}\text{tr}(f_1 f_2 f_3) \to -Q_3^3. \tag{2.41}$$

Since the trace of the f functions does not depend on u_k. Also, this happens with Q_3^2 because the existence of three $\dot{G}$ in all the terms, so we can convince ourself of the transformation $Q_3 \to -Q_3$. But here we are dealing with the same argument for the same integral, this means that the only possible result for the integral is zero. We can generalize this result for any odd N, but it does not work for even numbers, this is because an even number brings to us an even quantity of minus signs wich is equal to +1.

2.3.3 Four-photon Scattering

At this point, we have to treat the four-photon scattering problem. In order to do that, using the master formula with $N = 4$ and expanding P_4 from 2.19, integrating by parts to obtain Q_4, we shall get the following expression:

$$\begin{aligned}
Q_4 &= Q_4^4 + Q_4^3 + Q_4^2 + Q_4^{22} \\
Q_4^4 &= \dot{G}(1234) + \dot{G}(1243) + \dot{G}(1324) \\
Q_4^3 &= \dot{G}(123)T(4) + \dot{G}(234)T(1) + \dot{G}(341)T(2) + \dot{G}(412)T(3) \\
Q_4^2 &= \dot{G}(12)T(34) + \dot{G}(13)T(24) + \dot{G}(14)T(23) + \dot{G}(23)T(14) + \dot{G}(24)T(13) + \dot{G}(34)T(12) \\
Q_4^{22} &= \dot{G}(12)\dot{G}(34) + \dot{G}(13)\dot{G}(24) + \dot{G}(14)\dot{G}(23)
\end{aligned} \tag{2.42}$$

Where there are a new kind of structure called *N-cycle*, and the previously mentioned one-tail and a new kind of tail called *two-tail*:

$$\dot{G}(i_1, i_2 \ldots i_N) = \dot{G}_{i_1,i_2}\dot{G}_{i_2 i_3}\ldots\dot{G}_{i_N i_1}Z(i_1 i_2 \cdots {}_N) \tag{2.43}$$

$$T(a) = \dot{G}_{ai}\varepsilon_a \cdot k_i \tag{2.44}$$

$$T(ab) = \sum_{r,s;(r,s)\neq(b,a)} \dot{G}_{ar}\varepsilon_a \cdot k_r \dot{G}_{bs}\varepsilon_b \cdot k_s + \frac{1}{2}\dot{G}_{ab}\varepsilon_a \cdot \varepsilon_b(\sum_{r\neq b}\dot{G}_{ar}k_a \cdot k_r - \sum_{s\neq a}\dot{G}_{bs}k_b \cdot k_s) \tag{2.45}$$

However, despite the fact that this representation is compact and very organized, it could be optimized into a better structure where we separate the information of the tails and Lorentz-cycles into more general objects and the information about the worldline Green's function in another one. We shall discuss this in the next chapter.

Chapter 3

Four-photon Scattering in the Worldline Formalism

3.1 A First Example: The Scalar Case ϕ^3

In order to show how the method works, we shall begin with a similar calculation for $N = 4$ in the scalar case ϕ^3. In this theory, the master formula is simpler than scalar QED, but for educative purpose it will be enough. First of all, the amplitude representation for the one-loop scalar N-point is given by:

$$\begin{aligned} I_N(k_1 \cdots k_N) &= \frac{(2\pi)^D}{2(4\pi)^{D/2}} \delta\left(\sum_{i=1}^{N} k_i\right) \hat{I}_N(k_1 \cdots k_N) \\ \hat{I}_N(k_1 \cdots k_N) &= \int_0^\infty \frac{dT}{T} T^{-D/2} e^{-m^2 T} \int_0^T d\tau_1 \cdots \int_0^T d\tau_N exp\left[\sum_{i,j=1}^{N} \frac{1}{2} G_{ij} k_i \cdot k_j\right]. \end{aligned} \tag{3.1}$$

Here we must make an important clarification, regarding the integrand for the scalar case, we can appreciate that $\dot{G}_{ij}$'s do not appear in it. This makes the calculations easier to perform, unlike the case for scalar QED, in which, as we will see later on, we will have the appearance of these functions. Doing the rescaling $\tau_i = u_i T$, we get:

$$\hat{I}_N = \int_0^\infty \frac{dT}{T} T^{N-D/2} e^{-m^2 T} \int_0^1 du_1 \ldots du_N exp\left[T \sum_{i,j=1}^{N} \frac{1}{2} G_{ij} k_i \cdot k_j\right]. \tag{3.2}$$

Henceforth, we shall denote the exponential argument as:

$$\Lambda = \frac{T}{2} \sum_{i,j=1}^{4} G_{ij} k_i \cdot k_j. \tag{3.3}$$

Choosing $N = 4$ in equation 3.2, we get:

$$\hat{I}_4 = \int_0^\infty \frac{dT}{T} T^{4-D/2} e^{-m^2 T} \int_0^1 du_1 \ldots du_4 e^{\Lambda}. \tag{3.4}$$

3.1.1 The Mandelstam Variables

For the current calculations, we shall study the on-shell case, thus it will be useful to introduce the Mandelstam variables:

$$\begin{aligned} s &= -(k_1 + k_2)^2, \\ t &= -(k_1 + k_3)^2, \\ u &= -(k_1 + k_4)^2. \end{aligned} \tag{3.5}$$

In order to have only photons as external particles, the Mandelstam variables must satisfy

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2 = 0. \tag{3.6}$$

Recalling the Green functions G_{ij}:

$$G_{ij} \equiv |u_i - u_j| - (u_i - u_j)^2 \tag{3.7}$$

Writing Λ in terms of the variables u_i and the Mandelstam variables:

$$\Lambda = -\frac{T}{2} \left[(G_{12} + G_{23})s + (G_{13} + G_{24})t + (G_{14} + G_{23})u \right]. \tag{3.8}$$

In terms of the explicit functions G_{ij} and the Mandelstam variables:

$$\begin{aligned} \Lambda = -\frac{T}{2}[(|u_{12}| + 2u_1u_2 + |u_{34}| + 2u_3u_4)s &+ (|u_{13}| + 2u_1u_3 + |u_{24}| + 2u_2u_4)t \\ &+ (|u_{14}| + 2u_1u_4 + |u_{23}| + 2u_2u_3)u]. \end{aligned} \tag{3.9}$$

Here is necessary to remark the notation $u_{ij} = u_i - u_j$. Also, it is important to notice that the condition for the Mandelstam variables removes the quadratic terms for u_i. This is good for our integrals, because the quadratic integrals in a finite interval of integration give us error functions and they are not good functions for our calculations. Instead, we have linear terms in u_i and the integral is tractable in our interval.

3.2 Integration of the Amplitude

Our task is now to perform the integrals in equation 3.4, with the exponent Λ in equation 3.9. In order to do that, we shall start integrating over u_4, fixing the ordering for the legs $u_1 > u_2 > u_3$ and split the integral over u_4 into four contributions:

$$\int_0^1 du_4 = \int_0^{u_3} du_4 + \int_{u_3}^{u_2} du_4 + \int_{u_2}^{u_1} du_4 + \int_{u_1}^1 du_4. \tag{3.10}$$

Is then easy to see the result for the integrals:

$$\begin{aligned}
\int_0^{u_3} e^{\Lambda} du_4 &= \frac{1}{T[u_{23}s - u_{12}u]} \left\{ e^{(1-u_{13})u_{23}sT} - e^{(1-u_1)u_{23}sT + u_3 u_{12}uT} \right\} \\
\int_{u_3}^{u_2} e^{\Lambda} du_4 &= \frac{1}{T[(u_{23}-1)s - u_{12}u]} \left\{ e^{u_{12}u_{23}tT} - e^{(1-u_{13})u_{23}sT} \right\} \\
\int_{u_2}^{u_1} e^{\Lambda} du_4 &= \frac{1}{T[u_{23}s + (1-u_{12})u]} \left\{ e^{(1-u_{13})u_{12}uT} - e^{u_{12}u_{23}tT} \right\} \\
\int_{u_2}^{u_1} e^{\Lambda} du_4 &= \frac{1}{T[u_{23}s - u_{12}u]} \left\{ e^{(1-u_1)u_{23}sT + u_3 u_{12}uT} - e^{(1-u_{13})u_{12}uT} \right\},
\end{aligned} \tag{3.11}$$

we can notice an important fact on the integrals, the contributions from $u_4 = 0$ and $u_4 = 1$ vanishes, this is something expected because we are integrating a periodic function around the loop, whereby $u_4 = 1$ and $u_4 = 0$ are identified as the same point. We can add the four contributions and group by the same exponential factor, re-writing in terms of G_{ij} and $\dot{G}_{ij}$ as follows:

$$\begin{aligned}
\int_0^1 e^{\Lambda} du_4 &= \left\{ \frac{2}{T[s - \dot{G}_{31}u - \dot{G}_{32}t]} + \frac{2}{T[s + \dot{G}_{31}u + \dot{G}_{32}t]} \right\} e^{\frac{1}{2}(G_{13}+G_{23}-G_{12})sT} \\
&+ \left\{ \frac{2}{T[t - \dot{G}_{23}s - \dot{G}_{21}u]} + \frac{2}{T[t + \dot{G}_{23}s + \dot{G}_{21}u]} \right\} e^{\frac{1}{2}(G_{12}+G_{23}-G_{13})tT} \\
&+ \left\{ \frac{2}{T[u - \dot{G}_{12}t - \dot{G}_{13}s]} + \frac{2}{T[u + \dot{G}_{12}t + \dot{G}_{13}s]} \right\} e^{\frac{1}{2}(G_{12}+G_{13}-G_{23})uT}.
\end{aligned} \tag{3.12}$$

Writing our result in terms of G_{ij} and $\dot{G}_{ij}$ we manifest translation invariance and we can take advantage of it, by doing $u_3 = 0$ and choosing one of the remaining orderings $u_1 > u_2$ or $u_2 > u_1$. It is important to notice that we can pass from one to another by the change:

$$u_i \rightarrow 1 - u_i. \tag{3.13}$$

This transformation reverses the flow of proper time in the loop. We can take only one of the orderings and multiply by a factor of two, instead of calculate both options. Doing $u_3 = 0$ and performing the integral over T:

$$\begin{aligned} &\frac{1}{m^2 - \frac{1}{2}(G_{13} + G_{23} - G_{12})s}\left\{\frac{2}{[s - \dot{G}_{31}u - \dot{G}_{32}t]} + \frac{2}{[s + \dot{G}_{31}u + \dot{G}_{32}t]}\right\} \\ + &\frac{1}{m^2 - \frac{1}{2}(G_{12} + G_{23} - G_{13})t}\left\{\frac{2}{[t - \dot{G}_{23}s - \dot{G}_{21}u]} + \frac{2}{[t + \dot{G}_{23}s + \dot{G}_{21}u]}\right\} \\ + &\frac{1}{m^2 - \frac{1}{2}(G_{12} + G_{13} - G_{23})u}\left\{\frac{2}{[u - \dot{G}_{12}t - \dot{G}_{13}s]} + \frac{2}{[u + \dot{G}_{12}t + \dot{G}_{13}s]}\right\}. \end{aligned} \tag{3.14}$$

Doing $u_3 = 0$ and choosing $u_1 > u_2$, we can integrate over u_2, from 0 to u_1. We shall calculate only the integral for the s-sector and we can obtain the rest of results from it by the cyclic permutations $s \to t \to u \to s$. Using partial fraction decomposition, we get:

$$\begin{aligned} \int_0^{u_1} \frac{du_2}{(m^2 - s(1-u_1)u_2)\,(su_2 - u_{12}u)} &= \frac{\ln\left(1 - \frac{s(1-u_1)u_1}{m^2}\right) - \ln\left(-\frac{s}{u}\right)}{m^2 t + (1-u_1)u_1 su} \\ \int_0^{u_1} \frac{1}{(m^2 - s(1-u_1)u_2)\,(s(1-u_2) + u_{12}u)} &= \frac{\ln\left(\frac{(1-u_1)s}{s+u_1 u}\right) - \ln\left(1 - \frac{s(u_1-1)u_1}{m^2}\right)}{m^2 t + (1-u_1)^2 s^2 - (1-u_1)u_1 st} \end{aligned} \tag{3.15}$$

The last integral can be performed by completing the square into the denominator and again splitting in partial fractions, after that we can apply the formula:

$$\int dx \frac{\ln(x+b)}{x+a} = \ln(x+b)\ln\left(\frac{x+a}{a-b}\right) + Li_2\left(\frac{x+b}{b-a}\right). \tag{3.16}$$

Where $Li_2(x)$ is known as the Dilogarithm function*.

3.3 Optimized Representation of the Four Photon Scattering Amplitude

As we saw in the last chapter, there is a Q-representation for the four-photon scattering amplitude. But it could be improved by defining a new set of structures where we can detach the information of the tails and cycles and also define another structure, where we have the worldline Green's function information. In order to do this, we need to define a

*See appendix A.

new kind of tail, called *improved one-tail*:

$$T_R(i)e^{(\cdot)} = T(i)e^{(\cdot)} - \partial_i\left(\frac{r_i \cdot \varepsilon_i}{r_i \cdot k_i}e^{(\cdot)}\right), \tag{3.17}$$

where r_i is a *reference vector* and $e^{(\cdot)}$ is the same as in equation 2.19. It is important to realize that the introduction of the total derivative integrates to zero around the loop, which is why we can introduce it without changing our answer. It can be demostrated that we can express this improved tail as:

$$T_R(i) = \sum_{j\neq i}\dot{G}_{ij}\frac{r_i \cdot f_i \cdot k_j}{r_i \cdot k_i}. \tag{3.18}$$

Also, by adding total derivative terms to the two-tail, we can define a *short two-tail*:

$$\begin{aligned} T_{sh}(ij)e^{(\cdot)} &= T(ij)e^{(\cdot)} + \frac{1}{k_i \cdot k_j}\Big[\varepsilon_i \cdot \varepsilon_j\partial_i\partial_j e^{(\cdot)} - \varepsilon_i \cdot k_j\varepsilon_j \cdot k_s\partial_i(\dot{G}_{js}e^{(\cdot)}) \\ &- \varepsilon_j \cdot k_i\varepsilon_i \cdot k_r\partial_j(\dot{G}_{ir}e^{(\cdot)}) + \left(\frac{1}{2}\varepsilon_i \cdot \varepsilon_j k_i \cdot k_j - \varepsilon_i \cdot k_j\varepsilon_j \cdot k_i\right)(\partial_i - \partial_j)(\dot{G}_{ij}e^{(\cdot)})\Big]. \end{aligned} \tag{3.19}$$

And also, it can be demostrated that we can write it as:

$$T_{sh}(ij) = \sum_{r,s\neq a,b}\dot{G}_{ra}\dot{G}_{bs}\frac{k_r \cdot f_a \cdot f_b \cdot k_s}{k_a \cdot k_b}. \tag{3.20}$$

We can notice that now the tails split into a factorized part which only involves photon momentum and the photon field strength tensor, and other with the derivatives of the Green's functions. With the help of this new information, we can re-write the Q-representation into a new one with a better organization:

$$\Gamma_{scal} = \Gamma^{(1)}_{scal} + \Gamma^{(2)}_{scal} + \Gamma^{(3)}_{scal} + \Gamma^{(4)}_{scal} + \Gamma^{(5)}_{scal}, \tag{3.21}$$

$$\begin{aligned}
\Gamma^{(1)}_{scal} &= \Gamma^{scal}_{(1234)}T^{(1)}_{(1234)} + \Gamma^{scal}_{(1243)}T^{(1)}_{(1243)} + \Gamma^{scal}_{(1324)}T^{(1)}_{(1324)},\\
\Gamma^{(2)}_{scal} &= \Gamma^{scal}_{(12)(34)}T^{(2)}_{(12)(34)} + \Gamma^{scal}_{(13)(24)}T^{(2)}_{(13)(24)} + \Gamma^{scal}_{(14)(23)}T^{(2)}_{(14)(23)},\\
\Gamma^{(3)}_{scal} &= \sum_{i=1,2,3}\Gamma^{scal}_{(123)i}T^{(3)r_4}_{(123)i} + \sum_{i=2,3,4}\Gamma^{scal}_{(234)i}T^{(3)r_1}_{(234)i} + \sum_{i=3,4,1}\Gamma^{scal}_{(341)i}T^{(3)r_2}_{(341)i} + \sum_{i=4,1,2}\Gamma^{scal}_{(412)i}T^{(3)r_3}_{(412)i},\\
\Gamma^{(4)}_{scal} &= \sum_{i<j}\Gamma^{scal}_{(ij)ii}T^{(4)}_{(ij)ii} + \sum_{i<j}\Gamma^{scal}_{(ij)jj}T^{(4)}_{(ij)jj},\\
\Gamma^{(5)}_{scal} &= \sum_{i<j}\Gamma^{scal}_{(ij)ij}T^{(5)}_{(ij)ij} + \sum_{i<j}\Gamma^{scal}_{(ij)ji}T^{(5)}_{(ij)ji}.
\end{aligned} \tag{3.22}$$

The tensors T are given by:

$$\begin{aligned}
T^{(1)}_{(1234)} &= Z(1234),\\
T^{(2)}_{(12)(34)} &= Z(12)Z(34),\\
T^{(3)r_4}_{(123)i} &= Z(123)\frac{r_4\cdot f_4\cdot k_i}{r_4\cdot k_4},\\
T^{(4)}_{(12)11} &= Z(12)\frac{k_1\cdot f_3\cdot f_4\cdot k_1}{k_3\cdot k_4},\\
T^{(5)}_{(12)12} &= Z(12)\frac{k_1\cdot f_3\cdot f_4\cdot k_2}{k_3\cdot k_4}.
\end{aligned} \tag{3.23}$$

Each $\Gamma^{scal}_{...}$ is given by:

$$\Gamma^{scal}_{...} = \frac{e^4}{(4\pi)^{\frac{D}{2}}}\int_0^\infty \frac{dT}{T}T^{4-\frac{D}{2}}e^{-m^2T}\int_0^1 du_1\cdots\int_0^1 du_4\gamma^{scal}_{...}(\dot{G}_{ij})e^{\Lambda} \tag{3.24}$$

Where $\Lambda = \frac{T}{2}\sum_{i,j=1}^{4} G_{ij}k_i\cdot k_j$ and the terms $\gamma^{scal}_{...}(\dot{G}_{ij})$ are:

$$\begin{aligned}
\gamma^{scal}_{(1234)} &= \dot{G}_{12}\dot{G}_{23}\dot{G}_{34}\dot{G}_{41}\\
\gamma^{scal}_{(12)(34)} &= \dot{G}_{12}\dot{G}_{21}\dot{G}_{34}\dot{G}_{43}\\
\gamma^{scal}_{(123)i} &= \dot{G}_{12}\dot{G}_{23}\dot{G}_{31}\dot{G}_{4i}\\
\gamma^{scal}_{(12)11} &= \dot{G}_{12}\dot{G}_{21}\dot{G}_{13}\dot{G}_{41}\\
\gamma^{scal}_{(12)12} &= \dot{G}_{12}\dot{G}_{21}\dot{G}_{13}\dot{G}_{42}
\end{aligned} \tag{3.25}$$

Our task is to integrate the terms in equation 3.22 with the usage of equations 3.24 and 3.25.

3.4 Integration of the Amplitude

Now, we are going to integrate one structure of each $\Gamma^{scal}_{...}$. The procedure for the integration is as follows:

- Integrate over u_4, splitting the integral from 0 to 1 into four contributions.
- Sum all the previous contributions and re-write the result in terms of G_{ij} and $\dot{G}_{ij}$ to make clearer translational invariance around the circle.
- Integrate over T.
- Due to the translational invariance, we have the freedom to set any u_i variable equals to zero, this is not affecting the final result. In our case, to conserve the symmetry between sectors, we are going to do zero a different variable for each one, we shall discuss this point later.
- Integrate over one of the two remaining variables, depending on which integral is easier to perform.
- Integrate over the last variable. This last step shall be made numerically, this is due to the difficult to integrate analytically functions as *PolyLog*(n, x)[†].
- For this last numeric integration, we are going to attach a *Mathematica file* to this thesis, where the integration is done.

We shall begin from the easiest structure and we shall finish with the most difficult one.

3.4.1 Integration of $\gamma_{(123)1}$

For the current integration we begin by using the ordering $1 > u_1 > u_2 > u_3 > 0$. This ordering is not special from the rest of options, the particular reason to choose it, is the form of the structure and the quantity of u_4 appearing. As we mention before, we are going to split the integral into four contributions:

$$\int_0^1 du_4 = \int_0^{u_3} du_4 + \int_{u_3}^{u_2} du_4 + \int_{u_2}^{u_1} du_4 + \int_{u_1}^{1} du_4. \tag{3.26}$$

In this way, we are covering the whole circle. The first calculation we can perform is the u_4 integral for $\gamma^{scal}_{(123)1}$:

$$\dot{G}_{12}\dot{G}_{23}\dot{G}_{31} \int_0^1 \dot{G}_{41} e^{\Lambda} du_4 = \dot{G}_{12}\dot{G}_{23}\dot{G}_{31} \int_0^1 (\sigma_{41} - 2u_{41}) e^{\Lambda} du_4. \tag{3.27}$$

[†]See appendix A.

Which involve powers of u_4 in its structure, but they are linear in this variable. By writing explicitly the $\dot{G}_{ij}$, and by now dropping off the prefactor $\dot{G}_{12}\dot{G}_{23}\dot{G}_{31}$, it is straightforward to show that:

$$\begin{aligned}
\int_0^{u_3} \dot{G}_{41} e^{\Lambda} du_4 &= \left\{ \frac{2u_{13}-1}{T[u_{23}s-u_{12}u]} + \frac{2}{T^2[u_{23}s-u_{12}u]^2} \right\} e^{(1-u_{13})u_{23}sT} \\
&- \left\{ \frac{2u_1-1}{T[u_{23}s-u_{12}u]} + \frac{2}{T^2[u_{23}s-u_{12}u]^2} \right\} e^{(1-u_1)u_{23}sT+u_3u_{12}uT} \\
\int_{u_3}^{u_2} \dot{G}_{41} e^{\Lambda} du_4 &= \left\{ \frac{2u_{12}-1}{T[(u_{23}-1)s-u_{12}u]} + \frac{2}{T^2[(u_{23}-1)s-u_{12}u]^2} \right\} e^{u_{12}u_{23}tT} \\
&- \left\{ \frac{2u_{13}-1}{T[(u_{23}-1)s-u_{12}u]} + \frac{2}{T^2[(u_{23}-1)s-u_{12}u]^2} \right\} e^{(1-u_{13})u_{23}sT} \\
\int_{u_2}^{u_1} \dot{G}_{41} e^{\Lambda} du_4 &= \left\{ -\frac{2u_{12}-1}{T[u_{23}s+(1-u_{12})u]} - \frac{2}{T^2[u_{23}s+(1-u_{12})u]^2} \right\} e^{u_{12}u_{23}tT} \\
&+ \left\{ -\frac{1}{T[u_{23}s+(1-u_{12})u]} + \frac{2}{T^2[u_{23}s+(1-u_{12})u]^2} \right\} e^{(1-u_{13})u_{12}uT} \\
\int_{u_1}^{1} \dot{G}_{41} e^{\Lambda} du_4 &= \left\{ -\frac{1}{T[u_{23}s-u_{12}u]} - \frac{2}{T^2[u_{23}s-u_{12}u]^2} \right\} e^{(1-u_{13})u_{12}uT} \\
&+ \left\{ \frac{2u_1-1}{T[u_{23}s-u_{12}u]} + \frac{2}{T^2[u_{23}s-u_{12}u]^2} \right\} e^{(1-u_1)u_{23}sT+u_3u_{12}uT} .
\end{aligned} \tag{3.28}$$

As in the scalar case, we notice that the contributions from $u_4 = 0$ and $u_4 = 1$ vanishes. We can write this integral in terms of the Green's functions G_{ij} and $\dot{G}_{ij}$ as follows:

$$\begin{aligned}
\int_0^1 \dot{G}_{41} e^{\Lambda} du_4 \;=\; & \left\{ -\frac{2\dot{G}_{13}}{T[s-\dot{G}_{31}u-\dot{G}_{32}t]} + \frac{8}{T^2[s-\dot{G}_{31}u-\dot{G}_{32}t]^2} \right. \\
- & \left. \frac{2\dot{G}_{13}}{T[s+\dot{G}_{31}u+\dot{G}_{32}t]} - \frac{8}{T^2[s+\dot{G}_{31}u+\dot{G}_{32}t]^2} \right\} e^{\frac{1}{2}(G_{13}+G_{23}-G_{12})sT} \\
+ & \left\{ -\frac{2\dot{G}_{12}}{T[t-\dot{G}_{23}s-\dot{G}_{21}u]} + \frac{8}{T^2[t-\dot{G}_{23}s-\dot{G}_{21}u]^2} \right. \\
- & \left. \frac{2\dot{G}_{12}}{T[t+\dot{G}_{23}s+\dot{G}_{21}u]} - \frac{8}{T^2[t+\dot{G}_{23}s+\dot{G}_{21}u]^2} \right\} e^{\frac{1}{2}(G_{12}+G_{23}-G_{13})tT} \\
+ & \left\{ -\frac{2}{T[u-\dot{G}_{12}t-\dot{G}_{13}s]} + \frac{8}{T^2[u-\dot{G}_{12}t-\dot{G}_{13}s]^2} \right. \\
+ & \left. \frac{2}{T[u+\dot{G}_{12}t+\dot{G}_{13}s]} - \frac{8}{T^2[u+\dot{G}_{12}t+\dot{G}_{13}s]^2} \right\} e^{\frac{1}{2}(G_{12}+G_{13}-G_{23})uT}
\end{aligned} \tag{3.29}$$

From now, we are going to call to the first part of equation 3.29 s-sector, the second t-sector and to the third u-sector, this is due to the Mandelstam variable appearing in the exponential factor. Equation 3.29 makes explicit the translation invariance in proper-time. In the next step, we shall do a variable u_i equals to zero, and perform the integral over T. It is important to notice that the integrals with $\frac{1}{T}$ are trivial to do, but the integrals with factors $\frac{1}{T^2}$ need a special treatment, because if we attempt to integrate these terms directly, we get a divergent result. The general structure of the T integral is:

$$\int_0^\infty \frac{dT}{T} T^{4-\frac{D}{2}} \frac{B}{T^a} e^{-AT} \tag{3.30}$$

The procedure is to re-write the integral using the definition of the gamma function. By doing a variable change $y = AT$ and choosing $D = 4 - 2\epsilon$, to avoid and control the divergence near to a pole, we will have as result:

$$\int_0^\infty \frac{dT}{T} T^{4-\frac{D}{2}} \frac{B}{T^a} e^{-AT} = A^{a-\epsilon-2} B\Gamma(2-a+\epsilon) \tag{3.31}$$

In our case, for $a = 2$:

$$\int_0^\infty \frac{dT}{T} T^{4-\frac{D}{2}} \frac{B}{T^2} e^{-AT} = A^{-\epsilon} B\Gamma(\epsilon) \tag{3.32}$$

Expanding around the pole ϵ:

$$A^{-\epsilon}B\Gamma(\epsilon) \approx B[\frac{1}{\epsilon} - (\gamma_E + \ln A)] + O(\epsilon) \tag{3.33}$$

Where γ_E is the Euler constant. It is important to remark the fact that as result of evaluating at $\epsilon = 0$, all the divergences $1/\epsilon$ must vanish, since the amplitude is finite. Now we can do the integral over T for equation 3.29 we get:

$$\begin{aligned}
& \frac{-2\dot{G}_{13}}{m^2 - \frac{1}{2}(G_{13}+G_{23}-G_{12})s}\left\{\frac{1}{s-\dot{G}_{31}u-\dot{G}_{32}t} + \frac{1}{s+\dot{G}_{31}u+\dot{G}_{32}t}\right\} \\
+ \; & \frac{-2\dot{G}_{12}}{m^2 - \frac{1}{2}(G_{12}+G_{23}-G_{13})t}\left\{\frac{1}{t-\dot{G}_{23}s-\dot{G}_{21}u} + \frac{1}{t+\dot{G}_{23}s+\dot{G}_{21}u}\right\} \\
+ \; & \frac{-2}{m^2 - \frac{1}{2}(G_{12}+G_{13}-G_{23})u}\left\{\frac{1}{u-\dot{G}_{12}t-\dot{G}_{13}s} - \frac{1}{u+\dot{G}_{12}t+\dot{G}_{13}s}\right\} \\
+ \; & \left\{\frac{-8}{[s-\dot{G}_{31}u-\dot{G}_{32}t]^2} + \frac{8}{[s+\dot{G}_{31}u+\dot{G}_{32}t]^2}\right\}\ln(m^2 - \frac{1}{2}(G_{13}+G_{23}-G_{12})s) \\
+ \; & \left\{\frac{-8}{[t-\dot{G}_{23}s-\dot{G}_{21}u]^2} + \frac{8}{[t+\dot{G}_{23}s+\dot{G}_{21}u]^2}\right\}\ln(m^2 - \frac{1}{2}(G_{12}+G_{23}-G_{13})t) \\
+ \; & \left\{\frac{-8}{[u-\dot{G}_{12}t-\dot{G}_{13}s]^2} + \frac{8}{[u+\dot{G}_{12}t+\dot{G}_{13}s]^2}\right\}\ln(m^2 - \frac{1}{2}(G_{12}+G_{13}-G_{23})u).
\end{aligned} \tag{3.34}$$

Now, the next task is to perform another integral, but for any of the three remaining variables. Once, reinstated the missing $\dot{G}$'s, each of the lines is individually invariant under the transformation $u_i \to 1 - u_i$, giving us the two terms in brackets are exchanged, this is called inversion symmetry. As we saw before, we recover translational invariance and that let us choose one of the variables equals to zero. Also, the inversion symmetry above lets us fix $u_3 = 0$ and one of the orderings $u_1 > u_2$ or $u_2 > u_1$ and multiply by a global factor of 2 because of our choosing. Now our integral will be:

$$2\int_0^{u_1} f(u_1, u_2)du_2. \tag{3.35}$$

In our case, to conserve the symmetry between sectors, our choosing is set $u_3 = 0$ and $u_1 > u_2$ for s-sector, $u_2 = 0$ and $u_1 > u_3$ for t-sector and $u_1 = 0$ and $u_2 > u_3$ for u-sector. However, choosing any other ordering and variable that is made zero does not affect the result due to translational invariance. This way to perform the integrals will be used in all the remaining structures. Analogous to the scalar case, the integrals are made by decomposition into partial fractions and integrate term by term.

3.4.2 Spurious Poles

To calculate the penultimate integral, we will encounter some spurious poles and it is necessary to control them. The method used to avoid them is described below.

Consider from partial fraction decomposition of the t-sector in equation 3.34, we obtain the next term:

$$\frac{-8s^2u_1^3 + 12s^2u_1^2 - 6s^2u_1 + s^2 - 16stu_1^4 + 16stu_1^3 - 4stu_1^2 - 16t^2u_1^4 + 16t^2u_1^3 - 4t^2u_1^2}{s\left(m^2s + stu_1^2 - stu_1 + t^2u_1^2 - t^2u_1\right)(-su_1 + su_3 - tu_1)}. \tag{3.36}$$

We can notice that there is a pole in:

$$u_3 = \frac{s+t}{s}u_1 \tag{3.37}$$

To avoid it, we are going to split our integral into two intevarls surrounding the divergence:

$$\int_0^{u_1} du_3 = \int_0^{\frac{s+t}{s}u_1-\epsilon} du_3 + \int_{\frac{s+t}{s}u_1+\epsilon}^{u_1} du_3, \tag{3.38}$$

and after that we can expand the result in powers of ϵ around zero, obtaining with this the result controling the pole, this process is called Cauchy principal value method. If we decide not to control the divergence, we shall obtaing a divergent result or a complex number as result for the last integral, by avoiding it, the result will be a finite real number.

At this point, we are ready to compute the integral over the penultimate variable and due to the extent of the result, it is shown in Appendix B. Once done, we can take the expression for the last integral and calculate it by numerical methods, since the complexity of the functions we choose that *Mathematica* must decide which is the best method to integrate. We shall discuss the numeric part eventually.

3.4.3 Integration of $\gamma_{(12)11}^{scal}$

For the integration of $\gamma_{(12)11}^{scal}$, we are going to have:

$$\dot{G}_{12}\dot{G}_{21}\dot{G}_{13}\int_0^1 \dot{G}_{41}e^{\Lambda}du_4 = \dot{G}_{12}\dot{G}_{21}\dot{G}_{13}\int_0^1 (\sigma_{41} - 2u_{41})e^{\Lambda}du_4. \tag{3.39}$$

We can realize that the integrals for u_4 and T are exactly the same for the previous structure. This means that we do not need to do new integrals and the result is given by equation 3.34. To calculate the rest of integrals, we have to take back the prefactor $\dot{G}_{12}\dot{G}_{21}\dot{G}_{13}$, and perform the integral over one of the variables, following the same argument of conserve symmetry between sectors. We have to make it clear that the final result will be different from the previous structure because of the prefactor. Again, we present the result for this last integral in Appendix C.

3.4.4 Integration of $\gamma^{scal}_{(12)12}$

The first calculation we can perform is the u_4 integral for $\gamma^{scal}_{(123)2}$:

$$\dot{G}_{12}\dot{G}_{21}\dot{G}_{13}\int_0^1 \dot{G}_{42}e^{\Lambda}du_4 = \dot{G}_{12}\dot{G}_{21}\dot{G}_{13}\int_0^1 (\sigma_{42}-2u_{42})e^{\Lambda}du_4. \tag{3.40}$$

By writing explicitly the $\dot{G}_{ij}$ and by now dropping off the prefactor $\dot{G}_{12}\dot{G}_{21}\dot{G}_{13}$, it is straightforward to show that:

$$\begin{aligned}
\int_0^{u_3}\dot{G}_{42}e^{\Lambda}du_4 &= \left\{\frac{2u_{23}-1}{T[u_{23}s-u_{12}u]}+\frac{2}{T^2[u_{23}s-u_{12}u]^2}\right\}e^{(1-u_{13})u_{23}sT}\\
&- \left\{\frac{2u_2-1}{T[u_{23}s-u_{12}u]}+\frac{2}{T^2[u_{23}s-u_{12}u]^2}\right\}e^{(1-u_1)u_{23}sT+u_3u_{12}uT}\\
\int_{u_3}^{u_2}\dot{G}_{42}e^{\Lambda}du_4 &= \left\{\frac{-1}{T[(u_{23}-1)s-u_{12}u]}+\frac{2}{T^2[(u_{23}-1)s-u_{12}u]^2}\right\}e^{u_{12}u_{23}tT}\\
&- \left\{\frac{2u_{23}-1}{T[(u_{23}-1)s-u_{12}u]}+\frac{2}{T^2[(u_{23}-1)s-u_{12}u]^2}\right\}e^{(1-u_{13})u_{23}sT}\\
\int_{u_2}^{u_1}\dot{G}_{42}e^{\Lambda}du_4 &= \left\{\frac{1}{T[u_{23}s+(1-u_{12})u]}-\frac{2}{T^2[u_{23}s+(1-u_{12})u]^2}\right\}e^{u_{12}u_{23}tT}\\
&+ \left\{\frac{2u_{12}-1}{T[u_{23}s+(1-u_{12})u]}+\frac{2}{T^2[u_{23}s+(1-u_{12})u]^2}\right\}e^{(1-u_{13})u_{12}uT}\\
\int_{u_1}^{1}\dot{G}_{42}e^{\Lambda}du_4 &= \left\{\frac{2u_{12}-1}{T[u_{23}s-u_{12}u]}-\frac{2}{T^2[u_{23}s-u_{12}u]^2}\right\}e^{(1-u_{13})u_{12}uT}\\
&+ \left\{\frac{2u_2-1}{T[u_{23}s-u_{12}u]}+\frac{2}{T^2[u_{23}s-u_{12}u]^2}\right\}e^{(1-u_1)u_{23}sT+u_3u_{12}uT}.
\end{aligned} \tag{3.41}$$

Again, we can notice that the contributions from $u_4 = 0$ and $u_4 = 1$ cancel one to each other. Now, writing this integral in terms of G_{ij} and $\dot{G}_{ij}$:

$$\begin{aligned}
\int_0^1 \dot{G}_{42} e^{\Lambda} du_4 \;=\;& \left\{ -\frac{2\dot{G}_{23}}{T[s-\dot{G}_{31}u-\dot{G}_{32}t]} + \frac{8}{T^2[s-\dot{G}_{31}u-\dot{G}_{32}t]^2} \right. \\
&- \left. \frac{2\dot{G}_{23}}{T[s+\dot{G}_{31}u+\dot{G}_{32}t]} - \frac{8}{T^2[s+\dot{G}_{31}u+\dot{G}_{32}t]^2} \right\} e^{\frac{1}{2}(G_{13}+G_{23}-G_{12})sT} \\
&+ \left\{ -\frac{2}{T[t-\dot{G}_{23}s-\dot{G}_{21}u]} + \frac{8}{T^2[t-\dot{G}_{23}s-\dot{G}_{21}u]^2} \right. \\
&+ \left. \frac{2}{T[t+\dot{G}_{23}s+\dot{G}_{21}u]} - \frac{8}{T^2[t+\dot{G}_{23}s+\dot{G}_{21}u]^2} \right\} e^{\frac{1}{2}(G_{12}+G_{23}-G_{13})tT} \\
&+ \left\{ \frac{2\dot{G}_{12}}{T[u-\dot{G}_{12}t-\dot{G}_{13}s]} + \frac{8}{T^2[u-\dot{G}_{12}t-\dot{G}_{13}s]^2} \right. \\
&+ \left. \frac{2\dot{G}_{12}}{T[u+\dot{G}_{12}t+\dot{G}_{13}s]} - \frac{8}{T^2[u+\dot{G}_{12}t+\dot{G}_{13}s]^2} \right\} e^{\frac{1}{2}(G_{12}+G_{13}-G_{23})uT}
\end{aligned} \tag{3.42}$$

As we did before, we are going to integrate over T, following the same procedure, the integrals $\frac{1}{T}$ are easy to perform, but for integrals involving $\frac{1}{T^2}$ we shall use formula 3.33. Doing the calculations, we obtain:

$$\begin{aligned}
& \frac{-2\dot{G}_{23}}{m^2-\frac{1}{2}(G_{13}+G_{23}-G_{12})s} \left\{ \frac{1}{s-\dot{G}_{31}u-\dot{G}_{32}t} + \frac{1}{s+\dot{G}_{31}u+\dot{G}_{32}t} \right\} \\
+\;& \frac{-2}{m^2-\frac{1}{2}(G_{12}+G_{23}-G_{13})t} \left\{ \frac{1}{t-\dot{G}_{23}s-\dot{G}_{21}u} - \frac{1}{t+\dot{G}_{23}s+\dot{G}_{21}u} \right\} \\
+\;& \frac{2\dot{G}_{12}}{m^2-\frac{1}{2}(G_{12}+G_{13}-G_{23})u} \left\{ \frac{1}{u-\dot{G}_{12}t-\dot{G}_{13}s} + \frac{1}{u+\dot{G}_{12}t+\dot{G}_{13}s} \right\} \\
+\;& \left\{ \frac{-8}{[s-\dot{G}_{31}u-\dot{G}_{32}t]^2} + \frac{8}{[s+\dot{G}_{31}u+\dot{G}_{32}t]^2} \right\} \ln(m^2 - \frac{1}{2}(G_{13}+G_{23}-G_{12})s) \\
+\;& \left\{ \frac{-8}{[t-\dot{G}_{23}s-\dot{G}_{21}u]^2} + \frac{8}{[t+\dot{G}_{23}s+\dot{G}_{21}u]^2} \right\} \ln(m^2 - \frac{1}{2}(G_{12}+G_{23}-G_{13})t) \\
+\;& \left\{ \frac{-8}{[u-\dot{G}_{12}t-\dot{G}_{13}s]^2} + \frac{8}{[u+\dot{G}_{12}t+\dot{G}_{13}s]^2} \right\} \ln(m^2 - \frac{1}{2}(G_{12}+G_{13}-G_{23})u)
\end{aligned} \tag{3.43}$$

At this point, we are going to take back the prefactor and integrate over one of the variables. The results are shown in Appendix D.

3.4.5 Integration of $\gamma^{scal}_{(12)(34)}$

The next calculation we can perform is the u_4 integral for $\gamma^{scal}_{(12)(34)}$:

$$\dot{G}_{12}\dot{G}_{21}\int_0^1 \dot{G}_{34}\dot{G}_{43}e^{\Lambda}du_4 = (-1)^2(\dot{G}_{12})^2\int_0^1 (\sigma_{34}-2u_{34})^2 e^{\Lambda}du_4. \tag{3.44}$$

Writing explicitly the $\dot{G}_{ij}$ and by now dropping off the prefactor $(\dot{G}_{12})^2$, it is straightforward to show that:

$$\begin{aligned}
\int_0^{u_3} \dot{G}_{34}\dot{G}_{43}e^{\Lambda}du_4 &= \left\{\frac{1}{T[u_{23}s-u_{12}u]}+\frac{-4}{T^2[u_{23}s-u_{12}u]^2}+\frac{8}{T^3[u_{23}s-u_{12}u]^3}\right\}e^{(1-u_{13})u_{23}sT}\\
&+\left\{\frac{-(2u_3-1)^2}{T[u_{23}s-u_{12}u]}+\frac{-4(2u_3-1)}{T^2[u_{23}s-u_{12}u]^2}+\frac{-8}{T^3[u_{23}s-u_{12}u]^3}\right\}e^{(1-u_1)u_{23}sT+u_3u_{12}uT}\\
\int_{u_3}^{u_2} \dot{G}_{34}\dot{G}_{43}e^{\Lambda}du_4 &= \left\{\frac{(2u_{23}-1)^2}{T[(u_{23}-1)s-u_{12}u]}+\frac{-4(2u_{23}-1)}{T^2[(u_{23}-1)s-u_{12}u]^2}+\frac{8}{T^3[(u_{23}-1)s-u_{12}u]^3}\right\}e^{u_{12}u_{23}tT}\\
&+\left\{\frac{-1}{T[(u_{23}-1)s-u_{12}u]}+\frac{-4}{T^2[(u_{23}-1)s-u_{12}u]^2}+\frac{-8}{T^3[(u_{23}-1)s-u_{12}u]^3}\right\}e^{(1-u_{13})u_{23}sT}\\
\int_{u_2}^{u_1} \dot{G}_{34}\dot{G}_{43}e^{\Lambda}du_4 &= \left\{\frac{(2u_{23}-1)^2}{T[u_{23}s+(1-u_{12})u]}+\frac{4(2u_{23}-1)}{T^2[u_{23}s+(1-u_{12})u]^2}+\frac{8}{T^3[u_{23}s+(1-u_{12})u]^3}\right\}e^{u_{12}u_{23}tT}\\
&+\left\{-\frac{(2u_{13}-1)^2}{T[u_{23}s+(1-u_{12})u]}+\frac{-4(2u_{13}-1)}{T^2[u_{23}s+(1-u_{12})u]^2}-\frac{8}{T^3[u_{23}s+(1-u_{12})u]^3}\right\}e^{(1-u_{13})u_{12}uT}\\
\int_{u_1}^{1} \dot{G}_{34}\dot{G}_{43}e^{\Lambda}du_4 &= \left\{\frac{-(2u_{13}-1)^2}{T[u_{23}s-u_{12}u]}+\frac{4(2u_{13}-1)}{T^2[u_{23}s-u_{12}u]^2}+\frac{-8}{T^3[u_{23}s-u_{12}u]^3}\right\}e^{(1-u_{13})u_{12}uT}\\
&+\left\{\frac{(2u_3-1)^2}{T[u_{23}s-u_{12}u]}+\frac{4(2u_3-1)}{T^2[u_{23}s-u_{12}u]^2}+\frac{8}{T^3[u_{23}s-u_{12}u]^3}\right\}e^{(1-u_1)u_{23}sT+u_3u_{12}uT}.
\end{aligned} \tag{3.45}$$

Summing the contributions, we can write this integral in terms of the Green's functions G_{ij} and $\dot{G}_{ij}$ as follows:

$$\begin{aligned}
\int_0^1 \dot{G}_{34}\dot{G}_{43}e^{\Lambda}du_4 = &\left\{\frac{2}{T[s-\dot{G}_{31}u-\dot{G}_{32}t]}+\frac{-16}{T^2[s-\dot{G}_{31}u-\dot{G}_{32}t]^2}+\frac{64}{T^3[s-\dot{G}_{31}u-\dot{G}_{32}t]^3}\right.\\
&\left.+\frac{2}{T[s+\dot{G}_{31}u+\dot{G}_{32}t]}+\frac{-16}{T^2[s+\dot{G}_{31}u+\dot{G}_{32}t]^2}+\frac{64}{T^3[s+\dot{G}_{31}u+\dot{G}_{32}t]^3}\right\}e^{\frac{1}{2}(G_{13}+G_{23}-G_{12})sT}\\
&+\left\{\frac{2(\dot{G}_{23})^2}{T[t-\dot{G}_{23}s-\dot{G}_{21}u]}+\frac{16\dot{G}_{23}}{T^2[t-\dot{G}_{23}s-\dot{G}_{21}u]^2}+\frac{64}{T^3[t-\dot{G}_{23}s-\dot{G}_{21}u]^3}\right.\\
&\left.+\frac{2(\dot{G}_{23})^2}{T[t+\dot{G}_{23}s+\dot{G}_{21}u]}+\frac{-16\dot{G}_{23}}{T^2[t+\dot{G}_{23}s+\dot{G}_{21}u]^2}+\frac{64}{T^3[t+\dot{G}_{23}s+\dot{G}_{21}u]^3}\right\}e^{\frac{1}{2}(G_{12}+G_{23}-G_{13})tT}\\
&+\left\{\frac{2(\dot{G}_{13})^2}{T[u-\dot{G}_{12}t-\dot{G}_{13}s]}+\frac{16\dot{G}_{13}}{T^2[u-\dot{G}_{12}t-\dot{G}_{13}s]^2}+\frac{64}{T^3[u-\dot{G}_{12}t-\dot{G}_{13}s]^3}\right.\\
&\left.+\frac{2(\dot{G}_{13})^2}{T[u+\dot{G}_{12}t+\dot{G}_{13}s]}+\frac{-16\dot{G}_{13}}{T^2[u+\dot{G}_{12}t+\dot{G}_{13}s]^2}+\frac{64}{T^3[u+\dot{G}_{12}t+\dot{G}_{13}s]^3}\right\}e^{\frac{1}{2}(G_{12}+G_{13}-G_{23})uT}.
\end{aligned} \tag{3.46}$$

Equation 3.46 is taking us back in the case where we have not fixed which of the permutations of the photons we are working on. In the next step, we shall perform the integral over T, as before, we are going to use equation 3.31 but now with $a = 2, 3$:

$$\begin{aligned}
\int_0^\infty \frac{dT}{T}T^{4-\frac{D}{2}}\frac{B}{T^2}e^{-AT} &= A^{-\epsilon}B\Gamma(\epsilon)\\
\int_0^\infty \frac{dT}{T}T^{4-\frac{D}{2}}\frac{B}{T^3}e^{-AT} &= A^{1-\epsilon}B\Gamma(\epsilon-1)
\end{aligned} \tag{3.47}$$

Expanding around the pole ϵ:

$$\begin{aligned}
A^{-\epsilon}B\Gamma(\epsilon) &\approx B[\frac{1}{\epsilon}-(\gamma+\ln A)]+O(\epsilon)\\
A^{1-\epsilon}B\Gamma(\epsilon-1) &\approx AB[-\frac{1}{\epsilon}+\gamma-1+\ln A]+O(\epsilon)
\end{aligned} \tag{3.48}$$

Now we can do the integral over T for equation 3.46 we get:

$$
\begin{aligned}
& \frac{2}{m^2-\frac{1}{2}(G_{13}+G_{23}-G_{12})s}\left\{\frac{1}{s-\dot{G}_{31}u-\dot{G}_{32}t}+\frac{1}{s+\dot{G}_{31}u+\dot{G}_{32}t}\right\} \\
+ & \frac{2(\dot{G}_{23})^2}{m^2-\frac{1}{2}(G_{12}+G_{23}-G_{13})t}\left\{\frac{1}{t-\dot{G}_{23}s-\dot{G}_{21}u}+\frac{1}{t+\dot{G}_{23}s+\dot{G}_{21}u}\right\} \\
+ & \frac{2(\dot{G}_{13})^2}{m^2-\frac{1}{2}(G_{12}+G_{13}-G_{23})u}\left\{\frac{1}{u-\dot{G}_{12}t-\dot{G}_{13}s}+\frac{1}{u+\dot{G}_{12}t+\dot{G}_{13}s}\right\} \\
+ & \frac{32(G_{13}+G_{23}-G_{12})s}{[s-\dot{G}_{31}u-\dot{G}_{32}t]^3}+\frac{32(G_{13}+G_{23}-G_{12})s}{[s+\dot{G}_{31}u+\dot{G}_{32}t]^3} \\
+ & \frac{32(G_{12}+G_{23}-G_{13})t}{[t-\dot{G}_{23}s-\dot{G}_{21}u]^3}+\frac{32(G_{12}+G_{23}-G_{13})t}{[t+\dot{G}_{23}s+\dot{G}_{21}u]^3} \\
+ & \frac{32(G_{12}+G_{13}-G_{23})u}{[u-\dot{G}_{12}t-\dot{G}_{13}s]^3}+\frac{32(G_{12}+G_{13}-G_{23})u}{[u+\dot{G}_{12}t+\dot{G}_{13}s]^3} \\
+ & \left\{\frac{16}{[s-\dot{G}_{31}u-\dot{G}_{32}t]^2}+\frac{16}{[s+\dot{G}_{31}u+\dot{G}_{32}t]^2}+\frac{64(m^2-\frac{1}{2}(G_{13}+G_{23}-G_{12})s)}{[s-\dot{G}_{31}u-\dot{G}_{32}t]^3}\right. \\
+ & \left.\frac{64(m^2-\frac{1}{2}(G_{13}+G_{23}-G_{12})s)}{[s+\dot{G}_{31}u+\dot{G}_{32}t]^3}\right\}\ln(m^2-\frac{1}{2}(G_{13}+G_{23}-G_{12})s) \\
+ & \left\{\frac{-16\dot{G}_{23}}{[t-\dot{G}_{23}s-\dot{G}_{21}u]^2}+\frac{16\dot{G}_{23}}{[t+\dot{G}_{23}s+\dot{G}_{21}u]^2}+\frac{64(m^2-\frac{1}{2}(G_{12}+G_{23}-G_{13})t)}{[t+\dot{G}_{23}s+\dot{G}_{21}u]^3}\right. \\
+ & \left.\frac{64(m^2-\frac{1}{2}(G_{12}+G_{23}-G_{13})t)}{[t-\dot{G}_{23}s-\dot{G}_{21}u]^3}\right\}\ln(m^2-\frac{1}{2}(G_{12}+G_{23}-G_{13})t) \\
+ & \left\{\frac{-16\dot{G}_{13}}{[u-\dot{G}_{12}t-\dot{G}_{13}s]^2}+\frac{16\dot{G}_{13}}{[u+\dot{G}_{12}t+\dot{G}_{13}s]^2}+\frac{64(m^2-\frac{1}{2}(G_{12}+G_{13}-G_{23})u)}{[u+\dot{G}_{12}t+\dot{G}_{13}s]^3}\right. \\
+ & \left.\frac{64(m^2-\frac{1}{2}(G_{12}+G_{13}-G_{23})u)}{[u-\dot{G}_{12}t-\dot{G}_{13}s]^3}\right\}\ln(m^2-\frac{1}{2}(G_{12}+G_{13}-G_{23})u) \qquad (3.49)
\end{aligned}
$$

And again, we are going to do a variable u_i equals to zero conserving the symmetry between sectors and performing the integral. Due to the size of the result, we show it in Appendix E.

3.4.6 Integration of $\gamma^{scal}_{(1234)}$

The first calculation we can perform is the u_4 integral for $\gamma^{scal}_{(1234)}$:

$$\dot{G}_{12}\dot{G}_{23}\int_0^1 \dot{G}_{34}\dot{G}_{41}e^{\Lambda}du_4 = \dot{G}_{12}\dot{G}_{23}\int_0^1 (\sigma_{34}-2u_{34})(\sigma_{41}-2u_{41})e^{\Lambda}du_4. \tag{3.50}$$

Writing explicitly the $\dot{G}_{ij}$ and by now dropping off the prefactor $\dot{G}_{12}\dot{G}_{23}$, it is straightforward to show that:

$$\begin{aligned}
\int_0^{u_3} \dot{G}_{34}\dot{G}_{41}e^{\Lambda}du_4 &= \left\{\frac{2u_{13}-1}{T[u_{23}s-u_{12}u]} + \frac{4(1-u_{13})}{T^2[u_{23}s-u_{12}u]^2} - \frac{8}{T^3[u_{23}s-u_{12}u]^3}\right\} e^{(1-u_{13})u_{23}sT}\\
&+\left\{\frac{(2u_1-1)(2u_3-1)}{T[u_{23}s-u_{12}u]} + \frac{4(u_1+u_3-1)}{T^2[u_{23}s-u_{12}u]^2} + \frac{8}{T^3[u_{23}s-u_{12}u]^3}\right\} e^{(1-u_1)u_{23}sT+u_3u_{12}uT}\\
\int_{u_3}^{u_2} \dot{G}_{34}\dot{G}_{41}e^{\Lambda}du_4 &= \left\{\frac{(2u_{12}-1)(2u_{23}-1)}{T[(u_{23}-1)s-u_{12}u]} + \frac{4(u_{23}-u_{12})}{T^2[(u_{23}-1)s-u_{12}u]^2} - \frac{8}{T^3[(u_{23}-1)s-u_{12}u]^3}\right\} e^{u_{12}u_{23}tT}\\
&+\left\{\frac{2u_{13}-1}{T[(u_{23}-1)s-u_{12}u]} + \frac{4u_{13}}{T^2[(u_{23}-1)s-u_{12}u]^2} + \frac{8}{T^3[(u_{23}-1)s-u_{12}u]^3}\right\} e^{(1-u_{13})u_{23}sT}\\
\int_{u_2}^{u_1} \dot{G}_{34}\dot{G}_{41}e^{\Lambda}du_4 &= \left\{-\frac{(2u_{23}-1)(2u_{12}-1)}{T[u_{23}s+(1-u_{12})u]} + \frac{4(u_{12}-u_{23})}{T^2[u_{23}s+(1-u_{12})u]^2} + \frac{8}{T^3[u_{23}s+(1-u_{12})u]^3}\right\} e^{u_{12}u_{23}tT}\\
&+\left\{\frac{1-2u_{13}}{T[u_{23}s+(1-u_{12})u]} + \frac{4u_{13}}{T^2[u_{23}s+(1-u_{12})u]^2} - \frac{8}{T^3[u_{23}s+(1-u_{12})u]^3}\right\} e^{(1-u_{13})u_{12}uT}\\
\int_{u_1}^{1} \dot{G}_{34}\dot{G}_{41}e^{\Lambda}du_4 &= \left\{\frac{1-2u_{13}}{T[u_{23}s-u_{12}u]} + \frac{4(1-u_{13})}{T^2[u_{23}s-u_{12}u]^2} + \frac{8}{T^3[u_{23}s-u_{12}u]^3}\right\} e^{(1-u_{13})u_{12}uT}\\
&+\left\{-\frac{(2u_1-1)(2u_3-1)}{T[u_{23}s-u_{12}u]} + \frac{4(1-u_1-u_3)}{T^2[u_{23}s-u_{12}u]^2} - \frac{8}{T^3[u_{23}s-u_{12}u]^3}\right\} e^{(1-u_1)u_{23}sT+u_3u_{12}uT}.
\end{aligned} \tag{3.51}$$

Adding up all contributions, we can write last equation in terms of the Green's functions G_{ij} and $\dot{G}_{ij}$ as follows:

$$
\begin{aligned}
&\left\{-\frac{2\dot{G}_{13}}{T[s-\dot{G}_{31}u-\dot{G}_{32}t]}+\frac{8(1+\dot{G}_{13})}{T^2[s-\dot{G}_{31}u-\dot{G}_{32}t]^2}-\frac{64}{T^3[s-\dot{G}_{31}u-\dot{G}_{32}t]^3}\right.\\
+&\left.\frac{2\dot{G}_{13}}{T[s+\dot{G}_{31}u+\dot{G}_{32}t]}+\frac{8(1-\dot{G}_{13})}{T^2[s+\dot{G}_{31}u+\dot{G}_{32}t]^2}-\frac{64}{T^3[s+\dot{G}_{31}u+\dot{G}_{32}t]^3}\right\}e^{\frac{1}{2}(G_{13}+G_{23}-G_{12})sT}\\
+&\left\{\frac{2\dot{G}_{12}\dot{G}_{23}}{T[t-\dot{G}_{23}s-\dot{G}_{21}u]}+\frac{8(\dot{G}_{12}-\dot{G}_{23})}{T^2[t-\dot{G}_{23}s-\dot{G}_{21}u]^2}-\frac{64}{T^3[t-\dot{G}_{23}s-\dot{G}_{21}u]^3}\right.\\
+&\left.\frac{2\dot{G}_{12}\dot{G}_{23}}{T[t+\dot{G}_{23}s+\dot{G}_{21}u]}-\frac{8(\dot{G}_{12}-\dot{G}_{23})}{T^2[t+\dot{G}_{23}s+\dot{G}_{21}u]^2}-\frac{64}{T^3[t+\dot{G}_{23}s+\dot{G}_{21}u]^3}\right\}e^{\frac{1}{2}(G_{12}+G_{23}-G_{13})tT}\\
+&\left\{\frac{2\dot{G}_{13}}{T[u-\dot{G}_{12}t-\dot{G}_{13}s]}+\frac{8(1-\dot{G}_{13})}{T^2[u-\dot{G}_{12}t-\dot{G}_{13}s]^2}-\frac{64}{T^3[u-\dot{G}_{12}t-\dot{G}_{13}s]^3}\right.\\
-&\left.\frac{2\dot{G}_{13}}{T[u+\dot{G}_{12}t+\dot{G}_{13}s]}+\frac{8(1+\dot{G}_{13})}{T^2[u+\dot{G}_{12}t+\dot{G}_{13}s]^2}-\frac{64}{T^3[u+\dot{G}_{12}t+\dot{G}_{13}s]^3}\right\}e^{\frac{1}{2}(G_{12}+G_{13}-G_{23})uT}
\end{aligned}
\tag{3.52}
$$

By using equations 3.48, we can do the integral over T for equation 3.52 we get:

$$
\begin{aligned}
& \frac{2u_{13}-1}{m^2-u_{23}(1-u_{13})s}\left\{\frac{1}{u_{23}s-u_{12}u}+\frac{1}{(u_{23}-1)s-u_{12}u}\right\} \\
+ \quad & \frac{(2u_{12}-1)(2u_{23}-1)}{m^2-u_{12}u_{23}t}\left\{\frac{1}{u_{12}t-(1-u_{13})s}+\frac{1}{(1-u_{13})s+(1-u_{12})t}\right\} \\
+ \quad & \frac{2u_{13}-1}{m^2-u_{12}(1-u_{13})u}\left\{\frac{1}{u_{23}t-(1-u_{13})u}+\frac{1}{u_{23}t+u_{13}u}\right\} \\
- \quad & \frac{8(1-u_{13})}{[u_{23}s-u_{12}u]^2}-\frac{8u_{23}}{[(u_{23}-1)s-u_{12}u]^2}-\frac{8u_{12}}{[u_{23}s+(1-u_{12})u]^2} \\
+ \quad & \left\{-\frac{4(1-u_{13})}{[u_{23}s-u_{12}u]^2}-\frac{8(m^2-(1-u_{13})u_{23}s)}{[u_{23}s-u_{12}u]^3}-\frac{4u_{13}}{[(u_{23}-1)s-u_{12}u]^2}\right. \\
+ \quad & \left.\frac{8(m^2-(1-u_{13})u_{23}s)}{[(u_{23}-1)s-u_{12}u]^3}\right\}\ln(m^2-(1-u_{13})u_{23}s) \\
+ \quad & \left\{-\frac{4(u_{12}-u_{23})}{[u_{23}s+(1-u_{12})u]^2}+\frac{8(m^2-u_{12}u_{23}t)}{[u_{23}s+(1-u_{12})u]^3}+\frac{4(u_{12}-u_{23})}{[(u_{23}-1)s-u_{12}u]^2}\right. \\
- \quad & \left.\frac{8(m^2-u_{12}u_{23}t)}{[(u_{23}-1)s-u_{12}u]^3}\right\}\ln(m^2-u_{12}u_{23}t) \\
+ \quad & \left\{-\frac{4(1-u_{13})}{[u_{23}s-u_{12}u]^2}+\frac{8(m^2-(1-u_{13})u_{12}u)}{[u_{23}s-u_{12}u]^3}-\frac{4u_{13}}{[u_{23}s+(1-u_{12})u]^2}\right. \\
- \quad & \left.\frac{8(m^2-(1-u_{13})u_{12}u)}{[u_{23}s+(1-u_{12})u]^3}\right\}\ln(m^2-(1-u_{13})u_{12}u)
\end{aligned}
\tag{3.53}
$$

We can express last equation in terms of G_{ij} and $\dot{G}_{ij}$:

$$\begin{aligned}
& \frac{-2\dot{G}_{13}}{m^2-\frac{1}{2}(G_{13}+G_{23}-G_{12})s}\left\{\frac{1}{s-\dot{G}_{31}u-\dot{G}_{32}t}-\frac{1}{s+\dot{G}_{31}u+\dot{G}_{32}t}\right\} \\
+ & \frac{2\dot{G}_{12}\dot{G}_{23}}{m^2-\frac{1}{2}(G_{12}+G_{23}-G_{13})t}\left\{\frac{1}{t-\dot{G}_{23}s-\dot{G}_{21}u}+\frac{1}{t+\dot{G}_{23}s+\dot{G}_{21}u}\right\} \\
+ & \frac{2\dot{G}_{13}}{m^2-\frac{1}{2}(G_{12}+G_{13}-G_{23})u}\left\{\frac{1}{u-\dot{G}_{12}t-\dot{G}_{13}s}-\frac{1}{u+\dot{G}_{12}t+\dot{G}_{13}s}\right\} \\
+ & \frac{-32(G_{13}+G_{23}-G_{12})s}{[s-\dot{G}_{31}u-\dot{G}_{32}t]^3}+\frac{-32(G_{13}+G_{23}-G_{12})s}{[s+\dot{G}_{31}u+\dot{G}_{32}t]^3} \\
+ & \frac{-32(G_{12}+G_{23}-G_{13})t}{[t-\dot{G}_{23}s-\dot{G}_{21}u]^3}+\frac{-32(G_{12}+G_{23}-G_{13})t}{[t+\dot{G}_{23}s+\dot{G}_{21}u]^3} \\
+ & \frac{-32(G_{12}+G_{13}-G_{23})u}{[u-\dot{G}_{12}t-\dot{G}_{13}s]^3}+\frac{-32(G_{12}+G_{13}-G_{23})u}{[u+\dot{G}_{12}t+\dot{G}_{13}s]^3} \\
- & \left\{\frac{8(1+\dot{G}_{13})}{[s-\dot{G}_{31}u-\dot{G}_{32}t]^2}+\frac{8(1-\dot{G}_{13})}{[s+\dot{G}_{31}u+\dot{G}_{32}t]^2}+\frac{64(m^2-\frac{1}{2}(G_{13}+G_{23}-G_{12})s)}{[s-\dot{G}_{31}u-\dot{G}_{32}t]^3}\right. \\
+ & \left.\frac{64(m^2-\frac{1}{2}(G_{13}+G_{23}-G_{12})s)}{[s+\dot{G}_{31}u+\dot{G}_{32}t]^3}\right\}\ln(m^2-\frac{1}{2}(G_{13}+G_{23}-G_{12})s) \\
+ & \left\{\frac{-8(\dot{G}_{12}-\dot{G}_{23})}{[t-\dot{G}_{23}s-\dot{G}_{21}u]^2}+\frac{8(\dot{G}_{12}-\dot{G}_{23})}{[t+\dot{G}_{23}s+\dot{G}_{21}u]^2}+\frac{-64(m^2-\frac{1}{2}(G_{12}+G_{23}-G_{13})t)}{[t+\dot{G}_{23}s+\dot{G}_{21}u]^3}\right. \\
+ & \left.\frac{-64(m^2-\frac{1}{2}(G_{12}+G_{23}-G_{13})t)}{[t-\dot{G}_{23}s-\dot{G}_{21}u]^3}\right\}\ln(m^2-\frac{1}{2}(G_{12}+G_{23}-G_{13})t) \\
- & \left\{\frac{8(1-\dot{G}_{13})}{[u-\dot{G}_{12}t-\dot{G}_{13}s]^2}+\frac{8(1+\dot{G}_{13})}{[u+\dot{G}_{12}t+\dot{G}_{13}s]^2}+\frac{64(m^2-\frac{1}{2}(G_{12}+G_{13}-G_{23})u)}{[u+\dot{G}_{12}t+\dot{G}_{13}s]^3}\right. \\
+ & \left.\frac{64(m^2-\frac{1}{2}(G_{12}+G_{13}-G_{23})u)}{[u-\dot{G}_{12}t-\dot{G}_{13}s]^3}\right\}\ln(m^2-\frac{1}{2}(G_{12}+G_{13}-G_{23})u) \qquad (3.54)
\end{aligned}$$

With equation 3.54, we can now take back the prefactor and integrate over the penultimate variable. The result is shown in Appendix F.

3.5 Numerical Integration

As mentioned above, integrating over the last variable is the best strategy because we would analytically obtain the complete result for the amplitude. Unfortunately, the complexity of functions such as Li_2, the size of the results obtained and the possible appearance of spurious poles are impediments to such integration. This is why we have decided to perform the last integral numerically. Once the results with false poles had been regularised, we decided that *Mathematica* should decide the best numerical method to calculate each integral. At this point, spurious poles appear again, but this time on the last variable. These spurious poles are not removed in the same way as the previous ones, on the contrary, they are removed by matching terms containing the same degree of divergence but with opposite sign. To see this, the divergent results must be expanded in series around the two possible points of divergence, *i.e.* $u_i = 0$ or $u_i = 1$ which in effect, due to the inversion symmetry, are identified as the same point. Looking which terms contain the same degree of divergence we can add them and the divergence will be removed. In the *Mathematica file* attached to the thesis, we have cancelled the divergences for all structures.

As a last comment for the numerical part, we have to mention that *Mathematica* has limitations in its computational power when performing some operations and that is why the results shown, although correct, cannot cover all the values for the mass and the Mandelstam variables.

3.6 Expansion of the Weisskopf Lagrangian

At this point, it is possible that we have the natural question, *How do we know that the calculations performed so far are correct?* To answer this question, we have a interesting check, the case of the scalar QED in a constant electromagnetic background. As we mention in the introduction, there is a version for the scalar QED lagrangian, also called the Weisskopf lagrangian. This lagrangian can be expanded in the low-energy limit. Doing it, we obtain the corresponding factors to the contributions from four photon scattering as well, vacuum polarization and the tadpole. We shall focus on the coeficients for four photon scattering. First of all, we have to expand the lagrangian. Recalling the re-normalized Weisskopf lagrangian for scalar QED:

$$\mathcal{L}_W^{re} = \frac{1}{16\pi^2}\int_0^\infty \frac{dT}{T^3} e^{-m^2 T}\left[\frac{(eaT)(ebT)}{\sin(eaT)\sinh(ebT)} + \frac{1}{6}(eT)^2\mathcal{F} - 1\right]. \qquad (3.55)$$

The two last terms in brackets from last equation, re-normalize the lagrangian, without them the lagrangian is un-normalized and it is the lagrangian we are going to work out. It can be demostrated that for a constant external field in four dimensions, we can re-write the un-normalized Weisskopf lagrangian in the worldline formalism as [32]:

$$\mathcal{L}_W^{un} = \int_0^\infty \frac{dT}{T}(4\pi T)^{-2} e^{-m^2 T} det^{-1/2}\left[\frac{\sin(eFT)}{eFT}\right]. \qquad (3.56)$$

Now, we have to remember the identity $\ln(det(\cdot)) = \mathrm{tr}(\ln(\cdot))$, this identity allows us to wite the previous *det* as:

$$det^{-1/2}\left[\frac{\sin(eFT)}{eFT}\right] = exp\left[-\frac{1}{2}\mathrm{tr}\left(\ln\left(\frac{\sin(eFT)}{eFT}\right)\right)\right]. \tag{3.57}$$

The next step is to expand the argument of the ln into series in powers of eFT:

$$\ln\left(\frac{\sin(eFT)}{eFT}\right) = \ln\left(1 - \frac{(eFT)^2}{6} + \frac{(eFT)^4}{120} + \dots\right). \tag{3.58}$$

Neglecting the higher terms than $(eFT)^4$ and expanding the ln:

$$\ln\left(1 - \frac{(eFT)^2}{6} + \frac{(eFT)^4}{120}\right) \approx \left(-\frac{(eFT)^2}{6} + \frac{(eFT)^4}{120}\right) - \frac{1}{2}\left(\frac{(eFT)^2}{6} - \frac{(eFT)^4}{120}\right)^2. \tag{3.59}$$

Taking only the terms untill 4th order:

$$exp\left[-\frac{1}{2}\mathrm{tr}\left(\ln\left(\frac{\sin(eFT)}{eFT}\right)\right)\right] \approx exp\left[-\frac{1}{2}\mathrm{tr}\left(-\frac{(eFT)^2}{6} - \frac{(eFT)^4}{180}\right)\right]. \tag{3.60}$$

Expanding the exponential:

$$exp\left[\frac{1}{2}\mathrm{tr}\left(\frac{(eFT)^2}{6} + \frac{(eFT)^4}{180}\right)\right] \approx 1 + \frac{1}{12}\mathrm{tr}[(eFT)^2] + \frac{1}{360}\mathrm{tr}[(eFT)^4] + \frac{1}{288}(\mathrm{tr}[(eFT)^2])^2. \tag{3.61}$$

Substituting last equation into 3.6, we obtain:

$$\int_0^\infty dT(4\pi)^{-2}e^{-m^2T}\left(\frac{1}{T^3} + \frac{e^2}{12}\frac{\mathrm{tr}[F^2]}{T} + \frac{e^4}{36}\left[\frac{1}{10}\mathrm{tr}[F^4] + \frac{1}{8}(\mathrm{tr}[F^2])^2\right]T\right). \tag{3.62}$$

Taking only the term of order e^{4}[‡], and performing the integral over T:

$$\frac{e^4}{36(4\pi)^2m^4}\left[\frac{1}{10}\mathrm{tr}[F^4] + \frac{1}{8}(\mathrm{tr}[F^2])^2\right] \tag{3.63}$$

We obtain the result for the low-energy limit. In order to compare with our calculation in worldline formalism, we shall decompose the field F in terms of the field strength tensors of the individual photons as follows:

$$F^{\mu\nu} = f_1^{\mu\nu} + f_2^{\mu\nu} + f_3^{\mu\nu} + f_4^{\mu\nu}. \tag{3.64}$$

‡These terms correspond to four-photon scattering.

From now on, we shall omit the indices μ and ν. The next step is to calculate the traces appearing in equation 3.62, computing $tr(F^2)$:

$$\begin{aligned} \mathrm{tr}[F^2] &= \mathrm{tr}[(f_1 + f_2 + f_3 + f_4) \cdot (f_1 + f_2 + f_3 + f_4)] \\ &= 2\left(\mathrm{tr}[f_1 \cdot f_2] + \mathrm{tr}[f_1 \cdot f_3] + \mathrm{tr}[f_1 \cdot f_4] + \mathrm{tr}[f_2 \cdot f_4] + \mathrm{tr}[f_3 \cdot f_4]\right). \end{aligned} \tag{3.65}$$

Computing the square of last equation and taking the multilinear terms:

$$(\mathrm{tr}[F^2])^2 = 8\mathrm{tr}[f_1 \cdot f_2]\mathrm{tr}[f_3 \cdot f_4] + 8\mathrm{tr}[f_1 \cdot f_3]\mathrm{tr}[f_2 \cdot f_4] + 8\mathrm{tr}[f_1 \cdot f_4]\mathrm{tr}[f_2 \cdot f_3]. \tag{3.66}$$

Recalling the Lorentz-cycle we have defined in the Chapter 2, we can write the traces as follows:

$$\mathrm{tr}(f_i f_j) = 2Z_2(ij), \tag{3.67}$$

substituting into equation 3.66, we get:

$$(\mathrm{tr}[F^2])^2 = 32Z_2(12)Z_2(34) + 32Z_2(13)Z_2(24) + 32Z_2(14)Z_2(23) \tag{3.68}$$

Replacing in the last term of equation 3.63, we obtain:

$$\frac{e^4}{36(4\pi)^2 m^4}\left[\frac{1}{8}(\mathrm{tr}[F^2])^2\right] = \frac{e^4}{9(4\pi)^2 m^4}\left[Z_2(12)Z_2(34) + Z_2(13)Z_2(24) + Z_2(14)Z_2(23)\right] \tag{3.69}$$

Following the same steps for $tr(F^4)$, we get:

$$\mathrm{tr}[F^4] = 8\mathrm{tr}[f_1 \cdot f_2 \cdot f_3 \cdot f_4] + 8\mathrm{tr}[f_1 \cdot f_3 \cdot f_2 \cdot f_4] + 8\mathrm{tr}[f_1 \cdot f_4 \cdot f_2 \cdot f_3] \tag{3.70}$$

For these traces, we have the following expression in terms of Lorentz-cycles:

$$\mathrm{tr}\left(f_i f_j f_k f_l\right) = Z_4(ijkl), \tag{3.71}$$

substituting in the trace:

$$\mathrm{tr}[F^4] = 8Z_4(1234) + 8Z_4(1324) + 8Z_4(1423) \tag{3.72}$$

Replacing in the first term of equation 3.63, we obtain:

$$\frac{e^4}{36(4\pi)^2 m^4}\left[\frac{1}{10}\mathrm{tr}[F^4]\right] = \frac{e^4}{45(4\pi)^2 m^4}\left[Z_4(1234) + Z_4(1324) + Z_4(1423)\right] \tag{3.73}$$

To compare this results, we have to go back into equation 3.22, and we shall notice, there are terms involving the same Lorentz-cycles with our structures $\Gamma^{scal}_{(1234)}$ and $\Gamma^{scal}_{(12)(34)}$ plus permutations. It is not necessary to calculate every permutation, it is enough to compare only one structure, in our case, we have already calculated those involving $Z(1234)$ and

$Z(12)Z(34)$, thus we are going to use them. In the next section, we shall compute the same limit for our previous results.

3.6.1 Expansion of $\gamma_{(12)(34)}$

To expand the structure $\Gamma_{(12)(34)}$, we have two options. The first one is to expand the result we found in equation 3.46 and then integrate over T. After that we have to choose the last ordering $u_1 > u_2$ or $u_2 > u_1$. Due to this choice, we have to multiply by a factor of 2 instead of calculate both cases. The second option is to take the result in equation 3.49 which is already integrated over T and the rest of the calculation is equal as before. However, these methods are equivalent and it is easier to perform the expansion from equation 3.49. Taking back the prefactor $(\dot{G}_{12})^2$, expanding equation 3.49 in the low-energy limit, *i.e.* $1/m \to 0$, and only taking the terms m^{-4}, we get:

$$\frac{(2u_{12}-1)^2}{3m^4}. \tag{3.74}$$

Integrating over u_2 from 0 to u_1:

$$\int_0^{u_1} \frac{(2u_{12}-1)^2}{3m^4} du_2 = \frac{4u_1^3 - 6u_1^2 + 3u_1}{9m^4}. \tag{3.75}$$

And finally integrating over u_1 from 0 to 1, we get:

$$\int_0^1 \frac{4u_1^3 - 6u_1^2 + 3u_1}{9m^4} du_1 = \frac{1}{18m^4} \tag{3.76}$$

Taking back all the terms comming from the optimized representation in equations 3.22 and 3.24, also taking into account the factor of 2 from the ordering choice, we get the result:

$$\frac{e^4}{9(4\pi)^2 m^4}\left[Z_2(12)Z_2(34)\right]. \tag{3.77}$$

This result is consistent with the expansion of Weisskopf lagrangian.

3.6.2 Expansion of $\gamma_{(1234)}$

Just as before, we shall follow the same strategy. Taking back the prefactor $\dot{G}_{12}\dot{G}_{23}$, expanding equation 3.54 in the low-energy limit and taking only the terms m^{-4}, we get:

$$\frac{-2(2u_{12}-1)(2u_2-1)(u_1^2 - u_1 + \frac{1}{6})}{m^4}. \tag{3.78}$$

Integrating over u_2 from 0 to u_1

$$\int_0^{u_1} \frac{-2(2u_{12}-1)(2u_2-1)(u_1^2-u_1+\frac{1}{6})}{m^4} du_2 = -\frac{u_1(6u_1^2-6u_1+1)(2u_1^2-6u_1+3)}{9m^4}. \quad (3.79)$$

The last integral over u_1 from 0 to 1:

$$\int_0^1 -\frac{u_1(6u_1^2-6u_1+1)(2u_1^2-6u_1+3)}{9m^4} du_1 = \frac{1}{90m^4}. \quad (3.80)$$

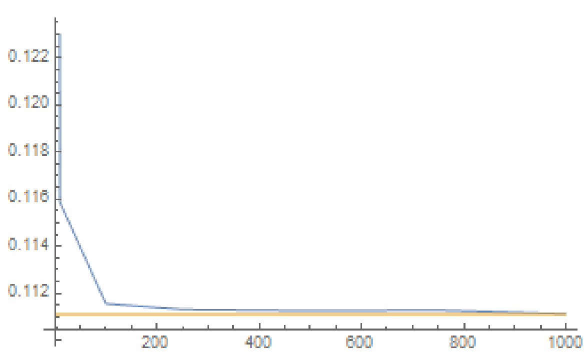
Again, with help of equations 3.22 and 3.24, also taking into account the factor of 2 from the ordering choice, we get the result:

$$\frac{e^4}{45(4\pi)^2 m^4}\left[Z_4(1234)\right], \quad (3.81)$$

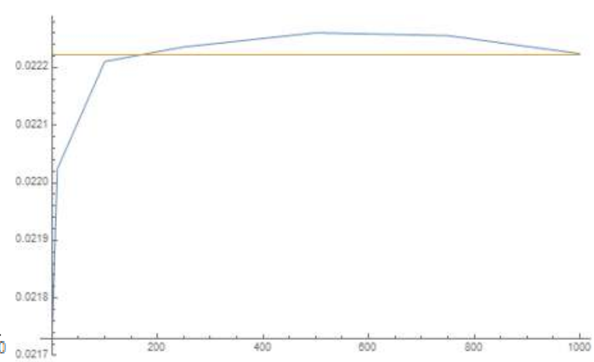
which is consistent with the calculated from the Weisskopf lagrangian.

3.6.3 Numerical Approximation

Into the *Mathematica file* attached to the thesis, we can approximate in the same limit, but it is necessary to do it properly. Instead of choosing m as a large number, we have chosen to make $m = 1$ and we have made the Mandelstam variables s, t and u very small, this choice is due to the computational power of mathematica, which does not allow to choose extremely high values for m. As a result, the analytically calculated coefficients are obtained except for a small error that is reduced depending on how small the values for the Mandelstam variables are.



(a) Approximation for $\gamma_{(12)(34)}$



(b) Approximation for $\gamma_{(1234)}$

Figure 3.1: Numeric approximation in the low-energy limit

The graphics in figure 3.1, we have chosen $s = 3/a$, $t = -2/a$ and $u = -1/a$, where a

is a changing parameter and as it gets larger, it gets closer to the low-energy limit. Again, due to the computational power of Mathematica, we have used a between 0 and 1000. In the graphic for $\gamma_{(12)(34)}$ we can notice the approximation to the factor of $\frac{1}{9}$, and for $\gamma_{(1234)}$ to the factor of $\frac{1}{45}$, both calculated before.

3.6.4 Higher Order Corrections

Furthermore, we can compute higher terms in the expansion, for example at order m^{-6}, for each structure:

Structure	Correction m^{-6}
$\gamma_{(123)1}$	$\frac{t-s}{3780}$
$\gamma_{(12)11}$	$\frac{s}{540}$
$\gamma_{(12)12}$	$\frac{s}{945}$
$\gamma_{(12)(34)}$	$\frac{s}{135}$
$\gamma_{(1234)}$	$\frac{t}{1890}$

These corrections of higher order correspond to derivatives of F terms in the Weisskopf lagrangian, particularly this terms m^{-6} are corrections for second derivatives of F. In our expansion, we cannot compare results, because we have used a constant background field and terms involving derivatives vanish.

Chapter 4

Conclusions and Perspectives

First of all, we would like to give a quick overview of the thesis. Throughout the development of this work, we have encountered a number of obstacles and challenges not originally expected. While it is true that the optimised representation for four photon scattering is more compact than the simple substitution in the Bern-Kosower master formula for scalar QED and better structured than the Q representation, it did not mean that the calculations were simpler or more straightforward than in the previous cases, this is to be expected, since the result is known to involve various hypergeometric functions and essentially we are rewriting these in terms of ln, Li_n and integrals thereof. We can start by noting that one of the biggest complications was the integration over T. We knew from naive analysis that this integral should be finite, but at first it seemed not to be, until after implementing the regularisation of the divergence with the help of the gamma function. Another important thing to note is the difficulty of some of the structures. In a way, we can say that although the $\gamma_{(123)1}$ structure was the simplest to realise, symmetry was lost due to the appearance of $\dot{G}_{ij}$ functions in the integrand, a weight that quickly reflected the inability to use symmetry between the Mandelstam variables. This effect disappear when we sum over the permutations for each tensor structure, doing that in the final result we recover the symmetry between Mandelstam variables by the cyclic permutations $s \to t \to u$.

Eventually, with integration over the remaining u_i variables, the problem was to choose the most efficient strategy. Initially the idea was that due to the symmetry of the structures, following the same method as for the scalar case ϕ^3 would be sufficient. Sooner or later, we realised that this strategy would not work because of the number of times the different variables u_i appeared in each integral. Many combinations were tried and the one that was presented was the one that seemed to be the most efficient of all the possibilities, always trying to preserve the symmetry between sectors.

Moving on to the numerical integration part, we had difficulties with the methods for performing such integrations. Some algorithms proved to be more efficient than others in performing some integrals, but there was no single method that integrated the whole amplitude. Within this same section, we first obtained complex numbers as a result of the numerical integrals, which is not consistent with the definition of the amplitude, this is relate to the fact that we were not dealing correctly with the divergences. This was because within the workings of numerical integration, we attributed this to the presence of ln and $Li_2(x)$ in our results, causing the software to have to choose different branches for each integral. Solving this problem was even more difficult than the rest of the work.

For purposes of getting the most out of this thesis and as a good check, it was decided to perform the expansion of the Weisskopf lagrangian and check that the more complicated structures, $\gamma_{(1233)}$ and $\gamma_{(12)(34)}$, were correctly realised. In that respect, we can mention that indeed, our calculations are in agreement with the low energy expansion of the Lagrangian.

With all this in our hand, we can conclude a few things. One of the questions possibly on the table is, why calculate this amplitude? It is true that it has been approximated and studied in other theories as spinor QED and approaches to quantum field theory, but not into the framework of scalar QED and not using the same representation and above all, under the worldline formalism. Also, previously, it was not known how difficult it would be to perform the rough calculations that have been done here. We now know that using this representation for the amplitude is not entirely efficient and this opens the door to present new proposals for the way in which the amplitude is organized in terms of the master formula and the worldline Green's functions. Within this time, but remaining out of the scope of this work, the use of generating functions has been considered, thus avoiding such a direct calculation involving integrals containing $\dot{G}_{ij}$ in the integrand and thus trying that in the end the result is in terms of partial derivatives with respect to the sources Λ_i. Now with the appreciation of the results of this thesis, it is considered that it is a better option and it is worth exploring that possibility.

Within the future work, it is clear that it is interesting to study the step to spinor QED with the use of the replacement formula given in equation 2.20. We do not yet know what kind of implications this has on the integrals we have already calculated here, but we suspect that the introduction of the fermionic worldline Green's function, which is a sign function of the parameters, may simplify some of the integrals and remove some of the divergences we have faced. Even with this, we do not rule out that similar difficulties as the ones we have presented here may arise. In the other hand, it will be interesting to compute the last integral analytically, and explore different strategies to integrate difficult functions as $Li_2(x)$. About the expansion in the low-energy limit, it will be interesting to calculate higher orders, to get derivative corrections in the Weisskopf lagrangian. Thinking in the phenomenology and experiments coming in next years, it would be attractive to convert the amplitude into a cross section in order to compare with experimental results, even though scalar QED is not a physical theory it is interesting to investigate whether it would be possible to measure this process at current or future experimental facilities. And finally, it would be interesting to increase the number of loops, since everything here has been calculated for one loop order.

Appendix A

Polylogarithm Function

In this appendix, we show the definitions for $PolyLog(n, z)$. This function is called polylogarithm, and it can be defined as:

$$PolyLog(n, z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \tag{A.1}$$

In the case $n = 2$:

$$PolyLog(2, z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} = \int_z^0 \frac{\ln(1-t)}{t} dt \tag{A.2}$$

Another notation for this function is:

$$PolyLog(n, z) = Li_n(z) \tag{A.3}$$

A.1 Proof for $n = 2$

To proof that equation A.2 is correct, we are going to introduce the Taylor series for $\ln(1-t)$ in the integral. Recalling the series:

$$\ln(1-t) = -\sum_{k=1}^{\infty} \frac{t^k}{k} \tag{A.4}$$

Setting it into equation A.2, we shall get the desired result:

$$\int_z^0 \left(-\sum_{k=1}^{\infty} \frac{t^k}{k} \right) \frac{1}{t} dt = -\sum_{k=1}^{\infty} \frac{1}{k} \int_z^0 t^{k-1} dt = -\sum_{k=1}^{\infty} \frac{1}{k} \frac{t^k}{k} \bigg|_z^0 = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \tag{A.5}$$

Appendix B

Integral Over Penultimate Variable for $\gamma_{(123)1}$

In this appendix, we show the result of the integration over the penultimate parameter for $\gamma_{(123)1}$. Here we have changed the last variable for x, that remains to be integrated over, just to make it clear at the moment of reading.

$$
\begin{aligned}
& \frac{(1-2x)^2\left(4m^2s(x-1)x+4m^4+s^2(x-1)^2(2x-1)\right)\left(\ln\left(m^2+s(x-1)x\right)-\ln\left(m^2\right)\right)}{(x-1)\left(m^2(s+u)+su(x-1)x\right)\left(m^2(s+u)+s(x-1)(s+ux)\right)}+\frac{(1-2x)^2\left(s^2(1-2x)+2su\left(2x^2-2x+1\right)+u^2(1-2x)\right)\ln\left(-\frac{s}{u}\right)}{(s+u)^2\left(m^2(s+u)+su(x-1)x\right)} \\
- \; & \frac{(1-2x)^2\left(s^2(2x-3)+2su\left(2x^2-4x+1\right)+u^2(1-2x)\right)\ln\left(\frac{s-sx}{s+ux}\right)}{(s+u)^2\left(m^2(s+u)+s(x-1)(s+ux)\right)}-\frac{(1-2x)^2\left(s^2(2x-1)+4stx^2+4t^2x^2\right)\ln\left(-\frac{t}{s+t}\right)}{s^2\left(m^2s+t(x-1)x(s+t)\right)} \\
+ \; & \frac{(1-2x)^2\left(s^2(2x-1)+4st(x-1)x+4t^2(x-1)^2\right)\ln\left(\frac{t(x-1)}{sx+t(x-1)}\right)}{s^2\left(m^2s+t(x-1)(sx+t(x-1))\right)}+\frac{(1-2x)^2\left(4m^2t(x-1)x+4m^4+t^2(x-1)^2(2x-1)\right)\left(\ln\left(m^2\right)-\ln\left(m^2+t(x-1)x\right)\right)}{(x-1)\left(m^2s+t(x-1)x(s+t)\right)\left(m^2s+t(x-1)(sx+t(x-1))\right)} \\
+ \; & \frac{8x(2x-1)}{u(x-1)(t+u)}+\frac{(2x-1)\left(t^2(1-2x)+2tu\left(2x^2-2x+1\right)+u^2(1-2x)\right)(\ln(-t)-\ln(-u))}{(t+u)^2\left(m^2(t+u)+tu(x-1)x\right)}+\frac{(2x-1)\left(t^2(1-2x)+2tu\left(2x^2-4x+1\right)+u^2(2x-3)\right)(\ln(tx+u)-\ln(u-ux))}{(t+u)^2\left(m^2(t+u)+u(x-1)(tx+u)\right)} \\
+ \; & \frac{(2x-1)\left(4m^2u(x-1)x+4m^4+u^2(x-1)^2(2x-1)\right)\left(2m^2(t+u)+u(x-1)(2tx+u)\right)\left(\ln\left(m^2\right)-\ln\left(m^2+u(x-1)x\right)\right)}{u^2(x-1)^2\left(m^2(t+u)+tu(x-1)x\right)\left(m^2(t+u)+u(x-1)(tx+u)\right)} \\
+ \; & \frac{8x(2x-1)(s-u)\left(\mathrm{Li}_2\left(\frac{m^2(s+u)}{(s+u)m^2+su(x-1)x}\right)-\mathrm{Li}_2\left(\frac{\left(m^2+s(x-1)x\right)(s+u)}{(s+u)m^2+su(x-1)x}\right)+\left(-\ln\left(m^2+s(x-1)x\right)\right)\ln\left(-\frac{s^2(x-1)x}{m^2(s+u)+su(x-1)x}\right)+\ln\left(m^2\right)\ln\left(\frac{su(x-1)x}{m^2(s+u)+su(x-1)x}\right)\right)}{(s+u)^3} \\
+ \; & \frac{8(2x-1)(s(x-2)-ux)\left(-\mathrm{Li}_2\left(\frac{m^2(s+u)}{(s+u)m^2+s(s+ux)(x-1)}\right)+\mathrm{Li}_2\left(\frac{\left(m^2+s(x-1)x\right)(s+u)}{(s+u)m^2+s(s+ux)(x-1)}\right)+\ln\left(m^2+s(x-1)x\right)\ln\left(-\frac{s^2(x-1)^2}{m^2(s+u)+s(x-1)(s+ux)}\right)-\ln\left(m^2\right)\ln\left(\frac{s(x-1)(s+ux)}{m^2(s+u)+s(x-1)(s+ux)}\right)\right)}{(s+u)^3} \\
+ \; & \frac{2(2x-1)\left(s^2(2x-3)+2su\left(2x^2-4x+1\right)+u^2(1-2x)\right)\left(-\frac{\ln\left(m^2\right)}{s+ux}+\frac{s(x-1)\left(\ln\left(m^2\right)-\ln(s+ux)\right)}{m^2(s+u)+s(x-1)(s+ux)}+\frac{s(x-1)\left(\ln(s-sx)-\ln\left(m^2+s(x-1)x\right)\right)}{m^2(s+u)+s(x-1)(s+ux)}+\frac{\ln\left(m^2+s(x-1)x\right)}{s-sx}\right)}{(s+u)^3} \\
+ \; & \frac{2(2x-1)\left(s^2(1-2x)+2su\left(2x^2-2x+1\right)+u^2(1-2x)\right)\left(u\left((s+u)\left(m^2+s(x-1)x\right)\ln\left(m^2+s(x-1)x\right)+s^2(x-1)x\ln(u(-x))-s^2(x-1)x\ln(sx)\right)+m^2s\ln\left(m^2\right)(s+u)\right)}{sux(s+u)^3\left(m^2(s+u)+su(x-1)x\right)} \\
+ \; & \frac{2(2x-1)\left(s^2(2x-1)+4stx^2+4t^2x^2\right)\left(-s^2tx^2\ln\left(m^2+t(x-1)x\right)+s^2tx\ln\left(m^2+t(x-1)x\right)-m^2s^2\ln\left(m^2+t(x-1)x\right)-st^2x^2\ln\left(m^2+t(x-1)x\right)\right)}{s^3tx(s+t)\left(m^2s+t(x-1)x(s+t)\right)} \\
+ \; & \frac{2(2x-1)\left(s^2(2x-1)+4stx^2+4t^2x^2\right)\left(st^2x\ln\left(m^2+t(x-1)x\right)-m^2st\ln\left(m^2+t(x-1)x\right)+m^2st\ln\left(m^2\right)+t^3x^2\ln(x(s+t))-st^2x^2\ln(-tx)+st^2x^2\ln(x(s+t))\right)}{s^3tx(s+t)\left(m^2s+t(x-1)x(s+t)\right)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{2(2x-1)\left(s^2(2x-1)+4stx^2+4t^2x^2\right)\left(-t^3x\ln(x(s+t))+st^2x\ln(-tx)-st^2x\ln(x(s+t))+t^3\left(-x^2\right)\ln(-tx)+t^3x\ln(-tx)\right)}{s^3tx(s+t)\left(m^2s+t(x-1)x(s+t)\right)} \\
&- \frac{8x(2x-1)(s+2t)\left(-\mathrm{Li}_2\left(\frac{m^2s}{sm^2+t(s+t)(x-1)x}\right)+\mathrm{Li}_2\left(\frac{s\left(m^2+t(x-1)x\right)}{sm^2+t(s+t)(x-1)x}\right)+\ln\left(m^2+t(x-1)x\right)\ln\left(-\frac{t^2(x-1)x}{m^2s+t(x-1)x(s+t)}\right)-i\pi\ln\left(m^2+\frac{t(x-1)x(s+t)}{s}\right)\right)}{s^3} \\
&+ \frac{8x(2x-1)(s+2t)\left(-\ln\left(m^2\right)\ln\left(\frac{t(x-1)x(s+t)}{m^2s+t(x-1)x(s+t)}\right)+i\pi\ln\left(m^2+t(x-1)x\right)\right)}{s^3} \\
&+ \frac{2(2x-1)\left(s^2(2x-1)+4st(x-1)x+4t^2(x-1)^2\right)\left(\frac{m^2s\ln\left(m^2\right)}{-sx+t(-x)+t}+\frac{s\left(m^2+t(x-1)x\right)\ln\left(m^2+t(x-1)x\right)}{t(x-1)}-t(x-1)\ln(-sx+t(-x)+t)+t(x-1)\ln(t-tx)\right)}{s^3\left(m^2s+t(x-1)(sx+t(x-1))\right)} \\
&+ \frac{8(2x-1)(sx+2t(x-1))\left(-\mathrm{Li}_2\left(\frac{m^2s}{sm^2+t(x-1)(t(x-1)+sx)}\right)+\mathrm{Li}_2\left(\frac{s\left(m^2+t(x-1)x\right)}{sm^2+t(x-1)(t(x-1)+sx)}\right)+\ln\left(m^2+t(x-1)x\right)\ln\left(\frac{t^2(x-1)^2}{m^2s+t(x-1)(sx+t(x-1))}\right)-\ln\left(m^2\right)\ln\left(\frac{t(x-1)(sx+t(x-1))}{m^2s+t(x-1)(sx+t(x-1))}\right)\right)}{s^3} \\
&+ -\frac{8x(2x-1)(t-u)\left(\mathrm{Li}_2\left(\frac{m^2(t+u)}{(t+u)m^2+tu(x-1)x}\right)-\mathrm{Li}_2\left(\frac{(t+u)\left(m^2+u(x-1)x\right)}{(t+u)m^2+tu(x-1)x}\right)+\ln\left(m^2+u(x-1)x\right)\left(-\ln\left(\frac{u^2(x-1)x}{m^2(t+u)+tu(x-1)x}\right)\right)+i\pi\ln\left(m^2+\frac{tu(x-1)x}{t+u}\right)\right)}{(t+u)^3} \\
&+ \frac{8x(2x-1)(t-u)\left(\ln\left(m^2\right)\ln\left(\frac{tu(x-1)x}{m^2(t+u)+tu(x-1)x}\right)-i\pi\ln\left(m^2+u(x-1)x\right)\right)}{(t+u)^3} \\
&+ \frac{8(2x-1)(tx-u(x-2))\left(\mathrm{Li}_2\left(\frac{m^2(t+u)}{(t+u)m^2+u(u+tx)(x-1)}\right)-\mathrm{Li}_2\left(\frac{(t+u)\left(m^2+u(x-1)x\right)}{(t+u)m^2+u(u+tx)(x-1)}\right)-\ln\left(m^2+u(x-1)x\right)\ln\left(-\frac{u^2(x-1)^2}{m^2(t+u)+u(x-1)(tx+u)}\right)+\ln\left(m^2\right)\ln\left(\frac{u(x-1)(tx+u)}{m^2(t+u)+u(x-1)(tx+u)}\right)\right)}{(t+u)^3} \\
&+ \frac{2(2(1-x)-1)\left(t^2(1-2(1-x))+2tu\left(2(1-x)^2-4(1-x)+1\right)+u^2(2(1-x)-3)\right)\left(-\frac{\ln\left(m^2\right)}{t(1-x)+u}+\frac{u((1-x)-1)\left(\ln\left(m^2\right)-\ln(t(1-x)+u)\right)}{m^2(t+u)+u((1-x)-1)(t(1-x)+u)}\right)}{(t+u)^3} \\
&+ \frac{2(2(1-x)-1)\left(t^2(1-2(1-x))+2tu\left(2(1-x)^2-4(1-x)+1\right)+u^2(2(1-x)-3)\right)\left(\frac{u((1-x)-1)\left(\ln(u-u(1-x))-\ln\left(m^2+u((1-x)-1)(1-x)\right)\right)}{m^2(t+u)+u((1-x)-1)(t(1-x)+u)}+\frac{\ln\left(m^2+u((1-x)-1)(1-x)\right)}{u-u(1-x)}\right)}{(t+u)^3} \\
&+ \frac{2(2x-1)\left(t^2(1-2x)+2tu\left(2x^2-2x+1\right)+u^2(1-2x)\right)\left(t^2ux^2\ln\left(m^2+u(x-1)x\right)+m^2t^2\ln\left(m^2+u(x-1)x\right)-t^2ux\ln\left(m^2+u(x-1)x\right)+tu^2x^2\ln\left(m^2\right)+tu^2x^2\ln\left(-\frac{tx}{m^2}\right)\right)}{tux(t+u)^3\left(m^2t+m^2u+tux^2-tux\right)} \\
&+ \frac{2(2x-1)\left(t^2(1-2x)+2tu\left(2x^2-2x+1\right)+u^2(1-2x)\right)\left(tu^2x^2\ln\left(m^2+u(x-1)x\right)-tu^2x\ln\left(m^2\right)-tu^2x\ln\left(-\frac{tx}{m^2}\right)-tu^2x\ln\left(m^2+u(x-1)x\right)\right)}{tux(t+u)^3\left(m^2t+m^2u+tux^2-tux\right)} \\
&+ \frac{2(2x-1)\left(t^2(1-2x)+2tu\left(2x^2-2x+1\right)+u^2(1-2x)\right)\left(m^2tu\ln\left(m^2+u(x-1)x\right)+m^2tu\ln\left(m^2\right)+m^2u^2\ln\left(m^2\right)-tu^2x^2\ln(u(-x))+tu^2x\ln(u(-x))\right)}{tux(t+u)^3\left(m^2t+m^2u+tux^2-tux\right)}
\end{aligned}
\tag{B.1}
$$

Appendix C

Integral Over Penultimate Variable for $\gamma_{(12)11}$

In this appendix, we show the result of the integration over the penultimate parameter for u_2 for $\gamma_{(12)11}$:

$$
\begin{aligned}
& \frac{(1-2x)^2\left(2m^2+2sx^2-3sx+s\right)^2\left(\ln\left(m^2\right)-\ln\left(m^2+s(x-1)x\right)\right)}{(x-1)\left(m^2(s+u)+su(x-1)x\right)\left(m^2(s+u)+s(x-1)(s+ux)\right)}+\frac{\left(s(1-2x)^2-2ux+u\right)^2\ln\left(-\frac{s}{u}\right)}{(s+u)^2\left(m^2(s+u)+su(x-1)x\right)}-\frac{(1-2x)^2(s(3-2x)+u)^2\ln\left(\frac{s-sx}{s+ux}\right)}{(s+u)^2\left(m^2(s+u)+s(x-1)(s+ux)\right)}+\frac{(2x-1)^3(s+2tx)\ln\left(-\frac{t}{s+t}\right)}{s\left(m^2s+t(x-1)x(s+t)\right)} \\
+ & \frac{(2x-1)^3(s+2t(x-1))(\ln(sx+t(x-1))-\ln(t(x-1)))}{s\left(m^2s+t(x-1)(sx+t(x-1))\right)}+\frac{t(2x-1)^3\left(2m^2+2tx^2-3tx+t\right)\left(\ln\left(m^2\right)-\ln\left(m^2+t(x-1)x\right)\right)}{\left(m^2s+t(x-1)x(s+t)\right)\left(m^2s+t(x-1)(sx+t(x-1))\right)}+\frac{(1-2x)^2(t(2x-1)-u)(\ln(-u)-\ln(-t))}{(t+u)\left(m^2(t+u)+tu(x-1)x\right)} \\
- & \frac{(1-2x)^2(t(2x-1)+u)(\ln(tx+u)-\ln(u-ux))}{(t+u)\left(m^2(t+u)+u(x-1)(tx+u)\right)}+\frac{(1-2x)^2\left(2m^2+u(x-1)\right)\left(2m^2(t+u)+u(x-1)(2tx+u)\right)\left(\ln\left(m^2+u(x-1)x\right)-\ln\left(m^2\right)\right)}{u(x-1)\left(m^2(t+u)+tu(x-1)x\right)\left(m^2(t+u)+u(x-1)(tx+u)\right)} \\
+ & \frac{8\left(s(1-2x)^2-2ux+u\right)\left(-\mathrm{Li}_2\left(\frac{m^2(s+u)}{(s+u)m^2+su(x-1)x}\right)+\mathrm{Li}_2\left(\frac{\left(m^2+s(x-1)x\right)(s+u)}{(s+u)m^2+su(x-1)x}\right)+\ln\left(m^2+s(x-1)x\right)\ln\left(-\frac{s^2(x-1)x}{m^2(s+u)+su(x-1)x}\right)-\ln\left(m^2\right)\ln\left(\frac{su(x-1)x}{m^2(s+u)+su(x-1)x}\right)\right)}{(s+u)^3} \\
- & \frac{8(2x-1)(s(2x-3)-u)\left(-\mathrm{Li}_2\left(\frac{m^2(s+u)}{(s+u)m^2+s(s+ux)(x-1)}\right)+\mathrm{Li}_2\left(\frac{\left(m^2+s(x-1)x\right)(s+u)}{(s+u)m^2+s(s+ux)(x-1)}\right)+\ln\left(m^2+s(x-1)x\right)\ln\left(-\frac{s^2(x-1)^2}{m^2(s+u)+s(x-1)(s+ux)}\right)-\ln\left(m^2\right)\ln\left(\frac{s(x-1)(s+ux)}{m^2(s+u)+s(x-1)(s+ux)}\right)\right)}{(s+u)^3} \\
+ & \frac{2(2x-1)(-2sx+s+u)^2\left(u\left((s+u)\left(m^2+s(x-1)x\right)\ln\left(m^2+s(x-1)x\right)+s^2(x-1)x\ln(u(-x))-s^2(x-1)x\ln(sx)\right)+m^2s\ln\left(m^2\right)(s+u)\right)}{sux(s+u)^3\left(m^2(s+u)+su(x-1)x\right)} \\
+ & \frac{2(2x-1)(s(3-2x)+u)^2\left(-\frac{\ln\left(m^2\right)}{s+ux}+\frac{s(x-1)\ln\left(\frac{m^2}{s+ux}\right)}{m^2(s+u)+s(x-1)(s+ux)}+\frac{s(x-1)\left(\ln(s-sx)-\ln\left(m^2+s(x-1)x\right)\right)}{m^2(s+u)+s(x-1)(s+ux)}+\frac{\ln\left(m^2+s(x-1)x\right)}{s-sx}\right)}{(s+u)^3} \\
+ & \frac{2(1-2x)^2(s+2tx)\left(s^2tx^2\ln\left(m^2+t(x-1)x\right)-s^2tx\ln\left(m^2+t(x-1)x\right)+m^2s^2\ln\left(m^2+t(x-1)x\right)+st^2x^2\ln\left(m^2+t(x-1)x\right)-st^2x\ln\left(m^2+t(x-1)x\right)\right)}{s^2tx(s+t)\left(m^2s+t(x-1)x(s+t)\right)} \\
+ & \frac{2(1-2x)^2(s+2tx)\left(m^2st\ln\left(m^2+t(x-1)x\right)-m^2st\ln\left(m^2\right)-t^3x^2\ln(x(s+t))+st^2x^2\ln(-tx)\right)}{s^2tx(s+t)\left(m^2s+t(x-1)x(s+t)\right)} \\
+ & \frac{2(1-2x)^2(s+2tx)\left(-st^2x^2\ln(x(s+t))+t^3x\ln(x(s+t))-st^2x\ln(-tx)+st^2x\ln(x(s+t))+t^3x^2\ln(-tx)-t^3x\ln(-tx)\right)}{s^2tx(s+t)\left(m^2s+t(x-1)x(s+t)\right)}+\frac{4(1-2x)^2\left(-\mathrm{Li}_2\left(\frac{m^2s}{sm^2+t(s+t)(x-1)x}\right)\right)}{s^2} \\
+ & \frac{4(1-2x)^2\left(\mathrm{Li}_2\left(\frac{s\left(m^2+t(x-1)x\right)}{sm^2+t(s+t)(x-1)x}\right)+\ln\left(m^2+t(x-1)x\right)\ln\left(-\frac{t^2(x-1)x}{m^2s+t(x-1)x(s+t)}\right)-i\pi\ln\left(m^2+\frac{t(x-1)x(s+t)}{s}\right)-\ln\left(m^2\right)\ln\left(\frac{t(x-1)x(s+t)}{m^2s+t(x-1)x(s+t)}\right)+i\pi\ln\left(m^2+t(x-1)x\right)\right)}{s^2} \\
+ & \frac{2(1-2x)^2(s+2t(x-1))\left(\frac{m^2s\ln\left(m^2\right)}{sx+t(x-1)}-\frac{s\left(m^2+t(x-1)x\right)\ln\left(m^2+t(x-1)x\right)}{t(x-1)}+t(x-1)\ln(-sx+t(-x)+t)-t(x-1)\ln(t-tx)\right)}{s^2\left(m^2s+t(x-1)(sx+t(x-1))\right)} \\
- & \frac{4(1-2x)^2\left(-\mathrm{Li}_2\left(\frac{m^2s}{sm^2+t(x-1)(t(x-1)+sx)}\right)+\mathrm{Li}_2\left(\frac{s\left(m^2+t(x-1)x\right)}{sm^2+t(x-1)(t(x-1)+sx)}\right)+\ln\left(m^2+t(x-1)x\right)\ln\left(\frac{t^2(x-1)^2}{m^2s+t(x-1)(sx+t(x-1))}\right)-\ln\left(m^2\right)\ln\left(\frac{t(x-1)(sx+t(x-1))}{m^2s+t(x-1)(sx+t(x-1))}\right)\right)}{s^2}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2(2x-1)^2(2tx-t-u)\left(t^2ux^2\ln\left(m^2+u(x-1)x\right)+m^2t^2\ln\left(m^2+u(x-1)x\right)-t^2ux\ln\left(m^2+u(x-1)x\right)+tu^2x^2\ln\left(m^2\right)+tu^2x^2\ln\left(-\frac{tx}{m^2}\right)\right)}{tux(t+u)^2\left(m^2t+m^2u+tux^2-tux\right)} \\
&+ \frac{2(2x-1)^2(2tx-t-u)\left(tu^2x^2\ln\left(m^2+u(x-1)x\right)-tu^2x\ln\left(m^2\right)-tu^2x\ln\left(-\frac{tx}{m^2}\right)-tu^2x\ln\left(m^2+u(x-1)x\right)+m^2tu\ln\left(m^2+u(x-1)x\right)\right)}{tux(t+u)^2\left(m^2t+m^2u+tux^2-tux\right)} \\
&+ \frac{2(2x-1)^2(2tx-t-u)\left(m^2tu\ln\left(m^2\right)+m^2u^2\ln\left(m^2\right)-tu^2x^2\ln(u(-x))+tu^2x\ln(u(-x))\right)}{tux(t+u)^2\left(m^2t+m^2u+tux^2-tux\right)} + \frac{4(1-2x)^2\left(\mathrm{Li}_2\left(\frac{m^2(t+u)}{(t+u)m^2+tu(x-1)x}\right)-\mathrm{Li}_2\left(\frac{(t+u)\left(m^2+u(x-1)x\right)}{(t+u)m^2+tu(x-1)x}\right)\right)}{(t+u)^2} \\
&- \frac{4(1-2x)^2\left(+\ln\left(m^2+u(x-1)x\right)\left(-\ln\left(\frac{u^2(x-1)x}{m^2(t+u)+tu(x-1)x}\right)\right)+i\pi\ln\left(m^2+\frac{tu(x-1)x}{t+u}\right)+\ln\left(m^2\right)\ln\left(\frac{tu(x-1)x}{m^2(t+u)+tu(x-1)x}\right)-i\pi\ln\left(m^2+u(x-1)x\right)\right)}{(t+u)^2} \\
&+ \frac{2(1-2(1-x))^2(t(2(1-x)-1)+u)\left(\frac{\ln\left(m^2\right)}{t(1-x)+u}+\frac{u((1-x)-1)\ln\left(\frac{t(1-x)+u}{m^2}\right)}{m^2(t+u)+u((1-x)-1)(t(1-x)+u)}-\frac{u((1-x)-1)\left(\ln(u-u(1-x))-\ln\left(m^2+u((1-x)-1)(1-x)\right)\right)}{m^2(t+u)+u((1-x)-1)(t(1-x)+u)}+\frac{\ln\left(m^2+u((1-x)-1)(1-x)\right)}{u((1-x)-1)}\right)}{(t+u)^2} \\
&+ \frac{4(1-2x)^2\left(\mathrm{Li}_2\left(\frac{m^2(t+u)}{(t+u)m^2+u(u+tx)(x-1)}\right)-\mathrm{Li}_2\left(\frac{(t+u)\left(m^2+u(x-1)x\right)}{(t+u)m^2+u(u+tx)(x-1)}\right)-\ln\left(m^2+u(x-1)x\right)\ln\left(-\frac{u^2(x-1)^2}{m^2(t+u)+u(x-1)(tx+u)}\right)+\ln\left(m^2\right)\ln\left(\frac{u(x-1)(tx+u)}{m^2(t+u)+u(x-1)(tx+u)}\right)\right)}{(t+u)^2}
\end{aligned}
\tag{C.1}
$$

Appendix D

Integral Over Penultimate Variable for $\gamma_{(12)12}$

In this appendix, we show the result of the integration over the penultimate parameter for $\gamma_{(12)12}$:

$$\frac{8(1-2x)x}{(x-1)(s+u)^2}+\frac{(2x-1)\left(2m^2+s(x-1)\right)\left(2m^2+2sx^2-3sx+s\right)^2\left(\log\left(m^2+s(x-1)x\right)-\log\left(m^2\right)\right)}{s(x-1)^2\left(m^2(s+u)+su(x-1)x\right)\left(m^2(s+u)+s(x-1)(s+ux)\right)}+\frac{(2x-1)(-2sx+s+u)^2(s-2ux+u)\log\left(-\frac{u}{s}\right)}{(s+u)^3\left(m^2(s+u)+su(x-1)x\right)}$$

$$-\quad\frac{(2x-1)(s(3-2x)+u)^2(s+u(2x-1))\log\left(\frac{s-sx}{s+ux}\right)}{(s+u)^3\left(m^2(s+u)+s(x-1)(s+ux)\right)}+\frac{(1-2x)^2(s+2tx)(\log(s+t)-\log(-t))}{s\left(m^2s+t(x-1)x(s+t)\right)}+\frac{(1-2x)^2(s+2t(x-1))(\log(sx+t(x-1))-\log(t(x-1)))}{s\left(m^2s+t(x-1)(sx+t(x-1))\right)}$$

$$+\quad\frac{(1-2x)^2\left(2m^2+2tx^2-3tx+t\right)\left(2m^2s+t(x-1)(2sx+t(2x-1))\right)\left(\log\left(m^2\right)-\log\left(m^2+t(x-1)x\right)\right)}{t(x-1)\left(m^2s+t(x-1)x(s+t)\right)\left(m^2s+t(x-1)(sx+t(x-1))\right)}+\frac{(2x-1)^3(t(2x-1)-u)(\log(-u)-\log(-t))}{(t+u)\left(m^2(t+u)+tu(x-1)x\right)}$$

$$+\quad\frac{(2x-1)^3(t(2x-1)+u)(\log(tx+u)-\log(u-ux))}{(t+u)\left(m^2(t+u)+u(x-1)(tx+u)\right)}+\frac{u(2x-1)^3\left(2m^2+u(x-1)\right)\left(\log\left(m^2+u(x-1)x\right)-\log\left(m^2\right)\right)}{\left(m^2(t+u)+tu(x-1)x\right)\left(m^2(t+u)+u(x-1)(tx+u)\right)}$$

$$+\quad\frac{8\left(s(1-2x)^2-2ux+u\right)\left(-\text{Li}_2\left(\frac{m^2(s+u)}{(s+u)m^2+su(x-1)x}\right)+\text{Li}_2\left(\frac{\left(m^2+s(x-1)x\right)(s+u)}{(s+u)m^2+su(x-1)x}\right)+\log\left(m^2+s(x-1)x\right)\log\left(-\frac{s^2(x-1)x}{m^2(s+u)+su(x-1)x}\right)-\log\left(m^2\right)\log\left(\frac{su(x-1)x}{m^2(s+u)+su(x-1)x}\right)\right)}{(s+u)^3}$$

$$-\quad\frac{8(2x-1)(s(2x-3)-u)\left(-\text{Li}_2\left(\frac{m^2(s+u)}{(s+u)m^2+s(s+ux)(x-1)}\right)+\text{Li}_2\left(\frac{\left(m^2+s(x-1)x\right)(s+u)}{(s+u)m^2+s(s+ux)(x-1)}\right)+\log\left(m^2+s(x-1)x\right)\log\left(-\frac{s^2(x-1)^2}{m^2(s+u)+s(x-1)(s+ux)}\right)-\log\left(m^2\right)\log\left(\frac{s(x-1)(s+ux)}{m^2(s+u)+s(x-1)(s+ux)}\right)\right)}{(s+u)^3}$$

$$+\quad\frac{2(2x-1)(-2sx+s+u)^2\left(u\left((s+u)\left(m^2+s(x-1)x\right)\log\left(m^2+s(x-1)x\right)+s^2(x-1)x\log(u(-x))-s^2(x-1)x\log(sx)\right)+m^2s\log\left(m^2\right)(s+u)\right)}{sux(s+u)^3\left(m^2(s+u)+su(x-1)x\right)}$$

$$+\quad\frac{2(2x-1)(s(3-2x)+u)^2\left(-\frac{\log\left(m^2\right)}{s+ux}+\frac{s(x-1)\left(\log\left(m^2\right)-\log(s+ux)\right)}{m^2(s+u)+s(x-1)(s+ux)}+\frac{s(x-1)\left(\log(s-sx)-\log\left(m^2+s(x-1)x\right)\right)}{m^2(s+u)+s(x-1)(s+ux)}+\frac{\log\left(m^2+s(x-1)x\right)}{s-sx}\right)}{(s+u)^3}$$

$$+\quad\frac{2(1-2x)^2(s+2tx)\left(s^2tx^2\log\left(m^2+t(x-1)x\right)-s^2tx\log\left(m^2+t(x-1)x\right)+m^2s^2\log\left(m^2+t(x-1)x\right)+st^2x^2\log\left(m^2+t(x-1)x\right)-st^2x\log\left(m^2+t(x-1)x\right)\right)}{s^2tx(s+t)\left(m^2s+t(x-1)x(s+t)\right)}$$

$$+\quad\frac{2(1-2x)^2(s+2tx)\left(m^2st\log\left(m^2+t(x-1)x\right)-m^2st\log\left(m^2\right)-i\pi st^2x^2-t^3x^2\log(x(s+t))+st^2x^2\log(tx)-st^2x^2\log(x(s+t))+i\pi st^2x+t^3x\log(x(s+t))-st^2x\log(tx)\right)}{s^2tx(s+t)\left(m^2s+t(x-1)x(s+t)\right)}$$

$$+\quad\frac{2(1-2x)^2(s+2tx)\left(st^2x\log(x(s+t))-i\pi t^3x^2+t^3x^2\log(tx)+i\pi t^3x-t^3x\log(tx)\right)}{s^2tx(s+t)\left(m^2s+t(x-1)x(s+t)\right)}+\frac{4(1-2x)^2\left(-\text{Li}_2\left(\frac{m^2s}{sm^2+t(s+t)(x-1)x}\right)+\text{Li}_2\left(\frac{s\left(m^2+t(x-1)x\right)}{sm^2+t(s+t)(x-1)x}\right)\right)}{s^2}$$

$$+\quad\frac{4(1-2x)^2\left(\log\left(m^2+t(x-1)x\right)\log\left(-\frac{t^2(x-1)x}{m^2s+t(x-1)x(s+t)}\right)-i\pi\log\left(m^2+\frac{t(x-1)x(s+t)}{s}\right)-\log\left(m^2\right)\log\left(\frac{t(x-1)x(s+t)}{m^2s+t(x-1)x(s+t)}\right)+i\pi\log\left(m^2+t(x-1)x\right)\right)}{s^2}$$

$$+\quad\frac{2(1-2x)^2(s+2t(x-1))\left(\frac{m^2s\log\left(m^2\right)}{sx+t(x-1)}-\frac{s\left(m^2+t(x-1)x\right)\log\left(m^2+t(x-1)x\right)}{t(x-1)}+t(x-1)\log(-sx+t(-x)+t)-t(x-1)\log(t-tx)\right)}{s^2\left(m^2s+t(x-1)(sx+t(x-1))\right)}$$

$$
\begin{aligned}
&- \frac{4(1-2x)^2\left(-\mathrm{Li}_2\left(\frac{m^2 s}{sm^2+t(x-1)(t(x-1)+sx)}\right)+\mathrm{Li}_2\left(\frac{s\left(m^2+t(x-1)x\right)}{sm^2+t(x-1)(t(x-1)+sx)}\right)+\log\left(m^2+t(x-1)x\right)\log\left(\frac{t^2(x-1)^2}{m^2 s+t(x-1)(sx+t(x-1))}\right)-\log\left(m^2\right)\log\left(\frac{t(x-1)(sx+t(x-1))}{m^2 s+t(x-1)(sx+t(x-1))}\right)\right)}{s^2}\\
&- \frac{2(2x-1)^2(2tx-t-u)\left(t^2ux^2\log\left(m^2+u(x-1)x\right)+m^2t^2\log\left(m^2+u(x-1)x\right)-t^2ux\log\left(m^2+u(x-1)x\right)+tu^2x^2\log\left(m^2\right)+tu^2x^2\log\left(-\frac{tx}{m^2}\right)+tu^2x^2\log\left(m^2+u(x-1)x\right)\right)}{tux(t+u)^2\left(m^2t+m^2u+tux^2-tux\right)}\\
&- \frac{2(2x-1)^2(2tx-t-u)\left(-tu^2x\log\left(m^2\right)-tu^2x\log\left(-\frac{tx}{m^2}\right)-tu^2x\log\left(m^2+u(x-1)x\right)+m^2tu\log\left(m^2+u(x-1)x\right)+m^2tu\log\left(m^2\right)+m^2u^2\log\left(m^2\right)-tu^2x^2\log(u(-x))+tu^2x\log(u(-x))\right)}{tux(t+u)^2\left(m^2t+m^2u+tux^2-tux\right)}\\
&- \frac{4(1-2x)^2\left(\mathrm{Li}_2\left(\frac{m^2(t+u)}{(t+u)m^2+tu(x-1)x}\right)-\mathrm{Li}_2\left(\frac{(t+u)\left(m^2+u(x-1)x\right)}{(t+u)m^2+tu(x-1)x}\right)+\log\left(m^2+u(x-1)x\right)\left(-\log\left(\frac{u^2(x-1)x}{m^2(t+u)+tu(x-1)x}\right)\right)+i\pi\log\left(m^2+\frac{tu(x-1)x}{t+u}\right)+\log\left(m^2\right)\log\left(\frac{tu(x-1)x}{m^2(t+u)+tu(x-1)x}\right)\right)}{(t+u)^2}\\
&+ \frac{4(1-2x)^2\left(i\pi\log\left(m^2+u(x-1)x\right)\right)}{(t+u)^2}\\
&+ \frac{2(1-2(1-x))^2(t(2(1-x)-1)+u)\left(\frac{\log\left(m^2\right)}{t(1-x)+u}+\frac{u((1-x)-1)\log\left(\frac{t(1-x)+u}{m^2}\right)}{m^2(t+u)+u((1-x)-1)(t(1-x)+u)}-\frac{u((1-x)-1)\left(\log(u-u(1-x))-\log\left(m^2+u((1-x)-1)(1-x)\right)\right)}{m^2(t+u)+u((1-x)-1)(t(1-x)+u)}+\frac{\log\left(m^2+u((1-x)-1)(1-x)\right)}{u((1-x)-1)}\right)}{(t+u)^2}\\
&+ \frac{4(1-2x)^2\left(\mathrm{Li}_2\left(\frac{m^2(t+u)}{(t+u)m^2+u(u+tx)(x-1)}\right)-\mathrm{Li}_2\left(\frac{(t+u)\left(m^2+u(x-1)x\right)}{(t+u)m^2+u(u+tx)(x-1)}\right)-\log\left(m^2+u(x-1)x\right)\log\left(-\frac{u^2(x-1)^2}{m^2(t+u)+u(x-1)(tx+u)}\right)+\log\left(m^2\right)\log\left(\frac{u(x-1)(tx+u)}{m^2(t+u)+u(x-1)(tx+u)}\right)\right)}{(t+u)^2}
\end{aligned}
\tag{D.1}
$$

Appendix E

Integral Over Penultimate Variable for $\gamma_{(12)(34)}$

In this appendix, we show the result of the integration over the penultimate parameter for $\gamma_{(12)(34)}$:

$$
\begin{aligned}
& \frac{\left(2m^2+2sx^2-3sx+s\right)^2\left(\ln\left(m^2\right)-\ln\left(m^2+s(x-1)x\right)\right)}{(x-1)\left(m^2(s+u)+su(x-1)x\right)\left(m^2(s+u)+s(x-1)(s+ux)\right)}+\frac{(-2sx+s+u)^2\ln\left(-\frac{s}{u}\right)}{(s+u)^2\left(m^2(s+u)+su(x-1)x\right)}+-\frac{(s(3-2x)+u)^2\ln\left(\frac{s-sx}{s+ux}\right)}{(s+u)^2\left(m^2(s+u)+s(x-1)(s+ux)\right)} \\
+ \quad & \frac{(1-2x)^2(s(2x-1)+2tx)^2(\ln(s+t)-\ln(-t))}{s^2\left(m^2s+t(x-1)x(s+t)\right)}+\frac{(1-2x)^2(s(2x-1)+2t(x-1))^2\ln\left(\frac{t(x-1)}{sx+t(x-1)}\right)}{s^2\left(m^2s+t(x-1)(sx+t(x-1))\right)}+\frac{(1-2x)^2\left(2m^2+t(x-1)\right)^2\left(\ln\left(m^2\right)-\ln\left(m^2+t(x-1)x\right)\right)}{(x-1)\left(m^2s+t(x-1)x(s+t)\right)\left(m^2s+t(x-1)(sx+t(x-1))\right)} \\
+ \quad & \frac{\left(t(1-2x)^2-2ux+u\right)^2(\ln(-u)-\ln(-t))}{(t+u)^2\left(m^2(t+u)+tu(x-1)x\right)}+\frac{(1-2x)^2(t(2x-1)+u)^2(\ln(tx+u)-\ln(u-ux))}{(t+u)^2\left(m^2(t+u)+u(x-1)(tx+u)\right)}+\frac{(1-2x)^2\left(2m^2+u(x-1)\right)^2\left(\ln\left(m^2\right)-\ln\left(m^2+u(x-1)x\right)\right)}{(x-1)\left(m^2(t+u)+tu(x-1)x\right)\left(m^2(t+u)+u(x-1)(tx+u)\right)} \\
+ \quad & \frac{32s(x-1)(s(2x-1)-u(x+1))\ln\left(-\frac{s}{u}\right)}{(s+u)^4}-\frac{32s(x-1)(2s(x-2)-u(x+1))\ln\left(\frac{s-sx}{s+ux}\right)}{(s+u)^4}+\frac{4x(s(3-2x)+u)^2(s(x-2)-ux)}{s(x-1)(s+u)^3(s+ux)}+\frac{8(x-1)\left(s^2(1-2x)^2+su\left(2-8x^2\right)+u^2(4x+1)\right)}{ux(s+u)^3} \\
- \quad & \frac{8x\left(s^2\left(4x^2-20x+21\right)+2su\left(-4x^2+4x+5\right)+u^2(4x+1)\right)}{(s+u)^3(s+ux)}-\frac{4(x-1)(s-u)(-2sx+s+u)^2}{sux(s+u)^3}+\frac{8(1-2x)^2(x-1)}{sx(s+t)}+\frac{4(1-2x)^2x(sx+2t(x-1))}{st(x-1)(sx+t(x-1))}-\frac{8(1-2x)^2x}{s(sx+t(x-1))} \\
+ \quad & \frac{8(1-2x)^2(x-1)}{tx(t+u)}-\frac{8(1-2x)^2x}{(t+u)(tx+u)}+\frac{4(1-2x)^2x(u(x-2)-tx)}{u(x-1)(t+u)(tx+u)}-\frac{32\left(-\left(m^2+s(x-1)x\right)\ln\left(m^2+s(x-1)x\right)+m^2\ln\left(m^2\right)+s(x-1)x\right)}{s(x-1)(s+u)^2} \\
+ \quad & \frac{16\left(2m^2(s+u)-s^2(1-2x)^2+2su(x-1)x+u^2\right)\left(-\mathrm{Li}_2\left(\frac{m^2(s+u)}{(s+u)m^2+su(x-1)x}\right)+\mathrm{Li}_2\left(\frac{\left(m^2+s(x-1)x\right)(s+u)}{(s+u)m^2+su(x-1)x}\right)+\ln\left(m^2+s(x-1)x\right)\ln\left(-\frac{s^2(x-1)x}{m^2(s+u)+su(x-1)x}\right)-\ln\left(m^2\right)\ln\left(\frac{su(x-1)x}{m^2(s+u)+su(x-1)x}\right)\right)}{(s+u)^4} \\
+ \quad & \frac{16\left(-2m^2(s+u)+s^2\left(4x^2-14x+11\right)-2su\left(x^2+x-3\right)+u^2\right)\left(-\mathrm{Li}_2\left(\frac{m^2(s+u)}{(s+u)m^2+s(s+ux)(x-1)}\right)+\mathrm{Li}_2\left(\frac{\left(m^2+s(x-1)x\right)(s+u)}{(s+u)m^2+s(s+ux)(x-1)}\right)+\ln\left(m^2+s(x-1)x\right)\ln\left(-\frac{s^2(x-1)^2}{m^2(s+u)+s(x-1)(s+ux)}\right)\right)}{(s+u)^4} \\
+ \quad & \frac{16\left(-2m^2(s+u)+s^2\left(4x^2-14x+11\right)-2su\left(x^2+x-3\right)+u^2\right)\left(-\ln\left(m^2\right)\ln\left(\frac{s(x-1)(s+ux)}{m^2(s+u)+s(x-1)(s+ux)}\right)\right)}{(s+u)^4} \\
+ \quad & \frac{4(s(2x-3)-u)\left(-8m^2(s+u)+s^2\left(4x^2-20x+17\right)+2su\left(-4x^2+2x+3\right)+u^2\right)\left(\frac{\ln\left(m^2\right)}{s+ux}+\frac{s(x-1)\left(\ln\left(m^2+s(x-1)x\right)-\ln(s-sx)\right)}{m^2(s+u)+s(x-1)(s+ux)}-\frac{s(x-1)\ln\left(\frac{m^2}{s+ux}\right)}{m^2(s+u)+s(x-1)(s+ux)}+\frac{\ln\left(m^2+s(x-1)x\right)}{s(x-1)}\right)}{(s+u)^4} \\
- \quad & \frac{4(s(2x-1)-u)\left(-8m^2(s+u)+s^2(1-2x)^2-8su(x-1)x-u^2\right)\left(u\left((s+u)\left(m^2+s(x-1)x\right)\ln\left(m^2+s(x-1)x\right)+s^2(x-1)x\ln(u(-x))-s^2(x-1)x\ln(sx)\right)+m^2s\ln\left(m^2\right)(s+u)\right)}{sux(s+u)^4\left(m^2(s+u)+su(x-1)x\right)} \\
+ \quad & \frac{4(-2sx+s+u)^2\left(m^2(s+u)+su(x-1)x\right)\left(-\frac{\ln\left(m^2+s(x-1)x\right)}{s^2x^2}+\frac{s(x-1)\left(-\frac{m^2(s+u)}{sx}+s(x-1)\ln\left(m^2+s(x-1)x\right)-s(x-1)\ln(-sx)-ux+u\right)}{\left(m^2(s+u)+su(x-1)x\right)^2}-\frac{s(x-1)\left(\frac{m^2(s+u)}{ux}+s(x-1)\ln\left(m^2\right)-s(x-1)\ln(ux)+s(x-1)\right)}{\left(m^2(s+u)+su(x-1)x\right)^2}+\frac{\ln\left(m^2\right)}{u^2x^2}\right)}{(s+u)^4} \\
+ \quad & \frac{4(s(3-2x)+u)^2\left(m^2(s+u)+s(x-1)(s+ux)\right)\left(\frac{-s^2(x-1)^2\ln\left(m^2+s(x-1)x\right)+m^2s+m^2u+s^2x+s^2(x-1)^2\ln(s-sx)-s^2+sux^2-sux}{\left(m^2(s+u)+s(x-1)(s+ux)\right)^2}+\frac{\ln\left(m^2+s(x-1)x\right)}{s^2(x-1)^2}-\frac{\ln\left(m^2\right)}{(s+ux)^2}\right)}{(s+u)^4}
\end{aligned}
$$

$$+\frac{4(-2sx+s+u)^2\left(m^2(s+u)+su(x-1)x\right)\left(\frac{s(x-1)\left(\frac{m^2(s+u)}{s+ux}+s(x-1)\ln\left(m^2\right)-s(x-1)\ln(s+ux)+s(x-1)\right)}{\left(m^2(s+u)+s(x-1)(s+ux)\right)^2}\right)}{(s+u)^4}$$

$$+\frac{4(2x-1)^2\left(x^4\ln(-tx)t^6-2x^3\ln(-tx)t^6+x^2\ln(-tx)t^6-x^4\ln((s+t)x)t^6+2x^3\ln((s+t)x)t^6-x^2\ln((s+t)x)t^6-sx^4t^5+2sx^3t^5-sx^2t^5+2sx^4\ln(-tx)t^5\right)}{s^2t^2(s+t)^2x^2\left(sm^2+t^2x^2+stx^2-t^2x-stx\right)}$$

$$+\frac{4(2x-1)^2\left(-4sx^3\ln(-tx)t^5+2sx^2\ln(-tx)t^5-2sx^4\ln((s+t)x)t^5+4sx^3\ln((s+t)x)t^5-2sx^2\ln((s+t)x)t^5+2sx^4\ln\left(m^2+t(x-1)x\right)t^5-4sx^3\ln\left(m^2+t(x-1)x\right)t^5\right)}{s^2t^2(s+t)^2x^2\left(sm^2+t^2x^2+stx^2-t^2x-stx\right)}$$

$$+\frac{4(2x-1)^2\left(2sx^2\ln\left(m^2+t(x-1)x\right)t^5-2s^2x^4t^4+4s^2x^3t^4-2s^2x^2t^4-2m^2sx^2\ln\left(m^2\right)t^4+2m^2sx\ln\left(m^2\right)t^4+s^2x^4\ln(-tx)t^4-2s^2x^3\ln(-tx)t^4\right)}{s^2t^2(s+t)^2x^2\left(sm^2+t^2x^2+stx^2-t^2x-stx\right)}$$

$$+\frac{4(2x-1)^2\left(s^2x^2\ln(-tx)t^4-s^2x^4\ln((s+t)x)t^4+2s^2x^3\ln((s+t)x)t^4-s^2x^2\ln((s+t)x)t^4+5s^2x^4\ln\left(m^2+t(x-1)x\right)t^4-10s^2x^3\ln\left(m^2+t(x-1)x\right)t^4\right)}{s^2t^2(s+t)^2x^2\left(sm^2+t^2x^2+stx^2-t^2x-stx\right)}$$

$$+\frac{4(2x-1)^2\left(5s^2x^2\ln\left(m^2+t(x-1)x\right)t^4+2m^2sx^2\ln\left(m^2+t(x-1)x\right)t^4-2m^2sx\ln\left(m^2+t(x-1)x\right)t^4-s^3x^4t^3+2s^3x^3t^3-s^3x^2t^3-m^2s^2x^2t^3\right)}{s^2t^2(s+t)^2x^2\left(sm^2+t^2x^2+stx^2-t^2x-stx\right)}$$

$$+\frac{4(2x-1)^2\left(m^2s^2xt^3-2m^2s^2x^2\ln\left(m^2\right)t^3+2m^2s^2x\ln\left(m^2\right)t^3+4s^3x^4\ln\left(m^2+t(x-1)x\right)t^3-8s^3x^3\ln\left(m^2+t(x-1)x\right)t^3+4s^3x^2\ln\left(m^2+t(x-1)x\right)t^3\right)}{s^2t^2(s+t)^2x^2\left(sm^2+t^2x^2+stx^2-t^2x-stx\right)}$$

$$+\frac{4(2x-1)^2\left(6m^2s^2x^2\ln\left(m^2+t(x-1)x\right)t^3-6m^2s^2x\ln\left(m^2+t(x-1)x\right)t^3-m^2s^3x^2t^2+m^2s^3xt^2-m^4s^2\ln\left(m^2\right)t^2+s^4x^4\ln\left(m^2+t(x-1)x\right)t^2-2s^4x^3\ln\left(m^2+t(x-1)x\right)t^2\right)}{s^2t^2(s+t)^2x^2\left(sm^2+t^2x^2+stx^2-t^2x-stx\right)}$$

$$+\frac{4(2x-1)^2\left(m^4s^2\ln\left(m^2+t(x-1)x\right)t^2+s^4x^2\ln\left(m^2+t(x-1)x\right)t^2+6m^2s^3x^2\ln\left(m^2+t(x-1)x\right)t^2-6m^2s^3x\ln\left(m^2+t(x-1)x\right)t^2+2m^4s^3\ln\left(m^2+t(x-1)x\right)t\right)}{s^2t^2(s+t)^2x^2\left(sm^2+t^2x^2+stx^2-t^2x-stx\right)}$$

$$+\frac{4(2x-1)^2\left(2m^2s^4x^2\ln\left(m^2+t(x-1)x\right)t-2m^2s^4x\ln\left(m^2+t(x-1)x\right)t+m^4s^4\ln\left(m^2+t(x-1)x\right)\right)}{s^2t^2(s+t)^2x^2\left(sm^2+t^2x^2+stx^2-t^2x-stx\right)}$$

$$+\frac{4(2x-1)^3(s+2t)\left(-s^2tx^2\ln\left(m^2+t(x-1)x\right)+s^2tx\ln\left(m^2+t(x-1)x\right)-m^2s^2\ln\left(m^2+t(x-1)x\right)-st^2x^2\ln\left(m^2+t(x-1)x\right)+st^2x\ln\left(m^2+t(x-1)x\right)\right)}{s^2tx(s+t)\left(m^2s+t(x-1)x(s+t)\right)}$$

$$+\frac{4(2x-1)^3(s+2t)\left(-m^2st\ln\left(m^2+t(x-1)x\right)+2m^2st\ln(m)+t^3x^2\ln(x(s+t))-st^2x^2\ln(-tx)+st^2x^2\ln(x(s+t))-t^3x\ln(x(s+t))\right)}{s^2tx(s+t)\left(m^2s+t(x-1)x(s+t)\right)}$$

$$+\frac{4(2x-1)^3(s+2t)\left(st^2x\ln(-tx)-st^2x\ln(x(s+t))+t^3\left(-x^2\right)\ln(-tx)+t^3x\ln(-tx)\right)}{s^2tx(s+t)\left(m^2s+t(x-1)x(s+t)\right)}-\frac{8(1-2x)^2\left(-\mathrm{Li}_2\left(\frac{m^2s}{sm^2+t(s+t)(x-1)x}\right)+\mathrm{Li}_2\left(\frac{s\left(m^2+t(x-1)x\right)}{sm^2+t(s+t)(x-1)x}\right)\right)}{s^2}$$

$$-\frac{8(1-2x)^2\left(\ln\left(m^2+t(x-1)x\right)\ln\left(-\frac{t^2(x-1)x}{m^2s+t(x-1)x(s+t)}\right)-i\pi\ln\left(m^2+\frac{t(x-1)x(s+t)}{s}\right)-\ln\left(m^2\right)\ln\left(\frac{t(x-1)x(s+t)}{m^2s+t(x-1)x(s+t)}\right)+i\pi\ln\left(m^2+t(x-1)x\right)\right)}{s^2}$$

$$+\frac{4(1-2x)^2\left(m^2s+t(x-1)(sx+t(x-1))\right)\left(\frac{\ln\left(m^2\right)}{(sx+t(x-1))^2}+\frac{t(x-1)\left(\frac{m^2s}{t(x-1)}+t(x-1)\ln\left(m^2+t(x-1)x\right)+sx+t(x-1)-t(x-1)\ln(t-tx)\right)}{\left(m^2s+t(x-1)(sx+t(x-1))\right)^2}-\frac{t(x-1)\left(\frac{m^2s}{sx+t(x-1)}+t(x-1)\ln\left(m^2\right)-t(x-1)\ln(-sx+t(-x)+t)+t(x-1)\right)}{\left(m^2s+t(x-1)(sx+t(x-1))\right)^2}\right)}{s^2}$$

$$+\frac{4(1-2x)^2\left(-\frac{\ln\left(m^2+t(x-1)x\right)}{t^2(x-1)^2}\right)}{s^2}+\frac{4(1-2x)^2(s(2x-1)+4t(x-1))\left(\frac{m^2s\ln\left(m^2\right)}{-sx+t(-x)+t}+\frac{s\left(m^2+t(x-1)x\right)\ln\left(m^2+t(x-1)x\right)}{t(x-1)}-t(x-1)\ln(-sx+t(-x)+t)+t(x-1)\ln(t-tx)\right)}{s^2\left(m^2s+t(x-1)(sx+t(x-1))\right)}$$

$$+\frac{8(1-2x)^2\left(-\mathrm{Li}_2\left(\frac{m^2s}{sm^2+t(x-1)(t(x-1)+sx)}\right)+\mathrm{Li}_2\left(\frac{s\left(m^2+t(x-1)x\right)}{sm^2+t(x-1)(t(x-1)+sx)}\right)+\ln\left(m^2+t(x-1)x\right)\ln\left(\frac{t^2(x-1)^2}{m^2s+t(x-1)(sx+t(x-1))}\right)-\ln\left(m^2\right)\ln\left(\frac{t(x-1)(sx+t(x-1))}{m^2s+t(x-1)(sx+t(x-1))}\right)\right)}{s^2}$$

$$-\frac{4(1-2x)^2\left(t\left((t+u)\left(u^2(x-1)x\left(m^2(t+u)+tu(x-1)x\right)+t\left(m^2+u(x-1)x\right)\left(m^2(t+u)+u(x-1)x(t-u)\right)\ln\left(m^2+u(x-1)x\right)\right)-tu^4(x-1)^2x^2\ln(-tx)+tu^4(x-1)^2x^2\ln(-ux)\right)\right)}{t^2u^2x^2(t+u)^2\left(m^2(t+u)+tu(x-1)x\right)}$$

$$+\frac{4(1-2x)^2\left(-m^2u^2\ln\left(m^2\right)(t+u)\left(m^2(t+u)+2tu(x-1)x\right)\right)}{t^2u^2x^2(t+u)^2\left(m^2(t+u)+tu(x-1)x\right)}$$

$$+\frac{4(2x-1)^3(t-u)\left(t\left(u^2(x-1)x\ln\left(-\frac{tx}{m^2}\right)+(t+u)\left(m^2+u(x-1)x\right)\ln\left(m^2+u(x-1)x\right)-u^2(x-1)x\ln(-ux)\right)+u\ln\left(m^2\right)\left(m^2(t+u)+tu(x-1)x\right)\right)}{tux(t+u)^2\left(m^2(t+u)+tu(x-1)x\right)}$$

$$
\begin{aligned}
&+ \frac{8(1-2x)^2\left(\mathrm{Li}_2\left(\frac{m^2(t+u)}{(t+u)m^2+tu(x-1)x}\right)-\mathrm{Li}_2\left(\frac{(t+u)\left(m^2+u(x-1)x\right)}{(t+u)m^2+tu(x-1)x}\right)+\ln\left(m^2+u(x-1)x\right)\left(-\ln\left(\frac{u^2(x-1)x}{m^2(t+u)+tu(x-1)x}\right)\right)+i\pi\ln\left(m^2+\frac{tu(x-1)x}{t+u}\right)+\ln\left(m^2\right)\ln\left(\frac{tu(x-1)x}{m^2(t+u)+tu(x-1)x}\right)\right)}{(t+u)^2} \\
&+ \frac{8(1-2x)^2\left(-i\pi\ln\left(m^2+u(x-1)x\right)\right)}{(t+u)^2}+\frac{4(1-2x)^2(t(2x-1)+u(3-2x))\left(-\frac{\ln\left(m^2\right)}{tx+u}+\frac{u(x-1)\left(\ln\left(m^2\right)-\ln(tx+u)\right)}{m^2(t+u)+u(x-1)(tx+u)}+\frac{u(x-1)\left(\ln(u-ux)-\ln\left(m^2+u(x-1)x\right)\right)}{m^2(t+u)+u(x-1)(tx+u)}+\frac{\ln\left(m^2+u(x-1)x\right)}{u-ux}\right)}{(t+u)^2} \\
&+ \frac{8(1-2x)^2\left(-\mathrm{Li}_2\left(\frac{m^2(t+u)}{(t+u)m^2+u(u+tx)(x-1)}\right)+\mathrm{Li}_2\left(\frac{(t+u)\left(m^2+u(x-1)x\right)}{(t+u)m^2+u(u+tx)(x-1)}\right)+\ln\left(m^2+u(x-1)x\right)\ln\left(-\frac{u^2(x-1)^2}{m^2(t+u)+u(x-1)(tx+u)}\right)-\ln\left(m^2\right)\ln\left(\frac{u(x-1)(tx+u)}{m^2(t+u)+u(x-1)(tx+u)}\right)\right)}{(t+u)^2} \\
&+ \frac{4(1-2x)^2\left(m^2(t+u)+u(x-1)(tx+u)\right)\left(\frac{m^2t-u^2(x-1)^2\ln\left(m^2+u(x-1)x\right)+m^2u+tux^2-tux+u^2x+u^2(x-1)^2\ln(u-ux)-u^2}{\left(m^2(t+u)+u(x-1)(tx+u)\right)^2}-\frac{\ln\left(m^2\right)}{(tx+u)^2}+\frac{u(x-1)\left(\frac{m^2(t+u)}{tx+u}+u(x-1)\ln\left(m^2\right)-u(x-1)\ln(tx+u)+u(x-1)\right)}{\left(m^2(t+u)+u(x-1)(tx+u)\right)^2}\right)}{(t+u)^2} \\
&+ \frac{4(1-2x)^2\left(\frac{\ln\left(m^2+u(x-1)x\right)}{(u-ux)^2}\right)}{(t+u)^2} \qquad \text{(E.1)}
\end{aligned}
$$

Appendix F

Integral Over Penultimate Variable for $\gamma_{(1234)}$

In this appendix, we show the result of the integration over the penultimate parameter for $\gamma_{(1234)}$:

$$
\begin{aligned}
& \frac{8(1-2x)x}{s(x-1)(s+u)} + \frac{(2x-1)\left(4m^2 s(x-1)x+4m^4+s^2(x-1)^2(2x-1)\right)\left(2m^2(s+u)+s(x-1)(s+2ux)\right)\left(\log\left(m^2+s(x-1)x\right)-\log\left(m^2\right)\right)}{s^2(x-1)^2\left(m^2(s+u)+su(x-1)x\right)\left(m^2(s+u)+s(x-1)(s+ux)\right)} \\
+\quad & \frac{(2x-1)\left(s^2(1-2x)+2su\left(2x^2-2x+1\right)+u^2(1-2x)\right)\log\left(-\frac{s}{u}\right)}{(s+u)^2\left(m^2(s+u)+su(x-1)x\right)} + \frac{(2x-1)\left(s^2(2x-3)+2su\left(2x^2-4x+1\right)+u^2(1-2x)\right)\log\left(\frac{s-sx}{s+ux}\right)}{(s+u)^2\left(m^2(s+u)+s(x-1)(s+ux)\right)} + \frac{(1-2x)^2(s(2x-1)+2tx)^2(\log(s+t)-\log(-t))}{s^2\left(m^2 s+t(x-1)x(s+t)\right)} \\
+\quad & \frac{(1-2x)^2(s(2x-1)+2t(x-1))^2\log\left(\frac{t(x-1)}{sx+t(x-1)}\right)}{s^2\left(m^2 s+t(x-1)(sx+t(x-1))\right)} + \frac{(1-2x)^2\left(2m^2+t(x-1)\right)^2\left(\log\left(m^2\right)-\log\left(m^2+t(x-1)x\right)\right)}{(x-1)\left(m^2 s+t(x-1)x(s+t)\right)\left(m^2 s+t(x-1)(sx+t(x-1))\right)} + \frac{8x(2x-1)}{u(x-1)(t+u)} \\
+\quad & \frac{(2x-1)\left(t^2(1-2x)+2tu\left(2x^2-2x+1\right)+u^2(1-2x)\right)\log\left(\frac{t}{u}\right)}{(t+u)^2\left(m^2(t+u)+tu(x-1)x\right)} + \frac{(2x-1)\left(t^2(1-2x)+2tu\left(2x^2-4x+1\right)+u^2(2x-3)\right)(\log(tx+u)-\log(u-ux))}{(t+u)^2\left(m^2(t+u)+u(x-1)(tx+u)\right)} \\
+\quad & \frac{(2x-1)\left(4m^2u(x-1)x+4m^4+u^2(x-1)^2(2x-1)\right)\left(2m^2(t+u)+u(x-1)(2tx+u)\right)\left(\log\left(m^2\right)-\log\left(m^2+u(x-1)x\right)\right)}{u^2(x-1)^2\left(m^2(t+u)+tu(x-1)x\right)\left(m^2(t+u)+u(x-1)(tx+u)\right)} + \frac{32s(x-1)x(s-2u)\log\left(-\frac{s}{u}\right)}{(s+u)^4} \\
-\quad & \frac{32s(x-1)(s(x-3)-2ux)\log\left(\frac{s-sx}{s+ux}\right)}{(s+u)^4} - \frac{4x(s(x-2)-ux)\left(s^2(2x-3)+2su\left(2x^2-4x+1\right)+u^2(1-2x)\right)}{s(x-1)(s+u)^3(s+ux)} + \frac{8x\left(s^2(6x-11)+2su\left(4x^2-10x+1\right)+u^2\left(-4x^2-2x+1\right)\right)}{(s+u)^3(s+ux)} \\
-\quad & \frac{8(x-1)\left(s^2(1-2x)+2su\left(4x^2-2x+1\right)+u^2\left(-4x^2-2x+1\right)\right)}{ux(s+u)^3} + \frac{4(x-1)(s-u)\left(s^2(1-2x)+2su\left(2x^2-2x+1\right)+u^2(1-2x)\right)}{sux(s+u)^3} + \frac{8\left(2x^2-3x+1\right)(s(4x-1)+4tx)}{s^2x(s+t)} \\
+\quad & \frac{16t\left(2x^2-3x+1\right)(\log(-t)-\log(s+t))}{s^3} + \frac{4x(2x-1)\left(s^2x(2x-1)+2st\left(3x^2-4x+1\right)+4t^2(x-1)^2\right)}{s^2t(x-1)(sx+t(x-1))} - \frac{8x(2x-1)(s(4x-1)+4t(x-1))}{s^2(sx+t(x-1))} - \frac{16t\left(2x^2-3x+1\right)\log\left(\frac{t(x-1)}{sx+t(x-1)}\right)}{s^3} \\
-\quad & \frac{16u\left(2x^2-3x+1\right)(\log(tx)-\log(ux))}{(t+u)^3} + \frac{16u\left(2x^2-3x+1\right)(\log(tx+u)-\log(u-ux))}{(t+u)^3} - \frac{8\left(2x^2-3x+1\right)(2tx+t-2ux+u)}{tx(t+u)^2} + \frac{8x(2x-1)(2tx+t+u(5-2x))}{(t+u)^2(tx+u)} \\
+\quad & \frac{4x(2x-1)\left(t^2x+2tu\left(-x^2+x+1\right)+u^2\left(2x^2-7x+6\right)\right)}{u(x-1)(t+u)^2(tx+u)} - \frac{16\left(-\left(m^2+s(x-1)x\right)\log\left(m^2+s(x-1)x\right)+m^2\log\left(m^2\right)+s(x-1)x\right)}{s(x-1)(s+u)^2} \\
+\quad & \frac{16\left(-2m^2(s+u)+s^2\left(x^2-6x+6\right)+2sux(3-2x)+u^2x^2\right)\left(-\mathrm{Li}_2\left(\frac{m^2(s+u)}{(s+u)m^2+s(s+ux)(x-1)}\right)+\mathrm{Li}_2\left(\frac{\left(m^2+s(x-1)x\right)(s+u)}{(s+u)m^2+s(s+ux)(x-1)}\right)+\log\left(m^2+s(x-1)x\right)\log\left(-\frac{s^2(x-1)^2}{m^2(s+u)+s(x-1)(s+ux)}\right)\right)}{(s+u)^4} \\
+\quad & \frac{16\left(-2m^2(s+u)+s^2\left(x^2-6x+6\right)+2sux(3-2x)+u^2x^2\right)\left(-\log\left(m^2\right)\log\left(\frac{s(x-1)(s+ux)}{m^2(s+u)+s(x-1)(s+ux)}\right)\right)}{(s+u)^4} \\
+\quad & \frac{16\left(2m^2(s+u)-(x-1)x\left(s^2-4su+u^2\right)\right)\left(-\mathrm{Li}_2\left(\frac{m^2(s+u)}{(s+u)m^2+su(x-1)x}\right)+\mathrm{Li}_2\left(\frac{\left(m^2+s(x-1)x\right)(s+u)}{(s+u)m^2+su(x-1)x}\right)+\log\left(m^2+s(x-1)x\right)\log\left(-\frac{s^2(x-1)x}{m^2(s+u)+su(x-1)x}\right)\right)}{(s+u)^4} \\
+\quad & \frac{16\left(2m^2(s+u)-(x-1)x\left(s^2-4su+u^2\right)\right)\left(-\log\left(m^2\right)\log\left(\frac{su(x-1)x}{m^2(s+u)+su(x-1)x}\right)\right)}{(s+u)^4}
\end{aligned}
$$

$$- \frac{4(s-u)\left(-2s\left(4m^2x+u\left(6x^3-8x^2+3x-1\right)\right)+u\left(-8m^2x+2ux^2-3ux+u\right)+s^2\left(2x^2-3x+1\right)\right)}{sux(s+u)^4\left(m^2(s+u)+su(x-1)x\right)}\left(u\left((s+u)\left(m^2+s(x-1)x\right)\log\left(m^2+s(x-1)x\right)\right.\right.$$

$$\left.\left. + s^2(x-1)x\log(u(-x))-s^2(x-1)x\log(sx)\right)+m^2s\log\left(m^2\right)(s+u)\right)$$

$$+ \frac{4(s(x-2)-ux)\left(8m^2(s+u)+s^2(10x-11)+2su\left(6x^2-8x+1\right)+u^2(1-2x)\right)\left(-\frac{\log\left(m^2\right)}{s+ux}+\frac{s(x-1)\log\left(\frac{m^2}{s+ux}\right)}{m^2(s+u)+s(x-1)(s+ux)}+\frac{s(x-1)\left(\log(s-sx)-\log\left(m^2+s(x-1)x\right)\right)}{m^2(s+u)+s(x-1)(s+ux)}+\frac{\log\left(m^2+s(x-1)x\right)}{s-sx}\right)}{(s+u)^4}$$

$$+ \frac{4\left(s^2(2x-3)+2su\left(2x^2-4x+1\right)+u^2(1-2x)\right)\left(m^2(s+u)+s(x-1)(s+ux)\right)\left(-\frac{-s^2(x-1)^2\log\left(m^2+s(x-1)x\right)+m^2s+m^2u+s^2x+s^2(x-1)^2\log(s-sx)-s^2+sux^2-sux}{\left(m^2(s+u)+s(x-1)(s+ux)\right)^2}-\frac{\log\left(m^2+s(x-1)x\right)}{s^2(x-1)^2}\right)}{(s+u)^4}$$

$$+ \frac{4\left(s^2(2x-3)+2su\left(2x^2-4x+1\right)+u^2(1-2x)\right)\left(m^2(s+u)+s(x-1)(s+ux)\right)\left(\frac{\log\left(m^2\right)}{(s+ux)^2}-\frac{s(x-1)\left(\frac{m^2(s+u)}{s+ux}+s(x-1)\log\left(m^2\right)-s(x-1)\log(s+ux)+s(x-1)\right)}{\left(m^2(s+u)+s(x-1)(s+ux)\right)^2}\right)}{(s+u)^4}$$

$$+ \frac{4\left(s^2(2x-1)-2su\left(2x^2-2x+1\right)+u^2(2x-1)\right)\left(m^2(s+u)+su(x-1)x\right)\left(-\frac{\log\left(m^2+s(x-1)x\right)}{s^2x^2}+\frac{s(x-1)\left(-\frac{m^2(s+u)}{sx}+s(x-1)\log\left(m^2+s(x-1)x\right)-s(x-1)\log(-sx)-ux+u\right)}{\left(m^2(s+u)+su(x-1)x\right)^2}\right)}{(s+u)^4}$$

$$+ \frac{4\left(s^2(2x-1)-2su\left(2x^2-2x+1\right)+u^2(2x-1)\right)\left(m^2(s+u)+su(x-1)x\right)\left(-\frac{s(x-1)\left(\frac{m^2(s+u)}{ux}+s(x-1)\log\left(m^2\right)-s(x-1)\log(ux)+s(x-1)\right)}{\left(m^2(s+u)+su(x-1)x\right)^2}+\frac{\log\left(m^2\right)}{u^2x^2}\right)}{(s+u)^4}$$

$$+ \frac{4(s+t)(2x-1)(s(2x-1)+2tx)\left(-st^2(x-1)x\left(m^2s+t(x-1)x(s+t)\right)+s(s+t)\left(m^2+t(x-1)x\right)\left(m^2s+t(x-1)x(s+2t)\right)\log\left(m^2+t(x-1)x\right)\right)}{s^3t^2x^2(s+t)^2\left(m^2s+t(x-1)x(s+t)\right)}$$

$$+ \frac{4(s+t)(2x-1)(s(2x-1)+2tx)\left(t^4(x-1)^2x^2(s+t)\log(-tx)-t^4(x-1)^2x^2(s+t)\log(x(s+t))\right)}{s^3t^2x^2(s+t)^2\left(m^2s+t(x-1)x(s+t)\right)}+\frac{4(2x-1)(s(2x-1)+2tx)\left(-m^2st^2\log\left(m^2\right)\left(m^2s+2t(x-1)x(s+t)\right)\right)}{s^3t^2x^2(s+t)^2\left(m^2s+t(x-1)x(s+t)\right)}$$

$$- \frac{4(2x-1)\left(4m^2s+s^2(1-2x)^2+st\left(14x^2-13x+2\right)+2t^2x(5x-4)\right)\left((s+t)\left(s\left(m^2+t(x-1)x\right)\log\left(m^2+t(x-1)x\right)-t^2(x-1)x\log(x(s+t))+t^2(x-1)x\log(-tx)\right)-m^2st\log\left(m^2\right)\right)}{s^3tx(s+t)\left(m^2s+t(x-1)x(s+t)\right)}$$

$$+ \frac{4(1-2x)^2(3s+4t)\left(\mathrm{Li}_2\left(\frac{m^2s}{sm^2+t(s+t)(x-1)x}\right)-\mathrm{Li}_2\left(\frac{s\left(m^2+t(x-1)x\right)}{sm^2+t(s+t)(x-1)x}\right)-\log\left(m^2+t(x-1)x\right)\log\left(-\frac{t^2(x-1)x}{m^2s+t(x-1)x(s+t)}\right)+i\pi\log\left(m^2+\frac{t(x-1)x(s+t)}{s}\right)\right)}{s^3}$$

$$+ \frac{4(1-2x)^2(3s+4t)\left(\log\left(m^2\right)\log\left(\frac{t(x-1)x(s+t)}{m^2s+t(x-1)x(s+t)}\right)-i\pi\log\left(m^2+t(x-1)x\right)\right)}{s^3}$$

$$+ \frac{4(2x-1)(s(2x-1)+2t(x-1))\left(m^2s+t(x-1)(sx+t(x-1))\right)\left(\frac{\log\left(m^2\right)}{(sx+t(x-1))^2}+\frac{t(x-1)\left(\frac{m^2s}{t(x-1)}+t(x-1)\log\left(m^2+t(x-1)x\right)+sx+t(x-1)-t(x-1)\log(t-tx)\right)}{\left(m^2s+t(x-1)(sx+t(x-1))\right)^2}\right)}{s^3}$$

$$+ \frac{4(2x-1)(s(2x-1)+2t(x-1))\left(m^2s+t(x-1)(sx+t(x-1))\right)\left(-\frac{t(x-1)\left(\frac{m^2s}{sx+t(x-1)}+t(x-1)\log\left(m^2\right)-t(x-1)\log(-sx+t(-x)+t)+t(x-1)\right)}{\left(m^2s+t(x-1)(sx+t(x-1))\right)^2}-\frac{\log\left(m^2+t(x-1)x\right)}{t^2(x-1)^2}\right)}{s^3}$$

$$+ \frac{4(2x-1)\left(4m^2s+s^2(1-2x)^2+st\left(14x^2-19x+5\right)+10t^2(x-1)^2\right)\left(\frac{m^2s\log\left(m^2\right)}{-sx+t(-x)+t}+\frac{s\left(m^2+t(x-1)x\right)\log\left(m^2+t(x-1)x\right)}{t(x-1)}-t(x-1)\log(-sx+t(-x)+t)+t(x-1)\log(t-tx)\right)}{s^3\left(m^2s+t(x-1)(sx+t(x-1))\right)}$$

$$+ \frac{4(2x-1)(s(6x-3)+8t(x-1))\left(-\mathrm{Li}_2\left(\frac{m^2s}{sm^2+t(x-1)(t(x-1)+sx)}\right)+\mathrm{Li}_2\left(\frac{s\left(m^2+t(x-1)x\right)}{sm^2+t(x-1)(t(x-1)+sx)}\right)+\log\left(m^2+t(x-1)x\right)\log\left(\frac{t^2(x-1)^2}{m^2s+t(x-1)(sx+t(x-1))}\right)-\log\left(m^2\right)\log\left(\frac{t(x-1)(sx+t(x-1))}{m^2s+t(x-1)(sx+t(x-1))}\right)\right)}{s^3}$$

$$+ \frac{4(1-2x)^2(t-3u)\left(-\mathrm{Li}_2\left(\frac{m^2(t+u)}{(t+u)m^2+tu(x-1)x}\right)+\mathrm{Li}_2\left(\frac{(t+u)\left(m^2+u(x-1)x\right)}{(t+u)m^2+tu(x-1)x}\right)+\log\left(m^2+u(x-1)x\right)\log\left(\frac{u^2(x-1)x}{m^2(t+u)+tu(x-1)x}\right)-i\pi\log\left(m^2+\frac{tu(x-1)x}{t+u}\right)\right)}{(t+u)^3}$$

$$+ \frac{4(1-2x)^2(t-3u)\left(-\log\left(m^2\right)\log\left(\frac{tu(x-1)x}{m^2(t+u)+tu(x-1)x}\right)+i\pi\log\left(m^2+u(x-1)x\right)\right)}{(t+u)^3}$$

$$+ \frac{4(2x-1)(2tx+t+u(9-6x))\left(\mathrm{Li}_2\left(\frac{m^2(t+u)}{(t+u)m^2+u(x-1)(u+tx)}\right)-\mathrm{Li}_2\left(\frac{(t+u)\left(m^2+u(x-1)x\right)}{(t+u)m^2+u(x-1)(u+tx)}\right)-\log\left(m^2+u(x-1)x\right)\log\left(-\frac{u^2(x-1)^2}{m^2(t+u)+u(x-1)(tx+u)}\right)+\log\left(m^2\right)\log\left(\frac{u(x-1)(tx+u)}{m^2(t+u)+u(x-1)(tx+u)}\right)\right)}{(t+u)^3}$$

$$+ \frac{4(2x-1)\left(4m^2(t+u)-t^2x+tu\left(6x^2-5x-3\right)+u^2\left(-4x^2+16x-13\right)\right)\left(-\frac{\log\left(m^2\right)}{tx+u}+\frac{u(x-1)\left(\log\left(m^2\right)-\log(tx+u)\right)}{m^2(t+u)+u(x-1)(tx+u)}+\frac{u(x-1)\left(\log(u-ux)-\log\left(m^2+u(x-1)x\right)\right)}{m^2(t+u)+u(x-1)(tx+u)}+\frac{\log\left(m^2+u(x-1)x\right)}{u-ux}\right)}{(t+u)^3}$$

$$+ \frac{4(2x-1)\left(4m^2(t+u)-t^2(x-1)+tux(6x-5)-u^2(1-2x)^2\right)\left(t\left(u^2(x-1)x\log\left(-\frac{tx}{m^2}\right)+(t+u)\left(m^2+u(x-1)x\right)\log\left(m^2+u(x-1)x\right)-u^2(x-1)x\log(-ux)\right)+u\log\left(m^2\right)\left(m^2(t+u)+tu(x-1)x\right)\right)}{tux(t+u)^3\left(m^2(t+u)+tu(x-1)x\right)}$$

$$
\begin{aligned}
&+ \frac{4(2x-1)(t-2ux+u)t\left((t+u)\left(u^2(x-1)x\left(m^2(t+u)+tu(x-1)x\right)+t\left(m^2+u(x-1)x\right)\left(m^2(t+u)+u(x-1)x(t-u)\right)\log\left(m^2+u(x-1)x\right)\right)\right)}{t^2u^2x^2(t+u)^3\left(m^2(t+u)+tu(x-1)x\right)} \\
&+ \frac{4(2x-1)(t-2ux+u)t\left((t+u)\left(-tu^4(x-1)^2x^2\log(-tx)+tu^4(x-1)^2x^2\log(-ux)\right)\right)}{t^2u^2x^2(t+u)^3\left(m^2(t+u)+tu(x-1)x\right)} + \frac{4(2x-1)(t-2ux+u)\left(-m^2u^2\log\left(m^2\right)(t+u)\left(m^2(t+u)+2tu(x-1)x\right)\right)}{t^2u^2x^2(t+u)^3\left(m^2(t+u)+tu(x-1)x\right)} \\
&+ \frac{4(2x-1)(t+u(3-2x))\left(m^2(t+u)+u(x-1)(tx+u)\right)\left(\frac{\log\left(m^2\right)}{(tx+u)^2}+\frac{u(x-1)\left(\frac{m^2(t+u)+u(x-1)(tx+u)}{u-ux}+u(x-1)\log\left(m^2+u(x-1)x\right)-u(x-1)\log(u-ux)\right)}{\left(m^2(t+u)+u(x-1)(tx+u)\right)^2}\right)}{(t+u)^3} \\
&+ \frac{4(2x-1)(t+u(3-2x))\left(m^2(t+u)+u(x-1)(tx+u)\right)\left(-\frac{u(x-1)\left(\frac{m^2(t+u)}{tx+u}+u(x-1)\log\left(m^2\right)-u(x-1)\log(tx+u)+u(x-1)\right)}{\left(m^2(t+u)+u(x-1)(tx+u)\right)^2}-\frac{\log\left(m^2+u(x-1)x\right)}{(u-ux)^2}\right)}{(t+u)^3}
\end{aligned}
\tag{F.1}
$$

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